Electronic Journal of Qualitative Theory of Differential Equations
2026, No. **19**, 1–16; https://doi.org/10.14232/ejqtde.2026.1.19 www.math.u-szeged.hu/ejqtde/

Existence results for $p(x)$-curl systems emerging in electromagnetic theory

Fang-Fang Liao[1], **Soheila Valizadeh**[2], **Shapour Heidarkhani**[✉2] and **Mohammad Abolghasemi**[2]

[1]Department of Mathematics, Shanghai Normal University, Shanghai 200234, China
[2]Department of Mathematics, Faculty of Sciences, Razi University, 67149 Kermanshah, Iran

Received 27 October 2025, appeared 3 June 2026
Communicated by Roberto Livrea

Abstract. By using variational techniques and critical point theory, we study the existence and multiplicity of solutions for a class of $p(x)$-curl systems, which appear in the field of electromagnetism. In particular, we discuss the existence of one solution under an asymptotical behaviour of the nonlinear term at zero, and the existence of a sequence of solutions under suitable oscillating behaviour of the nonlinear term at infinity for the system.

Keywords: variational method, $p(x)$-curl systems, Sobolev spaces, variable exponents.

2020 Mathematics Subject Classification: 35J50, 47J20.

1 Introduction

In recent years, the study of $p(x)$-curl systems has attracted much interest, due to the relevance to modeling electromagnetic phenomena. Such models are particularly important in nonlinear electromagnetism, wherein the spatial variation in the permeability and permittivity of materials creates complex behaviors of electromagnetic fields. Let $w = (w_1, w_2, w_3)$ be a vector function defined on Ω. The divergence of w is represented by $\nabla \cdot w = \partial_{x_1} w_1 + \partial_{x_2} w_2 + \partial_{x_3} w_3$ and the curl of w is represented by $\nabla \times w = (\partial_{x_2} w_3 - \partial_{x_3} w_2, \partial_{x_3} w_1 - \partial_{x_1} w_3, \partial_{x_1} w_2 - \partial_{x_2} w_1)$. The curl and divergence obey the subsequent relation $-\Delta w = \nabla \times (\nabla \times w) - \nabla \cdot (\nabla \cdot w)$, where $\Delta w = (\Delta w_1, \Delta w_2, \Delta w_3)$ and $\Delta w_i = \nabla \cdot (\nabla w_i), i = 1, 2, 3$.

We are interested in the study of the existence of solutions for the following stationary $p(x)$-curl system:

$$\begin{cases} \nabla \times (|\nabla \times u|^{p(x)-2} \nabla \times u) + a(x)|u|^{p(x)-2} u = f(x, u), \ \nabla \cdot u = 0, & \text{in } \Omega, \\ |\nabla \times u|^{p(x)-2} \nabla \times u \times n = 0, \qquad u \cdot n = 0, & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where Ω is a restricted and simply connected area in $\mathbb{R}^3$, with a $C^{1,1}$ boundary $\partial\Omega$. Let n be the unit normal vector pointing outward from the boundary of Ω, and ∂_x represent the partial

✉Corresponding author. Email: sh.heidarkhani@razi.ac.ir

derivative of a function in relation to the variable x. Furthermore, $a(x)$ is a function in L^∞ and there exist $a_0, a_1 > 0$, so that

$$a_0 < a(x) < a_1, \qquad \forall x \in \Omega.$$

Throughout this work unless stated otherwise, we will assume that the exponent $p(x)$ remains continuous throughout Ω with

$$3 < p^- = \min_{x\in\Omega} p(x) \le p^+ = \max_{x\in\Omega} p(x) < \infty,$$

and $p(x)$ exhibits logarithmic continuity, which means that there is a function $\omega : \mathbb{R}_0^+ \to \mathbb{R}_0^+$ such that

$$\forall x, y \in \bar{\Omega}, \quad |x-y| < 1, \quad |p(x) - p(y)| \le \omega|x-y|, \quad \lim_{t\to 0^+} \omega(t) \log \frac{1}{t} = c < \infty. \tag{1.2}$$

The function $F : \Omega \times \mathbb{R}^3 \to \mathbb{R}$ is differentiable with respect to $t \in \mathbb{R}^3$, such that $f = \partial_t F(x,t) : \Omega \times \mathbb{R}^3 \to \mathbb{R}^3$ is a Carathéodory function. So, we get

$$F(x,t) = \int_0^t f(x,\epsilon) d\epsilon, \qquad \forall (x,t) \in \Omega \times \mathbb{R}^3.$$

One of the essential concepts in the study of partial differential equations and functional analysis is Sobolev spaces. We will use the variable exponent Lebesgue and Sobolev spaces, $L^p(x)$ and $W^{1,p(x)}$, where p is a real-valued function. There are many literature on Sobolev spaces with variable exponents and their applications, see for example [6–8,16]. The pioneering work of researchers such as [8], laid the groundwork for the analysis of variable exponent Sobolev spaces, which became the natural setting for studying $p(x)$-curl systems. One of the first contributions to the study of $p(x)$-curl systems was made by Fan and Zhang [9], who investigated the existence of weak solutions for a class of nonlinear elliptic equations with variable exponents. With these results, several authors studied the existence and multiplicity of solutions for $p(x)$-curl systems under various assumptions on the nonlinearity and the variable exponent $p(x)$. For instance, Mihăilescu and Rădulescu in [17] established the existence of at least one solution for a class of $p(x)$-curl systems using Mountain pass techniques. In [12], the existence and multiplicity of solutions for a $p(x)$-curl system were studied by utilizing a variant of the Mountain Pass Theorem, where the functional is required to satisfy the Cerami condition instead of the classical Palais–Smale condition. A class of curl systems arising in electromagnetism, with a nonlinear source term was studied in [2]. They introduced a suitable functional framework and a convenient basis that allow them to apply the Galerkin's method and proved existence of local or global solutions, depending on the values of λ and σ. Xiang, Wang, and Zhang in [20], examined whether solutions of the problem (1.1) exist and how many solutions there are when $\lambda = 1$. They explored the presence of ground state solutions and an infinite number of solutions for the problem (1.1) with $\lambda = 1$, where the nonlinearity f satisfying superlinear growth condition. This was achieved through the combination of the Mountain pass theorem with the Nehari manifold approach, in addition to a modified version of the Mountain pass theorem. Their approach relied on the application of abstract critical point theorems, combined with careful estimates in variable exponent Sobolev spaces. In [13] the existence and multiplicity of solutions for the problem (1.1) in the absence of Ambrosetti–Rabinowitz condition under superlinear case was obtained. In [14] by using critical point theory and the variational method, the existence of at least one, two and three solutions to the problem (1.1) was investigated.

In the present paper, we investigate the existence of solutions for the system (1.1) employing a variant of Ricceri's variational principle [19, Theorem 2.1] given by Bonanno and Molica Bisci in [5]. First, we establish the existence of at least one nontrivial solution under specific asymptotic conditions on the nonlinear term near the origin. We prove the nontriviality of the solution for problem (3.5) under specified conditions. The result also describes the solution's norm behavior as $\lambda \to 0$ and shows that the associated functional I_λ is negative and strictly decreasing in the obtained interval of the parameter λ. We then show that, when the nonlinear term exhibits suitable oscillatory behavior at infinity, the system admits an infinite sequence of distinct solutions. Our approach is based on variational techniques and critical point theory, adapted to the framework of Sobolev spaces with variable exponents, which provide the appropriate functional setting for handling the nonstandard growth conditions inherent in the problem.

2 Preliminary results

We begin by presenting some preliminary results on the generalized Lebesgue–Sobolev spaces $L^{p(x)}(\Omega)$ and $W^{p(x)}(\Omega)$, where Ω is a bounded and simply connected area in $\mathbb{R}^3$, with a boundary of class $C^{1,1}$ represented by $\partial\Omega$. Let $C_+(\bar{\Omega}) = \{g : g \in C(\bar{\Omega}), g(x) > 1, \forall x \in \bar{\Omega}\}$. For any $p(x) \in C_+(\bar{\Omega})$, the variable exponent Lebesgue space of measurable functions is defined by

$$L^{p(x)}(\Omega) = \{u : \Omega \to \mathbb{R} : \rho_{p(x)}(u) < \infty\},$$

where convex modular $\rho_{p(x)} : L^{p(x)} \to \mathbb{R}$ is defined by

$$\rho_{p(x)}(u) = \int_\Omega |u|^{p(x)} dx.$$

This space with the following so-called Luxemburg norm, is a separable, reflexive and Banach space,

$$\|u\|_{L^{p(x)}(\Omega)} = \inf \left\{ \mu > 0 : \rho_{p(x)}\left(\frac{u}{\mu}\right) \leq 1 \right\}.$$

When $p_1(x), p_2(x)$ are two variable exponents such that $p_1(x) \leq p_2(x)$, *a.e.* $\forall x \in \Omega$ with $0 < |\Omega| < \infty$, then there exists a continuous embedding $L^{p_2(x)}(\Omega) \hookrightarrow L^{p_1(x)}(\Omega)$, which means that there exists $C > 0$ such that $\|u\|_{L^{p_1(x)}(\Omega)} \leq C\|u\|_{L^{p_2(x)}(\Omega)}$. Let $L^{q(x)}(\Omega)$ be the dual space of $L^{p(x)}(\Omega)$, where $\frac{1}{q(x)} + \frac{1}{p(x)} = 1$. Then, for any $u \in L^{p(x)}(\Omega), v \in L^{q(x)}(\Omega)$, Hölder's inequality holds as follows:

$$\left| \int_\Omega uv dx \right| \leq \left(\frac{1}{p^-} + \frac{1}{q^-} \right) \|u\|_{L^{p(x)}(\Omega)} \|v\|_{L^{q(x)}(\Omega)}.$$

Proposition 2.1 (see [10]). *If $u, u_n \in L^{p(x)}(\Omega)$ and $p^+ < \infty$, then*

(i_1) $\|u\|_{L^{p(x)}(\Omega)} > 1$, *then* $\|u\|^{p^-}_{L^{p(x)}(\Omega)} \leq \rho_{p(x)}(u) \leq \|u\|^{p^+}_{L^{p(x)}(\Omega)}$,

(i_2) $\|u\|_{L^{p(x)}(\Omega)} < 1$, *then* $\|u\|^{p^+}_{L^{p(x)}(\Omega)} \leq \rho_{p(x)}(u) \leq \|u\|^{p^-}_{L^{p(x)}(\Omega)}$,

(i_3) $\lim_{n\to\infty} \|u_n - u\|_{p(x)} = 0 \Leftrightarrow \lim_{n\to\infty} \rho_{p(x)}(u_n - u) = 0$.

The variable exponent Sobolev space $W^{1,p(x)}(\Omega)$, which is defined by

$$W^{1,p(x)}(\Omega) = \{u \in L^{p(x)}(\Omega) : |\nabla u| \in L^{p(x)}(\Omega)\},$$

equipped with the following norm

$$\|u\|_{W^{1,p(x)}(\Omega)} = \|u\|_{L^{p(x)}(\Omega)} + \|\nabla u\|_{L^{p(x)}(\Omega)},$$

is a separable and reflexive Banach space. The space $W_0^{1,p(x)}(\Omega)$, which is the closure of $C_0^\infty(\Omega)$ with regard to $\|.\|_{W^{1,p(x)}(\Omega)}$, is defined as follows

$$W_0^{1,p(x)}(\Omega) = \{u \in W^{1,p(x)}(\Omega) : u|_{\partial\Omega} = 0\},$$

equipped with the following norm $\|u\|_{W_0^{1,p(x)}(\Omega)} = \|\nabla u\|_{L^{p(x)}(\Omega)}$. The Poincaré inequality for the space $W^{1,p(x)}(\Omega)$ holds, that is, there exists a positive constant c such that

$$\|u\|_{p(x)} \leq c\|\nabla u\|_{p(x)}, \ \forall u \in W^{1,p(x)}(\Omega).$$

Moreover, the embedding $W^{1,p(x)}(\Omega) \hookrightarrow L^{q(x)}(\Omega)$ is compact for every $q \in C(\bar{\Omega})$ with $1 \leq q(x) \leq \frac{3p(x)}{3-p(x)}$.

Let

$$\mathcal{L}^{p(x)}(\Omega) = L^{p(x)}(\Omega) \times L^{p(x)}(\Omega) \times L^{p(x)}(\Omega).$$

Now, we define

$$\mathcal{W}^{p(x)}(\Omega) = \{\mathbf{v} \in \mathcal{L}^{p(x)}(\Omega) : \nabla \times \mathbf{v} \in \mathcal{L}^{p(x)}(\Omega), \nabla \cdot \mathbf{v} = 0, \mathbf{v} \cdot n|_{\partial\Omega} = 0\},$$

with the following norm

$$\|\mathbf{v}\|_{\mathcal{W}^{p(x)}(\Omega)} = \|\mathbf{v}\|_{\mathcal{L}^{p(x)}(\Omega)} + \|\nabla \times \mathbf{v}\|_{\mathcal{L}^{p(x)}(\Omega)}, \tag{2.1}$$

where n is the outward unitary normal vector to $\partial\Omega$.

If $p^- > 1$, by [2, Theorem 2.1], $\mathcal{W}^{p(x)}(\Omega)$ is a closed subspace of $\mathcal{W}_n^{1,p(x)}(\Omega)$, where

$$\mathcal{W}_n^{1,p(x)}(\Omega) = \{\mathbf{v} \in \mathcal{W}^{1,p(x)}(\Omega) : \mathbf{v} \cdot n|_{\partial\Omega} = 0\},$$

with

$$\mathcal{W}^{1,p(x)}(\Omega) = W^{1,p(x)}(\Omega) \times W^{1,p(x)}(\Omega) \times W^{1,p(x)}(\Omega).$$

Theorem 2.2 ([2, Theorem 2.1])**.** *Suppose that $1 < p^- \leq p^+ < \infty$ and p satisfies (1.2). Then $\mathcal{W}^{p(x)}(\Omega)$ is a closed subspace of $\mathcal{W}_n^{1,p(x)}(\Omega)$. Furthermore, if $p^- > \frac{6}{5}$, then $\|\nabla \times .\|_{\mathcal{L}^{p(x)}(\Omega)}$ is a norm in $\mathcal{W}^{p(x)}(\Omega)$ and there exists $C = C(N, p^-, p^+) > 0$, such that $\|\mathbf{v}\|_{\mathcal{W}^{p(x)}(\Omega)} \leq C\|\nabla \times \mathbf{v}\|_{\mathcal{L}^{p(x)}(\Omega)}$.*

It follows from Theorem 2.2 that we have the compact embedding $\mathcal{W}^{p(x)}(\Omega) \hookrightarrow C_0^\infty(\Omega)$, with $3 < p^- \leq p^+ < \infty$, for any $x \in \bar{\Omega}$. Furthermore, $(\mathcal{W}^{p(x)}(\Omega), \|.\|_{\mathcal{W}^{p(x)}(\Omega)})$ is a reflexive Banach space. Set

$$c_0 = \sup_{u \in \mathcal{W}^{p(x)}(\Omega)} \frac{\|u\|_\infty}{\|u\|_{\mathcal{W}^{p(x)}(\Omega)}}.$$

Definition 2.3 ([14])**.** A function $u \in \mathcal{W}^{p(x)}(\Omega)$ is a weak solution for the problem (1.1) if for every $v \in \mathcal{W}^{p(x)}(\Omega)$,

$$\int_\Omega |\nabla \times u|^{p(x)-2}\nabla \times u \cdot \nabla \times v dx + \int_\Omega a(x)|u|^{p(x)-2}u \cdot v dx = \int_\Omega f(x, u) \cdot v dx.$$

Let us introduce the functionals Φ and Ψ by

$$\Phi(u) = \int_\Omega \frac{|\nabla \times u|^{p(x)} + a(x)|u|^{p(x)}}{p(x)} dx, \tag{2.2}$$

$$\Psi(u) = \int_\Omega F(x,u)dx, \tag{2.3}$$

and $I(u) = \Phi(u) - \Psi(u)$ for $u \in \mathcal{W}^{p(x)}(\Omega)$.

Lemma 2.4 ([20, Lemmas 3.1 and 3.2])**.** *The functionals Φ, Ψ and I are differentiable in sense of Gâteaux and for any $u, v \in \mathcal{W}^{p(x)}(\Omega)$, we have*

$$\Phi'(u)(v) = \int_\Omega |\nabla \times u|^{p(x)-2} \nabla \times u \cdot \nabla \times v dx + \int_\Omega a(x)|u|^{p(x)-2} u \cdot v dx,$$

$$\Psi'(u)(v) = \int_\Omega f(x,u) \cdot v dx, \tag{2.4}$$

and thus

$$I'(u)(v) = \int_\Omega |\nabla \times u|^{p(x)-2} \nabla \times u \cdot \nabla \times v dx + \int_\Omega a(x)|u|^{p(x)-2} u \cdot v dx - \int_\Omega f(x,u) \cdot v dx.$$

Therefore every critical point of I is a weak solution of the problem (1.1).

Our results are proved by utilizing the subsequent version of Ricceri's variational principle [19, Theorem 2.1], as presented in the specific form by Bonanno and Molica Bisci in [5], whose the initial version was derived in [4]. We refer the readers to [1, 3, 11, 15] for some results, whose proofs are based on Theorem 2.5.

Theorem 2.5. *Let X be a reflexive real Banach space and $\Phi, \Psi : X \to \mathbb{R}$ be two Gâteaux differentiable functionals such that Φ is sequentially weakly lower semicontinuous, strongly continuous and coercive in X and Ψ is sequentially weakly upper semicontinuous in X. Suppose that I_λ be the functional defined as $I_\lambda = \Phi - \lambda\Psi, \lambda \in \mathbb{R}$, and for every $r > inf_X \Phi$, let ϕ be the function defined as*

$$\phi(r) := \inf_{u \in \Phi^{-1}(-\infty, r)} \frac{\sup_{v \in \Phi^{-1}(-\infty,r)} \Psi(v) - \Psi(u)}{r - \Phi(u)},$$

$$\alpha = \liminf_{r \to \infty} \phi(r), \quad \beta = \liminf_{r \to (inf_X \Phi)^+} \phi(r).$$

Then the following results hold.

(I_1) *For every $r > \inf_X \Phi$ and every $\lambda \in (0, \frac{1}{\phi(r)})$, the restriction of the functional I_λ to $\Phi^{-1}(-\infty, r)$ admits a global minimum, which is a critical point (precisely a local minimum) of I_λ in X.*

(I_2) *For every $\lambda \in (0, \frac{1}{\alpha})$ with $\alpha < \infty$, the subsequent conclusion is valid only for one:*

- (j_1) *the functional I_λ has a global minimum, or*
- (j_2) *a sequence $\{u_m\}$ of critical points (local minima) of I_λ exists, where $\lim_{m\to\infty} \Phi(u_m) = \infty$.*

(I_3) *For every $\lambda \in (0, \frac{1}{\beta})$ with $\beta < \infty$, the subsequent conclusion is valid only for one:*

- (k_1) *there exists a global minimum of Φ that also is a local minimum of I_λ, or*
- (k_2) *a sequence $\{u_n\}$ of pairwise distinct critical points (local minima) of I_λ exists, where $\lim_{n\to\infty} \Phi(u_n) = \inf_X \Phi(u)$ and this sequence converges to a global minimum of Φ.*

3 Main results

In this section, we present and prove our main results.

Theorem 3.1. *Suppose that*

$$\sup_{\tau>0}\frac{\tau^{p^-}}{F_\tau}>\frac{c_0^{p^-}\,p^+}{\min\{1,a_0\}},\tag{3.1}$$

where

$$F_\tau=\int_\Omega \max_{|t|\le\tau}F(x,t)dx.$$

Then the problem (1.1) *has at least one solution in* $\mathcal{W}^{p(x)}(\Omega)$.

Proof. We shall employ Theorem 2.5 (I_1) in connection with the problem (1.1). To this end, let us consider the real Banach space $\mathcal{W}^{p(x)}(\Omega)$, endowed with the norm specified in (2.1). Furthermore, let Φ and Ψ denote the functionals introduced in (2.2) and (2.3), respectively. It can be observed that Φ and Ψ are of class C^1, as demonstrated in Lemma 2.4. For every $u\in\mathcal{W}^{p(x)}(\Omega)$, we find

$$\frac{\min\{1,a_0\}}{p^+}\|u\|^{p(x)}_{\mathcal{W}^{p(x)}(\Omega)}\le\Phi(u)\le\frac{\max\{1,a_1\}}{p^-}\|u\|^{p(x)}_{\mathcal{W}^{p(x)}(\Omega)}.\tag{3.2}$$

We shall verify that the functionals Φ and Ψ satisfy the assumptions required in Theorem 2.5. The functional Ψ is Gâteaux differentiable and the Gâteaux derivative of Ψ at $u\in\mathcal{W}^{p(x)}(\Omega)$ is represented as $\Psi'(u)\in(\mathcal{W}^{p(x)}(\Omega))^*$, defined as in (2.4), and Ψ is sequentially weakly upper semicontinuous. Furthermore, Φ is also Gâteaux differentiable and the Gâteaux derivative of Φ at $u\in\mathcal{W}^{p(x)}(\Omega)$ is represented as $\Phi'(u)\in(\mathcal{W}^{p(x)}(\Omega))^*$. Furthermore, Φ is sequentially weakly lower semicontinuous. Based on the inequality in (3.2), it can be concluded that Φ is coercive.

By (3.1), we are able to choose $\bar\tau>0$ fulfilling in

$$\frac{\bar\tau^{p^-}}{F_{\bar\tau}}>\frac{c_0^{p^-}\,p^+}{\min\{1,a_0\}}.\tag{3.3}$$

Now, we define

$$r=\frac{\min\{1,a_0\}}{p^+}\left(\frac{\bar\tau}{c_0}\right)^{p^-}.$$

If $u\in\Phi^{-1}(-\infty,r)$, according to Proposition 2.1 (i_1) and (3.2), for every $u\in\mathcal{W}^{p(x)}(\Omega)$ where $\|u\|_{\mathcal{W}^{p(x)}(\Omega)}>1$, we have

$$\frac{\min\{1,a_0\}}{p^+}\|u\|^{p^-}_{\mathcal{W}^{p(x)}(\Omega)}\le\Phi(u)\le\frac{\min\{1,a_0\}}{p^+}\left(\frac{\bar\tau}{c_0}\right)^{p^-}.$$

In a comparable manner, according to Proposition 2.1 (i_2) and (3.2), for every $u\in\mathcal{W}^{p(x)}(\Omega)$ where $\|u\|_{\mathcal{W}^{p(x)}(\Omega)}<1$, we can derive

$$\frac{\min\{1,a_0\}}{p^+}\|u\|^{p^+}_{\mathcal{W}^{p(x)}(\Omega)}\le\Phi(u)\le\frac{\min\{1,a_0\}}{p^+}\left(\frac{\bar\tau}{c_0}\right)^{p^-}.$$

Thus,

$$\|u\|_{\mathcal{W}^{p(x)}(\Omega)} \leq \frac{\bar{\tau}}{c_0}.$$

Then we have

$$|u(x)| \leq \|u\|_{L^\infty(\Omega)} \leq c_0\|u\|_{\mathcal{W}^{p(x)}(\Omega)} \leq \bar{\tau},$$

and thus

$$\Phi^{-1}(-\infty, r) \subset \{u \in \mathcal{W}^{p(x)}(\Omega) \mid |u(x)| \leq \bar{\tau}\}.$$

Since $\Phi(0) = \Psi(0) = 0$, it follows that

$$\begin{aligned}
\phi(r) &:= \inf_{u\in\Phi^{-1}(-\infty,r)} \frac{\sup_{v\in\Phi^{-1}(-\infty,r)} \Psi(v) - \Psi(u)}{r - \Phi(u)} \\
&\leq \frac{\sup_{t\in\Phi^{-1}(-\infty,r)} \Psi(t)}{r} = \frac{\sup_{t\in\Phi^{-1}(-\infty,r)} \int_\Omega F(x,t)dx}{r} \\
&\leq \frac{\int_\Omega \max_{|t|\leq\bar{\tau}} F(x,t)dx}{r} = \frac{F_{\bar{\tau}}}{r} = \frac{F_{\bar{\tau}} p^+}{\min\{1, a_0\}} \left(\frac{c_0}{\bar{\tau}}\right)^{p^-}.
\end{aligned}$$

Then, by (3.3) we have $\phi(r) < 1$. Therefore, since $1 \in \left(0, \frac{1}{\phi(r)}\right)$, Theorem 2.5 (I_1) ensures that the functional I possesses at least one critical point, namely a local minimum $\tilde{u}$, which lies in $\Phi^{-1}(-\infty, r)$. Consequently, the existence of a solution is guaranteed. □

Example 3.2. Consider the following problem:

$$\begin{cases} \nabla \times (|\nabla \times u|^{p(x)-2}\nabla \times u) + a(x)|u|^{p(x)-2}u = \dfrac{1}{10^8 c_0^9} t^8 e^{-t}(9-t), \quad \nabla \cdot u = 0, & \text{in } \Omega, \\ |\nabla \times u|^{p(x)-2}\nabla \times u \times n = 0, \quad u \cdot n = 0, & \text{on } \partial\Omega, \end{cases} \tag{3.4}$$

where $\Omega = \{x = (x_1, x_2, x_3) \in \mathbb{R}^3, |x_1^2 + x_2^2 + x_3^2 \leq 9\}$, $a(x) = 1$ and $p(x) = x_1^2 + x_2^2 + x_3^2 + 9$, for every $x \in \Omega$. By direct calculations, we obtain $p^- = 9$ and $p^+ = 18$, and and the nonlinear term satisfies $F(t) = 10^{-8}c_0^{-9}t^9e^{-t}$. Hence,

$$\sup_{\tau>0} \frac{\tau^9}{\max_{|t|\leq\tau} F(t)} > 648\pi c_0^9.$$

Now all the conditions of Theorem 3.1 are satisfied and thus the problem (3.4) has at least one solution in $\mathcal{W}^{p(x)}(\Omega)$.

Theorem 3.1 can also be utilized to verify the existence of at least one solution for the following parametric problem:

$$\begin{cases} \nabla \times (|\nabla \times u|^{p(x)-2}\nabla \times u) + a(x)|u|^{p(x)-2}u = \lambda f(x,u), \quad \nabla \cdot u = 0, & \text{in } \Omega, \\ |\nabla \times u|^{p(x)-2}\nabla \times u \times n = 0, \quad u \cdot n = 0, & \text{on } \partial\Omega, \end{cases} \tag{3.5}$$

where $\lambda > 0$ is a parameter.

Theorem 3.3. *For any*

$$\lambda \in \Lambda = \left(0, \frac{\min\{1, a_0\}}{c_0^{p^-} p^+} \sup \frac{\tau^{p^-}}{F_\tau}\right),$$

the problem (3.5) *has a solution* $u_\lambda \in \mathcal{W}^{p(x)}(\Omega)$.

Proof. Fix λ as mentioned in the assertion. Consider Φ and Ψ as defined in (2.2) and (2.3), and define $I_\lambda(u) = \Phi(u) - \lambda\Psi(u)$ for every $u \in \mathcal{W}^{p(x)}(\Omega)$. Let us choose

$$0 < \lambda < \frac{\min\{1, a_0\}}{c_0^{p^-} p^+} \sup \frac{\tau^{p^-}}{F_\tau}.$$

As a consequence, we deduce the existence of a constant $\bar{\tau} > 0$ such that the following condition holds:

$$\frac{\lambda c_0^{p^-} p^+}{\min\{1, a_0\}} < \frac{\bar{\tau}^{p^-}}{F_{\bar{\tau}}}.$$

Set,

$$r = \frac{\min\{1, a_0\}}{p^+} \left(\frac{\bar{\tau}}{c_0}\right)^{p^-}.$$

By employing the notation introduced in Theorem 3.1, we obtain

$$\phi(r) < \frac{F_{\bar{\tau}} p^+}{\min\{1, a_0\}} \left(\frac{c_0}{\bar{\tau}}\right)^{p^-} < \frac{1}{\lambda}.$$

Since $\lambda \in \left(0, \frac{1}{\phi(r)}\right)$, Theorem 2.5 (I_1) confirms that the functional I_λ has a minimum point (local minima) $u_\lambda \in \Phi^{-1}(-\infty, r)$. We arrive at the conclusion. $\square$

Remark 3.4. In Theorem 3.1, if $f(x, u) \geq 0, a.e. \forall x \in \Omega$, then (3.1) turns into the form

$$\sup_{\tau>0} \frac{\tau^{p^-}}{\int_\Omega F(x, \tau)dx} > \frac{c_0^{p^-} p^+}{\min\{1, a_0\}}. \tag{3.6}$$

Moreover, if

$$\limsup_{\tau\to\infty} \frac{\tau^{p^-}}{\int_\Omega F(x, \tau)dx} > \frac{c_0^{p^-} p^+}{\min\{1, a_0\}},$$

then (3.6) holds.

Remark 3.5. Suppose that $\bar{\tau} > 0$ is constant and

$$\frac{c_0^{p^-} p^+}{\min\{1, a_0\}} < \frac{\bar{\tau}^{p^-}}{F_{\bar{\tau}}}.$$

Consequently, the outcome of Theorem 3.3 is applicable, with $\|u_\lambda\|_\infty \leq \bar{\tau}$.

Remark 3.6. In general, I_λ can be unbounded in below. For example, when $f(x, v) = 1 + |v|^{s-p^+} v^{p^+-1}, \forall(x, v) \in \Omega \times \mathbb{R}^3$, by the condition $s > p^+$, for any $u \in \mathcal{W}^{p(x)}(\Omega) \setminus \{0\}$ and $t \in \mathbb{R}$, we have

$$\begin{aligned} I_\lambda(tu) &= \Phi(tu) - \lambda \int_\Omega F(x, tu)dx \\ &\leq \frac{t^{p^+}}{p^-} \int_\Omega (|\nabla \times u|^{p(x)} + a(x)|u|^{p(x)})dx - \lambda t\|u\|_{L^1} - \lambda \frac{t^s}{s}\|u\|_{L^s}^s \to -\infty. \end{aligned}$$

as $t \to +\infty$. Hence, the condition [18, Theorem 2.2] is not satisfied, so we cannot apply direct minimization to find critical points of the functional I_λ.

The functional I_λ, in general is not coercive, for instance when $F(x,v) = |v|^s, s > p^+$, $\forall (x,v) \in \Omega \times \mathbb{R}^3$, for any $u \in \mathcal{W}^{p(x)}(\Omega) \setminus \{0\}$ and $t \in \mathbb{R}$, we have

$$\begin{aligned} I_\lambda(tu) &= \Phi(tu) - \lambda \int_\Omega F(x,tu)dx \\ &\leq \frac{t^{p^+}}{p^-} \int_\Omega (|\nabla \times u|^{p(x)} + a(x)|u|^{p(x)})dx - \lambda t^s \|u\|^s_{\mathcal{W}^{p(x)}(\Omega)} \to -\infty, \quad t \to +\infty. \end{aligned}$$

Remark 3.7. If f is a nonnegative function, then the solution obtained by Theorem 3.3 is also nonnegative. More precisely, if u^* is a nontrivial solution of (3.5), then u^* is nonnegative almost everywhere in Ω.

To see this, assume that the set $X = \{x \in \Omega : u^*(x) < 0\} \neq \emptyset$. Define $\bar{u}(x) = \min\{u^*(x), 0\}, \forall x \in \Omega$. We have $\bar{u}(x) \in \mathcal{W}^{p(x)}(\Omega)$ and

$$\begin{aligned} I^{'}(u^*)(\bar{u}) = &\int_\Omega |\nabla \times u^*|^{p(x)-2} \nabla \times u^* \cdot \nabla \times \bar{u} dx \\ &+ \int_\Omega a(x)|u^*|^{p(x)-2}u^* \cdot \bar{u} dx - \lambda \int_\Omega f(x,u^*) \cdot \bar{u} dx = 0. \end{aligned}$$

By taking $\bar{u} = u^*$ and noting that f is nonnegative, we have

$$\begin{aligned} 0 \leq \min\{1, a_0\} \|u^*\|_X^{p^-} \leq &\int_\Omega |\nabla \times u^*|^{p(x)-2} \nabla \times u^* \cdot \nabla \times u^* dx \\ &+ \int_\Omega a(x)|u^*|^{p(x)-2}u^* \cdot u^* dx = \lambda \int_\Omega f(x,u^*) \cdot u^* dx \leq 0. \end{aligned}$$

Thus $u^* = 0$, which is a contradiction. Therefore u^* is positive.

We now present the following theorem, which guarantees the nontriviality of solutions to problem (3.5) under the specified conditions. Furthermore, it characterizes the asymptotic behavior of the solution norm as the parameter λ approaches zero, and establishes that the associated functional I_λ is negative and strictly decreasing on the interval $(0, \lambda^*)$.

Theorem 3.8. *Suppose that there exist a nonempty open set $A \subseteq \Omega$ and a subset $B \subset A$ with positive Lebesgue measure such that*

$$\limsup_{t \to 0^+} \frac{\inf_{x \in B} F(x,t)}{|t|^{p^+}} = +\infty, \tag{3.7}$$

and

$$\liminf_{t \to 0^+} \frac{\inf_{x \in A} F(x,t)}{|t|^{p^+}} > -\infty. \tag{3.8}$$

Let

$$0 < \bar{\lambda} < \lambda^* = \frac{\min\{1, a_0\}}{c_0^{p^-} p^+} \sup \frac{\tau^{p^-}}{F_\tau}. \tag{3.9}$$

Then, for every $\lambda \in (0, \lambda^*)$ the solutions of the problem (3.5) are nontrivial. Moreover,

$$\lim_{\lambda \to 0^+} \|u_\lambda\|_{\mathcal{W}^{p(x)}(\Omega)} = 0,$$

and the real function $\lambda \to I_\lambda$ is negative and strictly decreasing.

Proof. we want to apply Theorem 3.3. In the case where $f(x,0) \neq 0$, $\forall x \in \Omega$, the solution ensured by Theorem 3.3 is clearly nontrivial. So, we suppose that $f(x,0) \neq 0$, $\forall x \in \Omega$. By (3.9) there exists $\bar{\tau} > 0$ such that

$$\frac{\bar{\lambda} c_0^{p^-} p^+}{\min\{1, a_0\}} < \frac{\bar{\tau}^{p^-}}{F_{\bar{\tau}}}.$$

By Theorem 2.5 (I_1), for any $\lambda \in (0, \bar{\lambda})$, there is a critical point $u_\lambda \in \Phi^{-1}(-\infty, r_\lambda)$, where $r_\lambda = \frac{\min\{1,a_0\}}{p^+} \left(\frac{\bar{\tau}}{c_0}\right)^{p^-}$. Indeed, u_λ is a global minimum of I_λ in $\Phi^{-1}(-\infty, r_\lambda)$.

Now, we shall show u_λ is nontrivial. To begin, we first show

$$\limsup_{\|u\|_{\mathcal{W}^{p(x)}} \to 0^+} \frac{\Psi(u)}{\Phi(u)} = \infty.$$

By (3.7) and (3.8), we can consider a sequence $\{t_n\} \subset \mathbb{R}^+$ converging to zero, two constants $\epsilon > 0$ and μ, such that

$$\lim_{n \to \infty} \frac{\inf_{x \in B} F(x, t_n)}{|t_n|^{p^+}} = +\infty,$$

and

$$\inf_{x \in A} F(x,t) \geq \mu |t|^{p^+}, \quad \forall t \in [0, \epsilon].$$

We can take an open set $E \subset B$, by positive measure and a function $\omega \in \mathcal{W}^{p(x)}(\Omega)$, such that

(m_1) $\omega(x) \in [0,1]$, for every $x \in \Omega$.

(m_2) $\omega(x) = 1$, for every $x \in E$.

(m_3) $\omega(x) = 0$, for every $x \in \Omega \setminus A$.

Now, we consider $T > 0$ and set $\eta > 0$, such that

$$T < \frac{\eta \operatorname{meas}(E) + \mu \int_{A \setminus E} |\omega(x)|^{p^+} dx}{\frac{\max\{1,a_1\}}{p^-} \|u\|^{p^+}_{\mathcal{W}^{p(x)}(\Omega)}}.$$

So, there is $n_0 \in \mathbb{N}$, such that for any $n > n_0$, we have $t_n < \epsilon$ and $\inf_{x \in B} F(x, t_n) \geq \eta |t_n|^{p^+}$. Now,

$$\begin{aligned} \frac{\Psi(t_n \omega)}{\Phi(t_n \omega)} &= \frac{\int_E F(x, t_n) dx + \int_{A \setminus E} F(x, t_n \omega(x)) dx}{\Phi(t_n \omega)} \\ &> \frac{\eta \operatorname{meas}(E) + \mu \int_{A \setminus E} |\omega(x)|^{p^+} dx}{\frac{\max\{1,a_1\}}{p^-} \|\omega\|^{p^+}_{\mathcal{W}^{p(x)}(\Omega)}} > T. \end{aligned}$$

Since T is arbitrary and may be taken large, we get

$$\lim_{n \to \infty} \frac{\Psi(t_n \omega)}{\Phi(t_n \omega)} = \infty.$$

Thus, there exist a sequence $\{s_n\} \subset \mathcal{W}^{p(x)}(\Omega)$, that $s_n \in \Phi^{-1}(-\infty, r)$ and

$$I_\lambda(s_n) = \Phi(s_n) - \lambda \Psi(s_n) < 0.$$

By this fact that, u_λ is a global minimum of the restriction of I_λ to $\Phi^{-1}(-\infty, r)$, we have

$$I_\lambda(u_\lambda) < 0, \tag{3.10}$$

hence, u_λ is nontrivial. By (3.10), we additionally have the following map

$$(0, \lambda^*) \ni \lambda \mapsto I_\lambda(u_\lambda),$$

is negative. Also, one has

$$\lim_{\lambda \to 0^+} \|u_\lambda\|_{\mathcal{W}^{p(x)}(\Omega)} = 0.$$

By this fact that Φ is coercive for every $\lambda \in (0, \lambda^*)$ and $u_\lambda \in \Phi^{-1}(-\infty, r)$, there exist a positive constant N with $\|u_\lambda\| \leq N$, $\forall \lambda \in (0, \lambda^*)$. This demonstrates the existence of positive constant $M > 0$, such that

$$|\Psi^{'}(u_\lambda)(u_\lambda)| \leq M\|u_\lambda\| \leq MN, \tag{3.11}$$

for all $\lambda \in (0, \lambda^*)$. By noting that u_λ is a critical point of I_λ, we have $I_\lambda^{'}(u_\lambda)(u_\lambda) = 0$, i.e.

$$\Phi^{'}(u_\lambda)(u_\lambda) = \lambda \Psi^{'}(u_\lambda)(u_\lambda), \qquad \forall \lambda \in (0, \lambda^*).$$

Since, $0 \leq \min\{1, a_0\}\|u_\lambda\|^{p^-}_{\mathcal{W}^{p(x)}(\Omega)} \leq \Phi^{'}(u_\lambda)(u_\lambda)$, we get

$$0 \leq \min\{1, a_0\}\|u_\lambda\|^{p^-}_{\mathcal{W}^{p(x)}(\Omega)} \leq \lambda \Psi^{'}(u_\lambda)(u_\lambda), \qquad \forall \lambda \in (0, \lambda^*). \tag{3.12}$$

By using (3.11), (3.12) and letting $\lambda \to 0^+$, we have

$$\lim_{\lambda \to 0^+} \|u_\lambda\|_{\mathcal{W}^{p(x)}(\Omega)} = 0.$$

Now, we exhibit that the map $\lambda \mapsto I_\lambda(u_\lambda)$ in $(0, \lambda^*)$ is strictly decreasing. For any $u \in \mathcal{W}^{p(x)}(\Omega)$, we have

$$I_\lambda(u) = \lambda \left(\frac{\Phi(u)}{\lambda} - \Psi(u) \right).$$

Now, fix $0 < \lambda_1 < \lambda_2 < \lambda^*$ and assume that u_{λ_j}, $j = 1, 2$, be the global minimum of the functional I_{λ_j} restricted to $\Phi^{-1}(-\infty, r)$. Moreover, put

$$l_{\lambda_j}(u) = \left(\frac{\Phi(u_{\lambda_j})}{\lambda_j} - \Psi(u_{\lambda_j}) \right) = \inf_{v \in \Phi^{-1}(-\infty, r)} \left(\frac{\Phi(v)}{\lambda_j} - \Psi(v) \right).$$

Obviously, one has

$$l_{\lambda_j} < 0, \quad j = 1, 2$$

and by this condition that $0 < \lambda_1 < \lambda_2$, we have $l_{\lambda_2} \leq l_{\lambda_1}$. By the mentioned relations, we have

$$I_{\lambda_2}(u_{\lambda_2}) \leq \lambda_2 l_{\lambda_2} \leq \lambda_2 l_{\lambda_1} < \lambda_1 l_{\lambda_1} = I_{\lambda_1}(u_{\lambda_1}).$$

So, the map $\lambda \mapsto l_\lambda(u_\lambda)$ is strictly decreasing and our preferred outcome is achieved. □

Now, we prove the existence of infinitely many solutions by applying Theorem 2.5.

Theorem 3.9. *Suppose that*

$$\int_\Omega F(x,t)dx \geq 0 \tag{3.13}$$

and

$$\mathfrak{A} < \mathfrak{T}\mathfrak{B}, \tag{3.14}$$

where

$$\mathfrak{A} = \liminf_{\tau\to+\infty} \frac{\int_\Omega \max_{|t|\leq\tau} F(x,t)dx}{\tau^{p^-}}, \quad \mathfrak{B} = \limsup_{\tau\to+\infty} \frac{\int_\Omega F(x,\tau)}{\tau^{p^-}},$$

and

$$\mathfrak{T} = \frac{\min\{1,a_0\}}{c_0^{p^-}\,\mathrm{meas}(\Omega)a_1}. \tag{3.15}$$

Then problem (1.1) has infinitely many nontrivial solutions in $\mathcal{W}^{p(x)}(\Omega)$, for every

$$\lambda \in \left(\frac{\mathrm{meas}(\Omega)a_1}{p^-\mathfrak{B}}, \frac{\min\{1,a_0\}}{c_0^{p^-}\,p^+\mathfrak{A}}\right).$$

Proof. We apply Theorem 2.5 (I_2) to the problem (1.1). In this context, fix

$$\lambda \in \left(\frac{\mathrm{meas}(\Omega)a_1}{p^-\mathfrak{B}}, \frac{\min\{1,a_0\}}{c_0^{p^-}\,p^+\mathfrak{A}}\right).$$

Let Φ and Ψ be the functionals outlined in (2.2) and (2.3). It can be observed that Φ is sequentially weakly lower semicontinuous, while Ψ is sequentially weakly upper semicontinuous. For every $u \in \mathcal{W}^{p(x)}(\Omega)$, one has

$$\frac{\min\{1,a_0\}}{p^+}\|u\|^{p(x)}_{\mathcal{W}^{p(x)}(\Omega)} \leq \Phi(u) \leq \frac{\max\{1,a_1\}}{p^-}\|u\|^{p(x)}_{\mathcal{W}^{p(x)}(\Omega)}, \tag{3.16}$$

that shows Φ is coercive. Now, we show that $\alpha < \infty$.

Let $\{\tau_n\}$ be a sequence that, $\lim_{n\to+\infty}\tau_n = +\infty$ and

$$\mathfrak{A} = \lim_{n\to+\infty} \frac{\int_\Omega \max_{|t|\leq\tau_n} F(x,t)dx}{\tau_n^{p^-}}.$$

For any $n \in \mathbb{N}$, set $r_n = \frac{\min\{1,a_0\}}{c_0^{p^-}\,p^+}\tau_n^{p^-}$. Obviously, $\lim_{n\to\infty} r_n = \infty$. If $u \in \Phi^{-1}(-\infty, r_n)$, according to Proposition 2.1 (i_1) and (3.16), for every $u \in \mathcal{W}^{p(x)}(\Omega)$ where $\|u\|_{\mathcal{W}^{p(x)}(\Omega)} > 1$, one has

$$\frac{\min\{1,a_0\}}{p^+}\|u\|^{p^-}_{\mathcal{W}^{p(x)}(\Omega)} \leq \Phi(u) \leq \frac{\min\{1,a_0\}}{p^+}\left(\frac{\tau_n}{c_0}\right)^{p^-}.$$

In a comparable manner, according to Proposition 2.1 (i_2) and (3.16), for every $u \in \mathcal{W}^{p(x)}(\Omega)$, that $u \in \Phi^{-1}(-\infty, r_n)$ and $\|u\|_{\mathcal{W}^{p(x)}(\Omega)} < 1$, one has

$$\frac{\min\{1,a_0\}}{p^+}\|u\|^{p^+}_{\mathcal{W}^{p(x)}(\Omega)} \leq \Phi(u) \leq \frac{\min\{1,a_0\}}{p^+}\left(\frac{\tau_n}{c_0}\right)^{p^-}.$$

Thus,

$$\|u\|_{\mathcal{W}^{p(x)}(\Omega)} \leq \frac{\tau_n}{c_0}.$$

We have

$$|u(x)| \leq \|u\|_{L^\infty(\Omega)} \leq c_0 \|u\|_{\mathcal{W}^{p(x)}(\Omega)} \leq \tau_n,$$

Thus $\Phi^{-1}(-\infty, r_n) \subset \{u \in \mathcal{W}^{p(x)}(\Omega) | |u(x)| \leq \tau_n\}$. Since $\Phi(0) = \Psi(0) = 0$, it follows that

$$\begin{aligned}
\phi(r_n) := & \inf_{u \in \Phi^{-1}(-\infty, r_n)} \frac{\sup_{v \in \Phi^{-1}(-\infty, r_n)} \Psi(v) - \Psi(u)}{r_n - \Phi(u)} \\
& \leq \frac{\sup_{t \in \Phi^{-1}(-\infty, r_n)} \Psi(t)}{r_n} = \frac{\sup_{t \in \Phi^{-1}(-\infty, r_n)} \int_\Omega F(x,t) dx}{r_n} \\
& \leq \frac{\int_\Omega \max_{|t| \leq \tau_n} F(x,t) dx}{r_n} = \frac{p^+ \int_\Omega \max_{|t| \leq \tau_n} F(x,t) dx}{\min\{1, a_0\}} \left(\frac{c_0}{\tau_n} \right)^{p^-}.
\end{aligned}$$

Thus, (3.14) yields

$$\alpha \leq \liminf_{n \to \infty} \phi(r_n) \leq \frac{p^+ c_0^{p^-} \mathfrak{A}}{\min\{1, a_0\}} < \infty.$$

Therefore, we have

$$\lambda \in \left(\frac{\text{meas}(\Omega) a_1}{p^- \mathfrak{B}}, \frac{\min\{1, a_0\}}{c_0^{p^-} p^+ \mathfrak{A}} \right) \subset \left(0, \frac{1}{\alpha} \right).$$

Now, we show (j_1) cannot hold.

By definition of $\mathfrak{B}$, there is a sequence $\{c_n\}$ of positive numbers, such that $\lim_{n \to \infty} c_n = \infty$ and

$$\mathfrak{B} = \lim_{n \to +\infty} \frac{\int_\Omega F(x, c_n)}{c_n^{p^-}}. \tag{3.17}$$

Now, we define $\omega_n \in \mathcal{W}^{p(x)}(\Omega)$, for every $n \in \mathbb{N}$ as follows:

$$\omega_n(x) = \begin{cases} c_n, & x \in \Omega, \\ 0, & o.w. \end{cases}$$

One has

$$\Phi(\omega_n) \leq \frac{\text{meas}(\Omega) a_1}{p^-} \max\{c_n^{p^+}, c_n^{p^-}\}.$$

Then,

$$\begin{aligned}
I_\lambda(\omega_n) &= \Phi(\omega_n) - \lambda \Psi(\omega_n) \\
&\leq \frac{\text{meas}(\Omega) a_1}{p^-} \max\{c_n^{p^+}, c_n^{p^-}\} - \lambda \int_\Omega F(x, \omega_n) dx.
\end{aligned} \tag{3.18}$$

We consider two cases to complete the proof, $\mathfrak{B} = \infty$ and $\mathfrak{B} < \infty$.

If $\mathfrak{B} = \infty$, then we choose K large enough, such that $K > \frac{\text{meas}(\Omega) a_1}{p^- \lambda}$. From (3.17), there exists $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$,

$$\int_\Omega F(x, c_n) dx > K c_n^{p^-}.$$

Then, from (3.18) we have

$$I_\lambda(\omega_n) \le \frac{\text{meas}(\Omega)a_1}{p^-}\max\{c_n^{p^+}, c_n^{p^-}\} - \lambda K c_n^{p^-} \le \left(\frac{\text{meas}(\Omega)a_1}{p^-} - \lambda K\right)\max\{c_n^{p^+}, c_n^{p^-}\},$$

which shows that $\lim_{n\to\infty} I_\lambda(\omega_n) = -\infty$.

In another case, if $\mathfrak{B} < \infty$, since $\lambda > \frac{\text{meas}(\Omega)a_1}{p^-\mathfrak{B}}$, there exists $\epsilon > 0$ such that $\epsilon < \mathfrak{B} - \frac{\text{meas}(\Omega)a_1}{p^-\lambda}$. Then, by (3.17) there exists $n_1 \in \mathbb{N}$ such that for all $n \ge n_1$,

$$\int_\Omega F(x, c_n)dx > (\mathfrak{B} - \epsilon)c_n^{p^-}.$$

By (3.18)

$$\begin{aligned} I_\lambda(\omega_n) &\le \frac{\text{meas}(\Omega)a_1}{p^-}\max\{c_n^{p^+}, c_n^{p^-}\} - \lambda(\mathfrak{B} - \epsilon)c_n^{p^-} \\ &\le \left(\frac{\text{meas}(\Omega)a_1}{p^-} - \lambda(\mathfrak{B} - \epsilon)\right)\max\{c_n^{p^+}, c_n^{p^-}\}. \end{aligned}$$

Then, we get $\lim_{n\to\infty} I_\lambda(\omega_n) = -\infty$. By considering the cases mentioned above, we observe that I_λ is unbounded from below. This shows that condition (j_1) does not hold, which implies that (j_2) must be satisfied. Consequently, I_λ admits an unbounded sequence of critical points, and the proof is finished. □

Remark 3.10. Suppose that $\mathfrak{B} = \infty$ and $\mathfrak{A} = 0$. It is clear that, all the assumptions required by Theorem 3.1 are fully satisfied. Thus, the problem (3.5) admits infinitely many solutions for every $\lambda \in (0, \infty)$.

Corollary 3.11. *Suppose that $f : \Omega \times \mathbb{R}^3 \to \mathbb{R}$ be a continuous function and (3.13) holds. Moreover,*

$$\limsup_{\tau\to+\infty} \frac{\int_\Omega F(x, \tau(x))}{\tau^{p^-}} > \frac{\text{meas}(\Omega)a_1}{p^-}$$

and

$$\liminf_{\tau\to+\infty} \frac{\int_\Omega \max_{|t|\le\tau} F(x,t)dx}{\tau^{p^-}} < \frac{\min\{1, a_0\}}{c_0^{p^-} p^+}.$$

Then problem (1.1) admits an unbounded sequence of weak solutions in $\mathcal{W}^{p(x)}(\Omega)$.

Theorem 3.12. *Suppose that $\int_\Omega F(x,t)dx \ge 0$ and $\mathfrak{A}^* < \mathfrak{T}\mathfrak{B}^*$, where*

$$\mathfrak{A}^* = \liminf_{\tau\to0^+} \frac{\int_\Omega \max_{|t|\le\tau} F(x,t)dx}{\tau^{p^-}}, \quad \mathfrak{B}^* = \limsup_{\tau\to0^+} \frac{\int_\Omega F(x,\tau)}{\tau^{p^-}}$$

and $\mathfrak{T}$ is defined in (3.15). Then problem (1.1) has a sequence of pairwise distinct critical points in $\mathcal{W}^{p(x)}(\Omega)$ for every $\lambda \in \left(\frac{\text{meas}(\Omega)a_1}{p^-\mathfrak{B}^*}, \frac{\min\{1,a_0\}}{c_0^{p^-} p^+\mathfrak{A}^*}\right)$.

Proof. By the same approach as in the previous proof and replacing Theorem 2.5 (I_2) with Theorem 2.5 (I_3), we obtain the desired result. □

Declaration of interests. The authors report no conflict of interest.

References

[1] G. A. Afrouzi, M. Bohner, G. Caristi, S. Heidarkhani, S. Moradi, An existence result for impulsive multi-point boundary value systems using a local minimization principle, *J. Optim. Theory Appl.* **177**(2018), 1–20. https://doi.org/10.1007/s10957-018-1253-1; MR3783643; Zbl 1391.34061

[2] S. Antontsev, F. Miranda, L. Santos, Blow-up and finite time extinction for $p(x,t)$-curl systems arising in electromagnetism, *J. Math. Anal. Appl.* **440**(2016), 300–322. https://doi.org/10.1016/j.jmaa.2016.03.045; MR3479601; Zbl 1339.35060

[3] M. Bohner, G. Caristi, S. Heidarkhani, S. Moradi, A critical point approach to boundary value problems on the real line, *Appl. Math. Lett.* **76**(2018), 215–220. https://doi.org/10.1016/j.aml.2017.08.017; MR3713519; Zbl 1388.35063

[4] G. Bonanno, P. Candito, Non-differentiable functionals and applications to elliptic problems with discontinuous nonlinearities, *J. Differential Equations* **244**(2008), 3031–3059. https://doi.org/10.1016/j.jde.2008.02.025; MR2420513; Zbl 1149.49007

[5] G. Bonanno, G. Molica Bisci, Infinitely many solutions for a boundary value problem with discontinuous nonlinearities, *Bound. Value Probl.* **2009**, Art. ID 670675, 1–20. https://doi.org/10.1155/2009/670675; MR2487254; Zbl 1177.34038

[6] M. M. Boureanu, V. D. Rădulescu, Anisotropic Neumann problems in Sobolev spaces with variable exponent, *Nonlinear Anal. TMA.* **75**(2012), 4471–4482. https://doi.org/10.1016/j.na.2011.09.033; MR2927115; Zbl 1262.35090

[7] Y. Chen, S. Levine, R. Rao, Variable exponent, linear growth functionals in image restoration, *SIAM J. Appl. Math.* **66**(2006), 1383–1406. https://doi.org/10.1137/050624522; MR2246061; Zbl 1102.49010

[8] L. Diening, P. Harjulehto, P. Hästö, M. Růžička, *Lebesgue and Sobolev spaces with variable exponents*, Lecture Notes in Mathematics, Vol. 2017, Springer, 2011. https://doi.org/10.1007/978-3-642-18363-8; MR2790542; Zbl 1222.46002

[9] X. L. Fan, Q. Zhang, Existence of solutions for $p(x)$-Laplacian Dirichlet problems, *Nonlinear Anal.* **52**(2003), 1843–1852. https://doi.org/10.1016/S0362-546X(02)00150-5; MR1954585; Zbl 1146.35353

[10] X. L. Fan, D. Zhao, On the spaces $L^{p(x)}(\Omega)$ and $W_{m,p(x)}(\Omega)$, *J. Math. Anal. Appl.* **263**(2001), No. 2, 424–446. https://doi.org/10.1006/jmaa.2000.7617; MR1866056; Zbl 1028.46041

[11] M. Galewski, G. Molica Bisci, Existence results for one-dimensional fractional equations, *Math. Meth. Appl. Sci.* **39**(2016), 1480–1492. https://doi.org/10.1002/mma.3582; MR3483509; Zbl 1381.34012

[12] B. Ge, J. F. Lu, Existence and multiplicity of solutions for $p(x)$-curl systems without the Ambrosetti–Rabinowitz condition, *Mediterr. J. Math.* **16**(2019), 1–17. https://doi.org/10.1007/s00009-019-1312-3; MR3918505; Zbl 1414.35050

[13] B. Ge, J. F. Lu, Multiple solutions for $p(x)$-curl systems with nonlinear boundary condition, *Math. Nachr.* **295**(2022), 512–535. https://doi.org/10.1002/mana.201900236; MR4394094; Zbl 1523.35128

[14] F. Gharehgazlouei, G. A. Afrouzi, Multiplicity of solutions for $p(x)$-curl systems arising in electromagnetism, *Electron. J. Qual. Theory Differ. Equ.* **2024**, No. 76, 1–14. https://doi.org/10.14232/ejqtde.2024.1.76; MR4863382; Zbl 1574.35328

[15] S. Heidarkhani, G. A. Afrouzi, S. Moradi, An existence result for discrete anisotropic equations, *Taiwanese J. Math.* **22**(2018), 725–739. https://doi.org/10.11650/TJM/170801; MR3807334; Zbl 1401.39002

[16] B. Kone, S. Ouaro, S. Traore, Weak solutions for anisotropic nonlinear elliptic equations with variable exponents, *Electron. J. Differential Equations* **2009**, No. 144, 1–11. MR2565886; Zbl 1182.35092

[17] M. Mihăilescu, V. Rădulescu, A multiplicity result for a nonlinear degenerate problem arising in the theory of electrorheological fluids, *Proc. R. Soc. A Math. Phys. Eng. Sci.* **462**(2006), 2625–2641. https://doi.org/10.1098/rspa.2005.1633; MR2253555; Zbl 1149.76692

[18] P. H. Rabinowitz, Minimax methods in critical point theory with applications to differential equations, *CBMS Reg. Conf. Ser. Math.* **65**(1986). https://doi.org/10.1090/cbms/065; MR0845785; Zbl 0609.58002

[19] B. Ricceri, A general variational principle and some of its applications, *J. Comput. Appl. Math.* **113**(2000), 401–410. https://doi.org/10.1016/S0377-0427(99)00269-1; MR1735837; Zbl 0946.49001

[20] M. Xiang, F. Wang, B. Zhang, Existence and multiplicity of solutions for $p(x)$-curl systems arising in electromagnetism, *J. Math. Anal. Appl.* **448**(2017), 1600–1617. https://doi.org/10.1016/j.jmaa.2016.11.086; MR3582298; Zbl 1358.35181