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Existence results for systems of abstract evolution equations with nonlocal initial conditions

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Abstract. This paper investigates the existence of solutions for systems of abstract evolution equations subject to nonlocal initial conditions. Our approach is based on a combination of the fixed point theorems due to Perov, Schauder and Avramescu, together with the semigroup method and a matrix-based technique involving vector-valued norms. The results presented here generalize and refine existing theorems in the literature.

Keywords: abstract evolution equation, nonlocal condition, fixed point, semigroup of operators.

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1 Introduction and preliminaries

In this paper we study the existence and the uniqueness of solutions to the system of abstract evolution equations

$$\begin{cases} x'(t) = A_1 x(t) + f_1(t, x(t), y(t)), \\ y'(t) = A_2 y(t) + f_2(t, x(t), y(t)), \end{cases} \qquad t \in [0,1] \tag{1.1}$$

with the constraints

$$\begin{cases} x(0) = \alpha[x, y], \\ y(0) = \beta[x, y]. \end{cases} \tag{1.2}$$

Here, the linear operator $A_i : D(A_i) \subseteq X \to X$ generates a strongly continuous semigroup $\{S_i(t) : t \geq 0\}$, $i = 1, 2$ on the Banach space $(X, |\cdot|_X)$, $f_i : [0,1] \times X^2 \to X$, $i = 1, 2$ is a given function and $\alpha, \beta : C([0,1], X)^2 \to X$.

Nonlocal initial problems for different classes of systems of equations, studied in the literature by a variety of methods (from semigroup theory, operator theory, and functional analysis), have attracted the attention of many researchers (see for instance [5–10, 16] and the

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references therein). Problems of this type arise in the mathematical modeling of real-world processes, such as heat transfer, fluid flow, and chemical or biological phenomena, where nonlocal conditions may be interpreted as feedback controls. Further discussions on the role of nonlocal conditions in various application areas, including examples of evolution equations and systems with or without delay subject to nonlocal initial conditions, as well as additional references, can be found in [1,7,9,11].

In the recent papers [4,5,13–15] a new method based on vector-valued norms and matrices convergent to zero was used for the treatment of first order differential systems under nonlocal conditions expressed by linear functionals. The aim of this paper is to extend the use of that technique to nonlocal conditions given by nonlinear functionals.

In what follows, we denote by $\mathcal{X} := C([0,1], X) \times X$. We consider the space $\mathcal{X} \times \mathcal{X}$ endowed with the vector-valued norm $|\cdot|_{\mathcal{X}\times\mathcal{X}}$,

$$|u|_{\mathcal{X}\times\mathcal{X}} = \begin{pmatrix} |x_a|_{\mathcal{X}} \\ |y_b|_{\mathcal{X}} \end{pmatrix},$$

for $u = (x_a, y_b)$ and

$$|x_a|_{\mathcal{X}} = |(x,a)|_{\mathcal{X}} = |x|_{C([0,1],X)} + \theta\, |a|_X, \quad \theta > 0 \tag{1.3}$$

which represents a norm on $\mathcal{X}$ with

$$|x|_{C([0,1],X)} = \max_{t\in[0,1]} |x(t)|_X .$$

We seek mild solutions of the problem (1.1)–(1.2), namely solutions to the following system

$$\begin{cases} x_a = \left(S_1(\cdot)a + \displaystyle\int_0^{(\cdot)} S_1(\cdot - s) f_1(s, x(s), y(s))ds, \alpha[x,y] \right), \\ y_b = \left(S_2(\cdot)b + \displaystyle\int_0^{(\cdot)} S_2(\cdot - s) f_2(s, x(s), y(s))ds, \beta[x,y] \right), \end{cases} \tag{1.4}$$

and by x_a, y_b we understand the pairs $(x,a), (y,b) \in \mathcal{X}$.

We consider the fixed point problem in $\mathcal{X} \times \mathcal{X}$ for the operator

$$T = (T_1, T_2) : \mathcal{X} \times \mathcal{X} \to \mathcal{X} \times \mathcal{X},$$

where T_1 and T_2 are given by

$$T_1[x_a, y_b] = \left(S_1(\cdot)a + \int_0^{(\cdot)} S_1(\cdot - s) f_1(s, x(s), y(s))ds, \alpha[x,y] \right), \tag{1.5}$$

$$T_2[x_a, y_b] = \left(S_2(\cdot)b + \int_0^{(\cdot)} S_2(\cdot - s) f_2(s, x(s), y(s))ds, \beta[x,y] \right). \tag{1.6}$$

In the study of systems of equations, a vector approach is particularly convenient. A square matrix $M \in \mathcal{M}_{n\times n}(\mathbb{R}_+)$ is said to be *convergent to zero* if M^k tends to the zero matrix as $k \to \infty$. The following lemma establishes equivalent conditions for a square matrix to be convergent to zero (see [3,12]).

Lemma 1.1. *Let $M \in \mathcal{M}_{n\times n}(\mathbb{R}_+)$ be a square matrix. The following statements are equivalent:*

1. *The matrix M is convergent to zero.*

2. *The spectral radius of M is less than 1, i.e., $\rho(M) < 1$.*

3. *The matrix $I - M$, where I is the unit matrix of the same size, is invertible and its inverse has nonnegative entries, i.e., $(I - M)^{-1} \in \mathcal{M}_{n\times n}(\mathbb{R}_+)$.*

In case $n = 2$, we have the following characterization.

Lemma 1.2. *A square matrix $M = [a_{ij}]_{1\le i,j\le 2} \in \mathcal{M}_{2\times 2}(\mathbb{R}_+)$ is convergent to zero if and only if*

$$a_{11}, a_{22} < 1$$

and

$$tr(M) < 1 + \det(M).$$

We end this section by presenting some fixed point theorems employed to support our results. The first result is Perov's fixed point theorem.

Theorem 1.3. *Let (X_i, d_i), $i = 1, 2$, be complete metric spaces and $T_i : X_1 \times X_2 \to X_i$ be two mappings for which there exists a square matrix M of size two with nonnegative entries and the spectral radius $\rho(M) < 1$ such that the following vector inequality*

$$\begin{pmatrix} d_1(T_1(x,y), T_1(\overline{x},\overline{y})) \\ d_2(T_2(x,y), T_2(\overline{x},\overline{y})) \end{pmatrix} \le M \begin{pmatrix} d_1(x,\overline{x}) \\ d_2(y,\overline{y}) \end{pmatrix}$$

holds, for all $(x, y), (\overline{x}, \overline{y}) \in X_1 \times X_2$. Then, there exists a unique point $(x, y) \in X_1 \times X_2$ with

$$(x, y) = (T_1(x, y), T_2(x, y)).$$

The next result is Schauder's fixed point theorem that guarantees the existence of a solution for our studied problem, but not also the uniqueness.

Theorem 1.4. *Let X be a Banach space, $D \subset X$ a nonempty closed bounded convex set and $T : D \to D$ a completely continuous operator (i.e., T is continuous and $T(D)$ is relatively compact). Then T has at least one fixed point.*

The last result is Avramescu's fixed point theorem (see [2]).

Theorem 1.5. *Let $\mathcal{B}_1$ be a closed convex subset of a normed space Y, $(\mathcal{B}_2, d)$ a complete metric space, and let $T_i : \mathcal{B}_1 \times \mathcal{B}_2 \to \mathcal{B}_i$, $i = 1, 2$, be continuous mappings. Assume that the following conditions are satisfied:*

(a) $T_1(\mathcal{B}_1 \times \mathcal{B}_2)$ *is a relatively compact subset of Y;*

(b) There is a constant $L \in [0, 1)$ such that

$$d(T_2(x, y), T_2(x, \overline{y})) \le L d(y, \overline{y}),$$

for all $x \in \mathcal{B}_1$ and $y, \overline{y} \in \mathcal{B}_2$.

Then, there exists $(x, y) \in \mathcal{B}_1 \times \mathcal{B}_2$ such that $T_1(x, y) = x$ and $T_2(x, y) = y$.

Some references related to these fixed point theorems are [1,4,5,11,13].

2 Main results

In order to prove the existence and uniqueness of the solution to system (1.4), we apply Perov's fixed point theorem in $\mathcal{X} \times \mathcal{X}$.

On the mappings f_1, f_2 and α, β we assume the next Lipschitz type conditions:

(H_1) there exist constants $a_1, a_2, b_1, b_2 > 0$ such that

$$\begin{aligned} |f_1(t,x,y) - f_1(t,\overline{x},\overline{y})|_X &\le a_1 |x-\overline{x}|_X + b_1 |y-\overline{y}|_X, \\ |f_2(t,x,y) - f_2(t,\overline{x},\overline{y})|_X &\le a_2 |x-\overline{x}|_X + b_2 |y-\overline{y}|_X \end{aligned} \tag{2.1}$$

for all $x, y, \overline{x}, \overline{y} \in X, t \in [0,1]$;

(H_2) there exist constants $A_1, A_2, B_1, B_2 > 0$ such that

$$\begin{aligned} |\alpha[x,y] - \alpha[\overline{x},\overline{y}]|_X &\le A_1 |x-\overline{x}|_{C([0,1],X)} + B_1 |y-\overline{y}|_{C([0,1],X)}, \\ |\beta[x,y] - \beta[\overline{x},\overline{y}]|_X &\le A_2 |x-\overline{x}|_{C([0,1],X)} + B_2 |y-\overline{y}|_{C([0,1],X)}, \end{aligned} \tag{2.2}$$

for all $x, y, \overline{x}, \overline{y} \in C([0,1],X)$.

For a given number, $\theta > 0$, denote

$$\begin{aligned} m_{11}(\theta) &= \max\left\{\frac{1}{\theta}C_1, C_1 a_1 + \theta A_1\right\}, \\ m_{12}(\theta) &= C_1 b_1 + \theta B_1, \\ m_{21}(\theta) &= C_2 a_2 + \theta A_2, \\ m_{22}(\theta) &= \max\left\{\frac{1}{\theta}C_2, C_2 b_2 + \theta B_2\right\}, \end{aligned} \tag{2.3}$$

where

$$C_i = \sup\{\|S_i(t)\| : t \in [0,1]\}, \ i = 1,2$$

and $\|\cdot\|$ stands for the norm of a continuous linear operator.

Assuming a vectorial condition involving the coefficients $a_i, \ b_i, \ A_i, \ B_i, \ i = 1,2$, we have the following result:

Theorem 2.1. *Let f_1, f_2 and α, β satisfy the conditions (H_1), (H_2), respectively. In addition, assume that for some $\theta > 0$, the spectral radius of the matrix*

$$M_\theta = \begin{bmatrix} m_{11}(\theta) & m_{12}(\theta) \\ m_{21}(\theta) & m_{22}(\theta) \end{bmatrix} \tag{2.4}$$

is less than one (here, m_{ij} are from (2.3)). Then, the problem (1.1)–(1.2) has a unique mild solution on $[0,1]$.

Proof. We prove that the operator $T = (T_1, T_2)$, where T_1 and T_2 are defined in (1.5) and (1.6), is a Perov contraction with respect to the convergent to zero matrix M_θ, more exactly that

$$|T(u) - T(\overline{u})|_{\mathcal{X}\times\mathcal{X}} \le M_\theta |u - \overline{u}|_{\mathcal{X}\times\mathcal{X}}, \tag{2.5}$$

for all $u = (x_a, y_b), \overline{u} = (\overline{x}_a, \overline{y}_b) \in \mathcal{X} \times \mathcal{X}$, where $\overline{x}_a = (\overline{x}, \overline{a}), \ \overline{y}_b = (\overline{y}, \overline{b})$.

By definition of the operator T_1 given in (1.5) and using the norm (1.3), we obtain

$$
\begin{aligned}
&|T_1[x_a, y_b] - T_1[\overline{x}_a, \overline{y}_b]|_{\mathcal{X}} \\
&\quad \le \|S_1(t)\| \, |a - \overline{a}|_X + \int_0^t \|S_1(t-s)\| \, |f_1(s, x(s), y(s)) - f_1(s, \overline{x}(s), \overline{y}(s))|_X \, ds \\
&\qquad + \theta \, |\alpha[x, y] - \alpha[\overline{x}, \overline{y}]|_X .
\end{aligned}
$$

Moreover, using the conditions (H_1) and (H_2) we get

$$
\begin{aligned}
&|T_1[x_a, y_b] - T_1[\overline{x}_a, \overline{y}_b]|_{\mathcal{X}} \\
&\quad \le C_1 \, |a - \overline{a}|_X + C_1 \int_0^t \left(a_1 \, |x(s) - \overline{x}(s)|_X + b_1 \, |y(s) - \overline{y}(s)|_X\right) ds \\
&\qquad + \theta \left(A_1 \, |x - \overline{x}|_{C([0,1],X)} + B_1 \, |y - \overline{y}|_{C([0,1],X)} \right) \\
&\quad \le C_1 \, |a - \overline{a}|_X + C_1 \int_0^t \left(a_1 \, |x - \overline{x}|_{C([0,1],X)} + b_1 \, |y - \overline{y}|_{C([0,1],X)} \right) ds \\
&\qquad + \theta \left(A_1 \, |x - \overline{x}|_{C([0,1],X)} + B_1 \, |y - \overline{y}|_{C([0,1],X)} \right) \\
&\quad \le C_1 \, |a - \overline{a}|_X + C_1 a_1 \, |x - \overline{x}|_{C([0,1],X)} + C_1 b_1 \, |y - \overline{y}|_{C([0,1],X)} \\
&\qquad + \theta A_1 \, |x - \overline{x}|_{C([0,1],X)} + \theta B_1 \, |y - \overline{y}|_{C([0,1],X)} \\
&\quad = (C_1 a_1 + \theta A_1) \, |x - \overline{x}|_{C([0,1],X)} + \frac{1}{\theta} \theta C_1 \, |a - \overline{a}|_X + (C_1 b_1 + \theta B_1) \, |y - \overline{y}|_{C([0,1],X)} .
\end{aligned}
$$

Following the definition of m_{ij} from (2.3) we have

$$
\begin{aligned}
&|T_1[x_a, y_b] - T_1[\overline{x}_a, \overline{y}_b]|_{\mathcal{X}} \\
&\quad \le \max \left\{ \frac{1}{\theta} C_1, C_1 a_1 + \theta A_1 \right\} |x_a - \overline{x}_a|_{\mathcal{X}} + (C_1 b_1 + \theta B_1) \, |y_b - \overline{y}_b|_{\mathcal{X}} \\
&\quad = m_{11}(\theta) \, |x_a - \overline{x}_a|_{\mathcal{X}} + m_{12}(\theta) \, |y_b - \overline{y}_b|_{\mathcal{X}} .
\end{aligned}
$$

Analogously, we have

$$
\begin{aligned}
&|T_2[x_a, y_b] - T_2[\overline{x}_a, \overline{y}_b]|_{\mathcal{X}} \\
&\quad \le (C_2 a_2 + \theta A_2) \, |x - \overline{x}|_{C([0,1],X)} + \left[(C_2 b_2 + \theta B_2) \, |y - \overline{y}|_{C([0,1],X)} + \frac{1}{\theta} \theta C_2 \left| b - \overline{b} \right|_X \right] \\
&\quad \le m_{21}(\theta) \, |x_a - \overline{x}_a|_{\mathcal{X}} + m_{22}(\theta) \, |y_b - \overline{y}_b|_{\mathcal{X}} .
\end{aligned}
$$

Now, we put both inequalities together and we obtain

$$
\begin{bmatrix} |T_1[x_a, y_b] - T_1[\overline{x}_a, \overline{y}_b]|_{\mathcal{X}} \\ |T_2[x_a, y_b] - T_2[\overline{x}_a, \overline{y}_b]|_{\mathcal{X}} \end{bmatrix} \le M_\theta \begin{bmatrix} |x_a - \overline{x}_a|_{\mathcal{X}} \\ |y_b - \overline{y}_b|_{\mathcal{X}} \end{bmatrix}
$$

Using the vector-valued norm we have (2.5), where M_θ is given by (2.4) and assumed to be convergent to zero (see Lemma 1.2). Thus T is a Perov contraction. The existence and uniqueness of the solution follows now from Perov's fixed point theorem. □

Now we establish the existence of solutions to the problem (1.1)–(1.2) using Schauder's fixed point theorem in case that f_1, f_2 satisfy a relaxed growth condition of the form

$$
\begin{aligned}
|f_1(t, x, y)|_X &\le a_1 \, |x|_X + b_1 \, |y|_X + d_1(t), \\
|f_2(t, x, y)|_X &\le a_2 \, |x|_X + b_2 \, |y|_X + d_2(t),
\end{aligned}
\tag{2.6}
$$

for all $x, y \in X, t \in [0,1]$ and some $d_1, d_2 \in L^1([0,1], \mathbb{R}_+)$. As regards the mappings α, β we assume that

$$
\begin{aligned}
|\alpha[x,y]|_X &\leq A_1 |x|_{C([0,1],X)} + B_1 |y|_{C([0,1],X)} + D_1, \\
|\beta[x,y]|_X &\leq A_2 |x|_{C([0,1],X)} + B_2 |y|_{C([0,1],X)} + D_2,
\end{aligned}
\tag{2.7}
$$

for all $x, y \in X$ and some nonnegative constants $A_1, A_2, B_1, B_2, D_1, D_2$.

For the next result consider the following condition:

(C) The semigroup $\{S(t) : t \geq 0\}$ is compact.

Theorem 2.2. *If the conditions (2.6), (2.7) and (C) hold and the matrix M_θ defined in (2.4) is convergent to zero for some $\theta > 0$, then the problem (1.1)–(1.2) has at least one solution.*

Proof. In order to apply Schauder's fixed point theorem, we prove that

$$T(\mathcal{B}_1 \times \mathcal{B}_2) \subset \mathcal{B}_1 \times \mathcal{B}_2,$$

where $\mathcal{B}_1 \times \mathcal{B}_2$ is a nonempty, bounded, closed and convex subset of $\mathcal{X} \times \mathcal{X}$. Using the norm (1.3) we have

$$
\begin{aligned}
|T_1[x_a, y_b]|_{\mathcal{X}} &= \left| \left(S_1(t)a + \int_0^t S_1(t-s) f_1(s, x(s), y(s)) ds, \alpha[x,y] \right) \right|_{\mathcal{X}} \\
&\leq \|S_1(t)\| \, |a|_X + \int_0^t \|S_1(t-s)\| \, |f_1(s, x(s), y(s))|_X \, ds + \theta \, |\alpha[x,y]|_X .
\end{aligned}
$$

In view of the conditions (2.6) and (2.7) we get

$$
\begin{aligned}
|T_1[x_a, y_b]|_{\mathcal{X}} &\leq C_1 |a|_X + C_1 \int_0^t \big(a_1 |x(s)|_X + b_1 |y(s)|_X + d_1(s)\big) ds \\
&\quad + \theta \big(A_1 |x|_{C([0,1],X)} + B_1 |y|_{C([0,1],X)} + D_1\big) \\
&\leq C_1 |a|_X + C_1 a_1 |x|_{C([0,1],X)} + C_1 b_1 |y|_{C([0,1],X)} + C_1 |d_1|_{L^1([0,1],\mathbb{R}_+)} \\
&\quad + \theta A_1 |x|_{C([0,1],X)} + \theta B_1 |y|_{C([0,1],X)} + \theta D_1 \\
&= (C_1 a_1 + \theta A_1) |x|_{C([0,1],X)} + \frac{1}{\theta} \theta C_1 |a|_X + (C_1 b_1 + \theta B_1) |y|_{C([0,1],X)} \\
&\quad + C_1 |d_1|_{L^1([0,1],\mathbb{R}_+)} + \theta D_1.
\end{aligned}
$$

Using the definition of m_{ij} from (2.3) we obtain

$$
\begin{aligned}
|T_1[x_a, y_b]|_{\mathcal{X}} &\leq \max \left\{ \frac{1}{\theta} C_1, C_1 a_1 + \theta A_1 \right\} |x_a|_{\mathcal{X}} + (C_1 b_1 + \theta B_1) |y_b|_{\mathcal{X}} + \widehat{d}_0 \\
&= m_{11}(\theta) |x_a|_{\mathcal{X}} + m_{12}(\theta) |y_b|_{\mathcal{X}} + \widehat{d}_0,
\end{aligned}
$$

where $\widehat{d}_0 = C_1 |d_1|_{L^1([0,1],\mathbb{R}_+)} + \theta D_1$.

Analogously

$$
\begin{aligned}
|T_2[x_a, y_b]|_{\mathcal{X}} &\leq (C_2 b_2 + \theta B_2) |x_a|_{\mathcal{X}} + \max \left\{ \frac{1}{\theta} C_2, C_2 a_2 + \theta A_2 \right\} |y_b|_{\mathcal{X}} + \widehat{D}_0 \\
&= m_{21}(\theta) |x_a|_{\mathcal{X}} + m_{22}(\theta) |y_b|_{\mathcal{X}} + \widehat{D}_0,
\end{aligned}
$$

where $\widehat{D}_0 = C_2 \, |d_2|_{L^1([0,1],\mathbb{R}_+)} + \theta D_2$.

Now, from the above estimations we have

$$\begin{bmatrix} |T_1[x_a, y_b]|_{\mathcal{X}} \\ |T_2[x_a, y_b]|_{\mathcal{X}} \end{bmatrix} \leq M_\theta \begin{bmatrix} |x_a|_{\mathcal{X}} \\ |y_b|_{\mathcal{X}} \end{bmatrix} + \begin{bmatrix} \widehat{d}_0 \\ \widehat{D}_0 \end{bmatrix},$$

where M_θ is given by (2.4) and is assumed to be convergent to zero. Next we look for two positive numbers $R_1,\ R_2$, such that if

$$\begin{bmatrix} |x_a|_{\mathcal{X}} \\ |y_b|_{\mathcal{X}} \end{bmatrix} \leq \begin{bmatrix} R_1 \\ R_2 \end{bmatrix},$$

then

$$\begin{bmatrix} |T_1[x_a, y_b]|_{\mathcal{X}} \\ |T_2[x_a, y_b]|_{\mathcal{X}} \end{bmatrix} \leq \begin{bmatrix} R_1 \\ R_2 \end{bmatrix}.$$

Thus it is sufficient that

$$M_\theta \begin{bmatrix} R_1 \\ R_2 \end{bmatrix} + \begin{bmatrix} \widehat{d}_0 \\ \widehat{D}_0 \end{bmatrix} \leq \begin{bmatrix} R_1 \\ R_2 \end{bmatrix},$$

whence

$$\begin{bmatrix} R_1 \\ R_2 \end{bmatrix} \geq (I - M_\theta)^{-1} \begin{bmatrix} \widehat{d}_0 \\ \widehat{D}_0 \end{bmatrix}.$$

Here we used the fact that $I - M_\theta$ is invertible and the matrix $(I - M_\theta)^{-1}$ has nonnegative elements since M_θ is convergent to zero.

Let

$$\mathcal{B}_1 = \{x_a \in \mathcal{X} : \ |x_a|_{\mathcal{X}} \leq R_1\}$$

and

$$\mathcal{B}_2 = \{y_b \in \mathcal{X} : \ |y_b|_{\mathcal{X}} \leq R_2\}.$$

Then $T_i(\mathcal{B}_1 \times \mathcal{B}_2) \subset \mathcal{B}_i,\ i = 1, 2$.

We know that the semigroup generated by $A_i, i = 1, 2$, is compact, so the operator T is completely continuous. Consequently, Schauder's fixed point theorem can be applied. Thus the existence of at least one fixed point of T in $\mathcal{B}_1 \times \mathcal{B}_2$ is guaranteed. □

Next we obtain the existence of a solution via Avramescu's fixed point theorem.

Theorem 2.3. *Assume that the following conditions are satisfied:*

$$\begin{aligned} |f_1(t,x,y)|_X &\leq a_1 \, |x|_X + b_1 \, |y|_X + d_1(t), \\ |f_2(t,x,y) - f_2(t,\overline{x},\overline{y})|_X &\leq a_2 \, |x - \overline{x}|_X + b_2 \, |y - \overline{y}|_X \,, \\ |\alpha[x,y]|_X &\leq A_1 \, |x|_{C([0,1],X)} + B_1 \, |y|_{C([0,1],X)} + D_1, \\ |\beta[x,y] - \beta[\overline{x},\overline{y}]|_X &\leq A_2 \, |x - \overline{x}|_{C([0,1],X)} + B_2 \, |y - \overline{y}|_{C([0,1],X)} \,, \end{aligned} \tag{2.8}$$

for all $x, \overline{x}, y, \overline{y} \in X$, $t \in [0,1]$, some nonnegative constants a_1, b_1, a_2, b_2, A_1, B_1, A_2, B_2, D_1 and $d_1 \in L^1([0,1], \mathbb{R}_+)$. In addition assume that the matrix M_θ defined in (2.4) is convergent to zero for some $\theta > 0$ and condition (C) holds.

Then, the problem (1.1)–(1.2) has at least one solution in $\mathcal{X} \times \mathcal{X}$.

Proof. As above, consider the sets $\mathcal{B}_1$ and $\mathcal{B}_2$ where R_1, R_2 are positive numbers which are chosen below.

From (2.8), the function f_2 satisfies the growth condition

$$|f_2(t,x,y)|_X \leq a_2 |x|_X + b_2 |y|_X + d_2(t),$$

where $d_2(t) = |f_2(t,0,0)|_X$, $d_2 \in L^1([0,1],\mathbb{R}_+)$. Following the proof of Theorem 2.2, we conclude that

$$|T_1[x_a,y_b]|_{\mathcal{X}} \leq R_1, \ |T_2[x_a,y_b]|_{\mathcal{X}} \leq R_2,$$

for all $(x_a,y_b) \in \mathcal{B}_1 \times \mathcal{B}_2$, where R_1, R_2 are obtained as in the proof of the previous theorem. This guarantees that

$$T_i(\mathcal{B}_1 \times \mathcal{B}_2) \subset \mathcal{B}_i, \qquad i = 1,2.$$

Since the matrix M_θ is convergent to zero, the operator $T_2(x_a,\cdot)$ is a contraction for all $x \in \mathcal{X}$. Indeed, using (1.3) and the conditions (2.8) one has

$$\begin{aligned}
&|T_2[x_a,y_b] - T_2[x_a,\overline{y}_b]|_{\mathcal{X}} \\
&\quad \leq \|S_2(t)\| \left|b - \overline{b}\right|_X + \int_0^t \|S_2(t-s)\| \, |f_2(s,x(s),y(s)) - f_2(s,x(s),\overline{y}(s))|_X \, ds \\
&\qquad + \theta \, |\beta[x,y] - \beta[x,\overline{y}]|_X \\
&\quad \leq C_2 \left|b - \overline{b}\right|_X + \int_0^t (a_2 |x(s) - x(s)|_X + b_2 |y(s) - \overline{y}(s)|_X) \, ds \\
&\qquad + \theta \left(A_2 |x - x|_{C([0,1],X)} + B_2 |y - \overline{y}|_{C([0,1],X)}\right) \\
&\quad \leq C_2 \left|b - \overline{b}\right|_X + C_2 a_2 |x - x|_{C([0,1],X)} + C_2 b_2 |y - \overline{y}|_{C([0,1],X)} \\
&\qquad + \theta A_2 |x - x|_{C([0,1],X)} + \theta B_2 |y - \overline{y}|_{C([0,1],X)} .
\end{aligned}$$

From the definition of m_{ij} given in (2.3) we have

$$\begin{aligned}
|T_2[x_a,y_b] - T_2[x_a,\overline{y}_b]|_{\mathcal{X}} &\leq C_2 \left|b - \overline{b}\right|_X + (C_2 b_2 + \theta B_2) \, |y - \overline{y}|_{C([0,1],X)} \\
&\leq m_{22}(\theta) \, |y_b - \overline{y}_b|_{\mathcal{X}} \, ,
\end{aligned}$$

for all $x \in \mathcal{B}_1$ and $y, \overline{y} \in \mathcal{B}_2$.

In addition, as above, $T_1(\mathcal{B}_1 \times \mathcal{B}_2)$ is relatively compact in $\mathcal{X}$. Therefore, Avramescu's Theorem applies and guarantees the existence of a pair $(x,y) \in \mathcal{B}_1 \times \mathcal{B}_2$ such that $T_1(x,y) = x$ and $T_2(x,y) = y$. □

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