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Nontrivial solutions for perturbed Schrödinger systems with pinching and mixed nonlinearities

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Abstract. In this paper, we prove the existence of nontrivial solutions for perturbed Schrödinger systems with pinching and mixed nonlinearities, where the nonlinearities are composed of sublinear terms, asymptotically linear terms (pinching terms) and superlinear terms. When the system is time-independent and perturbed, we obtain the existence of nontrivial homoclinic solutions. When the system is time-dependent and non-perturbed, we obtain the existence of nontrivial standing waves and homoclinic solutions. In addition, we also give some function examples that are suitable for our models.

Keywords: nonlinear Schrödinger systems, perturbed terms, mixed nonlinearities, homoclinic solutions, standing waves.

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1 Introduction and Main Results

Nonlinear Schrödinger systems play significant roles in many physical problems [1, 11, 17, 19, 25, 33], and their mathematical models can be abstracted as the following nonlinear Schrödinger system

$$\begin{cases} -\triangle u_i + V_i(x)u_i = f_i(x, u_1, \dots, u_m), & x \in \mathbb{R}^N, \\ u_i(x) \to 0 \in \mathbb{R}, & |x| \to \infty, \\ i = 1, \dots, m, \end{cases}$$

where $N \geq 1$, $\triangle$ denotes the Laplace operator and f_i $(i = 1, \dots, m)$ are the nonlinearities.

By using variational methods, the existence and multiplicity of solutions for Schrödinger systems have been studied by many authors, see [2, 4, 6–10, 12–16, 18, 21–24, 26–32, 34, 36–38]. These results are all about the case of *non-perturbed* nonlinearities, and most results are about the case of special nonlinearities, such as

$$f_i = \sum_{j=1}^{m} \beta_{ij} |u_j|^p |u_i|^{p-2} u_i, \quad \beta_{ij} \in \mathbb{R}, \quad 1 < p < \frac{2^*}{2}, \quad i = 1, \dots, m,$$

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where $2^* := \frac{2N}{N-2}$ if $N \geq 3$ and $2^* := \infty$ for $N = 1, 2$.

Motivated by the above papers, we will study the following Schrödinger system with perturbed term and general mixed nonlinearities

$$\begin{cases} -\triangle U + V(x)U = \nabla F(x,U), & x \in \mathbb{R}^N, \\ \nabla F(x,U) := -\nabla P(x,U) + \nabla W(x,U) + (\delta+1)h(x)|U|^{\delta-1}U + \chi(x), & 0 < \delta < 1, \\ U = (u_1(x),\dots,u_m(x))^\top \to 0 \in \mathbb{R}^m, & |x| \to \infty, \end{cases} \tag{1.1}$$

where $U \in \mathbb{R}^m$, $N \geq 1$, $\triangle$ denotes the Laplace operator, "$\top$" denotes the vector transpose, $\chi(x) = (\chi_1(x),\dots,\chi_m(x))^\top$ is the perturbed term, $\nabla P(x,\cdot)$ and $\nabla W(x,\cdot)$ are the gradients of $P(x,\cdot),\ W(x,\cdot) \in C^1(\mathbb{R}^N,\mathbb{R})$,

$$\triangle U = (\triangle u_1,\dots,\triangle u_m)^\top, \quad V(x) = \begin{pmatrix} V_1(x) & & \\ & \ddots & \\ & & V_m(x) \end{pmatrix}. \tag{1.2}$$

Here, our nonlinearity $\nabla F(x,U)$ is composed of three nonlinear vector functions and one perturbation vector function, i.e., $\nabla P(x,U)$, $\nabla W(x,U)$, $(\delta+1)h(x)|U|^{\delta-1}U$ and $\chi(x)$. Next, we will study the cases where these three nonlinear functions are sublinear, asymptotically linear (pinching), and superlinear, respectively.

The above system comes from the study of standing waves of the following time-dependent nonlinear Schrödinger equation

$$\iota\dot{\Phi} = -\triangle\Phi + \varepsilon(x)\Phi + \widetilde{p}(x,|\Phi|)\Phi - \widetilde{w}(x,|\Phi|)\Phi - (\delta+1)h(x)|\Phi|^{\delta-1}\Phi, \quad t > 0,\ x \in \mathbb{R}^N, \tag{1.3}$$

where ι is the imaginary unit, $\Phi = (\varphi_1(t,x),\dots,\varphi_m(t,x))^\top \in \mathbb{C}^m$,

$$\dot{\Phi} = \left(\frac{\partial\varphi_1}{\partial t},\dots,\frac{\partial\varphi_m}{\partial t}\right)^\top, \quad \triangle\Phi = (\triangle\varphi_1,\dots,\triangle\varphi_m)^\top, \quad \varepsilon(x) = \begin{pmatrix} \varepsilon_1(x) & & \\ & \ddots & \\ & & \varepsilon_m(x) \end{pmatrix}.$$

By the definition of the standing wave ($\Phi = (e^{\iota\omega_1 t}u_1(x),\dots,e^{\iota\omega_m t}u_m(x))^\top \to 0$ as $|x| \to \infty$), we have that (1.3) reduces to

$$\begin{cases} -\triangle U + \widetilde{V}(x)U = -\widetilde{p}(x,|U|)U + \widetilde{w}(x,|U|)U + (\delta+1)h(x)|U|^{\delta-1}U, & x \in \mathbb{R}^N, \\ U = (u_1(x),\dots,u_m(x))^\top \to 0 \in \mathbb{R}^m, & |x| \to \infty, \end{cases} \tag{1.4}$$

where

$$\widetilde{V}(x) = \begin{pmatrix} \varepsilon_1(x)+\omega_1 & & \\ & \ddots & \\ & & \varepsilon_m(x)+\omega_m \end{pmatrix}.$$

Therefore, the study of standing waves for the time-dependent nonlinear Schrödinger system (1.3) has been reduced to that of homoclinic solutions for the time-independent nonlinear Schrödinger system (1.4). Obviously, the system (1.4) is the special case of the system (1.1) with

$$V(x) \equiv \widetilde{V}(x), \quad \nabla P(x,U) \equiv \widetilde{p}(x,|U|)U, \quad \nabla W(x,U) \equiv \widetilde{w}(x,|U|)U, \quad \chi(x) \equiv 0. \tag{1.5}$$

Definition 1.1 ([35]). Let " $\cdot$ " denote the usual inner product and "$|\cdot|$" the norm in $\mathbb{R}^N$, respectively. Let $\|\cdot\|_{L^r}$ and $\|\cdot\|_r$ denote the norms of $L^r := L^r(\mathbb{R}^N, \mathbb{R})$ and $[L^r]^m$, respectively, $\forall r \geq 1$. Let $\langle\cdot,\cdot\rangle_{L^2}$ and $\langle\cdot,\cdot\rangle_2$ denote respectively the corresponding inner products of L^2 and $[L^2]^m$, where $\langle\cdot,\cdot\rangle_2$ is defined by $\langle U, V\rangle_2 = \langle u_1, v_1\rangle_{L^2} + \langle u_2, v_2\rangle_{L^2} + \cdots + \langle u_m, v_m\rangle_{L^2}$ for all $U = (u_1, u_2, \ldots, u_m)^\top$, $V = (v_1, v_2, \ldots, v_m)^\top \in [L^2]^m$.

To overcome the difficulties of the lack of periodic assumptions and the unbounded domain $\mathbb{R}^N$, we assume that

(**L1**) $V_j \in C(\mathbb{R}^N, \mathbb{R})$, $\inf_{x\in\mathbb{R}^N} V_j(x) > 0$, and there exists $m_0 > 0$ such that for any $M > 0$,
$$\operatorname{meas}(\{x \in \mathbb{R}^N : |x-y| \leq m_0,\ V_j(x) \leq M\}) \to 0, \quad |y| \to \infty, \quad j = 1, 2, \ldots, m,$$
where meas$(\cdot)$ denotes the Lebesgue measure in $\mathbb{R}^N$.

Let
$$\widetilde{W}(x,S) := \frac{1}{2}\nabla W(x,S)\cdot S - W(x,S), \quad \forall (x,S) \in \mathbb{R}^N \times \mathbb{R}^m.$$
For $h(x)$, $\chi(x)$, $P(x,U)$ and $W(x,U)$, we assume that

(**H1**) $h(x) \geq 0$, $\forall x \in \mathbb{R}^N$, $\|h\|_{L^{\frac{2}{1-\delta}}} \leq c_0$ and $\|\chi\|_2 \leq \bar{c}_0$ for some small $c_0, \bar{c}_0 > 0$.

(**P1**) There are $b_1, b_2 > 0$ such that $b_1|S|^2 \leq P(x,S) \leq b_2|S|^2$, $\quad \forall (x,S) \in \mathbb{R}^N \times \mathbb{R}^m$.

(**P2**) $P(x,S) \leq \nabla P(x,S)\cdot S \leq |\nabla P(x,S)||S| \leq 2P(x,S)$, $\quad \forall (x,S) \in \mathbb{R}^N \times \mathbb{R}^m$.

(**W0**) $W(x,S) \geq 0$, $|\nabla W(x,S)| \leq c|S|$ if $|S| < \infty$ for some $c > 0$ and all $(x,S) \in \mathbb{R}^N \times \mathbb{R}^m$, and
$$\lim_{|S|\to 0} \frac{|\nabla W(x,S)|}{|S|} = 0 \ \text{ uniformly in } x \in \mathbb{R}^N.$$

(**W1**) $\lim_{|S|\to\infty} \frac{|\nabla W(x,S)|}{|S|^{p-1}} = D_1$ uniformly in $x \in \mathbb{R}^N$ for some $p \in (2, 2^*)$ and $D_1 \geq 0$.

(**W2**) $\lim_{|S|\to\infty} \frac{W(x,S)}{|S|^2} = +\infty$ uniformly in $x \in \mathbb{R}^N$.

(**W3**) $C_a^b := \inf\left\{\frac{\widetilde{W}(x,S)}{|S|^2} : a \leq |S| < b,\ (x,S) \in \mathbb{R}^N \times \mathbb{R}^m\right\} > 0$, $\quad \forall b > a > 0$.

(**W4**) There exist $r_0, r_1 > 0$ such that $\nabla W(x,S)\cdot S \leq r_0\widetilde{W}(x,S)$, $\quad \forall |S| \geq r_1, \forall x \in \mathbb{R}^N$.

Now, our main results are as follows.

Theorem 1.2. *Suppose that* (L1), (H1), (P1), (P2) *and* (W0)–(W4) *hold, then we have the following two results.*

(1) *The time-independent system* (1.1) *has at least one nontrivial homoclinic solution*
$$U(x) = (u_1(x), \ldots, u_m(x))^\top.$$

(2) *If* (1.5) *and the assumptions of Theorem* 1.2 *hold, then the time-dependent system* (1.3) *has at least one nontrivial standing wave*
$$\Phi = \left(e^{i\omega_1 t}\tilde{u}_1(x), \ldots, e^{i\omega_m t}\tilde{u}_m(x)\right)^\top,$$
where $\widetilde{U} = (\tilde{u}_1(x), \ldots, \tilde{u}_m(x))^\top$ *is the nontrivial homoclinic solution of the non-perturbed system* (1.4) *obtained by* (1).

Remark 1.3. To the best of our knowledge, there is no result about nonlinear Schrödinger equations or systems with pinching terms, perturbed terms and mixed terms. By [20], $P(x,S)$ is called a pinching term if it satisfies the above condition $(P1)$. The following examples demonstrate that our assumptions are not merely reasonable, but also satisfied by a broad class of functions.

Example 1.4. We give two examples of the pinching terms $P(x,S)$. It is easy to verify that the following functions all satisfy our conditions $(P1)$–$(P2)$.

(1) Let $P(x,S) := \gamma(x)|S|^2$, then $\nabla P(x,S) := 2\gamma(x)S$, where $(x,S) \in \mathbb{R}^N \times \mathbb{R}^m$, $\inf_{x\in\mathbb{R}^N} \gamma(x) > 0$ and $\gamma(x) \in L^\infty$.

(2) Let $P(x,S) := |S|^2 \left(\Theta(x) + \frac{\Lambda(x)}{\Theta(x)+|S|^2}\right)$, then

$$\nabla P(x,S) := 2\left(\Theta(x) + \frac{\Lambda(x)}{\Theta(x)+|S|^2}\right) S - |S|^2 \frac{2\Lambda(x)S}{(\Theta(x)+|S|^2)^2},$$

where $(x,S) \in \mathbb{R}^N \times \mathbb{R}^m$, $\Lambda(x) \le 2\Theta(x)^2$ for all $x \in \mathbb{R}^N$, $\Theta(x) \in L^\infty$, $\Lambda(x) \in L^\infty$, $\inf_{x\in\mathbb{R}^N} \Theta(x) > 0$ and $\inf_{x\in\mathbb{R}^N} \Lambda(x) > 0$.

Example 1.5. We give two examples of the superlinear terms $W(x,S)$. It is not hard to verify that the following functions all satisfy our conditions $(W0)$–$(W4)$.

(1) Let $W(x,S) = |S|^p$, then$\nabla W(x,S) = p|S|^{p-2}S$, $\widetilde{W}(x,S) = \frac{p-2}{2}|S|^p$, where $(x,S) \in \mathbb{R}^N \times \mathbb{R}^m$ and $p > 2$.

(2) Let $W(x,S) = \xi(x)|S|^p \ln(1+|S|) + \beta(x)S^2$, then

$$\nabla W(x,S) = S\left[\xi(x)|S|^{p-2}\left(p\ln(1+|S|) + \frac{|S|}{1+|S|}\right) + 2\beta(x)\right],$$

$$\widetilde{W}(x,S) = \xi(x)|S|^p\left[\left(\frac{p}{2}-1\right)\ln(1+|S|) + \frac{|S|}{2(1+|S|)}\right],$$

where $p > 2$, $(x,S) \in \mathbb{R}^N \times \mathbb{R}^m$, $\xi(x) \in L^\infty$, $\beta(x) \in L^\infty$, $\inf_{x\in\mathbb{R}^N} \xi(x) > 0$ and $\inf_{x\in\mathbb{R}^N} \beta(x) > 0$.

2 Preliminary lemmas

Let

$$E_j := \left\{u \in H^1(\mathbb{R}^N, \mathbb{R}) : \int_{\mathbb{R}^N} V_j(x)u^2\,dx < +\infty\right\}, \quad j = 1, 2, \ldots, m,$$

and

$$E := E_1 \times E_2 \times \cdots \times E_m.$$

Under the condition $(L1)$, we introduce an operator $L : E \to E$ defined by $LU := (L_1u_1, \ldots, L_mu_m)^\top$ for any $U = (u_1, u_2, \ldots, u_m)^\top \in E$, where $L_j := -\Delta + V_j$ and $j = 1, 2, \ldots, m$.

For problem (1.1), we consider the following functional

$$\begin{aligned}\Phi(U) = \frac{1}{2}\langle LU, U\rangle_E &+ \int_{\mathbb{R}^N} P(x,U)dx - \int_{\mathbb{R}^N} W(x,U)dx \\ &- \int_{\mathbb{R}^N} h(x)|U|^{\delta+1}dx - \int_{\mathbb{R}^N} \chi(x)\cdot U dx, \quad \forall U \in E,\end{aligned} \tag{2.1}$$

where $\langle LU, U\rangle_E := \sum_{j=1}^{m}\int_{\mathbb{R}^N}(L_j u_j)u_j\,dx$ and $\delta \in (0,1)$. Now, the condition (L_1) implies the inner product $\langle\cdot,\cdot\rangle$ and the corresponding norm $\|\cdot\|$ on E are given, respectively, by

$$\langle U, V\rangle = \langle LU, V\rangle_E, \quad \|U\| = \langle U, U\rangle_E^{\frac{1}{2}},$$

where $U = (u_1, u_2, \dots, u_m)^\top, V = (v_1, v_2, \dots, v_m)^\top \in E$.

Therefore, we have that

$$\begin{aligned}\Phi(U) = \frac{1}{2}\|U\|^2 + \int_{\mathbb{R}^N} P(x,U)dx - \int_{\mathbb{R}^N} W(x,U)dx \\ - \int_{\mathbb{R}^N} h(x)|U|^{\delta+1}dx - \int_{\mathbb{R}^N}\chi(x)\cdot Udx, \quad \forall U \in E,\end{aligned} \tag{2.2}$$

and

$$\begin{aligned}\Phi'(U)V = \langle U, V\rangle + \int_{\mathbb{R}^N}\nabla P(x,U)\cdot V(x)dx - \int_{\mathbb{R}^N}\nabla W(x,U)\cdot V(x)dx \\ - \int_{\mathbb{R}^N}(\delta+1)h(x)|U|^{\delta-1}U\cdot V(x)dx - \int_{\mathbb{R}^N}\chi(x)\cdot V(x)dx, \quad \forall U, V \in E.\end{aligned} \tag{2.3}$$

Moreover, the nonzero critical points of (2.2) are nontrivial homoclinic solutions of (1.1).

Lemma 2.1 ([9]). *If $(L1)$ holds, then the embedding from E into $\left[L^s(\mathbb{R}^N, \mathbb{R})\right]^m$ is compact, and there exists a $\tau_s > 0$ such that*

$$\|U\|_s \le \tau_s\|U\|, \quad \forall U \in E, \quad \forall s \in [2, 2^*); \quad 2^* := +\infty \quad \text{if } N = 1, 2, \quad 2^* := \frac{2N}{N-2} \quad \text{if } N \ge 3.$$

Definition 2.2 ([5]). A sequence U^j is called a (C)-sequence for Φ if $\Phi(U^j)$ is bounded and $(1 + \|U^j\|)\Phi'(U^j) \to 0$, and a functional Φ satisfies the (C)-condition if and only if any (C)-sequence for Φ contains a convergent subsequence.

We shall use the following mountain pass theorem to prove our main results.

Lemma 2.3 ([3]). *Let E be a real Banach space. Suppose that*

(**h1**) *there are constants $\rho, \alpha > 0$ such that $\Phi|_{\partial B_\rho} \ge \alpha$,*

(**h2**) *there is an $e \in E \setminus B_\rho$ such that $\Phi(e) \le 0$,*

(**h3**) $\Phi(0) = 0$, (**h4**) *Φ satisfies (C)-condition.*

Then Φ possesses a critical value $c \ge \alpha$. Moreover, c can be characterizes as

$$c = \inf_{g\in\Gamma}\max_{u\in g([0,1])}\Phi(u), \quad \Gamma = \{g \in C([0,1], E) \mid g(0) = 0, g(1) = e\}.$$

Lemma 2.4. *If the assumptions of Theorem 1.2 hold, then Φ satisfies $(h1)$ of Lemma 2.3.*

Proof. By $(W0)$–$(W1)$, we have that for any $\varepsilon \in (0, b_1)$ there exists $C_\varepsilon > 0$ such that

$$|\nabla W(x,S)| \le \varepsilon|S| + C_\varepsilon|S|^{p-1}, \quad p \in (2, 2^*), \quad \forall (x,S) \in \mathbb{R}^N \times \mathbb{R}^m. \tag{2.4}$$

Note that

$$W(x,S) = \int_0^1 \frac{(\nabla W(x,tS), tS)}{t}\,dt.$$

It follows from $(H1)$, $(P1)$, (2.4), Hölder's inequality, $\varepsilon \in (0, b_1)$ and Lemma 2.1 that

$$\begin{aligned}
\Phi(U) &= \frac{1}{2}\|U\|^2 + \int_{\mathbb{R}^N} P(x,U)dx - \int_{\mathbb{R}^N} W(x,U)dx - \int_{\mathbb{R}^N} h(x)|U|^{\delta+1}dx - \int_{\mathbb{R}^N} \chi(x)\cdot U dx \\
&\geq \frac{1}{2}\|U\|^2 + b_1\|U\|_2^2 - \varepsilon\|U\|_2^2 - C_\varepsilon\|U\|_p^p - \|h\|_{L^{\frac{2}{1-\delta}}}\|U\|_2^{\delta+1} - \|\chi\|_2\|U\|_2 \\
&\geq \frac{1}{2}\|U\|^2 - C_\varepsilon\tau_p^p\|U\|^p - \|h\|_{L^{\frac{2}{1-\delta}}}\tau_2^{\delta+1}\|U\|^{\delta+1} - \tau_2\|\chi\|_2\|U\| \\
&\geq \|U\|\left[\frac{1}{2}\|U\| - C_\varepsilon\tau_p^p\|U\|^{p-1} - c_0\tau_2^{\delta+1}\|U\|^\delta - \tau_2\bar{c}_0\right] \\
&= \rho\left[\frac{1}{2}\rho - C_\varepsilon\tau_p^p\rho^{p-1} - c_0\tau_2^{\delta+1}\rho^\delta - \tau_2\bar{c}_0\right] \quad (\text{if } \|U\| := \rho).
\end{aligned} \tag{2.5}$$

Since $p-1>1$ by $p>2$, we can choose small enough ρ, c_0 and $\bar{c}_0$ such that

$$\alpha := \rho\left[\frac{1}{2}\rho - C_\varepsilon\tau_p^p\rho^{p-1} - c_0\tau_2^{\delta+1}\rho^\delta - \tau_2\bar{c}_0\right] > 0. \tag{2.6}$$

Therefore, (2.5) and (2.6) imply that

$$\Phi(U) \geq \alpha, \quad \forall U \in E, \quad \|U\| = \rho.$$

That is, the proof of this lemma is finished. □

Lemma 2.5. *If the assumptions of Theorem 1.2 hold, then Φ satisfies $(h2)$ of Lemma 2.3.*

Proof. Let $U^0(x) \in E \setminus \{0\}$. Then (H1), Hölder's inequality and Lemma 2.1 imply that

$$\begin{aligned}
0 &\leq \left|\int_{\mathbb{R}^N} \frac{h(x)|tU^0|^{\delta+1} + \chi(x)\cdot tU^0}{t^2}dx\right| \\
&\leq \frac{t^{\delta+1}\|h\|_{L^{\frac{2}{1-\delta}}}\tau_2^{\delta+1}\|U^0\|^{\delta+1} + t\|\chi\|_2\tau_2\|U^0\|}{t^2} \to 0, \quad t \to +\infty.
\end{aligned} \tag{2.7}$$

It follows from (2.2), (2.7), $(P1)$, $(W0)$, $(W2)$, Lemma 2.1 and Fatou's lemma that

$$\begin{aligned}
\lim_{t\to\infty}\frac{\Phi(tU^0)}{t^2} &= \frac{1}{2}\|U^0\|^2 + \lim_{t\to\infty}\int_{\mathbb{R}^N}\frac{P(x,tU^0)}{t^2}dx - \lim_{t\to\infty}\int_{\mathbb{R}^N}\frac{W(x,tU^0)}{t^2}dx \\
&\quad - \lim_{t\to\infty}\int_{\mathbb{R}^N}\frac{h(x)|tU^0|^{\delta+1} + \chi(x)\cdot tU^0}{t^2}dx \\
&\leq \frac{1}{2}\|U^0\|^2 + b_2\tau_2^2\|U^0\|^2 - \int_{\mathbb{R}^N}\liminf_{t\to\infty}\frac{W(x,tU^0)}{t^2|U^0|^2}|U^0|^2dx \\
&= -\infty.
\end{aligned}$$

Therefore, we can take $U := tU^0$ with $|t|$ big enough, then $\Phi(U) \leq 0$. □

Lemma 2.6. *If the assumptions of Theorem 1.2 hold, then Φ satisfies $(h3)$–$(h4)$ of Lemma 2.3.*

Proof. We can easily prove that the condition $(h3)$ holds.

Next, we will prove that $(h4)$ holds. Let $\{U^k\} \subset E$ be such that $\Phi(U^k)$ is bounded and

$$(1 + \|U^k\|)\Phi'(U^k) \to 0. \tag{2.8}$$

Then, by (2.8), $(P1)$, $(P2)$, $(W3)$, Hölder's inequality and Lemma 2.1, for k large enough, there exists $M_0 > 0$ such that

$$
\begin{aligned}
&M_0 + \|U^k\| \\
&\quad \geq \Phi(U^k) - \frac{1}{2}\Phi'(U^k)U^k \\
&\quad = \int_{\mathbb{R}^N} \widetilde{W}(x, U^k(x))dx + \int_{\mathbb{R}^N} P(x, U^k(x))dx - \frac{1}{2}\int_{\mathbb{R}^N} \nabla P(x, U^k(x)) \cdot U^k(x)dx \\
&\qquad + \left(\frac{\delta-1}{2}\right)\int_{\mathbb{R}^N} h(x)|U^k(x)|^{\delta+1}dx - \int_{\mathbb{R}^N} \frac{1}{2}\chi(x) \cdot U^k(x)dx \\
&\quad \geq \int_{\mathbb{R}^N} \widetilde{W}(x, U^k(x))dx + \left(\frac{\delta-1}{2}\right)\int_{\mathbb{R}^N} h(x)|U^k(x)|^{\delta+1}dx - \int_{\mathbb{R}^N} \frac{1}{2}\chi(x) \cdot U^k(x)dx \\
&\quad \geq \int_{\mathbb{R}^N} \widetilde{W}(x, U^k(x))dx - \frac{1-\delta}{2}\|h\|_{L^{\frac{2}{1-\delta}}}\tau_2^{\delta+1}\|U^k\|^{\delta+1} - \frac{1}{2}\tau_2\|\chi\|_2\|U^k\|.
\end{aligned}
\tag{2.9}
$$

First, we prove that $\{U^k\}$ is bounded in E. Suppose that $\|U^k\| \to \infty$. Let $V^k = \frac{U^k}{\|U^k\|}$. Then $\|V^k\| = 1$ and $V^k(x) = \frac{U^k(x)}{\|U^k\|}$, $x \in \mathbb{R}^N$. So there is a $V \in E$ such that

$$V^k \rightharpoonup V \quad \text{in } E, \quad V^k(x) \to V(x) \quad \text{a.e. for } x \in \mathbb{R}^N.$$

By (2.3) and $(P1)$–$(P2)$, we have

$$
\begin{aligned}
\Phi'(U^k)U^k = \|U^k\|^2 &+ \int_{\mathbb{R}^N} \nabla P(x, U^k(x)) \cdot U^k(x)dx - \int_{\mathbb{R}^N} \nabla W(x, U^k(x)) \cdot U^k(x)dx \\
&- \int_{\mathbb{R}^N} (\delta+1)h(x)|U^k(x)|^{\delta+1}dx - \int_{\mathbb{R}^N} \chi(x) \cdot U^k(x)dx.
\end{aligned}
$$

Hence,

$$
\begin{aligned}
\frac{\Phi'(U^k)U^k}{\|U^k\|^2} &= 1 + \int_{\mathbb{R}^N} \frac{\nabla P(x, U^k(x)) \cdot U^k(x)}{\|U^k(x)\|^2}dx - \int_{\mathbb{R}^N} \frac{\nabla W(x, U^k(x)) \cdot U^k(x)}{\|U^k(x)\|^2}dx \\
&\quad - \int_{\mathbb{R}^N} (\delta+1)\frac{h(x)|U^k(x)|^{\delta+1}}{\|U^k(x)\|^2}dx - \int_{\mathbb{R}^N} \frac{\chi(x) \cdot U^k(x)}{\|U^k(x)\|^2}dx \\
&\geq 1 - \int_{\mathbb{R}^N} \frac{\nabla W(x, U^k(x)) \cdot U^k(x)}{\|U^k(x)\|^2}dx \\
&\quad - \frac{\|h\|_{L^{\frac{2}{1-\delta}}}\tau_2^{\delta+1}\|U^k\|^{\delta+1} + \|\chi\|_2\tau_2\|U^k\|}{\|U^k\|^2}.
\end{aligned}
\tag{2.10}
$$

It follows from (2.8) and (2.10) that

$$\lim_{k\to\infty} \int_{\mathbb{R}^N} \frac{\nabla W(x, U^k(x)) \cdot U^k(x)}{\|U^k(x)\|^2}dx \geq 1.$$

Thus, there exists $K_0 > 0$ for all $k > K_0$ such that

$$\int_{\mathbb{R}^N} \frac{\nabla W(x, U^k(x)) \cdot U^k(x)}{\|U^k(x)\|^2}dx \geq 1. \tag{2.11}$$

By $(W3)$, we have

$$\widetilde{W}(x, U^k) \geq C_a^b|U^k|^2, \quad \forall x \in \Omega_k(a, b) := \{x \in \mathbb{R}^N : a \leq |U^k(x)| < b\}. \tag{2.12}$$

For k large enough, by (2.9) and $\|U^k\| \to \infty$, we have for some $M_1 > 0$ such that

$$\begin{aligned} &M_1(1+\|U^k\|^{1+\delta}) \\ &\quad \geq \int_{\mathbb{R}^N} \widetilde{W}(x, U^k(x))dx \\ &\quad = \int_{\Omega_k(0,a)} \widetilde{W}(x, U^k(x))dx + \int_{\Omega_k(a,b)} \widetilde{W}(x, U^k(x))dx + \int_{\Omega_k(b,\infty)} \widetilde{W}(x, U^k(x))dx. \end{aligned} \tag{2.13}$$

Let $\varepsilon \in \left(0, \frac{1}{6}\right)$. By $(W0)$, there is $a_\varepsilon > 0$ such that $\frac{|\nabla W(x,S)|}{|S|} \leq \frac{\varepsilon}{\tau_2^2}$ for all $|S| \leq a_\varepsilon$ and $x \in \mathbb{R}^N$. It follows from Lemma 2.1 and $\|V^k\| = 1$ that

$$\int_{\Omega_k(0,a_\varepsilon)} \frac{\nabla W(x, U^k(x)) \cdot U^k(x)}{\|U^k(x)\|^2} dx \leq \frac{\varepsilon}{\tau_2^2} \|V^k\|_2^2 \leq \varepsilon, \quad \forall k \in \mathbb{R}. \tag{2.14}$$

By $(W4)$, (2.13), Lemma 2.1 and $\|V^k\| = 1$, we can take $b_\varepsilon \geq r_1$ large enough such that

$$\begin{aligned} \int_{\Omega_k(b_\varepsilon,\infty)} \frac{\nabla W(U^k(x)) \cdot U^k(x)}{\|U^k\|^2} dx &\leq r_0 \int_{\Omega_k(b_\varepsilon,\infty)} \frac{\widetilde{W}(x, U^k(x))}{\|U^k\|^2} dx \\ &\leq r_0 \frac{M_1(1+\|U^k\|^{1+\delta})}{\|U^k\|^2} \\ &< \varepsilon \end{aligned} \tag{2.15}$$

for some $K_1 > 0$ and all $k \geq K_1$. By $(W3)$ and (2.13), we have

$$\begin{aligned} \int_{\Omega_k(a_\varepsilon,b_\varepsilon)} |V^k(x)|^2 dx = \frac{1}{\|U^k\|^2} \int_{\Omega_k(a_\varepsilon,b_\varepsilon)} |U^k(x)|^2 dx &\leq \frac{\int_{\Omega_k(a_\varepsilon,b_\varepsilon)} \widetilde{W}(x, U^k(x))dx}{C_{a_\varepsilon}^{b_\varepsilon} \|U^k\|^2} \\ &\leq \frac{M_1(1+\|U^k\|^{1+\delta})}{C_{a_\varepsilon}^{b_\varepsilon} \|U^k\|^2} \to 0, \quad k \to \infty. \end{aligned} \tag{2.16}$$

By $(W0)$, there is $\widetilde{q} = \varrho(\varepsilon) > 0$ independent of k such that $|\nabla W(x, U^k(x))| \leq \widetilde{q}|U^k(x)|$ for $x \in \Omega_k(a_\varepsilon, b_\varepsilon)$. It follows from (2.16) that there is a $K_2 > 0$ such that for all $k \geq K_2$ there holds

$$\int_{\Omega_k(a_\varepsilon,b_\varepsilon)} \frac{\nabla W(x, U^k(x)) \cdot U^k(x)}{\|U^k(x)\|^2} dx \leq \widetilde{q} \int_{\Omega_k(a_\varepsilon,b_\varepsilon)} |V^k(x)|^2 dx < \varepsilon. \tag{2.17}$$

Hence, by (2.14), (2.15) and (2.17), for all $k \geq \max\{K_0, K_1, K_2\}$, we have

$$\int_{\mathbb{R}^N} \frac{\nabla W(x, U^k(x)) \cdot U^k(x)}{\|U^k(x)\|^2} dx < 3\varepsilon < \frac{1}{2} < 1,$$

which contradicts (2.11). Thus, $\{U^k\}$ is bounded in E.

Last, we prove that $\{U^k\}$ has a strongly convergent subsequence in E. Since $\{U^k\}$ is bounded in E, we see that there is a renamed subsequence (it is also denoted by $\{U^k\}$) such that

$$U^k \rightharpoonup U \quad \text{in } E, \quad U^k(x) \to U(x) \quad \text{a.e. for } x \in \mathbb{R}^N. \tag{2.18}$$

Note that (2.3) and (2.8) imply

$$\begin{aligned} 0 &\leftarrow \Phi'(U^k)(U^k - U) \\ &= \langle U^k, (U^k - U) \rangle + \int_{\mathbb{R}^N} \nabla P(x, U^k(x)) \cdot (U^k(x) - U(x))dx \\ &\quad - \int_{\mathbb{R}^N} (\delta+1)h(x)|U^k(x)|^{\delta-1}U^k(x) \cdot (U^k(x) - U(x))dx \\ &\quad - \int_{\mathbb{R}^N} \nabla W(x, U^k(x)) \cdot (U^k(x) - U(x))dx - \int_{\mathbb{R}^N} \chi(x) \cdot (U^k(x) - U(x))dx. \end{aligned} \tag{2.19}$$

By (2.18), we have

$$\lim_{k\to\infty}\langle U^k, U^k - U\rangle = \lim_{k\to\infty}\|U^k\|^2 - \|U\|^2. \tag{2.20}$$

By $(P1)$–$(P2)$, Hölder's inequality, Lemma 2.1, $\|U^k\| < \infty$ and (2.18), we have

$$\begin{aligned}
&\left|\int_{\mathbb{R}^N} \nabla P(x, U^k(x))(U^k(x) - U(x))dx\right| \\
&\quad\le \int_{\mathbb{R}^N} 2b_2|U^k(x)|\cdot|U^k(x) - U(x)|dx \\
&\quad\le 2b_2\|U^k\|_2\|U^k - U\|_2 \\
&\quad\le 2b_2\tau_2\|U^k\|\|U^k - U\|_2 \to 0, \quad k\to\infty.
\end{aligned} \tag{2.21}$$

By (2.4), (2.18), Hölder's inequality, $\|U^k\| < \infty$ and Lemma 2.1, we get

$$\begin{aligned}
&\left|\int_{\mathbb{R}^N} \nabla W(x, U^k(x))(U^k(x) - U(x))dx\right| \\
&\quad\le \int_{\mathbb{R}^N} \left(\varepsilon|U^k(x)| + C_\varepsilon|U^k(x)|^{p-1}\right)|U^k(x) - U(x)|dx \\
&\quad\le \varepsilon\|U^k\|_2\|U^k - U\|_2 + C_\varepsilon\|U^k\|_p^{p-1}\|U^k - U\|_p \\
&\quad\le \varepsilon\tau_2\|U^k\|\|U^k - U\|_2 + C_\varepsilon\tau_p^{p-1}\|U^k\|^{p-1}\|U^k - U\|_p \to 0, \quad k\to\infty.
\end{aligned} \tag{2.22}$$

By (2.18), Hölder inequality, $\|U^k\| < \infty$ and Lemma 2.1, we have

$$\begin{aligned}
&\left|\int_{\mathbb{R}^N} \left[(\delta+1)h(x)|U^k(x)|^{\delta-1}U^k(x) + \chi(x)\right]\cdot(U^k(x) - U(x))dx\right| \\
&\quad\le (\delta+1)\int_{\mathbb{R}^N} h(x)\left|U^k(x)\right|^\delta|U^k(x) - U(x)|dx + \int_{\mathbb{R}^N}|\chi(x)||U^k(x) - U(x)|dx \\
&\quad\le (\delta+1)\left[\|h\|^2_{L^{\frac{2}{1-\delta}}}\|U^k\|_2^{2\delta}\right]^{\frac{1}{2}}\|U^k - U\|_2 + \|\chi\|_2\|U^k - U\|_2 \\
&\quad\le \left[(\delta+1)\|h\|_{L^{\frac{2}{1-\delta}}}\tau_2^\delta\|U^k\|^\delta\right]\|U^k - U\|_2 + \|\chi\|_2\|U^k - U\|_2 \to 0, \quad k\to+\infty.
\end{aligned} \tag{2.23}$$

Now, it follows from (2.19)–(2.23) that

$$\lim_{k\to\infty}\|U^k\|^2 = \|U\|^2.$$

Therefore, we have that $U^k \to U$ in E and Φ satisfies the (C)-condition. □

3 Proof of Theorem 1.2

Proof. By Lemmas 2.3–2.6, we have that there exists a critical value $c \ge \alpha$ for Φ, that is the perturbed system (1.1) has at least one nontrivial homoclinic solution

$$U = U(x) = (u_1(x), \dots, u_m(x))^\top.$$

Since the proofs still hold if the perturbed terms $\chi(x) \equiv 0$, thus, if (1.5) and the assumptions of Theorem 1.2 hold, then the non-perturbed system (1.4) has at least a nontrivial homoclinic solution $\widetilde{U} = (\widetilde{u}_1(x), \dots, \widetilde{u}_m(x))^\top$. By the relationships with (1.3) and (1.4), we get that the non-perturbed system (1.3) has at least one nontrivial standing wave

$$\Phi = \left(e^{i\omega_1 t}\widetilde{u}_1(x), \dots, e^{i\omega_m t}\widetilde{u}_m(x)\right)^\top.$$

The proof is finished. □

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