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Existence and multiplicity of positive solutions for Kirchhoff type equations at the critical frequency

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Abstract. We consider the following Kirchhoff equation with critical Sobolev exponent

$$\begin{cases} \left(a+b\int_{\mathbb{R}^N}|(-\Delta)^{\frac{s}{2}}u|^2dx\right)(-\Delta)^s u=|u|^{2_s^*-2}u+\mu g(x), & x\in\mathbb{R}^N,\\ u\in D^{s,2}(\mathbb{R}^N), \end{cases} \tag{K}$$

where $0<s\le 1$, $N>2s$, $a>0$, $b\in\mathbb{R}\backslash\{0\}$, μ is a non-negative parameter, $g\in L^{\frac{2N}{N+2s}}(\mathbb{R}^N)$ is a positive function, $2_s^*=\frac{2N}{N-2s}$ is the critical Sobolev exponent. We show that (K) has at least two positive solutions by the variational method. Particularly, in contrast to most existing results in the literature, our method can be applied to deal with wide ranges of s and N.

Keywords: nonlocal operator, positive solutions, critical exponent, variational methods.

2020 Mathematics Subject Classification: 35A15, 35J60, 35B09.

1 Introduction

Consider the existence of positive solutions for the following Kirchhoff equation

$$\begin{cases} \left(a+b\int_{\mathbb{R}^N}|(-\Delta)^{\frac{s}{2}}u|^2dx\right)(-\Delta)^s u=|u|^{2_s^*-2}u+\mu g(x), & x\in\mathbb{R}^N,\\ u\in D^{s,2}(\mathbb{R}^N), \end{cases} \tag{1.1}$$

where $0<s\le 1$, $N>2s$, $a>0$, $b\in\mathbb{R}\backslash\{0\}$, μ is a non-negative parameter, $g\in L^{\frac{2N}{N+2s}}(\mathbb{R}^N)$ is a positive function, $2_s^*=\frac{2N}{N-2s}$ is the critical Sobolev exponent and $(-\Delta)^s$ is the fractional Laplacian.

If $s=1$, (1.1) reduces to the following Kirchhoff equation

$$-\left(a+b\int_{\mathbb{R}^N}|\nabla u|^2dx\right)\Delta u=f(u),\qquad x\in\mathbb{R}^N, \tag{1.2}$$

[✉] Corresponding author. Email: hejuhua25@163.com where $f(u)=|u|^{2_s^*-2}u+\mu g(x)$. The Kirchhoff type equations are derived as models of several physical and biological phenomena. Physically, it is related to the stationary analogue equation proposed by Kirchhoff in [6] to describe the transversal oscillations of the stretched string. Moreover, models such as (1.2) can be used to describe the growth and movement of a particular species. In these models, both growth and movement are assumed to be dependent on the total population density within a domain or the energy of the entire population. Mathematically, this dependence is encoded in an integral term, where u represents the population density [13].

Recently, the Kirchhoff type problem has been studied by many researchers. In [10], Liu, Liao and Tang studied the nonlinear Kirchhoff-type equation

$$\begin{cases} -\left(a+b\int_{\mathbb{R}^N}|\nabla u|^2dx\right)\Delta u=|u|^{2^*-2}u+\mu h(x), & x\in\mathbb{R}^N, \\ u\in D^{1,2}(\mathbb{R}^N). \end{cases} \tag{1.3}$$

Under some assumptions on a,b,μ, they obtained the existence of positive solutions of (1.3) with $N\geq 3$ by the variational method. In [9], by establishing a local compactness splitting lemma of critical version, the authors obtained the existence of positive ground state solutions for Kirchhoff type elliptic problem with critical growth in $\mathbb{R}^4$. Zhang [24] obtained the existence of nontrivial solutions for Kirchhoff type problem with critical growth for $N\geq 3$.

Some authors also studied Kirchhoff type problems with negative modulus of the form

$$-\left(a-b\int_{\Omega}|\nabla u|^2dx\right)\Delta u=f(x,u), \tag{1.4}$$

where $a,b>0$. In [15], the authors proved that (1.4) has at least two positive solutions in the case $g\in L^{\frac{4}{3}}(\mathbb{R}^N)$ with $\mu\in(0,\mu_*]$ and has infinitely many positive solutions with $\mu=0$, where $f(x,u)=|u|^2u+\mu g(x)$ and $N=4$. In [23], by the variational method, Yin and Liu proved that (1.4) has at least a nontrivial non-negative solution and a nontrivial non-positive solution if $f(x,u)=|u|^{p-2}u$, $p\in(2,2^*)$. For the same nonlinearity $f(x,u)=|u|^{p-2}u$ but $p\in[2,2^*)$, by using combination of invariant sets of descent flow and the Ljusternik–Schnirelman type minimax method, Wang and Yang [17] obtained infinitely many sign-changing solutions of (1.4). For more information about Kirchhoff type problems with negative modulus, we refer the readers to [8,16,18].

If $0<s<1$, (1.1) is a bi-nonlocal problem. Especially for the fractional Kirchhoff equation with critical growth, because of the interaction by both the Kirchhoff nonlocal term $\int_{\mathbb{R}^N}|(-\Delta)^{\frac{s}{2}}u|^2dx$ and the critical term, it is difficult to derive the geometric structure of the functional and the boundedness, convergence of the Palais–Smale sequences if the critical exponent $2_s^*<4$. By contrast, there are many results on the existence of solutions for fractional Kirchhoff problems involving critical growth with $\frac{N}{4}<s<1$. Precisely, Gu, Tang and Zhang [4] obtained the existence of ground states for asymptotically periodic fractional Kirchhoff equation with critical Sobolev exponent in $\mathbb{R}^3$ and $s\in(\frac{3}{4},1)$. Still in $\mathbb{R}^3$, Gu, Tang and Yang [3] studied the fractional Kirchhoff equation with potential vanishing at infinity

$$\left(a+b\int_{\mathbb{R}^3}|(-\Delta)^{\frac{s}{2}}u|^2dx\right)(-\Delta)^su+V(x)u=Q(x)f(u)+|u|^{2_s^*-2}u, \quad x\in\mathbb{R}^3. \tag{1.5}$$

Under appropriate assumptions on f, they showed the existence of positive solutions for (1.5) by using the variational method.

In [12], the authors consider the following equation

$$\begin{cases} -\left(a+b\int_{\mathbb{R}^N}|(-\Delta)^{\frac{s}{2}}u|^2dx\right)(-\Delta)^s u+V(x)u=f(u), & \text{in } \mathbb{R}^N, \\ u\in H^s(\mathbb{R}^N), \quad u>0 \quad \text{in } \mathbb{R}^N. \end{cases} \tag{1.6}$$

By using the monotonicity trick and the profile decomposition, they proved that (1.6) has a positive ground state solution in the cases of $N=3$, $s\in\left(\frac{3}{4},1\right)$ and $N=2$, $s\in\left(\frac{1}{2},1\right)$, where the nonlinearity f does not satisfy the Ambrosetti–Rabinowitz type condition. Under more general conditions of nonlinearity, by establishing the Pohozǎev identity, the authors [25] proved that the critical Kirchhoff type fractional Schrödinger equation has ground state solutions with $s\in\left[\frac{3}{4},1\right)$, and has non-existence results with $s\in\left(0,\frac{3}{4}\right]$. In [7], by establishing equivalent results, Kong, Zhu and Deng concluded the nonexistence, existence and multiplicity of normalized solutions for the fractional Kirchhoff equation with $s\in\left(0,\frac{3}{4}\right]$. We refer readers to [11,14,20–22] for more studies of Kirchhoff equations.

Motivated by the work mentioned above, our aim of this paper is to investigate the existence of positive solutions for (1.1). Different from the classical fractional Kirchhoff equation (where $b>0$), we consider both cases of $b>0$ and $b<0$, which makes our research more interesting. Moreover, in the study of nonlocal Kirchhoff equations with critical growth, the estimate of critical values plays a crucial role. Different from [15], since the exponent s and the dimension N in our problem are not fixed, some techniques need to be employed in the process of proving compactness when $b<0$.

Our main result is the following theorem.

Theorem 1.1. *Assume that $a>0$, $\mu>0$ and $g\in L^{\frac{2N}{N+2s}}(\mathbb{R}^N)$ is a positive function. Then we have*

(i) if $b>0$ and $\frac{N}{4}<s\leq 1$, there exists a $\mu_*>0$ such that (1.1) has at least two positive solutions for any $\mu\in(0,\mu_*]$.

(ii) if $b<0$ and $N\geq 4$, $s\in(0,1)$ or $N=3$, $s\in\left(0,\frac{3}{4}\right)$, there exists a $\overline{\mu}_*>0$ such that (1.1) has at least two positive solutions for any $\mu\in(0,\overline{\mu}_*]$.

Remark 1.2. Due to the influence of both the Kirchhoff term and the critical term on equation (1.1), it is necessary to impose corresponding constraint conditions on the exponent s and the dimension N when verifying the mountain pass geometric structure and the compactness.

Remark 1.3. For the case $b>0$ and $s=1$, when $N=4$, it is clear that $2_s^*=4$. To our best knowledge, in this case, it is difficult to verify the functional satisfies the mountain pass geometry and compactness conditions, unless the constraint $b<\frac{1}{S^2}$ is imposed. The detailed arguments can be found in [10]. In our paper, under the same constraint $b<\frac{1}{S^2}$, we can also obtain the existence of two positive solutions for equation (1.1) with $2_s^*=4$. However, different from [10], we establish a uniform energy threshold, and prove that the functional satisfies the $(PS)_c$ condition. By combining the Ekeland variational principle with the mountain pass theorem, we obtain the existence of two positive solutions for equation (1.1).

In particular, for the case $N=3$ and $s=1$, our method is simpler than that proposed in [10].

Remark 1.4. When $N=4$, $s\in(0,1)$ or $N\geq 5$, $s\in(0,1]$, it is difficult to find a critical threshold to prove compactness in the case $b>0$.

Remark 1.5. For the case $b < 0$, in contrast to [15], our method does not depend on the specific properties of the exponents. More precisely, although the specific expressions for $\overline{\eta}$ and $\overline{l}$ (two constants we give in the proof of Lemma 2.7) cannot be determined because the exponent s and the dimension N are not fixed, we can still obtain the compactness result by using the relationship between these parameters. This enables our approach to handle more general cases. The detailed arguments are provided in Lemma 2.7.

2 Preliminaries and main lemmas

For convenience, we always assume that $g \in L^{\frac{2N}{N+2s}}(\mathbb{R}^N)$ throughout this paper.

Define the homogeneous fractional Sobolev space $D^{s,2}(\mathbb{R}^N)$ as the completion of the space $C_c^\infty(\mathbb{R}^N)$ with respect to the norm

$$\|u\| := \left(\int\int_{\mathbb{R}^N \times \mathbb{R}^N} \frac{|u(x)-u(y)|^2}{|x-y|^{N+2s}} dxdy \right)^{\frac{1}{2}}$$

for any $s \in (0,1)$.

For the fractional Laplacian, we see from [1] that

$$\|(-\Delta)^{\frac{s}{2}} u\|_2^2 = \frac{1}{2} C(s) \int\int_{\mathbb{R}^N \times \mathbb{R}^N} \frac{|u(x)-u(y)|^2}{|x-y|^{N+2s}} dxdy,$$

where

$$C(s) = \left(\int_{\mathbb{R}^N} \frac{1-\cos \xi_1}{|\xi|^{N+2s}} d\xi \right)^{-1}, \quad \xi = (\xi_1, \xi_2, \ldots, \xi_N).$$

Without loss of generality, we assume throughout this paper that $\frac{1}{2}C(s) = 1$.

Consider the functional defined on $D^{s,2}(\mathbb{R}^N)$:

$$I(u) = \frac{a}{2}\|u\|^2 + \frac{b}{4}\|u\|^4 - \frac{1}{2_s^*} \int_{\mathbb{R}^N} (u^+)^{2_s^*} - \mu \int_{\mathbb{R}^N} gu,$$

where $u^+ = \max\{0, u\}$. Since $g \in L^{\frac{2N}{N+2s}}(\mathbb{R}^N)$, it is clear that $I \in C^1(D^{s,2}(\mathbb{R}^N), \mathbb{R})$, and

$$\langle I'(u), v \rangle = (a + b\|u\|^2) \int_{\mathbb{R}^N} (-\Delta)^{\frac{s}{2}} u (-\Delta)^{\frac{s}{2}} v - \int_{\mathbb{R}^N} (u^+)^{2_s^*-1} v - \mu \int_{\mathbb{R}^N} gv$$

for any $u, v \in D^{s,2}(\mathbb{R}^N)$. Clearly, solutions of (1.1) are critical points of the functional I.

Define the mountain pass level c of I by

$$c = \inf_{\gamma \in \Gamma} \max_{t \in [0,1]} I(\gamma(t)), \tag{2.1}$$

where $\Gamma = \{\gamma \in C([0,1], D^{s,2}(\mathbb{R}^N)) : \gamma(0) = 0 \text{ and } I(\gamma(1)) < 0\}$.

In order to obtain nontrivial solutions of (1.1), We define the best constant S for the Sobolev embedding $D^{s,2}(\mathbb{R}^N) \hookrightarrow L^{2_s^*}(\mathbb{R}^N)$ by

$$S := \inf_{u \in D^{s,2}(\mathbb{R}^N) \setminus \{0\}} \frac{\|u\|^2}{\|u\|_{2_s^*}^2} > 0. \tag{2.2}$$

It is known that the best constant S is attained by the function

$$U_\varepsilon(x) = C_s \left(\frac{\varepsilon}{\varepsilon^2 + |x-y|^2} \right)^{\frac{N-2s}{2}}, \quad y \in \mathbb{R}^N, \ \varepsilon > 0, \tag{2.3}$$

(see [5]). Moreover, $\|U_\varepsilon\|^2 = \|U_\varepsilon\|_{2_s^*}^{2_s^*} = S^{\frac{N}{2s}}$ and U_ε satisfies the equation $(-\Delta)^s u = u^{2_s^*-1}$ in $\mathbb{R}^N$.

2.1 The case $b > 0$

The following lemma shows that the functional I has the mountain pass geometric structure.

Lemma 2.1. *Assume that $a > 0$, $b > 0$, $\frac{N}{4} < s \leq 1$ and $g \in L^{\frac{2N}{N+2s}}(\mathbb{R}^N)$ is a positive function. Then there exist $\rho, \alpha, \mu_1 > 0$ such that, for any $\mu \in (0, \mu_1]$*

(i) $\inf\limits_{\|u\|=\rho} I(u) \geq \alpha.$

(ii) $\inf\limits_{\|u\|<\rho} I(u) < 0.$

(iii) There exists $e \in D^{s,2}(\mathbb{R}^N)$ such that $\|e\| > \rho$ and $I(e) < 0$.

Proof. (i) By the Sobolev inequality and Hölder inequality, for any $u \in D^{s,2}(\mathbb{R}^N)$, we have

$$\begin{aligned} I(u) &= \frac{a}{2}\|u\|^2 + \frac{b}{4}\|u\|^4 - \frac{1}{2_s^*}\int_{\mathbb{R}^N}(u^+)^{2_s^*} - \mu\int_{\mathbb{R}^N} gu \\ &\geq \frac{a}{2}\|u\|^2 + \frac{b}{4}\|u\|^4 - \frac{1}{2_s^* S^{\frac{2_s^*}{2}}}\|u\|^{2_s^*} - \frac{\mu}{S^{\frac{1}{2}}}\|g\|_{\frac{2N}{N+2s}}\|u\| \\ &= \|u\|\left(\frac{a}{2}\|u\| + \frac{b}{4}\|u\|^3 - \frac{1}{2_s^* S^{\frac{2_s^*}{2}}}\|u\|^{2_s^*-1} - \frac{\mu}{S^{\frac{1}{2}}}\|g\|_{\frac{2N}{N+2s}}\right). \end{aligned}$$

Let $J(t) = \frac{a}{2}t + \frac{b}{4}t^3 - \frac{1}{2_s^* S^{\frac{2_s^*}{2}}}t^{2_s^*-1}$ for any $t \geq 0$. It is clear that $J(0) = 0$. Note that $2_s^* > 4$ when $\frac{N}{4} < s \leq 1$, then $J(t) > 0$ for small $t > 0$ and $J(t) < 0$ for large $t > 0$. Hence, there exists a $\rho > 0$ such that $\max_{t\geq 0} J(t) = J(\rho) > 0$. Set $\mu_1 = \frac{J(\rho)\, S^{\frac{1}{2}}}{2\,\|g\|_{\frac{2N}{N+2s}}}$. Then for any $\mu \in (0, \mu_1]$ with $\|u\| = \rho$, we have

$$\begin{aligned} I(u) &\geq \|u\|\left(\frac{a}{2}\|u\| + \frac{b}{4}\|u\|^3 - \frac{1}{2_s^* S^{\frac{2_s^*}{2}}}\|u\|^{2_s^*-1} - \frac{\mu}{S^{\frac{1}{2}}}\|g\|_{\frac{2N}{N+2s}}\right) \\ &= \frac{1}{2}\rho J(\rho). \end{aligned}$$

Hence, there exist $\rho, \alpha > 0$ such that $\inf_{\|u\|=\rho} I(u) \geq \alpha$.

(ii) Choose $u_0 \in D^{s,2}(\mathbb{R}^N)$ with $\|u_0\| = \rho$ and $\int_{\mathbb{R}^N} gu_0 > 0$, then

$$\lim_{t\to 0^+}\frac{I(tu_0)}{t} = -\mu\int_{\mathbb{R}^N} gu_0 < 0.$$

Hence, there exists a $u \in D^{s,2}(\mathbb{R}^N)$ such that $\|u\| < \rho$ and $I(u) < 0$. Therefore, $\inf_{\|u\|<\rho} I(u) < 0$.

(iii) Since

$$I(tu_0) = \frac{a}{2}\|u_0\|^2 t^2 + \frac{b}{4}\|u_0\|^4 t^4 - \frac{t^{2_s^*}}{2_s^*}\int_{\mathbb{R}^N}(u_0^+)^{2_s^*} - t\mu\int_{\mathbb{R}^N} gu_0,$$

We infer that

$$\lim_{t\to+\infty}\frac{I(tu_0)}{t^4} = \frac{b}{4}\|u_0\|^4 - \lim_{t\to+\infty}\frac{t^{2_s^*-4}}{2_s^*}\int_{\mathbb{R}^N}(u_0^+)^{2_s^*} \to -\infty,$$

which implies that there exists a large $t > 0$ such that $I(tu_0) < 0$. Let $e = tu_0$. Then $I(e) < 0$. The proof is completed. □

Lemma 2.2. *Assume that* $a > 0$, $b > 0$, $\frac{N}{4} < s \leq 1$ *and* $g \in L^{\frac{2N}{N+2s}}(\mathbb{R}^N)$ *is a positive function. Then for any* $\mu \in (0, \mu_1]$*, the functional* I *has a non-negative and bounded* $(PS)_c$ *sequence, where the constant* μ_1 *given in Lemma 2.1.*

Proof. For any $\mu \in (0, \mu_1]$, by the mountain pass theorem and Lemma 2.1, there exists a sequence $\{v_n\} \subset D^{s,2}(\mathbb{R}^N)$ such that $I(v_n) \to c$ and $I'(v_n) \to 0$ as $n \to \infty$. According to the Hölder inequality and Sobolev inequality, for sufficiently large n, one has

$$\begin{aligned} c + 1 + \|v_n\| &\geq I(v_n) - \frac{1}{2_s^*}\langle I'(v_n), v_n\rangle \\ &= \left(\frac{1}{2} - \frac{1}{2_s^*}\right) a\|v_n\|^2 + \left(\frac{1}{4} - \frac{1}{2_s^*}\right) b\|v_n\|^4 - \mu\left(1 - \frac{1}{2_s^*}\right)\int_{\mathbb{R}^N} g v_n \\ &\geq \left(\frac{1}{2} - \frac{1}{2_s^*}\right) a\|v_n\|^2 + \left(\frac{1}{4} - \frac{1}{2_s^*}\right) b\|v_n\|^4 - \mu\left(1 - \frac{1}{2_s^*}\right)\frac{1}{S^{\frac{1}{2}}}\|g\|_{\frac{2N}{N+2s}}\|v_n\|, \end{aligned}$$

which implies that $\{v_n\}$ is bounded in $D^{s,2}(\mathbb{R}^N)$.

Let $v_n^- = \min\{0, v_n\}$. Based on the fact that $g > 0$, we can conclude that

$$\begin{aligned} o(1) &= \langle I'(v_n), v_n^-\rangle \\ &= a\|v_n^-\|^2 + b\|v_n\|^2\|v_n^-\|^2 - \mu\int_{\mathbb{R}^N} g v_n^- \\ &\geq a\|v_n^-\|^2 + b\|v_n\|^2\|v_n^-\|^2 \\ &= (a + b\|v_n\|^2)\|v_n^-\|^2 \\ &\geq a\|v_n^-\|^2, \end{aligned}$$

which implies that $\|v_n^-\| \to 0$ in $D^{s,2}(\mathbb{R}^N)$. Hence, $\{v_n\}$ is a non-negative and bounded $(PS)_c$ sequence. $\square$

Lemma 2.3. *Assume that* $a > 0$, $b > 0$, $\frac{N}{4} < s \leq 1$ *and* $g \in L^{\frac{2N}{N+2s}}(\mathbb{R}^N)$ *is a positive function. If*

$$c < L - \xi\mu^2,$$

then the functional I *satisfies the* $(PS)_c$ *condition, where*

$$L := \max_{t \geq 0}\left\{\frac{a}{2}t^2 + \frac{b}{4}t^4 - \frac{1}{2_s^* S^{\frac{2_s^*}{2}}} t^{2_s^*}\right\},$$

and

$$\xi := \frac{(N+2s)^2}{16sN}\,\frac{\|g\|^2_{\frac{2N}{N+2s}}}{Sa}.$$

Proof. Let $\{v_n\}$ be a $(PS)_c$ sequence of the functional I. From Lemma 2.2, $\{v_n\}$ is bounded in $D^{s,2}(\mathbb{R}^N)$. Up to a subsequence, we may assume that $v_n \rightharpoonup v$ in $D^{s,2}(\mathbb{R}^N)$, $v_n \to v$ for a.e. $x \in \mathbb{R}^N$.

Set $u_n = v_n - v$. We next verify that $\|u_n\| \to 0$ as $n \to \infty$. Arguing by contradiction, we may assume that there exists a $\kappa > 0$ such that $\|v_n\|^2 \to \kappa^2 + \|v\|^2$.

For $\phi \in D^{s,2}(\mathbb{R}^N)$, it holds

$$\langle I'(v_n), \phi\rangle = (a + b\|v_n\|^2)\int_{\mathbb{R}^N}(-\Delta)^{\frac{s}{2}} v_n (-\Delta)^{\frac{s}{2}}\phi - \int_{\mathbb{R}^N}(v_n^+)^{2_s^*-1}\phi - \mu\int_{\mathbb{R}^N} g\phi = o(1).$$

By using the Brézis–Lieb lemma, we have

$$(a+b\kappa^2+b\|v\|^2)\int_{\mathbb{R}^N}(-\Delta)^{\frac{s}{2}}v(-\Delta)^{\frac{s}{2}}\phi-\int_{\mathbb{R}^N}(v^+)^{2_s^*-1}\phi-\mu\int_{\mathbb{R}^N}g\phi=0. \tag{2.4}$$

Particularly, taking $\phi=v$ in (2.4), we get

$$(a+b\kappa^2+b\|v\|^2)\|v\|^2-\int_{\mathbb{R}^N}(v^+)^{2_s^*}-\mu\int_{\mathbb{R}^N}gv=0. \tag{2.5}$$

By the definition of weak convergence for v_n, we have

$$\int_{\mathbb{R}^N}gv_n\to\int_{\mathbb{R}^N}gv\ (n\to\infty).$$

Since $\{v_n\}$ is bounded and $I'(v_n)=o(1)$ as $n\to\infty$, we get

$$\langle I'(v_n),v_n\rangle=(a+b\|v_n\|^2)\|v_n\|^2-\int_{\mathbb{R}^N}(v_n^+)^{2_s^*}-\mu\int_{\mathbb{R}^N}gv_n=o(1).$$

Using the Brézis–Lieb lemma again, we conclude from (2.5) that

$$o(1)=a\kappa^2+b\kappa^2\|v\|^2+b\kappa^4-\int_{\mathbb{R}^N}(u_n^+)^{2_s^*}. \tag{2.6}$$

Since $I(v_n)\to c$ as $n\to\infty$, then we have

$$\begin{aligned}c+o(1)&=I(v_n)\\&=\frac{a}{2}\|v_n\|^2+\frac{b}{4}\|v_n\|^4-\frac{1}{2_s^*}\int_{\mathbb{R}^N}(v_n^+)^{2_s^*}-\mu\int_{\mathbb{R}^N}gv_n\\&=\frac{a}{2}(\kappa^2+\|v\|^2)+\frac{b}{4}(\kappa^2+\|v\|^2)^2-\frac{1}{2_s^*}\int_{\mathbb{R}^N}(u_n^+)^{2_s^*}-\frac{1}{2_s^*}\int_{\mathbb{R}^N}(v^+)^{2_s^*}-\mu\int_{\mathbb{R}^N}gv.\end{aligned}$$

Applying (2.5) and (2.6), we obtain

$$\begin{aligned}c+o(1)=&\left(\frac12-\frac{1}{2_s^*}\right)a\|v\|^2+\left(\frac14-\frac{1}{2_s^*}\right)b\|v\|^4-\frac{b}{2_s^*}\kappa^2\|v\|^2-\frac{2_s^*-1}{2_s^*}\mu\int_{\mathbb{R}^N}gv\\&+\left(\frac12-\frac{1}{2_s^*}\right)(a\kappa^2+b\kappa^2\|v\|^2)+\left(\frac14-\frac{1}{2_s^*}\right)b\kappa^4.\end{aligned} \tag{2.7}$$

It follows from (2.6) and Sobolev inequality that

$$a\kappa^2+b\kappa^2\|v\|^2+b\kappa^4=\int_{\mathbb{R}^N}(u_n^+)^{2_s^*}\le\frac{1}{S^{\frac{2_s^*}{2}}}\kappa^{2_s^*}.$$

Let $f(t)=t^{2_s^*-2}-S^{\frac{2_s^*}{2}}bt^2$ for any $t>0$. It is clear that $S^{\frac{2_s^*}{2}}(a+b\|v\|^2)\le f(\kappa)$. Since $S^{\frac{2_s^*}{2}}(a+b\|v\|^2)>0$ and $2_s^*>4$ when $\frac{N}{4}<s\le1$, then there exists a $\eta>0$ such that

$$S^{\frac{2_s^*}{2}}(a+b\|v\|^2)=\eta^{2_s^*-2}-S^{\frac{2_s^*}{2}}b\eta^2 \tag{2.8}$$

and $\kappa\ge\eta$.

Set $l>0$ be the root of the following equation

$$S^{\frac{2_s^*}{2}}a=l^{2_s^*-2}-S^{\frac{2_s^*}{2}}bl^2. \tag{2.9}$$

Obviously, $\eta > l$ and

$$\begin{aligned} L := & \max_{t\geq 0}\left\{\frac{a}{2}S^{\frac{N}{2s}}t^2+\frac{b}{4}S^{\frac{N}{s}}t^4-\frac{1}{2_s^*}S^{\frac{N}{2s}}t^{2_s^*}\right\} \\ &= \max_{t\geq 0}\left\{\frac{a}{2}t^2+\frac{b}{4}t^4-\frac{1}{2_s^* S^{\frac{2_s^*}{2}}}t^{2_s^*}\right\} \\ &= \frac{a}{2}l^2+\frac{b}{4}l^4-\frac{1}{2_s^* S^{\frac{2_s^*}{2}}}l^{2_s^*}. \end{aligned} \tag{2.10}$$

By (2.7), (2.10) and $\kappa \geq \eta$, we have

$$\begin{aligned} c+o(1) \geq & \left(\frac{1}{2}-\frac{1}{2_s^*}\right)a\eta^2+\left(\frac{1}{2}-\frac{2}{2_s^*}\right)b\eta^2\|v\|^2+\left(\frac{1}{2}-\frac{1}{2_s^*}\right)a\|v\|^2+\left(\frac{1}{4}-\frac{1}{2_s^*}\right)b\|v\|^4 \\ & +\left(\frac{1}{4}-\frac{1}{2_s^*}\right)b\eta^4-\frac{2_s^*-1}{2_s^*}\mu\int_{\mathbb{R}^N} gv \\ = & L+\left(\frac{1}{2}-\frac{1}{2_s^*}\right)a(\eta^2-l^2)+\left(\frac{1}{2}-\frac{2}{2_s^*}\right)b\eta^2\|v\|^2+\left(\frac{1}{2}-\frac{1}{2_s^*}\right)a\|v\|^2 \\ & +\left(\frac{1}{4}-\frac{1}{2_s^*}\right)b\|v\|^4+\left(\frac{1}{4}-\frac{1}{2_s^*}\right)b(\eta^4-l^4)-\frac{2_s^*-1}{2_s^*}\mu\int_{\mathbb{R}^N} gv. \end{aligned} \tag{2.11}$$

From (2.8) and (2.9), we obtain

$$\|v\|^2=\frac{\eta^{2_s^*-2}-l^{2_s^*-2}}{bS^{\frac{2_s^*}{2}}}-(\eta^2-l^2). \tag{2.12}$$

It follows from (2.11) and (2.12) that

$$\begin{aligned} c+o(1) \geq & L+\left(\frac{1}{2}-\frac{1}{2_s^*}\right)\frac{a}{bS^{\frac{2_s^*}{2}}}(\eta^{2_s^*-2}-l^{2_s^*-2})+\left(\frac{1}{4}-\frac{1}{2_s^*}\right)\frac{1}{bS^{2_s^*}}(\eta^{2_s^*-2}-l^{2_s^*-2})^2 \\ & +\left(\frac{1}{2}-\frac{2}{2_s^*}\right)\frac{l^2}{S^{\frac{2_s^*}{2}}}(\eta^{2_s^*-2}-l^{2_s^*-2})-\frac{2_s^*-1}{2_s^*}\mu\int_{\mathbb{R}^N} gv. \end{aligned} \tag{2.13}$$

Applying (2.8) and (2.9) again, we get

$$\eta^{2_s^*-2}-l^{2_s^*-2}=S^{\frac{2_s^*}{2}}b\|v\|^2+S^{\frac{2_s^*}{2}}b(\eta^2-l^2)>S^{\frac{2_s^*}{2}}b\|v\|^2. \tag{2.14}$$

By (2.13) and (2.14), we have

$$c+o(1)\geq L+\left(\frac{1}{2}-\frac{1}{2_s^*}\right)a\|v\|^2-\frac{2_s^*-1}{2_s^*}\mu\int_{\mathbb{R}^N} gv. \tag{2.15}$$

Moreover, by Hölder, Sobolev and Young's inequalities, it holds

$$\begin{aligned} \frac{2_s^*-1}{2_s^*}\mu\int_{\mathbb{R}^N} gv & \leq \frac{2_s^*-1}{2_s^*}\mu\|g\|_{\frac{2N}{N+2s}}\|v\|_{2_s^*} \\ & \leq \frac{2_s^*-1}{2_s^*}\mu\|g\|_{\frac{2N}{N+2s}}\frac{1}{S^{\frac{1}{2}}}\|v\| \\ & \leq \left(\frac{1}{2}-\frac{1}{2_s^*}\right)a\|v\|^2+\frac{(N+2s)^2}{16sN}\frac{\|g\|^2_{\frac{2N}{N+2s}}}{Sa}\mu^2. \end{aligned} \tag{2.16}$$

Hence, we conclude from (2.15) and (2.16) that

$$c+o(1)\geq L-\frac{(N+2s)^2}{16sN}\frac{\|g\|^2_{\frac{2N}{N+2s}}}{Sa}\mu^2.$$

This is a contradiction. □

Taking $\varepsilon = 1$ and $y = 0$ in (2.3), we have the following estimate.

Lemma 2.4. *Assume that* $a > 0$, $b > 0$, $\frac{N}{4} < s \leq 1$ *and* $g \in L^{\frac{2N}{N+2s}}(\mathbb{R}^N)$ *is a positive function. Then there exists a* $\mu_* > 0$ *such that*

$$\sup_{t\geq 0} I(tU_1) < L - \xi\mu^2$$

for any $\mu \in (0, \mu_*]$, *where*

$$U_1 = C_s\left(\frac{1}{1+|x|^2}\right)^{\frac{N-2s}{2}}, \quad L = \max_{t\geq 0}\left\{\frac{a}{2}t^2 + \frac{b}{4}t^4 - \frac{1}{2_s^* S^{\frac{2_s^*}{2}}}t^{2_s^*}\right\} \quad \text{and} \quad \xi = \frac{(N+2s)^2}{16sN}\frac{\|g\|^2_{\frac{2N}{N+2s}}}{Sa}.$$

Proof. Let

$$\begin{aligned} k(t) &= I(tU_1) \\ &= \frac{a}{2}t^2\|U_1\|^2 + \frac{b}{4}t^4\|U_1\|^4 - \frac{t^{2_s^*}}{2_s^*}\int_{\mathbb{R}^N} U_1^{2_s^*} - \mu t\int_{\mathbb{R}^N} gU_1 \end{aligned}$$

and

$$\widetilde{k}(t) = \frac{a}{2}t^2\|U_1\|^2 + \frac{b}{4}t^4\|U_1\|^4 - \frac{t^{2_s^*}}{2_s^*}\int_{\mathbb{R}^N} U_1^{2_s^*}.$$

Since $\|U_1\|^2 = \|U_1\|_{2_s^*}^{2_s^*} = S^{\frac{N}{2s}}$, then $\widetilde{k}(t) = \frac{a}{2}t^2S^{\frac{N}{2s}} + \frac{b}{4}t^4S^{\frac{N}{s}} - \frac{t^{2_s^*}}{2_s^*}S^{\frac{N}{2s}}$. It is clear that the function $\widetilde{k}$ has a maximum value on $(0, +\infty)$ and $\max_{t\geq 0}\widetilde{k}(t) = L$.

Since $g \in L^{\frac{2N}{N+2s}}(\mathbb{R}^N)$ is a positive function, then there exists a $\mu_0 > 0$ such that

$$\mu t\int_{\mathbb{R}^N} gU_1 \leq \mu l\|g\|_{\frac{2N}{N+2s}}\|U_1\|_{2_s^*} < \mu_0 l\|g\|_{\frac{2N}{N+2s}}\|U_1\|_{2_s^*} \leq L$$

for any $\mu \in (0, \mu_0)$ and $t \in (0, l)$. Hence,

$$\begin{aligned} \max_{t\geq 0} k(t) &= \max_{t\geq 0}\left\{\widetilde{k}(t) - \mu t\int_{\mathbb{R}^N} gU_1\right\} \\ &\geq \widetilde{k}(l) - \mu l\int_{\mathbb{R}^N} gU_1 \\ &= L - \mu l\int_{\mathbb{R}^N} gU_1 \\ &> 0 \end{aligned}$$

for any $\mu \in (0, \mu_0)$.

Choose $\mu_1 \in (0, \mu_0)$ such that

$$L - \xi\mu_1^2 > 0,$$

then

$$L - \xi\mu^2 > L - \xi\mu_1^2 > 0$$

for any $\mu \in (0, \mu_1)$.

Since $\lim_{t\to 0^+} k(t) = 0 < L - \xi\mu_1^2$, then there exists a $l_0 \in (0, l)$ such that

$$k(t) < L - \xi\mu_1^2$$

for any $t \in (0, l_0)$. Hence,

$$\max_{0\leq t\leq l_0} k(t) \leq L - \xi\mu_1^2 < L - \xi\mu^2$$

for any $\mu \in (0, \mu_1)$.

Choosing $\mu_* \in (0, \mu_1)$ such that

$$l_0 \int_{\mathbb{R}^N} gU_1 > \xi\mu_* > \xi\mu, \quad \forall\ \mu \in (0, \mu_*),$$

then

$$\begin{aligned}\max_{t\geq l_0} k(t) &= \max_{t\geq l_0} \left\{ \widetilde{k}(t) - \mu t \int_{\mathbb{R}^N} gU_1 \right\} \\ &\leq \widetilde{k}(l) - \mu l_0 \int_{\mathbb{R}^N} gU_1 \\ &< L - \xi\mu^2\end{aligned}$$

for any $\mu \in (0, \mu_*)$.

Consequently, $\max_{t\geq 0} k(t) = \max\{\max_{0\leq t\leq l_0} k(t), \max_{t\geq l_0} k(t)\} < L - \xi\mu^2$. □

2.2 The case $b < 0$

For convenience, for the case $b < 0$, we here rewrite the functional $I(u)$ as the following form

$$\begin{aligned}I(u) &= \frac{a}{2}\|u\|^2 + \frac{|b|}{4}\|u\|^4 - \frac{1}{2_s^*}\int_{\mathbb{R}^N} (u^+)^{2_s^*} - \mu \int_{\mathbb{R}^N} gu \\ &= \frac{a}{2}\|u\|^2 - \frac{b}{4}\|u\|^4 - \frac{1}{2_s^*}\int_{\mathbb{R}^N} (u^+)^{2_s^*} - \mu \int_{\mathbb{R}^N} gu.\end{aligned}$$

where $u^+ = \max\{0, u\}$.

Lemma 2.5. *Assume that $a > 0$, $b < 0$, and $g \in L^{\frac{2N}{N+2s}}(\mathbb{R}^N)$ is a positive function. Then there exist $\overline{\rho}, \overline{\alpha}, \overline{\mu}_1 > 0$ such that, for any $\mu \in (0, \overline{\mu}_1]$*

(i) $\inf_{\|u\|=\overline{\rho}} I(u) \geq \overline{\alpha}$.

(ii) $\inf_{\|u\|<\overline{\rho}} I(u) < 0$.

(iii) There exists $e \in D^{s,2}(\mathbb{R}^N)$ such that $\|e\| > \rho$ and $I(e) < 0$.

The desired result can be deduced by the same arguments we used in the proof of Lemma 2.1 for the case $b > 0$.

Lemma 2.6. *Assume that $a > 0$, $b < 0$, and $g \in L^{\frac{2N}{N+2s}}(\mathbb{R}^N)$ is a positive function. If $c < T$, then for any $\mu \in (0, \overline{\mu}_1]$, the functional I has a non-negative and bounded $(PS)_c$ sequence, where*

$$T := \max_{t\geq 0} \left\{ \frac{a}{2}t^2 - \frac{b}{4}t^4 - \frac{1}{2_s^* S^{\frac{2_s^*}{2}}} t^{2_s^*} \right\}$$

and the constant $\overline{\mu}_1$ given in Lemma 2.5.

Proof. Let $\{v_n\}$ be a $(PS)_c$ sequence of the functional I. Since $2_s^* < 4$ when $N \geq 4$, $s \in (0,1)$ or $N = 3$, $s \in (0, \frac{3}{4})$, then we have

$$\begin{aligned}c + 1 + \|v_n\| &\geq I(v_n) - \frac{1}{2_s^*}\langle I'(v_n), v_n\rangle \\ &= \left(\frac{1}{2} - \frac{1}{2_s^*}\right) a\|v_n\|^2 + \left(\frac{1}{2_s^*} - \frac{1}{4}\right) b\|v_n\|^4 - \mu\left(1 - \frac{1}{2_s^*}\right)\int_{\mathbb{R}^N} gv_n \\ &\geq \left(\frac{1}{2} - \frac{1}{2_s^*}\right) a\|v_n\|^2 + \left(\frac{1}{2_s^*} - \frac{1}{4}\right) b\|v_n\|^4 - \mu\left(1 - \frac{1}{2_s^*}\right)\frac{1}{S^{\frac{1}{2}}}\|g\|_{\frac{2N}{N+2s}}\|v_n\|,\end{aligned}$$

which implies that $\{v_n\}$ is bounded in $D^{s,2}(\mathbb{R}^N)$.

Next, we claim that there exists a $\overline{\delta} > 0$ such that $(a - b\|v_n\|^2) \geq \overline{\delta}$ for sufficiently large n. Indeed, if the claim is not true. Then there exist a constant $d > 0$ and a subsequence (still denoted by $\{v_n\}$) such that

$$(a - b\|v_n\|^2) \to -d \leq 0 \qquad (n \to \infty).$$

Since

$$o(1) = \langle I'(v_n), v_n \rangle = (a - b\|v_n\|^2)\|v_n\|^2 - \int_{\mathbb{R}^N} (v_n^+)^{2_s^*} - \mu \int_{\mathbb{R}^N} g v_n,$$

then

$$\overline{\lim_{n\to\infty}} \mu \int_{\mathbb{R}^N} g v_n \leq 0.$$

Hence

$$\begin{aligned}
c &= \lim_{n\to\infty} \left(I(v_n) - \frac{1}{2_s^*} \langle I'(v_n), v_n \rangle \right) \\
&= \lim_{n\to\infty} \left(\left(\frac{1}{2} - \frac{1}{2_s^*}\right) a\|v_n\|^2 + \left(\frac{1}{2_s^*} - \frac{1}{4}\right) b\|v_n\|^4 - \frac{2_s^* - 1}{2_s^*} \mu \int_{\mathbb{R}^N} g v_n \right) \\
&\geq \lim_{n\to\infty} \left(\left(\frac{1}{2} - \frac{1}{2_s^*}\right) a\|v_n\|^2 + \left(\frac{1}{2_s^*} - \frac{1}{4}\right) b\|v_n\|^4 \right) \\
&= \left(\frac{1}{2} - \frac{1}{2_s^*}\right) a \lim_{n\to\infty} \|v_n\|^2 + \left(\frac{1}{2_s^*} - \frac{1}{4}\right) b \lim_{n\to\infty} \|v_n\|^4 \\
&= \left(\frac{1}{2} - \frac{1}{2_s^*}\right) a \frac{a+d}{b} + \left(\frac{1}{2_s^*} - \frac{1}{4}\right) b \left(\frac{a+d}{b}\right)^2 \\
&\geq \frac{a^2}{4b}.
\end{aligned}$$

On the other hand,

$$\begin{aligned}
T :&= \max_{t\geq 0} \left\{ \frac{a}{2}t^2 - \frac{b}{4}t^4 - \frac{1}{2_s^* S^{\frac{2_s^*}{2}}} t^{2_s^*} \right\} \\
&< \max_{t\geq 0} \left\{ \frac{a}{2}t^2 - \frac{b}{4}t^4 \right\} \\
&= \frac{a^2}{4b},
\end{aligned}$$

which contradicts $\frac{a^2}{4b} \leq c < T$. Hence the claim holds. Let $v_n^- = \min\{0, v_n\}$. Arguing in a similar way to Lemma 2.2, we can conclude that $\{v_n\}$ is a non-negative and bounded $(PS)_c$ sequence. ☐

Lemma 2.7. *Let $a > 0$, $b < 0$, $g \in L^{\frac{2N}{N+2s}}(\mathbb{R}^N)$ is a positive function. If*

$$c < T - \zeta\mu^{\frac{4}{3}}$$

then the functional I satisfies the $(PS)_c$ condition, where

$$T = \max_{t\geq 0} \left\{ \frac{a}{2}t^2 - \frac{b}{4}t^4 - \frac{1}{2_s^* S^{\frac{2_s^*}{2}}} t^{2_s^*} \right\}$$

and

$$\zeta := \frac{3}{4}\left(\frac{(2_s^*-1)^4\|g\|^4_{\frac{2N}{N+2s}}(\alpha+S^{\frac{2_s^*}{2}}b)^2}{(2_s^*)^3(4-2_s^*)\alpha^2 bS^2}\right)^{\frac{1}{3}}.$$

Proof. Let $\{v_n\}$ be a $(PS)_c$ sequence of the functional I. From Lemma 2.6, $\{v_n\}$ is bounded in $D^{s,2}(\mathbb{R}^N)$. Up to a subsequence, we may assume that $v_n \rightharpoonup v$ in $D^{s,2}(\mathbb{R}^N)$, $v_n \to v$ for a.e. $x \in \mathbb{R}^N$.

Set $u_n = v_n - v$. We verify that $\|u_n\| \to 0$ as $n \to \infty$. Arguing by contradiction, we may assume that there exists a $\overline{\kappa} > 0$ such that $\|v_n\|^2 \to \overline{\kappa}^2 + \|v\|^2$.

Arguing in a similar way to Lemma 2.3, we obtain

$$(a - b\overline{\kappa}^2 - b\|v\|^2)\|v\|^2 - \int_{\mathbb{R}^N}(v^+)^{2_s^*} - \mu\int_{\mathbb{R}^N} gv = 0, \tag{2.17}$$

and

$$o(1) = a\overline{\kappa}^2 - b\overline{\kappa}^2\|v\|^2 - b\overline{\kappa}^4 - \int_{\mathbb{R}^N}(u_n^+)^{2_s^*}. \tag{2.18}$$

Since $I(v_n) \to c$ as $n \to \infty$, applying (2.17) and (2.18), we obtain

$$\begin{aligned} c + o(1) &= I(v_n) \\ &= \left(\frac{1}{2}-\frac{1}{2_s^*}\right)a\|v\|^2 + \left(\frac{1}{2_s^*}-\frac{1}{4}\right)b\|v\|^4 + \frac{b}{2_s^*}\overline{\kappa}^2\|v\|^2 - \frac{2_s^*-1}{2_s^*}\mu\int_{\mathbb{R}^N} gv \\ &\quad + \left(\frac{1}{2}-\frac{1}{2_s^*}\right)(a\overline{\kappa}^2 - b\overline{\kappa}^2\|v\|^2) + \left(\frac{1}{2_s^*}-\frac{1}{4}\right)b\overline{\kappa}^4. \end{aligned} \tag{2.19}$$

It follows from (2.18) and Sobolev inequality that

$$a\overline{\kappa}^2 - b\overline{\kappa}^2\|v\|^2 - b\overline{\kappa}^4 = \int_{\mathbb{R}^N}(u_n^+)^{2_s^*} \le \frac{1}{S^{\frac{2_s^*}{2}}}\overline{\kappa}^{2_s^*}.$$

Since $a\overline{\kappa}^2 - b\overline{\kappa}^2\|v\|^2 - b\overline{\kappa}^4 \ge 0$ and $\overline{\kappa} > 0$, we have $a - b\|v\|^2 > 0$. Then there exists a $\overline{\eta} > 0$ such that

$$S^{\frac{2_s^*}{2}}(a - b\|v\|^2) = \overline{\eta}^{2_s^*-2} + S^{\frac{2_s^*}{2}}b\overline{\eta}^2 \tag{2.20}$$

and $\overline{\kappa} \ge \overline{\eta}$.

Let $\overline{l} > 0$ be the unique solution of the following equation

$$S^{\frac{2_s^*}{2}}a = \overline{l}^{2_s^*-2} + S^{\frac{2_s^*}{2}}b\overline{l}^2. \tag{2.21}$$

Clearly, it holds $\overline{l} > \overline{\eta}$ and

$$\begin{aligned} T :&= \max_{t\ge 0}\left\{\frac{a}{2}S^{\frac{N}{2s}}t^2 - \frac{b}{4}S^{\frac{N}{s}}t^4 - \frac{1}{2_s^*}S^{\frac{N}{2s}}t^{2_s^*}\right\} \\ &= \max_{t\ge 0}\left\{\frac{a}{2}t^2 - \frac{b}{4}t^4 - \frac{1}{2_s^*S^{\frac{2_s^*}{2}}}t^{2_s^*}\right\} \\ &= \frac{a}{2}\overline{l}^2 - \frac{b}{4}\overline{l}^4 - \frac{1}{2_s^*S^{\frac{2_s^*}{2}}}\overline{l}^{2_s^*}. \end{aligned} \tag{2.22}$$

Since $2_s^* < 4$ when $N \geq 4$, $s \in (0,1)$ or $N = 3$, $s \in (0, \frac{3}{4})$, hence by (2.19), (2.22) and $\bar{\kappa} \geq \bar{\eta}$, we have

$$\begin{aligned} c + o(1) \geq{}& \left(\frac{1}{2} - \frac{1}{2_s^*}\right) a\bar{\eta}^2 + \left(\frac{2}{2_s^*} - \frac{1}{2}\right) b\bar{\eta}^2\|v\|^2 + \left(\frac{1}{2} - \frac{1}{2_s^*}\right) a\|v\|^2 + \left(\frac{1}{2_s^*} - \frac{1}{4}\right) b\|v\|^4 \\ &+ \left(\frac{1}{2_s^*} - \frac{1}{4}\right) b\bar{\eta}^4 - \frac{2_s^* - 1}{2_s^*}\mu \int_{\mathbb{R}^N} gv \\ ={}& T + \left(\frac{1}{2} - \frac{1}{2_s^*}\right) a(\bar{\eta}^2 - \bar{l}^2) + \left(\frac{2}{2_s^*} - \frac{1}{2}\right) b\bar{\eta}^2\|v\|^2 + \left(\frac{1}{2} - \frac{1}{2_s^*}\right) a\|v\|^2 \\ &+ \left(\frac{1}{2_s^*} - \frac{1}{4}\right) b\|v\|^4 + \left(\frac{1}{2_s^*} - \frac{1}{4}\right) b(\bar{\eta}^4 - \bar{l}^4) - \frac{2_s^* - 1}{2_s^*}\mu \int_{\mathbb{R}^N} gv. \end{aligned} \tag{2.23}$$

From (2.20) and (2.21), we obtain

$$\|v\|^2 = \frac{\bar{l}^{2_s^*-2} - \bar{\eta}^{2_s^*-2}}{bS^{\frac{2_s^*}{2}}} + \bar{l}^2 - \bar{\eta}^2. \tag{2.24}$$

It follows from (2.23) and (2.24) that

$$\begin{aligned} c + o(1) \geq{}& T + \left(\frac{1}{2} - \frac{1}{2_s^*}\right) \frac{a}{bS^{\frac{2_s^*}{2}}}(\bar{l}^{2_s^*-2} - \bar{\eta}^{2_s^*-2}) + \left(\frac{1}{2_s^*} - \frac{1}{4}\right) \frac{1}{bS^{2_s^*}}(\bar{l}^{2_s^*-2} - \bar{\eta}^{2_s^*-2})^2 \\ &+ \left(\frac{2}{2_s^*} - \frac{1}{2}\right) \frac{\bar{l}^2}{S^{\frac{2_s^*}{2}}}(\bar{l}^{2_s^*-2} - \bar{\eta}^{2_s^*-2}) - \frac{2_s^* - 1}{2_s^*}\mu \int_{\mathbb{R}^N} gv. \end{aligned} \tag{2.25}$$

Applying (2.20) and (2.21) again, we get

$$\bar{l}^{2_s^*-2} - \bar{\eta}^{2_s^*-2} = S^{\frac{2_s^*}{2}} b\|v\|^2 - S^{\frac{2_s^*}{2}} b(\bar{l}^2 - \bar{\eta}^2). \tag{2.26}$$

Next, we claim that there exists a $\alpha > 0$ independent of v such that

$$\bar{l}^{2_s^*-2} - \bar{\eta}^{2_s^*-2} \geq \alpha(\bar{l}^2 - \bar{\eta}^2). \tag{2.27}$$

Indeed,

$$\frac{\bar{l}^{2_s^*-2} - \bar{\eta}^{2_s^*-2}}{\bar{l}^2 - \bar{\eta}^2} > \frac{1}{2\bar{l}} \, \frac{\bar{l}^{2_s^*-2} - \bar{\eta}^{2_s^*-2}}{\bar{l} - \bar{\eta}}.$$

On the one hand, if $\frac{\bar{l}}{2} \leq \bar{\eta} < \bar{l}$, then by the mean value theorem, we obtain

$$\begin{aligned} \bar{l}^{2_s^*-2} - \bar{\eta}^{2_s^*-2} &= (2_s^* - 2)\theta^{2_s^*-3}(\bar{l} - \bar{\eta}) \\ &\geq \begin{cases} (2_s^* - 2)\left(\dfrac{\bar{l}}{2}\right)^{2_s^*-3}(\bar{l} - \bar{\eta}), & 2_s^* - 3 \geq 0, \\ (2_s^* - 2)\bar{l}^{2_s^*-3}(\bar{l} - \bar{\eta}), & 2_s^* - 3 < 0, \end{cases} \end{aligned}$$

where $\theta \in (\bar{l}, \bar{\eta})$. On the other hand, if $\bar{\eta} < \frac{\bar{l}}{2}$, then

$$\bar{l}^{2_s^*-2} - \bar{\eta}^{2_s^*-2} \geq \left(1 - \frac{1}{2^{2_s^*-2}}\right)\bar{l}^{2_s^*-2} = \left(1 - \frac{1}{2^{2_s^*-2}}\right)\bar{l}^{2_s^*-3}\bar{l} \geq \left(1 - \frac{1}{2^{2_s^*-2}}\right)\bar{l}^{2_s^*-3}(\bar{l} - \bar{\eta}).$$

Consequently, the claim holds. It follows from (2.26) and (2.27) that

$$\bar{l}^{2_s^*-2} - \bar{\eta}^{2_s^*-2} \geq \frac{\alpha S^{\frac{2_s^*}{2}} b}{\alpha + S^{\frac{2_s^*}{2}} b}\|v\|^2. \tag{2.28}$$

By (2.25) and (2.28), we have

$$c + o(1) \geq T + \left(\frac{1}{2_s^*} - \frac{1}{4}\right) \frac{\alpha^2 b}{(\alpha + S^{\frac{2_s^*}{2}} b)^2}\|v\|^4 - \frac{2_s^*-1}{2_s^*}\mu \int_{\mathbb{R}^N} gv. \tag{2.29}$$

Moreover, by Hölder's, Sobolev's and Young's inequalities, it holds

$$\begin{aligned}\frac{2_s^*-1}{2_s^*}\mu \int_{\mathbb{R}^N} gv &\leq \frac{2_s^*-1}{2_s^*}\mu\|g\|_{\frac{2N}{N+2s}}\|v\|_{2_s^*} \\ &\leq \frac{2_s^*-1}{2_s^*}\mu\|g\|_{\frac{2N}{N+2s}}\frac{1}{S^{\frac{1}{2}}}\|v\| \\ &\leq \left(\frac{1}{2_s^*} - \frac{1}{4}\right)\frac{\alpha^2 b}{(\alpha + S^{\frac{2_s^*}{2}} b)^2}\|v\|^4 + \frac{3}{4}\left(\frac{(2_s^*-1)^4\|g\|^4_{\frac{2N}{N+2s}}(\alpha + S^{\frac{2_s^*}{2}} b)^2}{(2_s^*)^3(4-2_s^*)\alpha^2 b S^2}\right)^{\frac{1}{3}}\mu^{\frac{4}{3}}.\end{aligned} \tag{2.30}$$

Hence, we conclude from (2.29) and (2.30) that

$$c + o(1) \geq T - \frac{3}{4}\left(\frac{(2_s^*-1)^4\|g\|^4_{\frac{2N}{N+2s}}(\alpha + S^{\frac{2_s^*}{2}} b)^2}{(2_s^*)^3(4-2_s^*)\alpha^2 b S^2}\right)^{\frac{1}{3}}\mu^{\frac{4}{3}}.$$

This is a contradiction. □

Arguing in a similar way to Lemma 2.4, we have the following lemma.

Lemma 2.8. *Assume that* $a > 0$*,* $b < 0$ *and* $g \in L^{\frac{2N}{N+2s}}(\mathbb{R}^N)$ *is a positive function. Then there exists a* $\bar{\mu}_* > 0$ *such that*

$$\sup_{t \geq 0} I(tU_1) < T - \zeta\mu^{\frac{4}{3}}$$

for any $\mu \in (0, \bar{\mu}_*]$*, where*

$$U_1 = C_s\left(\frac{1}{1+|x|^2}\right)^{\frac{N-2s}{2}}, \qquad T = \max_{t\geq 0}\left\{\frac{a}{2}t^2 - \frac{b}{4}t^4 - \frac{1}{2_s^* S^{\frac{2_s^*}{2}}}t^{2_s^*}\right\}$$

and

$$\zeta = \frac{3}{4}\left(\frac{(2_s^*-1)^4\|g\|^4_{\frac{2N}{N+2s}}(\alpha + S^{\frac{2_s^*}{2}} b)^2}{(2_s^*)^3(4-2_s^*)\alpha^2 b S^2}\right)^{\frac{1}{3}}.$$

Remark 2.9. In Lemma 2.7, we obtain the compactness result of the functional I for the case $2_s^* < 4$. However, arguing in a similar way to Lemma 2.7, there seems no similar result for the case $2_s^* > 4$. Precisely, by (2.17), (2.18), (2.20), (2.21), $\bar{\kappa} \geq \bar{\eta}$ and $\bar{l} > \bar{\eta}$, we can conclude that

$$c + o(1) \geq T + \frac{1}{4bS^{2_s^*}}\left(\bar{l}^{2_s^*-2} - \bar{\eta}^{2_s^*-2}\right)^2 + \frac{\bar{l}^{2_s^*}}{2_s^* S^{\frac{2_s^*}{2}}} + \frac{\bar{\eta}^{2_s^*}}{4S^{\frac{2_s^*}{2}}} - \frac{\bar{\eta}^{2_s^*-2}\bar{l}^2}{2S^{\frac{2_s^*}{2}}} - \frac{\mu}{2}\int_{\mathbb{R}^N} gv.$$

By direct computation, we find

$$\begin{aligned}
&\frac{1}{4bS^{\frac{2_s^*}{2}}}\left(\bar{l}^{2_s^*-2}-\bar{\eta}^{2_s^*-2}\right)^2+\frac{\bar{l}^{2_s^*}}{2_s^*}+\frac{\bar{\eta}^{2_s^*}}{4}-\frac{\bar{\eta}^{2_s^*-2}\bar{l}^2}{2}\\
&\leq \frac{1}{4}bS^{\frac{2_s^*}{2}}\|v\|^4+\left(\frac{1}{2_s^*}+\frac{1}{4}\right)\bar{l}^{2_s^*}-\frac{\bar{\eta}^{2_s^*-2}\bar{l}^2}{2}\\
&= \frac{1}{4}bS^{\frac{2_s^*}{2}}\|v\|^4+\frac{1}{2}\bar{l}^2\left(\bar{l}^{2_s^*-2}-\bar{\eta}^{2_s^*-2}\right)+\left(\frac{1}{2_s^*}-\frac{1}{4}\right)\bar{l}^{2_s^*}\\
&\leq \frac{1}{4}bS^{\frac{2_s^*}{2}}\|v\|^4+\frac{1}{2}\bar{l}^2 bS^{\frac{2_s^*}{2}}\|v\|^2+\left(\frac{1}{2_s^*}-\frac{1}{4}\right)\bar{l}^{2_s^*}\\
&< 0,
\end{aligned}$$

if b is sufficiently small. Hence, we cannot obtain the compactness result of the functional I for the case $2_s^* > 4$.

3 Proof of the main result

Proof of Theorem 1.1. We distinguish two cases:

Case 1 : $b > 0$ and $\frac{N}{4} < s \leq 1$.

Existence of the first positive solution. Let ρ, μ_1 be the constants defined in Lemma 2.1, and set $B_\rho = \{u \in D^{s,2}(\mathbb{R}^N) : \|u\| \leq \rho\}$. It follows from Lemma 2.1 that

$$\sigma := \inf_{u \in B_\rho} I(u) < 0$$

for any $\mu \in (0, \mu_1)$. By Ekeland variational principle [2, Theorem 1.1], there exists $\{v_n\} \subset B_\rho$ such that

$$I(v_n) \leq \sigma + \frac{1}{n}, \qquad I(u) \geq I(v_n) - \frac{1}{n}\|v_n - v\|$$

for every integer n and for any $u \in B_\rho$. Hence, we conclude that $I(v_n) \to \sigma$, $I'(v_n) \to 0$ as $n \to \infty$. By Lemma 2.4, there exists a $\mu_* > 0$ such that

$$\sup_{t \geq 0} I(tU_1) < L - \xi\mu^2$$

for any $\mu \in (0, \mu_*)$, where L and ξ are given in Lemma 2.3. Assume that $\mu_* < \mu_1$, then for any $\mu \in (0, \mu_*)$, we have

$$\sigma = \inf_{u \in B_\rho} I(u) \leq \sup_{t \geq 0} I(tU_1) < L - \xi\mu^2.$$

It follows from Lemma 2.3 that $\{v_n\}$ has a convergent subsequence in $D^{s,2}(\mathbb{R}^N)$(still denoted by $\{v_n\}$). By Lemma 2.2, we know that $\{v_n\}$ is non-negative. Let $v_n \to v_*$ as $n \to \infty$, then $v_* \geq 0$, $I(v_*) = \sigma < 0$ and $I'(v_*) = 0$. Hence $v_* \neq 0$. By the strong maximum principle, we obtain $v_* > 0$. Therefore, v_* is a positive solution for (1.1).

Existence of the second positive solution. Let ρ, μ_1 be the constants defined in Lemma 2.1. From Lemma 2.1, there are

$$\inf_{\|u\| = \rho} I(u) \geq \alpha > 0$$

for any $\mu \in (0, \mu_1)$ and exists $e \in D^{s,2}(\mathbb{R}^N)$ such that $\|e\| > \rho$ and $I(e) < 0$.

Let c be the mountain pass level given in (2.1). By the mountain pass theorem [19, Theorem 1.15], there exists a sequence $\{v_n\} \subset D^{s,2}(\mathbb{R}^N)$ such that $I(v_n) \to c$ and $I'(v_n) \to 0$ as $n \to \infty$. Moreover $c \geq \alpha > 0$.

Note that $2_s^* > 4$, then we have

$$\lim_{t\to+\infty} I(tU_1) = \lim_{t\to+\infty} \left(\frac{at^2}{2} \|U_1\|^2 + \frac{bt^4}{4} \|U_1\|^4 - \frac{t^{2_s^*}}{2_s^*} \int_{\mathbb{R}^N} U_1^{2_s^*} - \mu t \int_{\mathbb{R}^N} gU_1 \right) = -\infty.$$

Hence, there exists a $t_0 > 0$ such that $I(t_0U_1) < 0$ and $\|t_0U_1\| > \rho$. Let $\gamma_0(t) = tt_0U_1$, $t \in [0,1]$. By Lemma 2.4, there exists $\mu_* \in (0, \mu_1)$ such that

$$c \leq \max_{t\in[0,1]} I(\gamma_0(t)) = \max_{t\in[0,1]} I(tt_0U_1) \leq \sup_{t\geq 0} I(tU_1) < L - \xi\mu^2$$

for any $\mu \in (0, \mu_*]$. It follows from Lemma 2.3 that $\{v_n\}$ has a convergent subsequence in $D^{s,2}(\mathbb{R}^N)$(still denoted by $\{v_n\}$). By Lemma 2.2, we know that $\{v_n\}$ is non-negative. Let $v_n \to v_{**}$ as $n \to \infty$, then $v_{**} \geq 0$, $I(v_{**}) = c \geq \alpha > 0$ and $I'(v_{**}) = 0$. Hence v_{**} is a non-negative and nontrivial solution for (1.1). By the strong maximum principle, we obtain $v_{**} > 0$. Note that $I(v_*) < 0 < I(v_{**})$. Therefore, v_{**} is a positive solution for (1.1), which different with v_*.

Case 2 : $b < 0$ and $N \geq 4$, $s \in (0,1)$ or $N = 3$, $s \in (0, \frac{3}{4})$.

The conclusion can be shown in the same way as used in the proof of case 1 in Theorem 1.1. The proof is completed. □

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