Electronic Journal of Qualitative Theory of Differential Equations
2026, No. **14**, 1–25; https://doi.org/10.14232/ejqtde.2026.1.14 www.math.u-szeged.hu/ejqtde/

Existence and multiplicity of radial sign-changing solutions for the Schrödinger–Born–Infeld system

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Received 28 October 2025, appeared 24 May 2026
Communicated by Gabriele Bonanno

Abstract. This paper is concerned with the existence and multiplicity of radial sign-changing solutions for a Schrödinger–Born–Infeld system in $\mathbb{R}^3$, involving a general subcritical nonlinearity $f(u)$. By employing variational methods, specifically the method of invariant sets of a descending flow, we establish the existence of at least one such solution under a general set of assumptions on f. Furthermore, when the nonlinearity f is odd, we prove that the system admits a sequence of radial sign-changing solutions whose energies are unbounded.

Keywords: Schrödinger–Born–Infeld equation, variational methods, invariant sets of descending flow, radial sign-changing solutions.

2020 Mathematics Subject Classification: 35J20, 35J60, 35B09, 35Q55.

1 Introduction

In this paper, we are concerned with the following Schrödinger–Born–Infeld system

$$\begin{cases} -\Delta u + u + \phi u = f(u) & \text{in } \mathbb{R}^3, \\ -\operatorname{div}\left(\dfrac{\nabla \phi}{\sqrt{1-|\nabla \phi|^2}}\right) = u^2 & \text{in } \mathbb{R}^3, \\ u(x) \to 0,\ \phi(x) \to 0, & \text{as } x \to \infty, \end{cases} \tag{1.1}$$

where $f \in \mathcal{C}(\mathbb{R}, \mathbb{R})$ is a suitable nonlinearity. A system like (1.1) is derived from the appropriate coupling of the Schrödinger Lagrangian and the Born–Infeld Lagrangian, which belongs to a class of models that describe the interaction between charged particles and electromagnetic fields. For a comprehensive review of recent developments concerning normalized solutions for related classes of nonlinear elliptic equations, such as Schrödinger and Schrödinger–Poisson systems, we refer the reader to [7]. The Schrödinger–Born–Infeld system is a typical

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nonlocal problem due to the presence of the term ϕ_u. In the past decade, various nonlocal operators have been extensively studied using variational methods. For instance, Kirchhoff-type equations with indefinite potentials or fractional operators have attracted significant attention (see, e.g., [14,15]).

In fact, within the field of classical electrodynamics, the Maxwell equations have long served as the core theoretical framework for describing the interaction between electromagnetic fields and charged particles (see, for instance, D'Aprile and Mugnai [8]). However, when studying the electrostatic field of a point charge, the electric field intensity diverges inversely with the square of the distance to the charge, which in turn causes the energy density of the field to tend to infinity. This divergence issue limits the applicability of classical electromagnetic theory at the microscopic scale, making it particularly difficult to describe physical processes dominated by quantum effects and strong-field interactions. In response, Born and Infeld proposed a nonlinear electromagnetic theory in [5,6], known as the Born–Infeld theory, which eliminates the energy divergence issue. The Lagrangian of the Born–Infeld theory can be expressed as

$$\mathcal{L}_{\mathrm{BI}} = b^2 \left(1 - \sqrt{1 - \frac{F_{\mu\nu}F^{\mu\nu}}{2b^2}} \right),$$

where $b > 0$ is the Born–Infeld parameter, which characterizes the strength of nonlinear effects.

This theory adopts a monistic perspective, asserting that both matter and electromagnetic fields are manifestations of a unique physical entity. By contrast, dualism holds that dynamics can be described through the appropriate combination of a Lagrangian associated with particles and a Lagrangian associated with electromagnetic fields.

From a dualistic standpoint, Yu [26] examined the coupling relationship between the Klein-Gordon equation and the Born–Infeld Lagrangian. Motivated by Yu's work, in [1] Azzollini, Pomponio, and Siciliano proposed and introduced a new model which replaces the Maxwell Lagrangian by the well-known Schrödinger–Maxwell system (proposed by Benci and Vieri in [3]) with the Born–Infeld Lagrangian, resulting in the Schrödinger–Born–Infeld system, which is as follows

$$\begin{cases} -\Delta u + u + \phi u = |u|^{p-1}u & \text{in } \mathbb{R}^3, \\ -\mathrm{div}\left(\dfrac{\nabla \phi}{\sqrt{1 - |\nabla \phi|^2}} \right) = u^2 & \text{in } \mathbb{R}^3, \\ u(x) \to 0, \ \phi(x) \to 0, & \text{as } x \to \infty. \end{cases} \tag{1.2}$$

In particular, they employed a slightly modified version of the monotonicity trick proposed by Jeanjean [9] and Struwe [21] to prove the existence of radial ground state solutions for system (1.2) when $p \in (5/2, 5)$.

In recent years, some authors began to focus on the system (1.2). Li, An and Wei analyzed the asymptotic behavior of solutions with general nonlinear terms in [10], which replaces the power-term with more general nonlinearities. Siciliano [19] proved the existence of minimal energy solutions by developing an approximation method, which improved the results obtained in [1]. Liu and Siciliano [12] adopted a perturbation method to prove the existence of radial ground state solutions and multiple solutions for the system (1.2) under subcritical and critical nonlinear conditions when $p \in (2, 5)$. Wang [23] discussed the existence of non-trivial solutions under a steep potential well. Wang, Sun and Chen in [22] obtained the existence of multiple solutions covering the case that $f(u) = |u|^{p-2}u$ for $p \in (2, 3]$ by Ekeland's variational principle and cutoff technique. In [29], Zhang focused on the variants of system (1.2)

$$\begin{cases} -\Delta u + u + \lambda \phi u = f(u) & \text{in } \mathbb{R}^3, \\ -\nabla \cdot \left(\frac{\nabla \phi}{1-\frac{1}{2}|\nabla \phi|^2} \right) = \lambda u^2 & \text{in } \mathbb{R}^3, \\ u(x) \to 0, \ \phi(x) \to 0, & \text{as } x \to \infty. \end{cases} \tag{1.3}$$

They established the existence of ground state solutions for system (1.3) via a truncation argument. To the best of our knowledge, the existence and multiplicity of sign-changing solutions for the Schrödinger–Born–Infeld system have not been previously studied. Motivated by the above works and inspired by [2, 11, 13, 27], in this paper, we study existence and multiplicity of radial sign-changing solutions to system (1.1). Recently, there has been significant interest in studying the existence of multiple solutions and their topological properties for various nonlinear elliptic problems. For instance, the existence of multiple solutions with sign information has been extensively investigated for Robin equations [17] and double-phase problems [16].

Formally, under some proper constraints on f, system (1.1) comes variationally from the action functional $\mathcal{F}$ defined by

$$\mathcal{F}(u,\phi) = \frac{1}{2}\int_{\mathbb{R}^3}(|\nabla u|^2 + u^2) + \frac{1}{2}\int_{\mathbb{R}^3}\phi u^2 - \frac{1}{2}\int_{\mathbb{R}^3}(1-\sqrt{1-|\nabla\phi|^2}) - \int_{\mathbb{R}^3}F(u),$$

where F is the primitive function of f and we omit dx to simplify the expression. It is obvious that the quantity $\sqrt{1-|\nabla\phi|^2}$ is well-defined only if the inequality $|\nabla\phi| \leqslant 1$ holds almost everywhere in $\mathbb{R}^3$. Thus, a necessary constraint is considered in the functional setting. Define

$$\mathcal{X} = \mathcal{D}^{1,2}(\mathbb{R}^3) \cap \{\phi \in \mathcal{C}^{0,1}(\mathbb{R}^3) : \|\nabla\phi\|_\infty \leqslant 1\},$$

where $\mathcal{D}^{1,2}(\mathbb{R}^3)$ is the completion of $\mathcal{C}_0^\infty(\mathbb{R}^3)$ with respect to the norm

$$\|\phi\|_{\mathcal{D}^{1,2}} = \left(\int_{\mathbb{R}^3}|\nabla\phi|^2\right)^{\frac{1}{2}}$$

and $\|\cdot\|_\infty$ denotes the usual norm of $L^\infty(\mathbb{R}^3)$. The functional $\mathcal{F}(u,\phi)$ defined on $H^1(\mathbb{R}^3) \times \mathcal{X}$ cannot directly apply the variational method, since $H^1(\mathbb{R}^3) \times \mathcal{X}$ is not a vector space. Fortunately, the strongly indefinite nature of the functional $\mathcal{F}$ can be smoothly removed by the classical reduction method. And the key is to find the relation between solutions of the minimizing problem and solutions of the second equation of system (1.1) for u fixed. Based on the work of Bonheure and Siciliano in [4], we obtained the uniqueness of the solution of the second equation in the radial setting. Let us define

$$H_r^1(\mathbb{R}^3) = \left\{u \in H^1(\mathbb{R}^3) : u \text{ is radially symmetric}\right\}$$

with the inner product and corresponding norm

$$(u,v) = \int_{\mathbb{R}^3}(\nabla u \cdot \nabla v + uv), \ \|u\| = \left(\int_{\mathbb{R}^3}(|\nabla u|^2 + |u|^2)\right)^{\frac{1}{2}}$$

and

$$\mathcal{X}_r(\mathbb{R}^3) = \{\phi \in \mathcal{X} \mid \phi \text{ is radially symmetric}\}.$$

Precisely, for $u \in H_r^1(\mathbb{R}^3)$ fixed, there exists a unique $\phi_u \in \mathcal{X}_r$ that is the unique solution to second equation of system (1.1). Then, the problem can be reduced to finding sign-changing critical points for the functional $\mathcal{F}(u,\phi) = \mathcal{F}(u,\phi_u)$, defined on $H_r^1(\mathbb{R}^3)$ (see Section 2 for more details). Recall that a solution (u,ϕ) to (1.1) is called a sign-changing solution if u changes its sign. Furthermore, due to the setting of the space, the sign-changing solutions we have found are radial.

In order to state our results, we assume that the nonlinearity f satisfies:

(f_1) $f \in \mathcal{C}(\mathbb{R},\mathbb{R})$ and $|f(t)| \leqslant c(1+|t|^{p-1})$ for some $c>0$ and $p \in (2,6)$;

(f_2) $\lim_{t\to 0} \frac{f(t)}{t} = 0$;

(f_3) there exists $\mu > 4$ such that $tf(t) \geqslant \mu F(t) > 0$ for all $t \neq 0$.

It should be noted that $F(t)$ is the same as mentioned earlier, and $F(t) = \int_0^t f(s)ds$.

The following is our first result in this paper.

Theorem 1.1. *Assume (f_1)–(f_3) hold, system (1.1) possesses at least one radial sign-changing solution. Moreover, if f is odd, then system (1.1) admits infinitely many radial sign-changing solutions $u_j \in H_r^1(\mathbb{R}^3)$, $j \in \mathbb{N}$. Moreover,*

$$\frac{1}{2}\int_{\mathbb{R}^3}(|\nabla u_j|^2 + u_j^2) + \frac{1}{2}\int_{\mathbb{R}^3}\phi_{u_j}u_j^2 - \int_{\mathbb{R}^3}\left(1-\sqrt{1-|\nabla\phi_{u_j}|^2}\right) - \int_{\mathbb{R}^3}F(u_j) \to \infty \quad \text{as } j \to \infty.$$

(f_3) is the well-known Ambrosetti–Rabinowitz condition ((AR) for short), which guarantees the compactness result when $\mu > 4$. Due to the presence of the nonlocal term, the compactness result becomes non-obvious when $\mu < 4$. Inspired by Liu and Siciliano's work in [12], we consider a new perturbation method to investigate the existence and multiplicity of sign-changing solutions for system (1.1) under conditions weaker than (f_3). Below is a new assumption on function f.

(f_4) there exists $\mu \in (3,4]$ such that $tf(t) \geqslant \mu F(t) > 0$ for all $t \neq 0$.

We state our second result as follows.

Theorem 1.2. *Assume (f_1)–(f_2) and (f_4) hold, system (1.1) possesses at least one radial sign-changing solution. Moreover, if f is odd, then system (1.1) admits infinitely many radial sign-changing solutions $u_j \in H_r^1(\mathbb{R}^3)$, $j \in \mathbb{N}$. Moreover,*

$$\frac{1}{2}\int_{\mathbb{R}^3}(|\nabla u_j|^2 + u_j^2) + \frac{1}{2}\int_{\mathbb{R}^3}\phi_{u_j}u_j^2 - \int_{\mathbb{R}^3}\left(1-\sqrt{1-|\nabla\phi_{u_j}|^2}\right) - \int_{\mathbb{R}^3}F(u_j) \to \infty \quad \text{as } j \to \infty.$$

It is worth to mentioning that after overcoming the obstacle of strong indefiniteness, we can directly apply the variational method. Subsequently, we have made a key observation: the nonlocal term ϕ_u neither has an exact expression nor can it be written as $\phi_u = \phi_{u^+} + \phi_{u^-}$ since the operator mapping u to ϕ_u is nonlinear. This implies that a classical constrained minimization method involves constructing a sign-changing Nehari manifold, proving its non-emptiness, demonstrating that the minimum value of the functional constrained on the manifold is attainable, and finally verifying the conclusion by combining with the Brouwer degree, such as those in [18,24,28], cannot be used to prove the existence of sign-changing solutions for system (1.1). To overcome the influence of the nonlocal term, inspired by [27], we consider using the abstract critical theorems proposed by Liu [11] to prove our main results.

For Theorem 1.1, the compactness condition is obtainable by conventional methods. We construct an auxiliary operator A (see Sect. 3), which serves as the starting point for constructing a pseudo-gradient vector field to ensure the existence of the desired invariant sets of the flow. Since A is merely continuous and cannot be directly used to construct invariant sets for the descending flow, we further construct a locally Lipschitz continuous operator B, which inherits the main properties of A, to define the flow. Combined with the abstract critical theorems, we draw the conclusion. For Theorem 1.2, the problem becomes more challenging due to the nonlocal term. To address the influence of the nonlocal term, we employ a new perturbation method by adding two terms with a small coefficient $\lambda > 0$, the first one grows faster than a degree-4 monomial, and the second one is the cube of the $L^2(\mathbb{R}^3)$ norm. After establishing results for the modified problem, we take limits to derive the existence and multiplicity of sign-changing solutions for the original system (1.1). To overcome the lack of compactness for $\mu \in (3,4]$, we employ a perturbation approach inspired by the works of Liu and Siciliano [12]. It is worth noting that a similar perturbative technique has been effectively utilized to obtain infinitely many solutions for gauged Schrödinger equations [25].

This paper is organized as follows: In Section 2, we construct the variational structure of system (1.1) and give some preliminary lemmas including abstract critical theorems. In Section 3, we construct auxiliary operators to prove the Theorem 1.1. In Section 4, we use a perturbation approach to prove Theorem 1.2.

Notation. Throughout this paper, we denote by $C, C_1, C_2, \dots, C', \dots$ various positive constants whose exact values may change from line to line. $|\cdot|_q$ denotes the usual norm of $L^q(\mathbb{R}^3)$ for $q \in [2, \infty)$. "$\to$" and "$\rightharpoonup$" denote the strong convergence and weak convergence in the related function space respectively.

2 Preliminary results

In this section, the variational setting for (1.1) is established and some preliminary lemmas have been listed. Then, we recall the abstract critical theorem. At the end of this section, we give the compactness condition under the assumptions (f_1)–(f_3).

Firstly, we present some properties of the ambient space of $\mathcal{X}$. For the proofs, see [4].

Lemma 2.1. *The following conclusions hold:*

(i) $\mathcal{X}$ *is continuously embedded in* $W^{1,p}(\mathbb{R}^3)$ *for all* $p \in [6, +\infty)$;

(ii) $\mathcal{X}$ *is continuously embedded in* $L^\infty(\mathbb{R}^3)$;

(iii) *if* $\phi \in \mathcal{X}$, *then* $\lim_{|x|\to\infty} \phi(x) = 0$;

(iv) $\mathcal{X}$ *is weakly closed;*

(v) *if* $\{\phi_n\} \subset \mathcal{X}$ *is bounded, there exists* $\phi \in \mathcal{X}$ *such that, up to subsequence,* $\phi_n \rightharpoonup \phi$ *in* $\mathcal{X}$ *and uniformly in compact sets in* $\mathbb{R}^3$.

Since the functional $\mathcal{F}$ is strongly indefinite on $H^1(\mathbb{R}^3) \times \mathcal{X}$ from above and from below, we consider a reduced one-variable functional, solving the second equation of system (1.1), which is successfully used with the classic Schrödinger–Poisson system. As mentioned in the introduction, we achieve this idea with the help of the following lemma.

Lemma 2.2 ([1, Lemma 2.2]). *For any $u \in H^1(\mathbb{R}^3)$ fixed, there exists a unique $\phi_u \in \mathcal{X}$ such that the following properties hold:*

(i) *ϕ_u is the unique minimizer of the functional $E(u, \cdot) : \mathcal{X} \to \mathbb{R}$ defined by*

$$E(u,\phi) = \int_{\mathbb{R}^3} \left(1 - \sqrt{1 - |\nabla \phi|^2}\right) - \int_{\mathbb{R}^3} \phi u^2 \tag{2.1}$$

and $E(u, \phi_u) \leqslant 0$, namely

$$\int_{\mathbb{R}^3} \phi_u u^2 \geqslant \int_{\mathbb{R}^3} (1 - \sqrt{1 - |\nabla \phi_u|^2}); \tag{2.2}$$

(ii) *$\phi_u \geqslant 0$ and $\phi_u = 0$ if and only if $u = 0$;*

(iii) *if ϕ is a weak solution of the second equation of system (1.1), then $\phi = \phi_u$ and it satisfies the following equality*

$$\int_{\mathbb{R}^3} \frac{|\nabla \phi_u|^2}{\sqrt{1 - |\nabla \phi_u|^2}} = \int_{\mathbb{R}^3} \phi_u u^2.$$

Moreover, if $u \in H_r^1(\mathbb{R}^3)$, then $\phi_u \in \mathcal{X}_r$ is the unique weak solution of the second equation of system (1.1).

In particular, since for all $t \in [0, 1)$, the following holds

$$1 - \sqrt{1 - t} \leqslant \frac{1}{2} \frac{t}{\sqrt{1 - t}}.$$

Recalling (iii) in Lemma 2.2, we can get the following inequality

$$\frac{1}{2} \int_{\mathbb{R}^3} \phi_u u^2 \geqslant \int_{\mathbb{R}^3} (1 - \sqrt{1 - |\nabla \phi_u|^2}). \tag{2.3}$$

By Lemma 2.2, system (1.1) can be reduced to the following one variable equation

$$-\Delta u + u + \phi_u u = f(u), \quad x \in \mathbb{R}^3. \tag{2.4}$$

Similarly, the functional $\mathcal{F}$ can be reduced to a one-variable functional defined on $u \in H_r^1(\mathbb{R}^3)$ as

$$\begin{aligned} J(u) &:= \mathcal{F}(u, \phi_u) \\ &= \frac{1}{2}\|u\|^2 + \frac{1}{2} \int_{\mathbb{R}^3} \phi_u u^2 - \frac{1}{2} \int_{\mathbb{R}^3} (1 - \sqrt{1 - |\nabla \phi_u|^2}) - \int_{\mathbb{R}^3} F(u) \\ &= \frac{1}{2}\|u\|^2 - \int_{\mathbb{R}^3} F(u) - \frac{1}{2} E(u, \phi_u). \end{aligned} \tag{2.5}$$

As shown in [1], the functional $J(u)$ is of class $\mathcal{C}^1$ and its Fréchet derivative at $u \in H_r^1(\mathbb{R}^3)$ is given by

$$\langle J'(u), v \rangle = \int_{\mathbb{R}^3} (\nabla u \cdot \nabla v + uv + \phi_u uv) - \int_{\mathbb{R}^3} f(u)v, \quad \text{for } v \in H_r^1(\mathbb{R}^3). \tag{2.6}$$

The following proposition establishes the relationship between the critical points of $J(u)$ and the weak solutions of system (1.1).

Proposition 2.3 ([1, Proposition 2.5]). *If $u \in H_r^1(\mathbb{R}^3) \setminus \{0\}$ is a critical point of $J(u)$, then (u, ϕ_u) is a weak nontrivial solution of (1.1).*

Thanks to Remark 5.5 in [4] and Proposition 2.4 in [1], the function ϕ_u satisfies

Lemma 2.4. *If $u_n \rightharpoonup u$ in $H_r^1(\mathbb{R}^3)$, then*

(i) $\phi_{u_n} \to \phi_u$ *in* $L^\infty(\mathbb{R}^3)$;

(ii) $\int_{\mathbb{R}^3} \phi_{u_n} u_n^2 \to \int_{\mathbb{R}^3} \phi_u u^2$, $\int_{\mathbb{R}^3} \phi_{u_n} u_n v \to \int_{\mathbb{R}^3} \phi_u uv$, $v \in H_r^1(\mathbb{R}^3)$.

The following lemma plays a crucial role in constructing invariant sets of functionals. For a proof, we refer to [1]

Lemma 2.5. *Let $s \in [2,3)$. Then there exist positive constants C and C' such that for any $u \in H_r^1(\mathbb{R}^3)$, we have*

$$|\nabla \phi_u|_2^{\frac{s-1}{s}} \leqslant C|u|_{2(s^*)'} \leqslant C'\|u\|,$$

where s^* is the critical Sobolev exponent related to s and $(s^*)'$ is its conjugate exponent, namely

$$s^* = \frac{3s}{3-s} \quad \text{and} \quad (s^*)' = \frac{3s}{4s-3}.$$

In particular, choosing $s = 2$, by Sobolev's embedding theorem and Hölder's inequality, we derive the following inequality

$$\int_{\mathbb{R}^3} \phi_u u^2 \leqslant |\phi_u|_6 |u|_{12/5}^2 \leqslant |\nabla \phi_u|_2 |u|_{12/5}^2 \leqslant C'|u|_{12/5}^4 \leqslant C''\|u\|^4. \tag{2.7}$$

Next, we present some definitions and the abstract critical theory. For additional details and proofs, we refer to [11].

Let X be a Banach space, $I \in \mathcal{C}^1(X, \mathbb{R})$, $P, Q \subset X$ be open sets, $M = P \cap Q$, $\Sigma = \partial P \cap \partial Q$ and $W = P \cup Q$. For $c \in \mathbb{R}$, define

$$K_c = \{x \in X : I(x) = c,\ I'(x) = 0\} \quad \text{and} \quad I^c = \{x \in X : I(x) \leqslant c\}.$$

Definition 2.6 ([11]). $\{P, Q\}$ is called an admissible family of invariant sets with respect to I at level c provided that the following deformation property holds: if $K_c \setminus W = \emptyset$, then, there exists $\varepsilon_0 > 0$ such that for $\varepsilon \in (0, \varepsilon_0)$, there exists $\eta \in \mathcal{C}(X, X)$ satisfying

(i) $\eta(\overline{P}) \subset \overline{P}$, $\eta(\overline{Q}) \subset \overline{Q}$;

(ii) $\eta \mid_{I^{c-2\varepsilon}} = \mathrm{id}$;

(iii) $\eta(I^{c+\varepsilon} \setminus W) \subset I^{c-\varepsilon}$.

Theorem 2.7 ([11, Theorem 2.4]). *Assume that $\{P, Q\}$ is an admissible family of invariant sets with respect to I at any level $c \geqslant c_* := \inf_{u \in \Sigma} I(u)$ and there exists a map $\varphi_0 : \Delta \to X$ satisfying*

(i) $\varphi_0(\partial_1 \Delta) \subset P$ *and* $\varphi_0(\partial_2 \Delta) \subset Q$;

(ii) $\varphi_0(\partial_0 \Delta) \cap M = \emptyset$;

(iii) $\sup_{u \in \varphi_0(\partial_0 \Delta)} I(u) < c_*$,

where $\Delta = \{(t_1,t_2) \in \mathbb{R}^2 : t_1,t_2 \geqslant 0, t_1+t_2 \leqslant 1\}$, $\partial_1\Delta = \{0\}\times[0,1]$, $\partial_2\Delta = [0,1]\times\{0\}$ *and* $\partial_0\Delta = \{(t_1,t_2)\in\mathbb{R}^2 : t_1,t_2 \geqslant 0, t_1+t_2 = 1\}$. *Define*

$$c = \inf_{\varphi\in\Gamma}\sup_{u\in\varphi(\Delta)\setminus W} I(u),$$

where $\Gamma := \{\varphi \in \mathcal{C}(\Delta, X) : \varphi(\partial_1\Delta) \subset P, \varphi(\partial_2\Delta) \subset Q, \varphi|_{\partial_0\Delta} = \varphi_0|_{\partial_0\Delta}\}$. *Then* $c \geqslant c_*$ *and* $K_c \setminus W \neq \emptyset$.

To obtain the second statement of Theorem 1.1, we present the following properties and theorems.

Assume $G : X \to X$ to be an isometric involution, that is, $G^2 = \mathrm{id}$ and $d(Gx, Gy) = d(x,y)$ for $x,y \in X$. We assume I is G-invariant on X in the sense that $I(Gx) = I(x)$ for any $x \in X$. We also assume $Q = GP$. A subset $F \subset X$ is said to be symmetric if $Gx \in F$ for any $x \in F$. The genus of a closed symmetric subset F of $X\setminus\{0\}$ is denoted by $\gamma(F)$.

Definition 2.8 ([11]). P is called a G-admissible invariant set with respect to I at level c, if the following deformation property holds: there exist $\varepsilon_0 > 0$ and a symmetric open neighborhood N of $K_c \setminus W$ with $\gamma(\overline{N}) < \infty$, such that for $\varepsilon \in (0, \varepsilon_0)$ there exists $\eta \in \mathcal{C}(X, X)$ satisfying

(*i*) $\eta(\overline{P}) \subset \overline{P}$, $\eta(\overline{Q}) \subset \overline{Q}$;

(*ii*) $\eta \circ G = G \circ \eta$;

(*iii*) $\eta\, |_{I^{c-2\varepsilon}} = id$;

(*iv*) $\eta(I^{c+\varepsilon} \setminus (N \cup W)) \subset I^{c-\varepsilon}$.

Theorem 2.9 ([11, Theorem 2.5]). *Assume that* P *is a* G*-admissible invariant set with respect to* I *at any level* $c \geqslant c_* := \inf_{u\in\Sigma} I(u)$ *and for any* $n \in \mathbb{N}$*, there exists a continuous map* $\varphi_n : B_n := \{x \in \mathbb{R}^n : |x| \leqslant 1\} \to X$ *satisfying*

(i) $\varphi_n(0) \in M := P \cap Q$, $\varphi_n(-t) = G\varphi_n(t)$ *for* $t \in B_n$;

(ii) $\varphi_n(\partial B_n) \cap M = \emptyset$;

(iii) $\sup_{u\in \mathrm{Fix}_G \cup \varphi_n(\partial B_n)} I(u) < c_*$, *where* $\mathrm{Fix}_G := \{u \in X : Gu = u\}$. *For* $j \in \mathbb{N}$*, define*

$$c_j = \inf_{B\in\Gamma_j}\sup_{u\in B\setminus W} I(u),$$

where

$$\Gamma_j := \left\{ B \;\middle|\; \begin{array}{l} B = \varphi(B_n \setminus Y) \text{ for some } \varphi \in G_n,\ n \geqslant j, \text{ and open } Y \subset B_n \\ \text{such that } -Y = Y \text{ and } \gamma(\bar{Y}) \leqslant n-j \end{array} \right\}$$

and

$$G_n := \{\varphi \in \mathcal{C}(B_n, X) \mid \varphi(-t) = G\varphi(t) \text{ for } t \in B_n, \varphi(0) \in M \text{ and } \varphi|_{\partial B_n} = \varphi_n|_{\partial B_n}\}.$$

Then for $j \geqslant 2$, $c_j \geqslant c_*$, $K_{c_j} \setminus W \neq \emptyset$ *and* $c_j \to \infty$ *as* $j \to \infty$.

To end this section, we give a compactness result.

Lemma 2.10. *Under the assumptions* (f_1)–(f_3) *of Theorem 1.1, the functional* J *satisfies the (PS) condition.*

Proof. Assume that there exists a sequence $\{u_n\} \subset H_r^1(\mathbb{R}^3)$ such that $\{J(u_n)\}$ is bounded and $J'(u_n) \to 0$ as $n \to \infty$.

The proof consists of two steps. First, we show that the (PS) sequence $\{u_n\}$ is bounded in $H_r^1(\mathbb{R}^3)$. By (f_3), (2.5), and (2.6), we have

$$\begin{aligned}|J(u_n)| + \frac{1}{\mu}\|J'(u_n)\|\|u_n\| &\geqslant J(u_n) - \frac{1}{\mu}\langle J'(u_n), u_n\rangle \\ &= \frac{1}{2}\|u_n\|^2 + \frac{1}{2}\int_{\mathbb{R}^3}\phi_{u_n}u_n^2 - \frac{1}{2}\int_{\mathbb{R}^3}(1-\sqrt{1-|\nabla\phi_{u_n}|^2}) - \int_{\mathbb{R}^3}F(u_n) \\ &\quad - \frac{1}{\mu}\|u_n\|^2 - \frac{1}{\mu}\int_{\mathbb{R}^3}\phi_{u_n}u_n^2 + \frac{1}{\mu}\int_{\mathbb{R}^3}f(u_n)u_n \\ &\geqslant \frac{\mu-2}{2\mu}\|u_n\|^2 + \frac{\mu-4}{4\mu}\int_{\mathbb{R}^3}\phi_{u_n}u_n^2 \\ &\geqslant \frac{\mu-2}{2\mu}\|u_n\|^2.\end{aligned}$$

Then, $\{u_n\}$ is bounded.

Second, we prove that $\{u_n\}$ contains a strongly convergent subsequence. Exploiting the Strauss's Lemma [20] for $H_r^1(\mathbb{R}^3)$, there exists a subsequence, still denoted as $\{u_n\}$ and $u \in H_r^1(\mathbb{R}^3)$ such that

$$\begin{aligned} u_n &\rightharpoonup u && \text{in } H_r^1(\mathbb{R}^3), \\ u_n &\to u && \text{in } L^q(\mathbb{R}^3),\ q \in (2,6), \\ u_n(x) &\to u(x) && \text{a.e. } x \in \mathbb{R}^3. \end{aligned}$$

By taking the limit, we have

$$\begin{aligned} 0 \leftarrow &\langle J'(u_n), u_n - u\rangle \\ &= \int_{\mathbb{R}^3}\nabla u_n \nabla(u_n-u) + \int_{\mathbb{R}^3}u_n(u_n-u) \\ &\quad + \int_{\mathbb{R}^3}\phi_{u_n}u_n(u_n-u) + \int_{\mathbb{R}^3}f((u_n)(u_n-u). \end{aligned} \tag{2.8}$$

Using (2.7) and Hölder's inequality, the following holds

$$\int_{\mathbb{R}^3}|\phi_{u_n}u_n||u_n-u| \leqslant |\phi_{u_n}|_6|u_n|_2|u_n-u|_3 \leqslant C''\|u_n\|^2|u_n|_2|u_n-u|_3 \to 0, n\to\infty. \tag{2.9}$$

By(f_1) and (f_2), for any small $\delta > 0$, there exists $C_\delta > 0$ such that

$$|f(t)| \leqslant \delta|t| + C_\delta|t|^{p-1}, t \in \mathbb{R}. \tag{2.10}$$

Then, from (2.10), using the fact $\delta > 0$ is arbitrary, we deduce that

$$\begin{aligned}\int_{\mathbb{R}^3}|f(u_n)||u_n-u| &\leqslant \delta\int_{\mathbb{R}^3}|u_n||u_n-u| + C_\delta\int_{\mathbb{R}^3}|u_n|^{p-1}|u_n-u| \\ &\leqslant \delta|u_n|_2|u_n-u|_2 + C_\delta|u_n|_p^{p-1}|u_n-u|_p \\ &\leqslant \delta\|u_n\||u_n-u|_2 + CC_\delta\|u_n\|^{p-1}|u_n-u|_p \\ &\to 0, \quad n\to\infty.\end{aligned} \tag{2.11}$$

Combining with (2.8), (2.9) and (2.11), we conclude that

$$\|u_n\|^2 - \|u\|^2 = o_n(1).$$

and then $u_n \to u$ in $H_r^1(\mathbb{R}^3)$. □

3 Proof of Theorem 1.1

In this section, we aim to apply the abstract critical theory to prove main results of system (1.1), which is produced by the minimax method combined with invariant sets of descending flow.

To derive the main results with the aforementioned theorems, we firstly construct the descending flow by means of an auxiliary operator $A : H_r^1(\mathbb{R}^3) \to H_r^1(\mathbb{R}^3)$. The auxiliary operator A is defined as follows: for any $u \in H_r^1(\mathbb{R}^3)$, $v = Au \in H_r^1(\mathbb{R}^3)$ is the unique solution to the equation

$$-\Delta v + v + \phi_u v = f(u), \quad v \in H_r^1(\mathbb{R}^3). \tag{3.1}$$

Lemma 3.1. *Suppose that (f_1)–(f_3) hold, then*

(i) the operator A is well-defined;

(ii) the operator$A$ maps bounded sets into bounded sets;

(iii) the operator A is continuous;

(iv) the operator A is compact;

(v) if f is odd then A is odd.

Proof. (i) For any given $u \in H_r^1(\mathbb{R}^3)$, we define

$$K(v) = \frac{1}{2}\int_{\mathbb{R}^3}(|\nabla v|^2 + v^2) + \frac{1}{2}\int_{\mathbb{R}^3}\phi_u v^2 - \int_{\mathbb{R}^3} f(u)v, \quad v \in H_r^1(\mathbb{R}^3). \tag{3.2}$$

It is obvious that $K \in C^1(H_r^1(\mathbb{R}^3), \mathbb{R})$.

Then, by Lemma 2.2, (2.10) and Hölder's inequality, the following inequality holds

$$\begin{aligned} K(v) &\geqslant \frac{1}{2}\|v\|^2 - \int_{\mathbb{R}^3} f(u)v \\ &\geqslant \frac{1}{2}\|v\|^2 - \delta\|u\|\|v\| - C_\delta\|u\|^{p-1}\|v\|. \end{aligned}$$

It means that as $\|v\| \to \infty$, $K(v) \to \infty$. Then, $K(v)$ is coercive, bounded from below, strictly convex, and weakly lower semi-continuous. Therefore, $K(v)$ admits a unique minimizer $v = Au \in H_r^1(\mathbb{R}^3)$, which is the unique solution of (3.1). That is, the operator A is well defined on $H_r^1(\mathbb{R}^3)$.

(ii) For $v = Au$, it follows from the definition of A and (2.10) that

$$\|v\|^2 \leqslant \|v\|^2 + \int_{\mathbb{R}^3}\phi_u v^2 = \int_{\mathbb{R}^3} f(u)v \leqslant \delta\|u\|\|v\| + C_\delta\|u\|^{p-1}\|v\|.$$

Hence, we conclude that, A maps bounded sets into bounded sets.

(iii) Let $u_n \to u$ in $H_r^1(\mathbb{R}^3)$ as $n \to \infty$, and we denote $v = Au$, $v_n = Au_n$. It follows from (3.1) and (3.2) that

$$\begin{aligned} \|v_n - v\|^2 &\leqslant \|v_n - v\|^2 + \int_{\mathbb{R}^3}\phi_{u_n}(v_n - v)^2 \\ &= \int_{\mathbb{R}^3}(f(u) - f(u_n))(v - v_n) + \int_{\mathbb{R}^3}(\phi_{u_n}v_n - \phi_u v)(v - v_n) + \int_{\mathbb{R}^3}\phi_{u_n}(v_n - v)^2 \\ &= \int_{\mathbb{R}^3}(f(u) - f(u_n))(v - v_n) + \int_{\mathbb{R}^3}(\phi_{u_n} - \phi_u)v(v - v_n). \end{aligned}$$

By Hölder's inequality and Lemma 2.4, it follows that

$$\int_{\mathbb{R}^3}(\phi_{u_n}-\phi_u)v(v-v_n)\leqslant C\|\phi_{u_n}-\phi_u\|_\infty\|v\|\|v_n-v\|.$$

Let $\kappa\in C_0^\infty(\mathbb{R})$ be such that $\kappa(t)\in[0,1]$ for $t\in\mathbb{R}$, $\kappa(t)=1$ for $|t|\leqslant 1$ and $\kappa(t)=0$ for $|t|\geqslant 2$. Setting

$$g_1(t)=\kappa(t)f(t),\ g_2(t)=f(t)-g_1(t).$$

By (f_1)–(f_2), there exists $C_1>0$ such that $|g_1(s)|\leqslant C_1|s|$ and $|g_2(s)|\leqslant C_1|s|^5$ for $s\in\mathbb{R}$. Then,

$$\begin{aligned}\int_{\mathbb{R}^3}(f(u)-f(u_n))(v-v_n)&=\int_{\mathbb{R}^3}(g_1(u)-g_1(u_n))(v-v_n)+\int_{\mathbb{R}^3}(g_2(u)-g_2(u_n))(v-v_n)\\&\leqslant|g_1(u_n)-g_1(u)|_2|v-v_n|_2+|g_2(u_n)-g_2(u)|_{\frac{6}{5}}|v-v_n|_6\\&\leqslant C_2\|v-v_n\|(|g_1(u_n)-g_1(u)|_2+|g_2(u_n)-g_2(u)|_{\frac{6}{5}}).\end{aligned}$$

Thus,

$$\|v-v_n\|\leqslant C_3(\|\phi_{u_n}-\phi_u\|_\infty\|v\|+|g_1(u_n)-g_1(u)|_2+|g_2(u_n)-g_2(u)|_{\frac{6}{5}}).$$

Therefore, by the dominated convergence theorem, $\|v-v_n\|\to 0$ as $n\to\infty$. That is, A is continuous.

(iv) Let $\{u_n\}\subset H_r^1(\mathbb{R}^3)$ be a bounded sequence and denote $v_n=Au_n$. It follows from (ii) that $\{v_n\}\subset H_r^1(\mathbb{R}^3)$ is a bounded sequence. Passing to a subsequence, we may assume that $u_n\rightharpoonup u$ and $v_n\rightharpoonup v$ in $H_r^1(\mathbb{R}^3)$. Consider the following identity

$$\int_{\mathbb{R}^3}(\nabla v_n\cdot\nabla\psi+v_n\psi)+\int_{\mathbb{R}^3}\phi_{u_n}v_n\psi=\int_{\mathbb{R}^3}f(u_n)\psi,\quad\psi\in H_r^1(\mathbb{R}^3).$$

It is obvious that $\int_{\mathbb{R}^3}f(u_n)\psi\to\int_{\mathbb{R}^3}f(u)\psi$. Furthermore, it can be derived from Lemma 2.4 that $\int_{\mathbb{R}^3}\phi_{u_n}v_n\psi\to\int_{\mathbb{R}^3}\phi_u v\psi$, for any $\psi\in H_r^1(\mathbb{R}^3)$. Then, as $n\to\infty$, we have

$$\int_{\mathbb{R}^3}(\nabla v\cdot\nabla\psi+v\psi)+\int_{\mathbb{R}^3}\phi_u v\psi=\int_{\mathbb{R}^3}f(u)\psi,\quad\psi\in H_r^1(\mathbb{R}^3).$$

That is $v=Au$ and thus

$$\|v_n-v\|^2=\int_{\mathbb{R}^3}(f(u)-f(u_n))(v-v_n)+\int_{\mathbb{R}^3}(\phi_{u_n}v_n-\phi_u v)(v-v_n).$$

Using the same proof method as in (iii), we can obtain $v_n\to v$, which means that the operator A is compact.

(v) if f is odd and $\phi_{-u}=\phi_u$, by (3.1) it is obvious that A is odd. We use proof by contradiction to verify $\phi_{-u}=\phi_u$. If $\phi_{-u}\neq\phi_u$, from (2.2) it is deduced that

$$E(-u,\phi)=\int_{\mathbb{R}^3}(1-\sqrt{1-|\nabla\phi|^2})-\int_{\mathbb{R}^3}\phi(-u)^2=\int_{\mathbb{R}^3}(1-\sqrt{1-|\nabla\phi|^2})-\int_{\mathbb{R}^3}\phi u^2=E(u,\phi).$$

According to Lemma 2.2, for any $u\in H_r^1(\mathbb{R}^3)$ fixed, (u,ϕ_u) is the unique minimizer of the functional $E(-u,\cdot)$. That is

$$E(u,\phi_{-u})=E(-u,\phi_{-u})\leqslant E(-u,\phi)=E(u,\phi),\quad\forall\phi\in\mathcal{X}_r,$$

which is a contradiction. □

Remark 3.2. From Lemma 3.1 it is obvious that the three statements are equivalent: u is a solution of (2.4), u is a critical point of the functional $J(u)$, and u is a fixed point of the operator A.

The following lemma is crucial for the construction of the descending flow.

Lemma 3.3. *The following statements hold:*

(i) $\langle J'(u), u - Au\rangle \geqslant \|u - Au\|^2$ *for all* $u \in H_r^1(\mathbb{R}^3)$*;*

(ii) $\|J'(u)\| \leqslant \|u - Au\|(1 + C\|u\|^2)$ *for some* $C > 0$ *and all* $u \in H_r^1(\mathbb{R}^3)$*;*

(iii) for $a < b$ *and* $\alpha > 0$*, there exists* $\beta > 0$ *such that* $\|u - Au\| \geqslant \beta$ *if* $u \in H_r^1(\mathbb{R}^3)$*,* $J(u) \in [a, b]$ *and* $\|J'(u)\| \geqslant \alpha$*.*

Proof. (i) Let $v = Au$. Combining with (2.6) and (3.1), it follows that

$$\begin{aligned}\langle J'(u), u - v\rangle &= \|u - v\|^2 + \int_{\mathbb{R}^3} \phi_u (u - v)^2 \\ &\geqslant \|u - v\|^2.\end{aligned}$$

(ii) Similarly, by (2.7), Sobolev's embedding theorem and Hölder's inequality, take any $\omega \in H_r^1(\mathbb{R}^3)$,

$$\begin{aligned}\langle J'(u), \omega\rangle &= \langle J'(u), \omega\rangle - (Au, \omega) + (Au, \omega) \\ &= (u - Au, \omega) + \int_{\mathbb{R}^3} \phi_u (u - Au)\omega \\ &\leqslant \|u - Au\|\|\omega\| + C\|u\|^2\|u - Au\|\|\omega\|.\end{aligned}$$

The conclusion follows from the definition of the operator norm.

(iii) By (f_3) , (2.3) and (3.1), it follows that

$$\begin{aligned}J(u) - \frac{1}{\mu}(u, u - Au) &= \frac{1}{2}\|u\|^2 + \frac{1}{2}\int_{\mathbb{R}^3} \phi_u u^2 - \frac{1}{2}\int_{\mathbb{R}^3}\left(1 - \sqrt{1 - |\nabla\phi_u|^2}\right) - \int_{\mathbb{R}^3} F(u) \\ &\quad - \frac{1}{\mu}\|u\|^2 - \frac{1}{\mu}\int_{\mathbb{R}^3} \phi_u u Au + \frac{1}{\mu}\int_{\mathbb{R}^3} f(u)u \\ &\geqslant \left(\frac{1}{2} - \frac{1}{\mu}\right)\|u\|^2 + \frac{1}{4}\int_{\mathbb{R}^3} \phi_u u^2 - \frac{1}{\mu}\int_{\mathbb{R}^3} \phi_u u Au \\ &= \left(\frac{1}{2} - \frac{1}{\mu}\right)\|u\|^2 + \left(\frac{1}{4} - \frac{1}{\mu}\right)\int_{\mathbb{R}^3} \phi_u u^2 + \frac{1}{\mu}\int_{\mathbb{R}^3} \phi_u u(u - Au).\end{aligned}$$

Then,

$$\|u\|^2 + \int_{\mathbb{R}^3} \phi_u u^2 \leqslant C(|J(u)| + \|u\|\|u - Au\| + |\int_{\mathbb{R}^3} \phi_u u(u - Au)|). \tag{3.3}$$

By (2.7) and Hölder's inequality, we can derive

$$\begin{aligned}\left|\int_{\mathbb{R}^3} \phi_u u(u - Au)\right| &\leqslant \left(\int_{\mathbb{R}^3} \phi_u (u - Au)^2\right)^{\frac{1}{2}} \left(\int_{\mathbb{R}^3} \phi_u u^2\right)^{\frac{1}{2}} \\ &\leqslant C_1\|u\|\|u - A\| \left(\int_{\mathbb{R}^3} \phi_u u^2\right)^{\frac{1}{2}}.\end{aligned}$$

Thus, by Young's inequality, it follows from (3.3) that

$$\|u\|^2 \leqslant C\left(|J(u)| + \|u\|\|u - Au\| + \|u\|^2\|u - Au\|^2\right). \tag{3.4}$$

If there exists $\{u_n\} \subset H_r^1(\mathbb{R}^3)$ with $J(u_n) \in [a, b]$ and $\|J'(u_n)\| \geqslant \alpha$ such that $\|u_n - Au_n\| \to 0$ as $n \to \infty$, then it follows from (3.4) that $\{\|u_n\|\}$ is bounded. According to (ii), $\|J'(u_n)\| \to 0$ as $n \to \infty$, which is a contradiction. □

To obtain sign-changing solutions, we define the positive and negative cones as follows

$$P^+ := \{u \in H_r^1(\mathbb{R}^3) : u \geqslant 0\} \quad \text{and} \quad P^- := \{u \in H_r^1(\mathbb{R}^3) : u \leqslant 0\}.$$

Set for $\varepsilon > 0$,

$$P_\varepsilon^+ := \{u \in H_r^1(\mathbb{R}^3) : \text{dist}(u, P^+) < \varepsilon\} \quad \text{and} \quad P_\varepsilon^- := \{u \in H_r^1(\mathbb{R}^3) : \text{dist}(u, P^-) < \varepsilon\},$$

where $\text{dist}(u, P^\pm) = \inf_{v \in P^\pm} \|u - v\|$. Obviously, $P_\varepsilon^- = -P_\varepsilon^+$. Let $W = P_\varepsilon^+ \cup P_\varepsilon^-$. Then, W is an open and symmetric subset of $H_r^1(\mathbb{R}^3)$ and $H_r^1(\mathbb{R}^3) \setminus W$ contains only sign-changing functions.

The next lemma shows that for ε small, all sign-changing solutions to (2.4) are contained in $H_r^1(\mathbb{R}^3) \setminus W$.

Lemma 3.4. *There exists $\varepsilon_0 > 0$ such that for $\varepsilon \in (0, \varepsilon_0)$,*

(i) $A(\partial P_\varepsilon^-) \subset P_\varepsilon^-$ and every nontrivial solution $u \in P_\varepsilon^-$ is negative,

(ii) $A(\partial P_\varepsilon^+) \subset P_\varepsilon^+$ and every nontrivial solution $u \in P_\varepsilon^+$ is positive.

Proof. The proof is similar to the proof of Lemma 3.4 in [13]. Since the proofs of (i) and (ii) are almost identical, we only present the proof of (i).

For any $u \in H_r^1(\mathbb{R}^3)$, it is known that $\text{dist}(u, P^\mp) = \|u^\pm\|$. Take $u \in H_r^1(\mathbb{R}^3)$ and $v = Au$. By Sobolev's embedding theorem, for any $q \in [2, 6]$ there exists a constant $\beta_q > 0$ such that $|w|_q \leq \beta_q \|w\|$ for all $w \in H_r^1(\mathbb{R}^3)$. Applying this to $w = u^\pm$, we get

$$\left|u^\pm\right|_q = \inf_{w \in P^\mp} |u - w|_q \leqslant \beta_q \inf_{w \in P^\mp} \|u - w\| = \beta_q \,\text{dist}\left(u, P^\mp\right). \tag{3.5}$$

Since $\text{dist}(v, P^-) \leqslant \|v^+\|$ holds, it follows from (3.1) and Lemma 2.2 that

$$\text{dist}(v, P^-)\|v^+\| \leqslant \|v^+\|^2 = \int_{\mathbb{R}^3} f(u)v^+ - \int_{\mathbb{R}^3} \phi_u v v^+ \leqslant \int_{\mathbb{R}^3} f(u)v^+.$$

It is obvious from (f_3) that

$$\int_{\mathbb{R}^3} f(u)v^+ \leqslant \int_{\{u>0\}} f(u)v^+ = \int_{\mathbb{R}^3} f\left(u^+\right)v^+.$$

Then, combining with (2.10), the following inequality holds

$$\begin{aligned} \text{dist}(v, P^-)\left\|v^+\right\| &\leqslant \delta \left|u^+\right|_2 \left|v^+\right|_2 + C_\delta \left|u^+\right|_p^{p-1} \left|v^+\right|_p \\ &\leqslant C\left(\delta \,\text{dist}\left(u, P^-\right) + C_\delta \,\text{dist}\left(u, P^-\right)^{p-1}\right)\left\|v^+\right\|. \end{aligned}$$

That is

$$\text{dist}(Au, P^-) \leqslant C\left(\delta \,\text{dist}\left(u, P^-\right) + C_\delta \,\text{dist}\left(u, P^-\right)^{p-1}\right).$$

Fix $\delta > 0$ small enough such that $C_\delta < 1/2$. Then, there exists $\varepsilon_0 > 0$ such that for $\varepsilon \in (0, \varepsilon_0)$, for any $u \in P_\varepsilon^-$, we have

$$\operatorname{dist}(Au, P^-) \leqslant \frac{1}{2} \operatorname{dist}\left(u, P^-\right).$$

This implies that $A\left(\partial P_\varepsilon^-\right) \subset P_\varepsilon^-$. If there exists $u \in P_\varepsilon^-$ such that $Au = u \neq 0$, then $u \in P^-$. If $u \not\equiv 0$, by the maximum principle, $u < 0$ in $\mathbb{R}^3$. □

Since A only has continuity, it cannot yet be used to construct a descending gradient flow. Therefore, in the following lemma, we shall construct a locally Lipschitz continuous operator B on $H := H_r^1\left(\mathbb{R}^3\right) \setminus H_0$ that satisfies the main properties as A, where H_0 is the set of fixed points of A.

Lemma 3.5. *There exists a locally Lipschitz continuous operator $B: \ H \to H_r^1\left(\mathbb{R}^3\right)$ such that*

(i) $B(\partial P_\varepsilon^+) \subset P_\varepsilon^+$ and $B(\partial P_\varepsilon^-) \subset P_\varepsilon^-$ for $\varepsilon \in (0, \varepsilon_0)$;

(ii) $\frac{1}{2}\|u - B(u)\| \leqslant \|u - A(u)\| \leqslant 2\|u - B(u)\|$ for all $u \in H$;

(iii) $\langle J'(u), u - B(u)\rangle \geqslant \frac{1}{2}\|u - A(u)\|^2$ for all $u \in H$;

(iv) if f is odd then B is odd.

Proof. The proof is similar to the proofs of Lemma 4.1 in [2] and Lemma 3.5 in [27]. For the sake of completeness, we give the details here. Let C be as in Lemma 3.3 , for any $u \in H$, set

$$\Delta_1(u) = \frac{1}{2}\|u - Au\|, \quad \Delta_2(u) = \frac{1}{2}\|u - Au\|(1 + C\|u\|^2)^{-1}, \tag{3.6}$$

where $\Delta_1, \Delta_2 \in \mathcal{C}(H, \mathbb{R})$. Since A is continuous on H, for each $u \in H$, we can choose $r(u) \in (0,1)$ small enough such that for every $v, w \in N(u) := \{z \in H_r^1\left(\mathbb{R}^3\right) : \|z - u\| \leq r(u)\}$, the following inequality holds:

$$\|Av - Aw\| \leqslant \min\left\{\Delta_1(v), \Delta_1(w), \Delta_2(v), \Delta_2(w)\right\}. \tag{3.7}$$

Let $\mathcal{V}$ be a locally finite open refinement of $\{N(u), u \in H\}$, and define

$$\mathcal{W} = \{V \in \mathcal{V} : V \cap \overline{P_\varepsilon^+} \neq \emptyset,\ V \cap \overline{P_\varepsilon^-} \neq \emptyset,\ V \cap \overline{P_\varepsilon^+} \cap \overline{P_\varepsilon^-} = \emptyset\} \tag{3.8}$$

and

$$\mathcal{U} := \bigcup_{V \in \mathcal{V} \setminus \mathcal{W}} V \cup \bigcup_{V \in \mathcal{W}} \{V \setminus \overline{P_\varepsilon^+}, V \setminus \overline{P_\varepsilon^-}\}.$$

Then, $\mathcal{U}$ is a locally finite open refinement of $\{N(u) : u \in H\}$, and for any $U \in \mathcal{U}$,

$$\text{if} \quad U \cap \overline{P_\varepsilon^+} \neq \emptyset \quad \text{and} \quad U \cap \overline{P_\varepsilon^-} \neq \emptyset, \quad \text{then} \quad U \cap \overline{P_\varepsilon^+} \cap \overline{P_\varepsilon^-} \neq \emptyset.$$

Define $\rho_U(u) = \operatorname{dist}(u, H \setminus U)$. Let $\{\pi_U : U \in \mathcal{U}\}$ be the standard partition of unity subordinated to $\mathcal{U}$ defined by

$$\pi_U(u) := \left(\sum_{V \in \mathcal{U}} \rho_V(u)\right)^{-1} \rho_U(u).$$

For any $U \in \mathcal{U}$, we choose $a_U \in U$ such that

$$\text{if } U \cap \overline{P_\varepsilon^+} \neq \emptyset, \text{ then } a_U \in U \cap \overline{P_\varepsilon^+} \cap \overline{P_\varepsilon^-}.$$

Now, we define the operator $B : H \to H^1_r(\mathbb{R}^3)$ by

$$B(u) = \sum_{U \in \mathcal{U}} \pi_U(u) A(a_U).$$

Then, by the Lipschitz continuity of π_U and the locally finiteness of the covering $\mathcal{U}$, the operator $B : H \to H^1_r(\mathbb{R}^3)$ is locally Lipschitz continuous. The conclusion (i) follows.

For (ii) and (iii), take $u \in H$, we observe that

$$\|Bu - Au\| \leqslant \sum_{U \in \mathcal{U}} \pi_U(u) \|A(a_U) - Au\|.$$

This together with (3.6), (3.7) implies that for $u \in H$,

$$\|Bu - Au\| < \sum_{U \in \mathcal{U}} \pi_U(u) \left(\frac{1}{2} \|u - Au\| \right) = \frac{1}{2} \|u - Au\| \tag{3.9}$$

and

$$\|Bu - Au\| < \frac{1}{2} \|u - Au\| (1 + C\|u\|^2)^{-1}. \tag{3.10}$$

Then, it follows from (3.9) that conclusion (ii) holds.

By (ii) of Lemma 3.3 and (3.10), for $u \in H$,

$$\begin{aligned} \langle J'(u), u - Bu \rangle &\geqslant \langle J'(u), u - Au \rangle - \|J'(u)\| \|Bu - Au\| \\ &\geqslant \frac{1}{2} \|u - Au\|^2. \end{aligned}$$

The conclusion (iii) follows.

For (iv), if f is odd, then J is even and A is odd. Since $P^+_\varepsilon = -P^-_\varepsilon$, we can replace $B(u)$ by $\widetilde{B}(u) = \frac{1}{2}(B(u) - B(-u))$ which is odd. It is a standard procedure to verify that $\widetilde{B}(u)$ is also locally Lipschitz and satisfies properties (i)–(iii). For simplicity, we still denote $\widetilde{B}(u)$ by $B(u)$. □

Next, we use Theorem 2.7 and Theorem 2.9 to prove Theorem 1.1. Take $X = H^1_r(\mathbb{R}^3)$, $P = P^+_\varepsilon$, $Q = P^-_\varepsilon$, $W = P^+_\varepsilon \cup P^-_\varepsilon$, $\Sigma = \partial P^-_\varepsilon \cap \partial P^+_\varepsilon$, and we first show that $\{P^+_\varepsilon, P^-_\varepsilon\}$ is an admissible family of invariant sets for the functional J at any level $c \in \mathbb{R}$.

Lemma 3.6. *$\{P^+_\varepsilon, P^-_\varepsilon\}$ is an admissible family of invariant sets for the functional J at any level $c \in \mathbb{R}$.*

Proof. We need to construct a descending flow invariant set that satisfies Definition 2.6. The proof relies on the properties of the locally Lipschitz operator B constructed in Lemma 3.5.

Firstly, we build a mapping that holds the following conditions, which is standard for functionals satisfying the (PS) condition as is stated in Lemma 2.10 and it is the same as the proof of Lemma 3.6 in [13].

There exists $\varepsilon_0 > 0$ such that for $0 < \varepsilon < \varepsilon' < \varepsilon_0$, there exists a continuous map $\sigma : [0,1] \times H^1_r(\mathbb{R}^3) \to H^1_r(\mathbb{R}^3)$ satisfying

(i) $\sigma(0, u) = u$ for $u \in H^1_r(\mathbb{R}^3)$;

(ii) $\sigma(t, u) = u$ for $t \in [0,1]$, $u \notin J^{-1}[c - \varepsilon', c + \varepsilon']$;

(iii) $\sigma(1, J^{c+\varepsilon} \setminus W) \subset J^{c-\varepsilon}$;

(iv) $\sigma(t, \overline{P^+_\varepsilon}) \subset \overline{P^+_\varepsilon}$ and $\sigma(t, \overline{P^-_\varepsilon}) \subset \overline{P^-_\varepsilon}$ for $t \in [0,1]$.

Then, take $\eta(\cdot) = \sigma(1, \cdot)$ and the proof is accomplished. □

Corollary 3.7. *If f is also odd, P_ε^+ is a G-admissible invariant set for the functional J at any level c.*

This conclusion can be obtained by using the same method in Lemma 3.6.

Lemma 3.8. *For $q \in [2,6]$, there exists $\alpha_q > 0$ independent of ε such that $|u|_q \leqslant \alpha_q \varepsilon$ for $u \in M = P_\varepsilon^+ \cap P_\varepsilon^-$.*

Proof. For $u \in M = P_\varepsilon^+ \cap P_\varepsilon^-$, we have $\text{dist}(u, P^+) < \varepsilon$ and $\text{dist}(u, P^-) < \varepsilon$. This implies $\|u\| = (\|u^+\|^2 + \|u^-\|^2)^{1/2} < \sqrt{2}\varepsilon$. As is discussed in (3.5), for any $q \in [2,6]$, there exists a constant $\alpha_q > 0$ independent of ε such that

$$|u|_q \leq \beta_q \|u\| < \alpha_q \varepsilon.$$ □

Lemma 3.9. *If $\varepsilon > 0$ is small enough, then $J(u) \geqslant \frac{\varepsilon^2}{8}$ for $u \in \Sigma = \partial P_\varepsilon^+ \cap \partial P_\varepsilon^-$, that is, $c_* \geqslant \frac{\varepsilon^2}{8}$.*

Proof. For $u \in \partial P_\varepsilon^+ \cap \partial P_\varepsilon^-$, we have $\|u^\pm\| \geqslant \text{dist}(u, P^\mp) = \varepsilon$. By (f_1)–(f_2), we have $F(t) \leqslant \frac{1}{4\alpha_2^2}|t|^2 + C_\alpha |t|^p$ for all $t \in \mathbb{R}$. Then, using Lemma 2.2 and Lemma 3.8, for ε small enough, it follows that

$$J(u) \geqslant \frac{1}{2}\|u\|^2 - \frac{1}{4\alpha_2^2}|u|_2^2 - C_\alpha |u|_p^p \geqslant \frac{\varepsilon^2}{8}.$$ □

Proof Theorem 1.1. Choose $u_1, u_2 \in C_0^\infty(\mathbb{R}^3) \setminus \{0\}$ satisfying $\text{supp}(u_1) \cap \text{supp}(u_2) = \emptyset$ and $u_1 \leqslant 0$, $u_2 \geqslant 0$. For $(t,s) \in \Delta$, we define $\varphi_0(t,s) := R(tu_1 + su_2)$. Obviously, for $t, s \in [0,1]$, $\varphi_0(0,s) = Rsu_2 \in P_\varepsilon^+$ and $\varphi_0(t,0) = Rtu_1 \in P_\varepsilon^-$. Hence, (i) of Theorem 2.7 holds.

For every $t \in [0,1]$, denote $u_t = \varphi_0(t, 1-t)$. Then,

$$\|u_t\|^2 = R^2(t^2\|u_1\|^2 + (1-t)^2\|u_2\|^2) \tag{3.11}$$

and

$$|u_t|_q^q = R^q(t^q|u_1|_q^q + (1-t)^q|u_2|_q^q), \quad q \in [2,6].$$

Then, $|u_t|_q^q \to \infty$, as $R \to \infty$ uniformly for $t \in [0,1]$. Together with Lemma 3.8, when R is sufficiently large, $\varphi_0(\partial_0\Delta) \cap M = \emptyset$ holds.

By (f_3), there exists $C_1, C_2 > 0$ such that $F(t) \geqslant C_1|t|^\mu - C_2$ for all $t \in \mathbb{R}$. Then, by (2.5), (2.7) and (3.11), we have

$$\begin{aligned} J(u_t) &\leqslant \frac{1}{2}\|u_t\|^2 + C''\|u_t\|^4 - \int_{\text{supp}(u_1)\cup\text{supp}(u_2)} F(u_t) \\ &\leqslant \frac{1}{2}\|u_t\|^2 + C''\|u_t\|^4 - C_1\|u_t\|^\mu + C_2. \end{aligned}$$

It is obvious that $J(u_t) \to -\infty$ as $R \to \infty$ uniformly for $t \in [0,1]$. According to Lemma 3.9, for R large enough and $\varepsilon > 0$ small enough, it follows

$$\sup_{u \in \varphi_0(\partial_0\Delta)} J(u) < 0 < c_*.$$

Then, by Theorem 2.7, $J(u)$ has at least one critical point u in $H_r^1(\mathbb{R}^3) \setminus (P_\varepsilon^+ \cup P_\varepsilon^-)$, which implies that (u, ϕ_u) is a radial sign-changing solution of (1.1).

In the following, we show that if f is also odd, the system (1.1) has infinitely many radial sign-changing solutions.

Choose $\{u_i\}_{i=1}^n \in C_0^\infty(\mathbb{R}^3)\setminus\{0\}$ satisfying $\operatorname{supp}(u_i)\cap\operatorname{supp}(u_j)=\emptyset$ for $i\neq j$. For any $n\in\mathbb{N}$, let $t=(t_1,t_2,\dots,t_n)\in B_n$ and we define $\varphi_n\in C(B_n,H_r^1(\mathbb{R}^3))$ by

$$\varphi_n(t)=R_n\sum_{i=1}^n t_iu_i,\quad R_n\geqslant 0.$$

By similar discussions as above, for R_n large enough, we can verify that all assumptions of Theorem 2.9 are satisfied. Then, $J(u)$ possesses infinitely many critical points $\{u_j\}\subset H_r^1(\mathbb{R}^3)\setminus(P_\varepsilon^+\cup P_\varepsilon^-)$ which yield infinitely many sign-changing solutions to system (1.1). Furthermore, we can obtain $J(u_j)\to\infty$ as $j\to\infty$, The proof is completed. □

4 Proof of Theorem 1.2

In this section, due to the restrictions of nonlocal term, we cannot directly obtain the compactness condition and some necessary properties of the auxiliary operator A. To overcome these difficulties, inspired by [12], we consider the perturbation approach. The method from Sec. 3 can be used for the perturbed problem. Finally, by taking the limit of the modified problem, we obtain the solution to the original system 1.1.

We first consider the following modified problem

$$\begin{cases}-\Delta u+u+\phi u+\lambda|u|_2u=f(u)+\lambda|u|^{q-1}u & \text{in } \mathbb{R}^3,\\ -\operatorname{div}\left(\dfrac{\nabla\phi}{\sqrt{1-|\nabla\phi|^2}}\right)=u^2 & \text{in } \mathbb{R}^3,\\ u(x)\to 0,\ \phi(x)\to 0, & \text{as } x\to\infty.\end{cases}\tag{4.1}$$

where $\lambda\in(0,1)$ and $q\in(\max\{p,4\},5)$. Thus, its associated functional is

$$J_\lambda(u)=J(u)+\frac{\lambda}{3}|u|_2^3-\frac{\lambda}{q+1}\int_{\mathbb{R}^3}|u|^{q+1}.\tag{4.2}$$

It is standard to show that $J_\lambda\in C^1(H_r^1(\mathbb{R}^3),\mathbb{R})$ and

$$\langle J_\lambda'(u),v\rangle=\langle J'(u),v\rangle+\lambda|u|_2\int_{\mathbb{R}^3}uv-\lambda\int_{\mathbb{R}^3}|u|^{q-1}uv,\quad \text{for } v\in H_r^1(\mathbb{R}^3).\tag{4.3}$$

For any $u\in H_r^1(\mathbb{R}^3)$, we denote by $v=A_\lambda(u)\in H_r^1(\mathbb{R}^3)$ the unique solution to the problem

$$-\Delta v+v+\phi_uv+\lambda|u|_2u=f(u)+\lambda|u|^{q-1}u,\quad v\in H_r^1(\mathbb{R}^3).\tag{4.4}$$

As in Sect. 3, one verifies that the operator A_λ: $H_r^1(\mathbb{R}^3)\to H_r^1(\mathbb{R}^3)$ is well defined and is continuous and compact. In the following, the proof of a result which is similar to its counterpart in Sect. 3 will be omitted.

Lemma 4.1. *The following statements hold:*

(i) $\langle J_\lambda'(u),u-A_\lambda(u)\rangle\geqslant\|u-A_\lambda(u)\|^2$ *for all* $u\in H_r^1(\mathbb{R}^3)$;

(ii) there exists $C>0$ *independent of* λ *such that* $\|J_\lambda'(u)\|\leqslant\|u-A_\lambda(u)\|(1+C\|u\|^2)$ *for all* $u\in H_r^1(\mathbb{R}^3)$.

Lemma 4.2. *For any* $\lambda\in(0,1)$, $a<b$ *and* $\alpha>0$, *there exists* $\beta_\lambda>0$ *such that* $\|u-A_\lambda(u)\|\geqslant\beta_\lambda$ *if* $u\in H_r^1(\mathbb{R}^3)$, $J_\lambda(u)\in[a,b]$ *and* $\|J_\lambda'(u)\|\geqslant\alpha$.

Proof. Fix a number $\theta \in (4, q)$. For $u \in H_r^1(\mathbb{R}^3)$, by (f_1) and (f_2), together with (2.3), (2.10), (4.2) and (4.4), there exist C_1, $C_2 > 0$ such that

$$\begin{aligned} J_\lambda(u) - \frac{1}{\theta}(u, u - A_\lambda(u)) &= \frac{\theta-2}{2\theta}\|u\|^2 + \frac{\theta-3}{3\theta}\lambda|u|_2^3 \\ &\quad + \frac{1}{2}\int_{\mathbb{R}^3}\phi_u u^2 - \frac{1}{2}\int_{\mathbb{R}^3}\left(1-\sqrt{1-|\nabla\phi_u|^2}\right) - \frac{1}{\theta}\int_{\mathbb{R}^3}\phi_u u A_\lambda(u) \\ &\quad + \int_{\mathbb{R}^3}\left(\frac{1}{\theta}f(u)u - F(u)\right) + \frac{q+1-\theta}{\theta(q+1)}\lambda\int_{\mathbb{R}^3}|u|^{q+1} \\ &\geqslant \frac{\theta-2}{2\theta}C_1\|u\|^2 + \frac{\theta-4}{4\theta}\int_{\mathbb{R}^3}\phi_u u^2 + \frac{1}{\theta}\int_{\mathbb{R}^3}\phi_u u(u - A_\lambda(u)) \\ &\quad + \frac{\theta-3}{3\theta}\lambda|u|_2^3 - C_2\int_{\mathbb{R}^3}|u|^p + \frac{q+1-\theta}{\theta(q+1)}\lambda\int_{\mathbb{R}^3}|u|^{q+1}. \end{aligned}$$

Then,

$$\begin{aligned} &|J_\lambda(u)| + \|u\|\|u - A_\lambda(u)\| + \frac{1}{\theta}\left|\int_{\mathbb{R}^3}\phi_u u(u - A_\lambda(u))\right| \\ &\geqslant \frac{\theta-2}{2\theta}C_1\|u\|^2 + \frac{\theta-4}{4\theta}\int_{\mathbb{R}^3}\phi_u u^2 + \frac{\theta-3}{3\theta}\lambda|u|_2^3 - C_2\int_{\mathbb{R}^3}|u|^p + \frac{q+1-\theta}{\theta(q+1)}\lambda\int_{\mathbb{R}^3}|u|^{q+1}. \end{aligned} \tag{4.5}$$

Observe that for any large $C_3 > 0$, there exists $C_4 > 0$ such that

$$\frac{\theta-3}{3\theta}|u|_2^3 \geqslant C_3|u_n|_2^2 - C_4.$$

Together with (4.5), it follows that

$$\begin{aligned} &|J_\lambda(u)| + \|u\|\|u - A_\lambda(u)\| + \lambda C_4 + \frac{1}{\theta}\left|\int_{\mathbb{R}^3}\phi_u u(u - A_\lambda(u))\right| \\ &\geqslant \frac{\theta-2}{2\theta}C_1\|u_n\|^2 + \frac{\theta-4}{4\theta}\int_{\mathbb{R}^3}\phi_u u^2 + \int_{\mathbb{R}^3}\left(\lambda C_3|u|^2 - C_2|u|^p + \frac{q+1-\theta}{\theta(q+1)}\lambda|u|^{q+1}\right). \end{aligned}$$

Since C_3 can be chosen arbitrary large, we can get $\lambda C_3|t|^2 - C_2|t|^p + \frac{q+1-\theta}{\theta(q+1)}\lambda|t|^{q+1} \geqslant 0$ for $t \geqslant 0$, which implies that

$$\begin{aligned} &|J_\lambda(u)| + \|u\|\|u - A_\lambda(u)\| + C_5 + \frac{1}{\theta}\left|\int_{\mathbb{R}^3}\phi_u u(u - A_\lambda(u))\right| \\ &\geqslant \frac{\theta-2}{2\theta}C_1\|u_n\|^2 + \frac{\theta-4}{4\theta}\int_{\mathbb{R}^3}\phi_u u^2. \end{aligned} \tag{4.6}$$

By (2.7) and Hölder's inequality, we can derive

$$\begin{aligned} \left|\int_{\mathbb{R}^3}\phi_u u(u - A_\lambda(u))\right| &\leqslant \left(\int_{\mathbb{R}^3}\phi_u(u - A_\lambda(u))^2\right)^{\frac{1}{2}}\left(\int_{\mathbb{R}^3}\phi_u u^2\right)^{\frac{1}{2}} \\ &\leqslant C_6\|u\|\|u - A_\lambda(u)\|\left(\int_{\mathbb{R}^3}\phi_u u^2\right)^{\frac{1}{2}}. \end{aligned}$$

Thus, by Young's inequality, it follows from (4.6) that

$$\|u\|^2 \leqslant C\left(|J_\lambda(u)| + C_7 + \|u\|\|u - A_\lambda(u)\| + \|u\|^2\|u - A_\lambda(u)\|^2\right). \tag{4.7}$$

If there exists a sequence $\{u_n\} \subset H_r^1(\mathbb{R}^3)$ with $J_\lambda(u_n) \in [a, b]$ and $\|J_\lambda'(u_n)\| \geqslant \alpha$ such that $\|u_n - A_\lambda(u_n)\| \to 0$ as $n \to \infty$, then it follows from (4.7) that $\{\|u_n\|\}$ is bounded. According to (ii) in Lemma 4.1, $\|J_\lambda'(u_n)\| \to 0$ as $n \to \infty$, which is a contradiction. $\square$

Lemma 4.3. *There exists $\varepsilon_1 > 0$ independent of λ such that for $\varepsilon \in (0, \varepsilon_1)$,*

(i) $A_\lambda(\partial P_\varepsilon^-) \subset P_\varepsilon^-$ and every nontrivial solution $u \in P_\varepsilon^-$ is negative;

(ii) $A_\lambda(\partial P_\varepsilon^+) \subset P_\varepsilon^+$ and every nontrivial solution $u \in P_\varepsilon^+$ is positive.

Lemma 4.4. *There exists a locally Lipschitz continuous operator $B_\lambda : H_r^1(\mathbb{R}^3) \setminus H_\lambda \to H_r^1(\mathbb{R}^3)$, where $H_\lambda := \mathrm{Fix}(A_\lambda)$, such that*

(i) $B_\lambda(\partial P_\varepsilon^+) \subset P_\varepsilon^+$, $B_\lambda(\partial P_\varepsilon^-) \subset P_\varepsilon^-$ for $\varepsilon \in (0, \varepsilon_1)$;

(ii) $\frac{1}{2}\|u - B_\lambda(u)\| \leqslant \|u - A_\lambda(u)\| \leqslant 2\|u - B_\lambda(u)\|$ for all $u \in H_r^1(\mathbb{R}^3) \setminus H_\lambda$;

(iii) $\langle J_\lambda'(u), u - B_\lambda(u)\rangle \geqslant \frac{1}{2}\|u - A_\lambda(u)\|^2$ for all $u \in H_r^1(\mathbb{R}^3) \setminus H_\lambda$;

(iv) if f is odd then B_λ is odd.

We use Theorem 2.7 for J_λ. We claim that $\{P_\varepsilon^+, P_\varepsilon^-\}$ is an admissible family of invariant sets for the functional J_λ at any level c_λ. In view of the approach in Sect. 3 and the fact that we have already had Lemmas 4.1–4.4, we need only to prove the compactness condition which is given in the following lemma.

Lemma 4.5. *For any fixed $\lambda \in (0, 1)$, J_λ satisfies the (PS) condition.*

Proof. Since the proof is identical to Lemma 3.3 in [12], here we outline the sketch. Assume that there exist $\{u_n\} \subset H_r^1(\mathbb{R}^3)$ and $c_\lambda \in \mathbb{R}$ such that $J_\lambda(u_n) \to c_\lambda$ and $J_\lambda'(u_n) \to 0$ as $n \to \infty$. For $\theta \in (4, q+1)$, we have

$$\begin{aligned}
C + \frac{1}{\theta}\|J_\lambda'(u_n)\|\|u_n\| &\geqslant J_\lambda(u_n) - \frac{1}{\theta}\langle J_\lambda'(u_n), u_n\rangle \\
&= \frac{\theta-2}{2\theta}\|u_n\|^2 + \frac{\theta-3}{3\theta}\lambda|u_n|_2^3 \\
&\quad + \frac{1}{2}\int_{\mathbb{R}^3}\phi_{u_n}u_n^2 - \frac{1}{2}\int_{\mathbb{R}^3}\left(1 - \sqrt{1-|\nabla\phi_{u_n}|^2}\right) - \frac{1}{\theta}\int_{\mathbb{R}^3}\phi_{u_n}u_n^2 \\
&\quad + \int_{\mathbb{R}^3}\left(\frac{1}{\theta}f(u_n)u_n - F(u_n)\right) + \frac{q+1-\theta}{\theta(q+1)}\lambda\int_{\mathbb{R}^3}|u_n|^{q+1} \\
&\geqslant C_1\|u_n\|^2 + \frac{\theta-3}{3\theta}\lambda|u_n|_2^3 - C_2\int_{\mathbb{R}^3}|u_n|^p + \frac{q+1-\theta}{\theta(q+1)}\lambda\int_{\mathbb{R}^3}|u_n|^{q+1}.
\end{aligned}$$

As in the proof of Lemma 4.2, we can conclude that

$$C + \frac{1}{\theta}\|J_\lambda'(u_n)\|\|u_n\| \geqslant C_1\|u_n\|^2,$$

which implies that $\{u_n\}$ is bounded in $H_r^1(\mathbb{R}^3)$. Then, using the proof method from the second part of Lemma 2.10, we can show that $\{u_n\}$ has a convergent subsequence, verifying the (PS) condition. □

Then, $\{P_\varepsilon^+, P_\varepsilon^-\}$ is an admissible family of invariant sets for the functional J_λ at any level c_λ. We are ready to prove Theorem 1.2.

Proof of Theorem 1.2. (Existence part) The final proof is divided into two steps. First, we prove that for each fixed $\lambda \in (0,1)$, the modified functional J_λ has a radial sign-changing solution. Then, by taking the limit of the modified functional, we prove that the original functional also has a radial sign-changing solution.

Step 1. Choose $u_1, u_2 \in C_0^\infty(B_1(0)) \setminus \{0\}$ satisfying $\mathrm{supp}(u_1) \cap \mathrm{supp}(u_2) = \emptyset$ and $u_1 \leqslant 0$, $u_2 \geqslant 0$, where $B_r(0) := \{x \in \mathbb{R}^3 : |x| < r\}$. For $(t,s) \in \Delta$, we define

$$\varphi_0(t,s)(\cdot) := R^2 \left(t u_1(R\cdot) + s u_2(R\cdot) \right).$$

Obviously, for $t, s \in [0,1]$, $\varphi_0(0,s)(\cdot) = R^2 s u_2(R\cdot) \in P_\varepsilon^+$ and $\varphi_0(t,0)(\cdot) = R^2 t u_1(R\cdot) \in P_\varepsilon^-$. Hence, (i) of Theorem 2.7 holds.

For any $\lambda \in (0,1)$, $u \in \Sigma := \partial P_\varepsilon^+ \cap \partial P_\varepsilon^-$, similar to Lemma 3.9, for small $\varepsilon > 0$, we have

$$J_\lambda(u) \geqslant \frac{1}{2}\|u\|^2 - \int_{\mathbb{R}^3} F(u) - \frac{\lambda}{q+1} \int_{\mathbb{R}^3} |u|^{q+1} \geqslant \frac{\varepsilon^2}{8}.$$

Then, for $\lambda \in (0,1)$, $c_\lambda^* := \inf\limits_{u \in \Sigma} J_\lambda(u) \geqslant \frac{\varepsilon^2}{8}$.

Let $u_t = \varphi_0(t, 1-t)$ for $t \in [0,1]$, then we have

$$\int_{\mathbb{R}^3} |\nabla u_t|^2 = R^3 \int_{\mathbb{R}^3} \left(t^2 |\nabla u_1|^2 + (1-t)^2 |\nabla u_2|^2 \right), \tag{4.8}$$

$$\int_{\mathbb{R}^3} |u_t|^2 = R \int_{\mathbb{R}^3} \left(t^2 u_1^2 + (1-t)^2 u_2^2 \right), \tag{4.9}$$

$$\int_{\mathbb{R}^3} |u_t|^\mu = R^{2\mu-3} \int_{\mathbb{R}^3} (t^\mu |u_1|^\mu + (1-t)^\mu |u_2|^\mu), \tag{4.10}$$

$$\int_{\mathbb{R}^3} |u_t|^{\frac{12}{5}} = R^{\frac{9}{5}} \int_{\mathbb{R}^3} (t^{\frac{12}{5}} |u_1|^{\frac{12}{5}} + (1-t)^{\frac{12}{5}} |u_2|^{\frac{12}{5}}). \tag{4.11}$$

Since $F(t) \geqslant C_1 |t|^\mu - C_2$ for any $t \in \mathbb{R}$, for $\lambda \in (0,1)$ and $t \in [0,1]$, by (2.3), (2.7) and (4.8)–(4.11), we have

$$\begin{aligned}
J_\lambda(u_t) &\leqslant \frac{1}{2}\|u_t\|^2 + \frac{1}{4}\int_{\mathbb{R}^3} \phi_{u_t} u_t^2 + \frac{1}{3}\left(\int_{\mathbb{R}^3} |u|^2 \right)^{\frac{3}{2}} - \int_{B_{R^{-1}}(0)} F(u_t) \\
&\leqslant \frac{1}{2}\|u_t\|^2 + \frac{C'}{4}\left(\int_{\mathbb{R}^3} |u_t|^{\frac{12}{5}} \right)^{\frac{5}{3}} + \frac{1}{3}\left(\int_{\mathbb{R}^3} |u_t|^2 \right)^{\frac{3}{2}} - \int_{B_{R^{-1}}(0)} F(u_t) \\
&\leqslant \frac{R^3}{2} \int_{\mathbb{R}^3} \left(t^2 |\nabla u_1|^2 + (1-t)^2 |\nabla u_2|^2 \right) + \frac{R}{2} \int_{\mathbb{R}^3} \left(t^2 u_1^2 + (1-t)^2 u_2^2 \right) \\
&\quad + \frac{R^3 C'}{4} \left(\int_{\mathbb{R}^3} (t^{\frac{12}{5}} |u_1|^{\frac{12}{5}} + (1-t)^{\frac{12}{5}} |u_2|^{\frac{12}{5}}) \right)^{\frac{5}{3}} + \frac{R^{\frac{3}{2}}}{2} \left(\int_{\mathbb{R}^3} \left(t^2 u_1^2 + (1-t)^2 u_2^2 \right) \right)^{\frac{3}{2}} \\
&\quad - C_1 R^{2\mu-3} \int_{\mathbb{R}^3} (t^\mu |u_1|^\mu + (1-t)^\mu |u_1|^\mu) + C_3 R^{-3}.
\end{aligned} \tag{4.12}$$

Since $\mu > 3$, one sees that $J_\lambda(u_t) \to -\infty$ as $R \to \infty$ uniformly for $\lambda \in (0,1)$, $t \in [0,1]$. Hence, for any $\lambda \in (0,1)$, choosing R independent of λ and large enough, we have

$$\sup_{u \in \varphi_0(\partial_0 \Delta)} J_\lambda(u) < 0 < c_\lambda^*.$$

Since $|u_t|_2 \to \infty$ as $R \to \infty$ uniformly for $t \in [0,1]$, it follows from Lemma 3.8 that $\varphi_0(\partial_0\Delta) \cap M = \emptyset$ for R large enough. Thus, φ_0 with a large R independent of λ satisfies the assumptions of Theorem 2.7. Therefore,

$$c_\lambda = \inf_{\varphi \in \Gamma} \sup_{u \in \varphi(\triangle) \setminus W} J_\lambda(u)$$

is a critical value of J_λ satisfying $c_\lambda \geqslant c_\lambda^*$, and there exists $u_\lambda \in H_r^1(\mathbb{R}^3) \setminus (P_\varepsilon^+ \cup P_\varepsilon^-)$ such that $J_\lambda(u_\lambda) = c_\lambda$ and $J_\lambda'(u_\lambda) = 0$.

Step 2. Passing to the limit as $\lambda \to 0^+$. We first claim that $\{u_\lambda\}_{\lambda\in(0,1)}$ is bounded in $H_r^1(\mathbb{R}^3)$ uniformly in $\lambda \in (0,1)$. Through the proof in Lemma 4.5, it is obvious that for each $\lambda \in (0,1)$, there exists $u_\lambda \in H_r^1(\mathbb{R}^3)$ such that

$$J_\lambda(u_\lambda) = c_\lambda \quad \text{and} \quad J_\lambda'(u_\lambda) = 0.$$

By $\langle J_\lambda'(u_\lambda), u_\lambda\rangle = 0$, we have

$$\int_{\mathbb{R}^3} (|\nabla u_\lambda|^2 + u_\lambda^2) + \int_{\mathbb{R}^3} \phi_{u_\lambda} u_\lambda^2 + \lambda|u_\lambda|_2^3 - \int_{\mathbb{R}^3} f(u_\lambda)u_\lambda - \lambda \int_{\mathbb{R}^3} |u_\lambda|^{q+1} = 0. \tag{4.13}$$

Recalling (f_3), it follows from (4.13) that for $b > 0$,

$$b\int_{\mathbb{R}^3} F(u_\lambda) \leqslant \frac{b}{\mu}\int_{\mathbb{R}^3} (|\nabla u_\lambda|^2 + u_\lambda^2) + \frac{b}{\mu}\int_{\mathbb{R}^3} \phi_{u_\lambda} u_\lambda^2 + \frac{b\lambda}{\mu}|u_\lambda|_2^3 - \frac{b\lambda}{\mu}\int_{\mathbb{R}^3} |u_\lambda|^{q+1}. \tag{4.14}$$

Moreover, the associated Pohozaev type identity for the modified problem (4.1) is

$$\begin{aligned} 3\int_{\mathbb{R}^3} F(u_\lambda) + \frac{3\lambda}{q+1}\int_{\mathbb{R}^3} |u_\lambda|^{q+1} &= \frac{1}{2}\int_{\mathbb{R}^3} |\nabla u_\lambda|^2 + \frac{3}{2}\int_{\mathbb{R}^3} u_\lambda^2 + \frac{3\lambda}{2}|u_\lambda|_2^3 \\ &\quad + 2\int_{\mathbb{R}^3} \phi_{u_\lambda} u_\lambda^2 - \frac{3}{2}\int_{\mathbb{R}^3} \left(1 - \sqrt{1 - |\nabla \phi_{u_\lambda}|^2}\right). \end{aligned} \tag{4.15}$$

Combining (4.14) with (4.15), we have for $a \in \mathbb{R}$

$$\begin{aligned} (a+b)\int_{\mathbb{R}^3} F(u_\lambda) \leqslant& \left(\frac{a}{6} + \frac{b}{\mu}\right)\int_{\mathbb{R}^3} |\nabla u_\lambda|^2 + \left(\frac{a}{2} + \frac{b}{\mu}\right)\int_{\mathbb{R}^3} |u_\lambda|^2 \\ &+ \left(\frac{2a}{3} + \frac{b}{\mu}\right)\int_{\mathbb{R}^3} \phi_{u_\lambda} u_\lambda^2 - \frac{a}{2}\int_{\mathbb{R}^3} \left(1 - \sqrt{1 - |\nabla u|^2}\right) \\ &+ \left(\frac{a}{2} + \frac{b}{\mu}\right)|u_\lambda|_2^3 - \left(\frac{a}{q+1} + \frac{b}{\mu}\right)|u_\lambda|_{q+1}^{q+1}. \end{aligned} \tag{4.16}$$

Choosing $a = 1 - b$ in (4.16), by (4.2) and (2.3), it follows that

$$\begin{aligned} c_\lambda = J_\lambda(u_\lambda) \geqslant& \left(\frac{1}{3} + \frac{b(\mu-6)}{6\mu}\right)\int_{\mathbb{R}^3} |\nabla u_\lambda|^2 + \left(\frac{b}{2} - \frac{b}{\mu}\right)\int_{\mathbb{R}^3} |u_\lambda|^2 \\ &+ \left(\frac{1}{2} - \frac{2(1-b)}{3} - \frac{b}{\mu}\right)\int_{\mathbb{R}^3} \phi_{u_\lambda} u_\lambda^2 - \frac{b}{2}\int_{\mathbb{R}^3} \left(1 - \sqrt{1 - |\nabla u|^2}\right) \\ &+ \left(\frac{1}{3} - \frac{1-b}{2} - \frac{b}{\mu}\right)|u_\lambda|_2^3 + \left(\frac{1-b}{q+1} + \frac{b}{\mu} - \frac{1}{q+1}\right)|u_\lambda|_{q+1}^{q+1} \\ \geqslant& \left(\frac{1}{3} + \frac{b(\mu-6)}{6\mu}\right)\int_{\mathbb{R}^3} |\nabla u_\lambda|^2 + \left(\frac{b}{2} - \frac{b}{\mu}\right)\int_{\mathbb{R}^3} |u_\lambda|^2 \\ &+ \left(\frac{1}{2} - \frac{2(1-b)}{3} - \frac{b}{\mu} - \frac{b}{4}\right)\int_{\mathbb{R}^3} \phi_{u_\lambda} u_\lambda^2 \\ &+ \left(\frac{1}{3} - \frac{1-b}{2} - \frac{b}{\mu}\right)|u_\lambda|_2^3 + \left(\frac{1-b}{q+1} + \frac{b}{\mu} - \frac{1}{q+1}\right)|u_\lambda|_{q+1}^{q+1}. \end{aligned} \tag{4.17}$$

It is easy to check that with the choice $b = 2$ all the coefficients above are positive, which implies that $c_\lambda = J_\lambda(u_\lambda) \geqslant C\|u_\lambda\|^2$.

By the definition of c_λ and (4.12), for $\lambda \in (0,1)$, we see that

$$0 < c_\lambda \leqslant \sup_{u \in \varphi_0(\Delta)} J_\lambda(u) < \infty.$$

Thus we finish the proof of the claim.

Then we can assume, as $\lambda \to 0^+$,

$$u_\lambda \rightharpoonup u_0 \text{ in } H_r^1(\mathbb{R}^3) \quad \text{and} \quad I_\lambda(u_\lambda) = c_\lambda \to c_0 > 0.$$

Moreover, for any $v \in H_r^1(\mathbb{R}^3)$, we have

$$\langle J'(u_\lambda), v\rangle = \langle J'_\lambda(u_\lambda), v\rangle - \lambda |u_\lambda|_2 \int_{\mathbb{R}^3} u_\lambda v + \lambda \int_{\mathbb{R}^3} |u_\lambda|^q v = o_\lambda(1)\|v\|$$

and

$$J(u_\lambda) = J_\lambda(u_\lambda) - \frac{\lambda}{3}|u_\lambda|_2^3 + \frac{\lambda}{q+1}\int_{\mathbb{R}^3} |u_\lambda|^{q+1} = c_0 + o_\lambda(1).$$

Thus, $\{u_\lambda\}_{\lambda\in(0,1)}$ is a bounded Palais–Smale sequence for the functional J at level c_0. Arguing similarly as in the second part of the proof of Lemma 2.10, by $\langle J'(u_0), u_\lambda - u_0\rangle = o_\lambda(1)$, we get that

$$u_\lambda \to u_0 \text{ on } H_r^1(\mathbb{R}^3), \quad J'(u_0) = 0 \quad \text{and} \quad J(u_0) = c_0 > 0.$$

The fact that $u_\lambda \in H_r^1(\mathbb{R}^3) \setminus (P_\varepsilon^+ \cup P_\varepsilon^-)$ and $c_\lambda \geqslant \frac{\varepsilon^2}{8}$ for $\lambda \in (0,1)$ implies $u_0 \in H_r^1(\mathbb{R}^3) \setminus (P_\varepsilon^+ \cup P_\varepsilon^-)$ and $J(u) \geqslant \frac{\varepsilon^2}{8}$. Therefore, u_0 is a radial sign-changing solution of (2.4).

In the following, we show that if f is odd, the system (1.1) has infinitely many radial sign-changing solutions when $\mu \in (3,4]$. Thanks to Lemmas 4.1–4.4, we have seen that for $0 < \lambda < 1$, P_ε^+ is a G-admissible invariant set for the functional J_λ at any level c_λ. □

Proof of Theorem 1.2. (Multiplicity part) Choose $\{u_i\}_{i=1}^n \in C_0^\infty(\mathbb{R}^3) \setminus \{0\}$ satisfying $\mathrm{supp}(u_i) \cap \mathrm{supp}(u_j) = \emptyset$ for $i \neq j$. For any $n \in \mathbb{N}$, let $t = (t_1, t_2, \ldots, t_n) \in B_n$ and we define $\varphi_n \in C(B_n, H_r^1(\mathbb{R}^3))$ by

$$\varphi_n(t)(\cdot) = R_n^2 \sum_{i=1}^n t_i v_i(R_n \cdot),$$

where R_n is a large number independent of λ. By similar discussions as above, for R_n large enough, we can verify that all assumptions of Theorem 2.9 are satisfied. Then, for any $0 < \lambda < 1$ and $j \geqslant 2$, we have

$$0 < \inf_{u\in\Sigma} J_\lambda(u) := c_\lambda^* \leqslant c_{\lambda,j} := \inf_{B\in\Gamma_j} \sup_{u\in B\setminus W} J_\lambda(u) \to \infty, \quad \text{as } j \to \infty$$

and there exists $\{u_{\lambda,j}\}_{j\geqslant 2} \subset H_r^1(\mathbb{R}^3) \setminus W$ such that $J_\lambda(u_{\lambda,j}) = c_{\lambda,j}$ and $J'_\lambda(u_{\lambda,j}) = 0$.

For any fixed $j \geqslant 2$, in a similar way to step 2 of existence part, there exists $u_j \in H_r^1(\mathbb{R}^3) \setminus W$ such that $u_{\lambda,j} \to u_j$, as $\lambda \to 0^+$ and $J'(u_j) = 0$, $J(u_j) = c_j$.

Finally, we prove that $c_j \to \infty$ as $j \to \infty$. Assume for contradiction that the sequence $\{c_j\}$ is bounded, i.e., $c_j \leq C$ for some constant $C > 0$. Then the sequence of solutions $\{u_j\}$ has bounded energy, $J(u_j) \leq C$. Repeating the computations from (4.13)–(4.17) for the functional J, it can be deduced that if there exists $u_j \in H_r^1(\mathbb{R}^3)$ such that $J'(u_j) = 0$, then $J(u_j) \geqslant C_1\|u_j\|^2$, with C_1 independent on u_j, which implies that the sequence $\{u_j\}$ is bounded in $H_r^1(\mathbb{R}^3)$. Since $\{u_j\}$ is a bounded sequence of critical points for J, and J satisfies the (PS) condition, the set $\{u_j\}$ must be pre-compact. This means it can only contain a finite number of distinct solutions.

However, the critical values c_j are constructed using min-max principles over sets with increasing Krasnoselskii genus. A standard result in critical point theory states that if the sequence of critical values is bounded, then there must be a critical level to which infinitely many c_j accumulate. If the critical points are isolated (which is often the case), this leads to a contradiction. More generally, the pre-compactness of the set of solutions $\{u_j\}$ contradicts the fact that the genus of the sublevel sets J^{c_j} must tend to infinity. Therefore, the assumption that $\{c_j\}$ is bounded must be false, and we have $\lim\_j \to \infty c_j = \infty$. The proof is completed. □

Acknowledgements

This work is supported by the Research Fund of the Natural Science Foundation of Chongqing, China (No. CSTB2024NSCQ-MSX0843), the National Natural Science Foundation of China (No. 12361024) and the Team Building Project for Graduate Tutors in Chongqing (No. yds223010).

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