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Existence and multiplicity of positive solutions for nonlinear polyharmonic Dirichlet problems

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Abstract. In this paper, we are concerned with the existence and multiplicity of positive solutions for the following polyharmonic equations with Dirichlet boundary conditions

$$\begin{cases} (-\Delta)^m u = \lambda f(x,u) + \mu g(x,u) & \text{in } B, \\ u = \partial_\nu u = \cdots = \partial_\nu^{m-1} u = 0 & \text{on } \partial B, \end{cases}$$

where B is the unit ball in $\mathbb{R}^N$, and ν is the exterior unit normal vector on the boundary. We employ degree theory combined with a continuation theorem and an unfolding of the bifurcations from zero and from infinity to prove our result.

Keywords: polyharmonic, multiparameters, positive solutions, continuation method, bifurcation theory.

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1 Introduction

Let m be a positive integer and $B = \{x \in \mathbb{R}^N : |x| < 1\}$ be the unit ball. This paper deals with the existence and multiplicity of positive solutions to the following semilinear problems

$$\begin{cases} (-\Delta)^m u = \lambda f(x,u) + \mu g(x,u) & \text{in } B, \\ u = \partial_\nu u = \cdots = \partial_\nu^{m-1} u = 0 & \text{on } \partial B, \end{cases} \tag{$P^m_{\lambda,\mu}$}$$

where $\partial_\nu := \nu(x) \cdot \nabla$ denotes the outer normal derivative on ∂B, $\lambda, \mu \geq 0$ are two real parameters, and f, g are nonlinearities which are to be specified later.

In recent years, there has been a great deal of work on elliptic equations, such as

$$(-\Delta)^m u = h(x,u) \quad \text{in } \Omega, \tag{1.1}$$

where Ω is a bounded smooth domain in $\mathbb{R}^N$, and h is a continuous function. When $\Omega \neq \mathbb{R}^N$, two types of boundary conditions for (1.1) are most frequently considered: Dirichlet boundary conditions (BCs)

$$u = \partial_\nu u = \cdots = \partial_\nu^{m-1} u = 0 \quad \text{on } \partial\Omega \tag{1.2}$$

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or Navier BCs

$$u = \Delta u = \cdots = \Delta^{m-1} u = 0 \quad \text{on } \partial\Omega. \tag{1.3}$$

The operators in equation (1.1) are usually called polyharmonic and are the prototypes of linear elliptic operators of order $2m$, we focus on the Dirichlet problem here, consequently, most of the references we provide are concerned with this problem.

In the second-order case $m = 1$, the major contributions in this direction have been given by Brezis and Turner [8], Gidas and Spruck [17] and de Figueiredo, Lions and Nussbaum [14]. They proved uniform bounds for positive classical solutions of the second-order semilinear Dirichlet problem (1.1)–(1.2). Further, using known topological fixed-point theorems, the existence of solutions is obtained.

In the higher-order case $m \geq 2$, studies have also been conducted on both equations like (1.1) and their systems, though to a lesser extent. One of the main reasons lies in the fact that it strongly depends on the type of boundary conditions considered. Specifically, some recent contributions focusing on either Dirichlet BCs (1.2) or Navier BCs (1.3) include papers by Feng and Zhang [13], Hajlaoui and Harrabi [18], Iturriaga and Massa [19], Mancini and Romani [24], Pucci [29], Quittner and Souplet [30], Reichel and Weth [33], Sirakov [34, 35], Soranzo [36] and references therein.

It is worth pointing out that Dirichlet BCs are physically relevant (for example, they appear in the modeling of the clamped plate) as well as mathematically challenging: indeed, differently from the Navier case, the polyharmonic operator combined with Dirichlet BCs is not equivalent to the iteration of the Laplace operator with Dirichlet BCs (see e.g., [36]). Furthermore, under Dirichlet BCs, the operator in equation (1.1) does not satisfy the maximum principle (in general) unless $m = 1$.

We emphasize that the equation in ($P^m_{\lambda,\mu}$) is more general than (1.1)–(1.2). When the nonlinearities f and g are concave and convex, the problem ($P^m_{\lambda,\mu}$) with $m = \mu = 1$ was first studied by Ambrosetti, Brezis and Cerami in their celebrated paper [3]. More precisely, in that article the authors studied the problem

$$\begin{cases} -\Delta u = \lambda u^q + u^p & x \in \Omega, \\ u = 0 & x \in \partial\Omega, \end{cases} \tag{$P^1_{\lambda,1}$}$$

with $0 < q < 1 < p$ and Ω a bounded domain in $\mathbb{R}^N$. They proved the following facts: there exists $\Lambda > 0$ such that: if $\lambda \in (0, \Lambda)$ then ($P^1_{\lambda,1}$) has at least two positive solutions, if $\lambda = \Lambda$ there is at least one positive solution, and if $\lambda > \Lambda$ then there are no positive solutions. Similar combined effects have been extended to more general operators in $\mathbb{R}^N$ via variational methods, yielding existence and multiplicity results for p-Laplacian and quasilinear Schrödinger equations, see Baldelli and Filippucci [5] and Li [21], and references therein; see also [22,27,28] for related Nehari-type and mountain-pass approaches. In addition, the case of the second order elliptic problem ($P^1_{\lambda,1}$) with $f(x,u) = f(u)$ and $g(x,u) = g(u)$, concerning the existence of positive solutions for $\lambda > 0$ and $\mu > 0$, has been studied in [10] based on variational methods.

A basic result goes from Boggio in [7], where he gave an explicit formula for the Green function $G_{m,N}$ of $(-\Delta)^m$ on the unit ball $B \subset \mathbb{R}^N$ with Dirichlet boundary conditions (1.2), see Lemma 2.2 below. Consequently, any solution of ($P^m_{\lambda,\mu}$) can be represented as an integral operator involving the Green function $G_{m,N}$. This reformulates the differential equation as a fixed-point problem in $C(\overline{B})$. The positivity of $G_{m,N}$ does not hold for the Green function of the polyharmonic operator in an arbitrary bounded domain (see, e.g., [15]). Only for the case

$m = 1$, we do not have this restriction. So in the rest this paper, we will always work in the special case $\Omega = B$.

In order to present the main results in this paper, we recall the results of an eigenvalue problem. It is well known (see [29]) that the linear problem

$$\begin{cases} (-\Delta)^m u = \eta u & \text{in } B, \\ u = \partial_\nu u = \cdots = \partial_\nu^{m-1} u = 0 & \text{on } \partial B \end{cases}$$

possesses lowest eigenvalue η_1, which is simple, that is, the associated eigenfunctions are merely multiples of each other. Moreover, they are radial, strictly monotone in $r = |x|$ and never change sign in B.

In [6], Boccuto and Filippucci considered the following eigenvalue problem

$$\begin{cases} (-\Delta)^m u = \eta \sigma(x) u & \text{in } B, \\ u = \partial_\nu u = \cdots = \partial_\nu^{m-1} u = 0 & \text{on } \partial B, \end{cases}$$

where $\sigma \in L^\infty(B)$, $\sigma > 0$ almost everywhere. Note that $\eta_1(\sigma(\cdot))$, the first eigenvalue, is characterized by

$$\eta_1(\sigma(\cdot)) = \min \left\{ E(v), v \in H_0^m(B), \int_B |v|^2 \sigma \, dx = 1 \right\},$$

with E the Dirichlet energy defined by

$$E(v) = \begin{cases} \displaystyle\int_B \left| (-\Delta)^k v \right|^2 dx, & \text{if } m = 2k \text{ is even}, \\ \displaystyle\int_B \left| \nabla (-\Delta)^k v \right|^2 dx, & \text{if } m = 2k+1 \text{ is odd}. \end{cases}$$

They proved that $\eta_1(\sigma(\cdot))$ is positive and simple. Furthermore, if σ is radial and in $C^{0,\alpha}(B)$ ($\alpha \in (0,1)$), they showed that the first eigenfunctions associated with $\eta_1(\sigma(\cdot))$ are radial and strictly monotone along the radius r.

We will use the following assumptions:

(A1) $f, g : \overline{B} \times [0, +\infty) \to [0, +\infty)$ are continuous functions with $f(x,0) = g(x,0) = 0$;

(A2) $$\lim_{t \to +\infty} \frac{f(x,t)}{t} = 0 \quad \text{uniformly in } x \in \overline{B};$$

(A3) $$\lim_{t \to +\infty} \frac{g(x,t)}{t} = +\infty \quad \text{uniformly in } x \in \overline{B};$$

(A4) Assume that $q > 1$ if $N \le 2m$ and $1 < q < \frac{N+2m}{N-2m}$ if $N > 2m$. Suppose further that there exists a continuous function $h : \overline{B} \to (0, \infty)$ such that

$$\lim_{t \to +\infty} \frac{g(x,t)}{t^q} = h(x)$$

uniformly with respect to $x \in \overline{B}$.

Our main existence results read as follows.

Theorem 1.1. *Suppose conditions* (A1)–(A4) *hold, and the function f satisfies*

$$\lim_{t\to 0^+} \frac{f(x,t)}{t} = +\infty \quad \text{uniformly in } x \in \overline{B}. \tag{1.4}$$

Then there exists a function $M_*^+ : (0,\infty) \to (0,\infty]$ such that the problem ($P^m_{\lambda,\mu}$) has

(i) at least one positive solution for $\lambda > 0$ and $\mu = 0$;

(ii) at least two positive solutions for $\lambda > 0$ and $0 < \mu < M_*^+(\lambda)$.

Theorem 1.2. *Let $\Lambda_1 := \eta_1(b_1(\cdot))$. Suppose conditions* (A1)–(A4) *hold, and the function f satisfies*

$$\liminf_{t\to 0^+} \frac{f(x,t)}{t} = b_1(x) \quad \text{uniformly in } x \in \overline{B}, \tag{1.5}$$

$$\limsup_{t\to 0^+} \frac{f(x,t)}{t} = b_2(x) \quad \text{uniformly in } x \in \overline{B}, \tag{1.6}$$

where $b_1, b_2 \in C(\overline{B},(0,\infty))$. Then there exists a function $M_{**}^+ : (0,\infty) \to (0,\infty]$ such that ($P^m_{\lambda,\mu}$) has

(i) at least one positive solution for $\lambda > \Lambda_1$ and $\mu = 0$;

(ii) at least two positive solutions for $\lambda > \Lambda_1$ and $0 < \mu < M_{**}^+(\lambda)$.

Remark 1.3. A main difficulty in studying ($P^m_{\lambda,\mu}$) is establishing the multiplicity of solutions in the two-parameter setting (λ,μ), which is more delicate than in the single-parameter case. We address this by combining Rabinowitz's global bifurcation theorems [31, Theorem 1.3], [32, Theorem 1.6] with the continuation theorem (Lemma 2.4).

For Theorem 1.1, condition (1.4) induces a jump of the Leray–Schauder index at $u = 0$, and hence by [31, Theorem 1.3] an unbounded continuum of positive solutions emanates from $(0,0)$. For fixed $\lambda > 0$, these branches are continued in μ by Lemma 2.4. By (A2)–(A3) and [32, Theorem 1.6], a second continuum bifurcates from infinity at $\mu = 0$, and (A4) provides a priori bounds precluding "blow up". These two components yield two positive solutions for all sufficiently small $\mu > 0$. Theorem 1.2 follows the same approach, except that the asymptotic conditions near zero involve the principal eigenvalues $\eta_1(b_1(\cdot))$ and $\eta_1(b_2(\cdot))$, so the continuum emanates from the interval $[\eta_1(b_1(\cdot)), \eta_1(b_2(\cdot))]$.

Example 1.4. Let $a, b \in C(\overline{B},(0,\infty))$ and let $q > 1$ satisfy $q < \frac{N+2m}{N-2m}$ whenever $N > 2m$. Set

$$f(x,t) = a(x)t^\alpha \quad \text{and} \quad g(x,t) = b(x)t^q, \qquad x \in \overline{B},\ t \geq 0,$$

where $0 < \alpha < 1$. Obviously $f(x,0) = g(x,0) = 0$. Straightforward calculations yield

$$\lim_{t\to 0^+} \frac{f(x,t)}{t} = +\infty, \qquad \lim_{t\to +\infty} \frac{f(x,t)}{t} = 0,$$

and

$$\lim_{t\to +\infty} \frac{g(x,t)}{t} = +\infty, \qquad \lim_{t\to +\infty} \frac{g(x,t)}{t^q} = b(x),$$

uniformly in $x \in \overline{B}$. Hence (A1)–(A4) and condition (1.4) are satisfied.

Example 1.5. Let a, b and q be as in Example 1.4. Set

$$f(x,t) = a(x)\,\frac{t}{1+t^{\alpha}}\big(2+\sin(1/t)\big), \qquad g(x,t) = b(x)t^q, \qquad x \in \overline{B},\ t \geq 0,$$

where $0 < \alpha < 1$. Then $f(x,0) = g(x,0) = 0$, and

$$\frac{f(x,t)}{t} = a(x)\,\frac{2+\sin(1/t)}{1+t^{\alpha}}.$$

Consequently,

$$\liminf_{t\to 0^+} \frac{f(x,t)}{t} = a(x), \qquad \limsup_{t\to 0^+} \frac{f(x,t)}{t} = 3a(x),$$

uniformly in $x \in \overline{B}$. In addition,

$$\lim_{t\to+\infty} \frac{f(x,t)}{t} = 0, \qquad \lim_{t\to+\infty} \frac{g(x,t)}{t} = +\infty, \qquad \lim_{t\to+\infty} \frac{g(x,t)}{t^q} = b(x),$$

uniformly in $x \in \overline{B}$. Thus (A1)–(A4), (1.5) and (1.6) hold with $b_1 = a$ and $b_2 = 3a$.

The rest of the paper is organized as follows. In Section 2, we list some existing lemmas that will be used in subsequent sections. Theorems 1.1 and 1.2 are proved in Sections 3 and 4, respectively. Finally, Section 5 deals with the special case $N = 1$.

2 Preliminary results

Let Ω be a bounded domain in $\mathbb{R}^N$ with smooth boundary $\partial\Omega$. We consider the following problem

$$\begin{cases} (-\Delta)^m u = f & \text{in } \Omega, \\ u = \partial_\nu u = \cdots = \partial_\nu^{m-1} u = 0 & \text{on } \partial\Omega. \end{cases} \tag{2.1}$$

In the following result, we prove the compactness of the solution operator.

Lemma 2.1. *For each $f \in C(\overline{\Omega})$, then there exists a solution u to (2.1) which is in $C^{2m,\gamma}(\overline{\Omega})$ for any $\gamma \in (0,1)$. Moreover, the solution operator $K_m : C(\overline{\Omega}) \to C^{2m}(\overline{\Omega})$ defined by*

$$K_m(f) =: u$$

is compact.

Proof. By [16, Theorem 2.20] and elliptic regularity we know that if $f \in W^{k-2m,p}(\Omega)$ for $k \geq 2m$ then there exists a unique solution to (2.1) such that

$$\|u\|_{W^{k,p}} \leq C\|f\|_{W^{k-2m,p}}. \tag{2.2}$$

Assume $f \in C(\overline{\Omega})$. Then $f \in W^{1,p}(\Omega)$ for any p. Hence, by (2.2) one has

$$\|u\|_{W^{1+2m,p}} \leq C\|f\|_{W^{1,p}}$$

for any p. By taking p large enough, one has $W^{1+2m,p}(\Omega) \hookrightarrow C^{2m,\gamma}(\overline{\Omega})$ compactly for any $\gamma \in (0,1)$ (see [1]) and $u \in C^{2m,\gamma}(\overline{\Omega}) \subset C^{2m}(\overline{\Omega})$. This completes the proof. □

The next generalization of Boggio's principle establishes a positivity preserving property due to Boggio [7].

Lemma 2.2 (See [16, Proposition 3.6]). *Let us assume $\Omega = B$ in (2.1). If $f \geq 0$, then $K_m(f) \geq 0$ as well.*

Remark 2.3. Lemma 2.2 is crucial for ensuring the positivity of solutions in the subsequent content, and this is also the main reason why we restrict ourselves to the case $\Omega = B$.

A useful tool in the study of global continua of solutions of nonlinear partial differential equations is a connectivity result on the fixed-points set of a parameterised map that goes back to Leray and Schauder [20] and it was proved in its full generality by Browder [9].

Lemma 2.4 (See [25]). *Let E be a Banach space, I be the unit interval $[-1,1] \subset \mathbb{R}$ and U be an open subset of $I \times E$. Let $\bar{f} : \overline{U} \to X$ be a compact map such that*

$$u \neq \bar{f}(\lambda, u) \quad \text{for } (\lambda, u) \in \partial U$$

$$\deg(I - \bar{f}(\lambda, \cdot), U_\lambda, 0) \neq 0 \text{ for some } \lambda \in [-1,1],$$

where $U_\lambda = \{u \in E : (\lambda, u) \in U\}$ is the λ-section of U in $\mathbb{R} \times E$. Then

(i) there exists a connected subset C of the solution set

$$\mathcal{S} = \{(\lambda, u) \in U : u - \bar{f}(\lambda, u) = 0\}$$

joining $\mathcal{S}_{-1} := \mathcal{S} \cap (\{-1\} \times E)$ with $\mathcal{S}_1 := \mathcal{S} \cap (\{1\} \times E)$.

(ii) $\mathcal{S}_{-1}$ cannot be separated from $\mathcal{S}_1$ in $\mathcal{S}$.

We now state some well-known definitions and lemmas for later use.

Definition 2.5 (See [37]). Let X be a Banach space and $\{C_n \mid n = 1, 2, \dots\}$ be a certain infinite collection of subsets of X. Then the superior limit $\mathcal{D}$ of $\{C_n\}$ is defined by

$$\mathcal{D} := \limsup_{n \to \infty} C_n = \{x \in X : \exists \{n_k\} \subset \mathbb{N}, x_{n_k} \in C_{n_k} \text{ such that } x_{n_k} \to x\}.$$

Definition 2.6 (See [37]). A component of a set M means a maximal connected subset of M, i.e., a connected subset of M which is not contained in any other connected subset of M.

Lemma 2.7 (See [23, Lemma 2.2]). *Let X be a Banach space with norm $\|\cdot\|_X$ and $\{C_n \mid n = 1, 2, \dots\}$ be a family of closed connected subsets of X. Assume that*

(i) there exist $z_n \in C_n$, $n = 1, 2, \dots$, and $z^* \in X$, such that $z_n \to z^*$;

(ii) $r_n = \sup\{\|x\|_X : x \in C_n\} = \infty$;

(iii) for every $R > 0$, $(\bigcup_{n=1}^{\infty} C_n) \cap B_R$ is a relatively compact set of X, where

$$B_R = \{x \in X : \|x\|_X \leq R\}.$$

Then there exists an unbounded component C in $\limsup_{n \to \infty} C_n$ and $z^* \in C$.

3 Proof of Theorem 1.1

First of all, as usual when dealing with positive solutions of a boundary value problem, we introduce a continuous extension $\hat{f}, \hat{g} : \overline{B} \times \mathbb{R} \to \mathbb{R}$ of f, g by setting

$$\hat{f}(x,s) := \begin{cases} f(x,s) & \text{if } s \geq 0, \\ f(x,-s) & \text{if } s < 0, \end{cases} \qquad \hat{g}(x,s) := \begin{cases} g(x,s) & \text{if } s \geq 0, \\ g(x,-s) & \text{if } s < 0. \end{cases}$$

We consider the Banach space $X := C(\overline{B})$ endowed with the sup norm $\|\cdot\|_\infty$, denote $B_\rho := \{u \in X : \|u\|_\infty < \rho\}$. Let us consider the following problem

$$\begin{cases} (-\Delta)^m u = \lambda \hat{f}(x,u) + \mu \hat{g}(x,u) & \text{in } B, \\ u = \partial_\nu u = \cdots = \partial_\nu^{m-1} u = 0 & \text{on } \partial B. \end{cases} \tag{$\widehat{P}^m_{\lambda,\mu}$}$$

Because of the smoothness assumptions imposed on the problem ($\widehat{P}^m_{\lambda,\mu}$) will, due to regularity theory, be equivalent to an operator equation

$$u = \mathcal{K}_m(\lambda F(u) + \mu G(u)), \tag{3.1}$$

where $\mathcal{K}_m : X \to X$ is a positive and compact linear operator induced by K_m and F, G are the Nemytskii operators induced by

$$F(u)(x) = \hat{f}(x, u(x)), \quad G(u)(x) = \hat{g}(x, u(x)),$$

both of which are continuous operators. Thanks to Lemma 2.2, solutions of ($\widehat{P}^m_{\lambda,\mu}$) are nonnegative and hence are solutions of our given problem ($P^m_{\lambda,\mu}$).

Define the map $T_{\lambda,\mu} : X \mapsto X$ by

$$T_{\lambda,\mu}(u) := u - \mathcal{K}_m(\lambda F(u) + \mu G(u)).$$

It is clear that u is a non-negative solution of ($\widehat{P}^m_{\lambda,\mu}$) if and only if, u is a zero of the map $T_{\lambda,\mu}$. Observe that $T_{\lambda,\mu}$ is compact.

In order to prove the main result of this section we use the Leray–Schauder degree of $T_{\lambda,\mu}$ on B_ρ with respect to zero, denoted by $\deg(T_{\lambda,\mu}, B_\rho, 0)$, and the index of the isolated zero u of $T_{\lambda,\mu}$, denoted by $i(T_{\lambda,\mu}, u)$.

The following lemmas play a fundamental role in the proof of the result.

Lemma 3.1. *Let* (A1)–(A4) *and* (1.4) *hold. Then for fixed* $\lambda < 0$, *one has* $i(T_{\lambda,0}, 0) = 1$.

Proof. Fix $\lambda < 0$ and define the map

$$\mathcal{H}_1 : [0,1] \times X \to X; \quad \mathcal{H}_1(t,u) := \mathcal{K}_m(t\lambda F(u)), \quad t \in [0,1].$$

We claim that there exists $\delta > 0$ such that

$$u \neq \mathcal{H}_1(t,u) \quad \text{for all } u \in \overline{B}_\delta \text{ and } t \in [0,1].$$

Indeed, suppose that there exist sequences $\{u_n\} \in X \setminus \{0\}$ with $\|u_n\|_\infty \to 0$ and $t_n \in [0,1]$ such that

$$u_n = \mathcal{H}_1(t_n, u_n),$$

that is

$$(-\Delta)^m u_n = t_n \lambda F(u_n) \leq 0.$$

By Lemma 2.2, we have $u_n \leq 0$, and going back to the equation, $u_n \equiv 0$, which is a contradiction.

Taking now $\epsilon \in (0, \delta]$, the homotopy defined by $\mathcal{H}_1$ is admissible and so,

$$\begin{aligned} i(T_{\lambda,0}, 0) &= \deg(T_{\lambda,0}, B_\epsilon, 0) \\ &= \deg(I - \mathcal{H}_1(1, \cdot), B_\epsilon, 0) \\ &= \deg(I - \mathcal{H}_1(0, \cdot), B_\epsilon, 0) \\ &= \deg(I, B_\epsilon, 0) = 1. \end{aligned}$$

This concludes the proof. □

Lemma 3.2. *Let* (A1)–(A4) *and* (1.4) *hold. Let* $\lambda > 0$ *be fixed, one has* $i(T_{\lambda,0}, 0) = 0$.

Proof. Fix $\lambda > 0$ and $\phi \in X$, $\phi > 0$ and define the map

$$\mathcal{H}_2 : [0,1] \times X \to X; \quad \mathcal{H}_2(t,u) := \mathcal{K}_m(\lambda F(u) + t\phi).$$

We will show that there exists $\delta > 0$ such that $u \neq \mathcal{H}_2(t,u)$ for all $u \in B_\delta$, $u \neq 0$ and $t \in [0,1]$. In fact, suppose the contrary there exist sequences $\{u_n\} \in X \setminus \{0\}$ with $\|u_n\|_\infty \to 0$ and $t_n \in [0,1]$ such that $u_n = \mathcal{H}_2(t_n, u_n)$, that is

$$(-\Delta)^m u_n = \lambda F(u_n) + t_n \phi.$$

Since $t_n\phi \geq 0$ and $t_n\phi \not\equiv 0$, from Lemma 2.2, we have that $u_n > 0$ in B.

On the other hand, since $\|u_n\|_\infty \to 0$, we have from (1.4) that $\lambda F(u_n) > \eta_1 u_n$ for n large enough. Therefore,

$$(-\Delta)^m u_n = \lambda F(u_n) + t_n \phi > \eta_1 u_n.$$

Thus, multiplying the above inequality by ψ_1, a positive eigenfunction associated to η_1, and integrating by parts, we get that

$$\eta_1 \int_B u_n \psi_1 \, dx > \eta_1 \int_B u_n \psi_1 \, dx,$$

which is a contradiction.

Now, if take $\epsilon \in (0, \delta]$, we have

$$i(T_{\lambda,0}, 0) = \deg(T_{\lambda,0}, B_\epsilon, 0) = \deg(I - \mathcal{H}_2(0, \cdot), B_\epsilon, 0) = \deg(I - \mathcal{H}_2(1, \cdot), B_\epsilon, 0) = 0.$$

This concludes the proof. □

In the following result, we show that from the trivial solution emanates an unbounded continuum of positive solution.

Lemma 3.3. *The value* $\lambda = 0$ *is the only bifurcation point from the trivial solution for* $(\widehat{P}^m_{\lambda,0})$. *Moreover, there exists a continuum* $\mathcal{C}_0$ *of positive solutions of* $(\widehat{P}^m_{\lambda,0})$ *unbounded in* $\mathbb{R} \times X$ *emanating from* $(0,0)$.

Proof. First, we show that $(0,0)$ is a bifurcation point from the line of trivial solutions $\{(\lambda,0)\in \mathbb{R}\times X : \lambda\in\mathbb{R}\}$ for the equation

$$u=\lambda\mathcal{K}_m(F(u)). \tag{3.2}$$

In fact, this can be obtained following from a simple modification of the global bifurcation theorem of Rabinowitz [31, Theorem 1.3], and the similar arguments have been used in [4, Proposition 3.5]. Suppose on the contrary that $(0,0)$ is not a bifurcation point for (3.2), then there is a neighborhood of $(0,0)$ containing no nontrivial solutions of (3.2). In particular there exists a small $\epsilon>0$ such that there are no solutions of (3.2) on $[-\epsilon,\epsilon]\times\partial B_\epsilon$. Then $\deg(T_{\lambda,0}(\cdot),B_\epsilon,0)$ is well defined for $\lambda\in[-\epsilon,\epsilon]$ and, by the homotopy invariance property of degree we have

$$\deg(T_{\lambda,0}(\cdot),B_\epsilon,0)\equiv \text{constant},\quad \forall\lambda\in[-\epsilon,\epsilon],$$

which is a contradiction with Lemmas 3.1 and 3.2.

Then according to Rabinowitz's global bifurcation theorem, a continuum $\mathcal{C}_0$ of positive solutions of (3.2) emanates from $(0,0)$, and either

(i) $\mathcal{C}_0$ is unbounded in $\mathbb{R}\times X$, or

(ii) $\mathcal{C}_0\cap[(\mathbb{R}\setminus\{0\})\times\{0\}]\neq\emptyset$.

To prove the unboundedness of $\mathcal{C}_0$, we only need to show that the case (ii) cannot occur, that is, $\mathcal{C}_0$ can not meet $(\lambda,0)$ for any $\lambda\neq 0$. It is easy to see that for $\lambda<0$ problem $(\widehat{P}^m_{\lambda,0})$ does not possess a positive solution. For the case $\lambda>0$, assume now that there exist $\lambda_0>0$ and a sequence of positive solutions of (3.2) such that $\lambda_n\to\lambda_0$ and $\|u_n\|_\infty\to 0$. Then, fixed $\varepsilon>0$ there exists $n_0\in\mathbb{N}$ such that for $n\geq n_0$ we get

$$(-\Delta)^m u_n=\lambda_n\hat{f}(x,u_n)>\eta_1 u_n.$$

Now, we arrive at a contradiction in a similar way to in the proof of Lemma 3.2. Hence, we conclude the existence of an unbounded continuum $\mathcal{C}_0$ of solutions of $(\widehat{P}^m_{\lambda,0})$ bifurcating from $(0,0)$. This completes the proof. □

Proof of Theorem 1.1(i). By Lemma 3.3 we know the existence of an unbounded continuum $\mathcal{C}_0$ of positive solution of $(\widehat{P}^m_{\lambda,0})$. On the other hand, assume there exists a sequence of solutions $\{(\lambda_n,u_n)\}$, $|\lambda_n|\leq\rho$, $\rho>0$ and $\|u_n\|_\infty\to\infty$ and

$$u_n=\lambda_n\mathcal{K}_m(F(u_n)). \tag{3.3}$$

Dividing (3.3) by $\|u_n\|_\infty$, letting $v_n=u_n\|u_n\|_\infty^{-1}$, we obtain

$$v_n=\lambda_n\mathcal{K}_m\left(\frac{F(u_n)}{u_n}v_n\right). \tag{3.4}$$

Hence, $\{v_n\}$ has a subsequence converging to $v\in X$ with $\|v\|_\infty=1$. However, by (A2) the limit satisfies $v=0$, which contradicts $\|v\|_\infty=1$, therefore $\|u\|_\infty$ is bounded for all $\lambda>0$. □

Proof of Theorem 1.1(ii). For fixed $\lambda\in(0,\infty)$, according to (i),

$$\mathcal{S}_\lambda:=\{(\eta,u)\in\mathcal{C}_0 : \eta=\lambda\}\neq\emptyset.$$

Moreover, Lemmas 3.1 and 3.2 imply that there exist two positive constants r_λ and R_λ with $r_\lambda<R_\lambda$ such that

$$i(T_{\lambda,0},B_{R_\lambda}\setminus B_{r_\lambda})=1.$$

By Lemma 2.4 and the standard continuation method, see for example, Ambrosetti, Arcoya and Buffoni [2, p. 663], Mawhin and Schmitt [26, p. 142–143], there exists a component $\mathcal{S}$ with

$$\mathcal{S}_\lambda \subset \mathcal{S},$$

and accordingly, there exist two constants $M_1^-(\lambda) < 0 < M_1^+(\lambda)$, such that

(i) $\mathrm{Proj}_{\mathbb{R}}\mathcal{S} \supseteq (M_1^-(\lambda), M_1^+(\lambda))$;

(ii) for $\mu \in (M_1^-(\lambda), M_1^+(\lambda))$, $(\widehat{P}^m_{\lambda,\mu})$ has a positive solution in $\mathcal{C}_0$.

Next, we show that 0 is a bifurcation point from infinity. For each $n \in \mathbb{N}^*$, let us define

$$g^{[n]}(x,s) := \begin{cases} \widehat{g}(x,s), & 0 < s \le n, \\ n(s-n) + \widehat{g}(x,n), & s > n. \end{cases}$$

Then

$$(g^{[n]})_\infty = \lim_{s\to\infty} \frac{g^{[n]}(x,s)}{s} = n.$$

To prove Theorem 1.1, let us consider the auxiliary family of the problems

$$\begin{cases} (-\Delta)^m u = \lambda \hat{f}(x,u) + \mu g^{[n]}(x,u) & \text{in } B, \\ u = \partial_\nu u = \cdots = \partial_\nu^{m-1} u = 0 & \text{on } \partial B. \end{cases} \qquad (\widehat{P}^m_{\lambda,\mu})^{[n]}$$

Let $\zeta^{[n]} \in C(\overline{B} \times \mathbb{R})$ be such that

$$g^{[n]}(x,s) = (g^{[n]})_\infty s + \zeta^{[n]}(x,s). \tag{3.5}$$

Note that

$$\lim_{|s|\to\infty} \frac{\zeta^{[n]}(x,s)}{s} = 0. \tag{3.6}$$

By (A2), one has

$$\lim_{\|u\|_\infty\to\infty} \frac{H(x,u,n,\mu,\lambda)}{\|u\|_\infty} = 0 \quad \text{uniformly for } \mu \text{ in any compact interval,}$$

where

$$H(x,u,n,\mu,\lambda) := \lambda \hat{f}(x,u) + \mu \zeta^{[n]}(x,u).$$

Let $\mu_\infty^{[n]} = \eta_1/n$, and let $\mathcal{S}^{[n]}$ denote the closure of the set of solutions of $(\widehat{P}^m_{\lambda,\mu})^{[n]}$. Applying the same method to prove Rabinowitz [32, Theorem 1.6] with obvious changes, we may get

Lemma 3.4. *Let* (A1)–(A4) *and* (1.4) *hold. Then* $\mu_\infty^{[n]}$ *is a bifurcation point from infinity for* $(\widehat{P}^m_{\lambda,\mu})^{[n]}$. *More precisely, there exists an unbounded closed connected set* $\mathcal{D}_\infty^{[n]} \subset \mathcal{S}^{[n]}$ *that bifurcates from infinity.*

Moreover, if $\{(\lambda, \mu_n, u_{\mu_n})\} \subset \mathcal{D}_\infty^{[n]}$, using (A4) and the a priori estimates in Reichel and Weth [33, Theorem 1], for each fixed $\mu > 0$ there exists $M_\mu > 0$ such that

$$\|u_{\mu_n}\|_\infty \le M_\mu, \qquad n \in \mathbb{N},$$

where M_μ is independent of n and u_{μ_n}. Taking $z_n = (\mu_\infty^{[n]}, \infty)$ and $z^* = (0, \infty)$, we have $z_n \to z^*$, and thus, condition (i) in Lemma 2.7 holds. Obviously,

$$r_n = \sup\left\{|\lambda| + |\mu| + \|u\|_\infty : (\lambda,\mu,u) \in \mathcal{D}_\infty^{[n]}\right\} = \infty,$$

i.e., condition (ii) in Lemma 2.7 is satisfied. Condition (iii) in Lemma 2.7 can be deduced from the compactness of $\mathcal{K}_m$. Therefore, by Lemma 2.7, the superior limit of $\mathcal{D}_\infty^{[n]}$ contains a component $\mathcal{D}_\infty$ of solutions of the problem $(\widehat{P}^m_{\lambda,\mu})$. Thus there exists $M_*^+(\lambda) > 0$ with

$$M_*^+(\lambda) \le M_1^+(\lambda),$$

such that for any $\mu \in (0, M_*^+(\lambda))$, $(\widehat{P}^m_{\lambda,\mu})$ has a solution. □

4 Proof of Theorem 1.2

Proof of Theorem 1.2(i). Conditions (1.5) and (1.6) imply there exists a component $\mathcal{C}$ of positive solutions of the problem $(\widehat{P}^m_{\lambda,0})$ which bifurcates from interval $[\eta_1(b_1(\cdot)), \eta_1(b_2(\cdot))]$. By the standard method, (A1) and (A2) guarantees that $\mathcal{C}$ to infinity in λ direction. Take

$$\Lambda_1 = \eta_1(b_1(\cdot)).$$

Then $(\widehat{P}^m_{\lambda,0})$ has at least one positive solution for $\lambda > \Lambda_1$. □

Proof of Theorem 1.2(ii). Fix $\lambda \in (\Lambda_1, \infty)$. By the same method to prove Theorem 1.1(ii), with obvious changes, we may deduce that there exist two functions $M_2^+ : (0,\infty) \to (0,\infty]$ and $M_2^- : (0,\infty) \to [-\infty, 0)$, such that the problem $(\widehat{P}^m_{\lambda,\mu})$ has at least two positive solutions in $\mathcal{S}$ for $\lambda > \Lambda_1$ and $M_2^-(\lambda) < \mu < M_2^+(\lambda)$. Thus there exists $M_{**}^+(\lambda) > 0$ with

$$M_{**}^+(\lambda) \le M_2^+(\lambda),$$

such that for any $\mu \in (0, M_{**}^+(\lambda))$, $(\widehat{P}^m_{\lambda,\mu})$ has a solution. □

5 Case $N = 1$

In this section, we will deal with the special case that $N = 1$, that is, we consider the two point boundary value problem

$$\begin{cases} (-1)^k u^{(2k)}(x) = \lambda f(x,u) + \mu g(x,u) & \text{in } (-1,1), \\ u^{(i)}(-1) = 0,\ u^{(i)}(1) = 0, & i = 0,1,\dots,k-1. \end{cases} \tag{$Q_{\lambda,\mu}$}$$

The main result of this section is as follows:

Theorem 5.1. *Assume that $f, g \in C([-1,1] \times [0,+\infty), [0,+\infty))$ with $f(x,u) > 0$, $g(x,u) > 0$ for $(x,u) \in (-1,1) \times (0,+\infty)$, $f(x,0) = g(x,0) = 0$, and that $f(\cdot,u)$, $g(\cdot,u)$ are even on $[-1,1]$ for every $u \ge 0$. Suppose further that the following conditions hold:*

(H1) $$\lim_{u\to 0^+} \frac{f(x,u)}{u} = 0 \quad \textit{uniformly in } x \in [-1,1];$$

(H2) $$\lim_{u\to +\infty} \frac{f(x,u)}{u} = 0 \quad \textit{uniformly in } x \in [-1,1];$$

(H3) $$\lim_{u\to +\infty} \frac{g(x,u)}{u} = +\infty \quad \textit{uniformly in } x \in [-1,1].$$

Then there exist $\lambda_0 > 0$, and a function $M : (0,\infty) \to (0,\infty]$ such that the problem $(Q_{\lambda,\mu})$ has

(i) at least two positive solutions when $\lambda > \lambda_0$ and $\mu = 0$;

(ii) at least three positive solutions when $\lambda > \lambda_0$ and $0 < \mu < M(\lambda)$.

In order to study problem ($Q_{\lambda,\mu}$), we recall that the Green's function associated to the problem $(-1)^k u^{(2k)}(x) = 0$ subject to the boundary conditions $u^{(i)}(-1) = u^{(i)}(1) = 0, i = 0, 1, \ldots, k-1$ is the function $G : [-1,1] \times [-1,1] \to [0,\infty)$ defined by (see [11, Theorem 2.1])

$$G(x,y) = \frac{1}{(2k-1)!} \begin{cases} \left[\frac{(1-x)(1+y)}{2}\right]^k \sum_{j=0}^{k-1} \binom{k-1+j}{j} (x-y)^{k-1-j} \left[\frac{(1+x)(1-y)}{2}\right]^j, & y \le x, \\ \left[\frac{(1+x)(1-y)}{2}\right]^k \sum_{j=0}^{k-1} \binom{k-1+j}{j} (y-x)^{k-1-j} \left[\frac{(1-x)(1+y)}{2}\right]^j, & x \le y. \end{cases}$$

The space $E = \{u \in C[-1,1] : u(-x) = u(x), x \in [-1,1], u \text{ is nonincreasing in } [0,1]\}$ will be endowed with the usual sup-norm $\|\cdot\|_\infty$. We denote $B_d := \{u \in E : \|u\|_\infty < d\}$ and define the cone in E by

$$K := \left\{ u \in E : u \ge 0 \text{ and } \min_{-\frac{1}{2} \le x \le \frac{1}{2}} u(x) \ge \gamma \|u\|_\infty \right\},$$

where $\gamma = \frac{1}{2^k}$, and for $r > 0$, let

$$K_r = \{u \in K : \|u\|_\infty < r\}.$$

It is not difficult to see that a nonnegative function u is a solution of problem ($Q_{\lambda,\mu}$) if and only if u is a fixed point of the operator $\mathcal{T}_{\lambda,\mu} : C[-1,1] \to C[-1,1]$ given by

$$\mathcal{T}_{\lambda,\mu} u(x) = \int_{-1}^{1} G(x,y)(\lambda f(y,u(y)) + \mu g(y,u(y)))dy, \quad x \in [-1,1].$$

It is a standard matter that the operator $\mathcal{T}_{\lambda,\mu}$ is completely continuous.

We shall state and prove a lemma that characterizes a generalized notion of concavity. The inequalities obtained here are very closely related to those obtained in [12], but for the sake of completeness and convenience in the subsequent discussion, we will present a proof sketch.

Lemma 5.2. *Suppose $u \in C^{2k-1}[-1,1] \cap C^{2k}(-1,1)$ satisfies $(-1)^k u^{(2k)}(x) > 0$ for $x \in (-1,1)$ with $u^{(i)}(-1) = u^{(i)}(1) = 0$, $0 \le i \le k-1$. Then u has one extreme point, say at $x_0 \in (-1,1)$. If we define*

$$p(x) = \begin{cases} \dfrac{\|u\|_\infty}{(1+x_0)^k}(1+x)^k, & -1 \le x \le x_0, \\ \dfrac{\|u\|_\infty}{(1-x_0)^k}(1-x)^k, & x_0 \le x \le 1. \end{cases}$$

Then

$$u(x) \ge p(x) \quad \text{for } x \in [-1,1].$$

Proof. We first note that u has precisely one extreme point, say $x_0 \in (-1,1)$. For if u' vanishes twice in $(-1,1)$, then by employing the boundary conditions and applying Rolle's theorem repeatedly, one obtains the contradiction that $u^{(2k)}$ vanishes in $(-1,1)$. We also note that since $(-1)^k u^{(2k)}(x) > 0, -1 < x < 1$, u' has a simple zero at x_0. Similarly, u'' has two simple zeros

in $(-1,1)$. We shall provide two sets of labels for the zeros of u''. We shall label these zeros by $x_{21} < x_{22}$ or by $\tau_{22} < \tau_{21}$.

So, $x_{21} = \tau_{22}$ and $x_{22} = \tau_{21}$. Similarly, u''' has at least two simple zeros in $(-1,1)$ and we label these zeros by $x_{31} < x_{32} < \cdots$ or by $\cdots < \tau_{32} < \tau_{31}$. Inductively, for $i = 2, \ldots, 2k-2$, $u^{(i)}$ has at least two simple zeros in $(-1,1)$ and we label these zeros by $x_{i1} < x_{i2} < \cdots$ or by $\cdots < \tau_{i2} < \tau_{i1}$. Note that

$$x_{k1} < x_{k+1,1} < \cdots < x_{2k-1,1} = \tau_{2k-1,1} < \cdots < \tau_{k+1,1} < \tau_{k,1}.$$

We shall now show that $u(x) - p(x) \geq 0, -1 \leq x \leq 1$.

Next we only present the case for $-1 \leq x \leq x_0$, the case for $x_0 \leq x \leq 1$ is analogous. Let $h(x) = u(x) - p(x)$. Note that

$$h(-1) = \cdots = h^{(k-1)}(-1) = h(x_0) = 0 \quad \text{and} \quad h'(x_0) = -p'(x_0) < 0.$$

Thus, $h(x_0^-) > 0$. Assume, for the sake of contradiction, that $h(c) = 0$ for some $c \in (-1, x_0)$. Apply Rolle's theorem and note that h' vanishes at $c_{1i}, i = 1,2$, where $-1 < c_{11} < c_{12} < x_0$. Apply Rolle's theorem again and note that h'' vanishes at $c_{2i}, i = 1,2$, where $-1 < c_{21} < c_{22} < c_{12}$. We now argue that, in fact, $-1 < c_{21} < c_{22} < x_{21}$. We have noted that for each $j = 1, \ldots, 2k-1$, each root of $u^{(j)}$ in $(-1,1)$ is simple, so u'' changes sign at x_{21} and at x_{22}. Note that $u^{(k)}(-1) > 0$. This follows since $u(x) = \int_{-1}^{1} G(x,s)(-1)^k u^{(2k)}(s)ds$ and $(\partial^k/\partial x^k)G(-1,s) > 0, -1 < s < 1$. So by Taylor's theorem, $u^{(j)}(x) > 0, -1 < x < x_{j1}, j = 2, \ldots, k-1$, and $u^{(j)}(x) < 0, x_{j1} < x < x_{j2}, j = 2, \ldots, k-1$. Now, $h''(c_{2i}) = 0, i = 1,2$, implies $u''(c_{2i}) = p''(c_{2i}) > 0, i = 1,2$. Since, $c_{22} < x_0 < x_{22}$, then $c_{2i} \in (-1, x_{21}), i = 1,2$. It now follows inductively for $j = 2, \ldots, k$, that $h^{(j)}$ vanishes at $-1 < c_{j1} < c_{j2} < x_{j1}$. To see this, it follows by Rolle's theorem that $-1 < c_{j1} < c_{j2} < c_{j-1,2}$. $h^{(j)}(c_{ji}) = 0, i = 1,2$, implies $u^{(j)}(c_{ji}) = p^{(j)}(c_{ji}) > 0, i = 1,2$. By the simplicity of the interior roots of $u^{(j)}$ and the fact that $c_{j2} < x_{j-1,1} < x_{j2}$ it follows that $-1 < c_{j1} < c_{j2} < x_{j1}, j = 2, \ldots, k$.

Now, apply Rolle's theorem to $h^{(k)}$ and note that $h^{(k+1)}$ vanishes at some c_{k+1} where $c_{k+1} < c_{k2} < x_{k1} < x_{k+1,1}$. This produces a contradiction since $h^{(k+1)} \equiv u^{(k+1)}$ and $x_{k+1,1}$ is the smallest positive root of $u^{(k+1)}$. Thus, h does not vanish on $(-1, x_0)$ and $h(x) > 0, -1 < x < x_0$. This completes the proof. □

Remark 5.3. Suppose $k \geq 1$ is an integer and $u \in E$ satisfies

$$\begin{cases} (-1)^k u^{(2k)} \geq 0, & -1 \leq x \leq 1, \\ u^{(i)}(-1) = u^{(i)}(1) = 0, & i = 0, \ldots, k-1. \end{cases} \tag{5.1}$$

Then, for $-1/2 \leq x \leq 1/2$, we have

$$u(x) \geq \frac{\|u\|_\infty}{2^k}. \tag{5.2}$$

Indeed, since $u \in E$, u is even and attains its maximum at $x_0 = 0$. Applying Lemma 5.2 with $x_0 = 0$, we obtain

$$p(x) = \|u\|_\infty (1 - |x|)^k, \qquad x \in [-1,1],$$

and consequently $u(x) \geq p(x)$ for all $x \in [-1,1]$. In particular, for $-1/2 \leq x \leq 1/2$,

$$u(x) \geq p(x) \geq p(1/2) = \frac{\|u\|_\infty}{2^k}.$$

Thus, inequality (5.2) holds in the case where $(-1)^k u^{(2k)}(x) > 0$, $-1 < x < 1$.

Now consider the case where $(-1)^k u^{(2k)}(x) \geq 0$, $-1 < x < 1$. For $\epsilon > 0$, define $u(\epsilon, x) = u(x) + \epsilon(1-x^2)^k$. Then $u(\epsilon, \cdot)$ satisfies the boundary conditions in (5.3) and $(-1)^k \frac{\partial^{2k}}{\partial x^{2k}} u(\epsilon, x) = (-1)^k u^{(2k)}(x) + \epsilon(2k)! > 0$. Hence, by the previous case, (5.2) holds for $u(\epsilon, \cdot)$, and by continuity with respect to ϵ, (5.2) also holds when $\epsilon = 0$.

Lemma 5.4. *Suppose $f(\cdot, u)$, $g(\cdot, u)$ are even on $[-1,1]$ for every $u \geq 0$. Then $\mathcal{T}_{\lambda,\mu} : K \to K$.*

Proof. Let $H(x,u) = \lambda f(x,u) + \mu g(x,u)$. To see that $\mathcal{T}_{\lambda,\mu} u(-x) = \mathcal{T}_{\lambda,\mu} u(x)$, we have

$$\begin{aligned}
\mathcal{T}_{\lambda,\mu} u(-x) &= \int_{-1}^{1} G(-x,s) H(s,u(s)) ds \\
&= -\int_{1}^{-1} G(-x,-\sigma) H(-\sigma, u(-\sigma)) d\sigma \\
&= \int_{-1}^{1} G(-x,-\sigma) H(-\sigma, u(-\sigma)) d\sigma \\
&= \int_{-1}^{1} G(x,\sigma) H(\sigma, u(\sigma)) d\sigma = \mathcal{T}_{\lambda,\mu} u(x).
\end{aligned}$$

Clearly, $\mathcal{T}_{\lambda,\mu} u(x) \geq 0$ on $[-1,1]$ since $G(x,y) \geq 0$ on $[-1,1] \times [-1,1]$ and $f, g : [-1,1] \times [0,+\infty) \to [0,+\infty)$.

To see that $\mathcal{T}_{\lambda,\mu} u$ is nonincreasing on $[0,1]$, assume for the moment that $H(x,u(x)) > 0$. For $i = 0, \ldots, k-1$, $(\mathcal{T}_{\lambda,\mu} u)^{(i)}$ has roots at $x = -1$ and at $x = 1$. Since $\mathcal{T}_{\lambda,\mu} u(-x) = \mathcal{T}_{\lambda,\mu} u(x)$, $(\mathcal{T}_{\lambda,\mu} u)'(0) = 0$. Moreover,

$$(-1)^k (\mathcal{T}_{\lambda,\mu} u)^{(2k)}(x) = H(x,u(x)) > 0, \quad x \in [-1,1],$$

and so $(\mathcal{T}_{\lambda,\mu} u)^{(2k)}$ has no roots in $[-1,1]$. One can then use repeated applications of Rolle's theorem and show that $(\mathcal{T}_{\lambda,\mu} u)'$ only has roots at $t = -1, 0$, or 1. In particular, $(\mathcal{T}_{\lambda,\mu} u)'$ is strictly negative on $(0,1)$.

We consider $u_\epsilon = \int_{-1}^{1} G(x,s)(H(s,u(s)) + \epsilon) ds$. Then u_ϵ' is strictly negative on $(0,1)$ and $u_\epsilon' \to (\mathcal{T}_{\lambda,\mu} u)'$ as $\epsilon \to 0^+$. In conclusion, $\mathcal{T}_{\lambda,\mu} u$ is nonincreasing on $[0,1]$. So by Lemma 5.2 and Remark 5.3, $\mathcal{T}_{\lambda,\mu} u(x) \geq \frac{\|\mathcal{T}_{\lambda,\mu} u\|_\infty}{2^k}$ and $\mathcal{T}_{\lambda,\mu} u$ satisfies the concavity condition. This completes the proof. □

Lemma 5.5. *Suppose conditions* (H1)–(H3) *hold. Then there exist positive constants λ_0, ρ_0, r_0, and R_0 satisfying $r_0 < \rho_0 < R_0$ such that for all $\lambda > \lambda_0$, the operator $\mathcal{T}_{\lambda,0}$ has at least two nonzero fixed points.*

Proof. Choose $\rho_0 > 0$. We define $m_f = \min_{\substack{-1/2 \leq x \leq 1/2 \\ \gamma\rho_0 \leq s \leq \rho_0}} f(x,s)$. We claim that there exists $\lambda_0 = \frac{\rho_0}{m_f \min\limits_{t \in [-1/2, 1/2]} \int_{-1/2}^{1/2} G(t,s) ds} > 0$ such that

$$\|\mathcal{T}_{\lambda,0} u\|_\infty > \|u\|_\infty \quad \text{for all } u \in \partial K_{\rho_0} \text{ and } \lambda > \lambda_0. \tag{5.3}$$

Indeed, for $u \in \partial K_{\rho_0}$, we have

$$\begin{aligned}
\min_{t \in [-\frac{1}{2}, \frac{1}{2}]} (\mathcal{T}_{\lambda,0} u)(t) &= \lambda \min_{t \in [-\frac{1}{2}, \frac{1}{2}]} \int_{-1}^{1} G(t,s) f(s, u(s)) ds \\
&\geq \lambda \min_{t \in [-\frac{1}{2}, \frac{1}{2}]} \int_{-\frac{1}{2}}^{\frac{1}{2}} G(t,s) m_f ds \\
&> \rho_0.
\end{aligned} \tag{5.4}$$

Let $K_0 = 1 + \sup_{t\in[-1,1]} \int_{-1}^{1} G(t,s)\mathrm{d}s$. By condition (H2), there exists $b > 0$ such that

$$f(x,u) \le \frac{u}{2\lambda K_0}, \quad \forall u \ge b,\ x \in [-1,1].$$

Let $\beta = \sup_{\substack{x\in[-1,1]\\ u\in[0,b]}} f(x,u)$. Then for all (x,u), we have

$$f(x,u) \le \frac{u}{2\lambda K_0} + \beta. \tag{5.5}$$

Condition (H1) implies that for $\lambda > \lambda_0$, there exists $r_0 \in (0,\rho_0)$ such that

$$0 \le f(x,u) \le \frac{u}{\lambda K_0}, \quad \forall 0 < u \le r_0,\ x \in [-1,1]. \tag{5.6}$$

From (5.5), choose $R_0 > \rho_0$ satisfying

$$\frac{R_0}{2} + \lambda K_0 \beta < R_0. \tag{5.7}$$

For $u \in \partial K_{r_0}$, (5.6) guarantees

$$\begin{aligned}
\|\mathcal{T}_{\lambda,0}u\|_\infty &= \lambda \sup_{t\in[-1,1]} \int_{-1}^{1} G(t,s)f(s,u(s))ds \\
&\le \lambda \sup_{t\in[-1,1]} \int_{-1}^{1} G(t,s)\frac{1}{\lambda K_0}u(s)ds \\
&\le \lambda \sup_{t\in[-1,1]} \int_{-1}^{1} G(t,s)ds\frac{1}{\lambda K_0}\|u\|_\infty \\
&< \|u\|_\infty.
\end{aligned}$$

Thus,

$$i(\mathcal{T}_{\lambda,0}, K_{r_0}, K) = 1. \tag{5.8}$$

For $u \in \partial K_{R_0}$, from (5.5) and (5.7), we have

$$\begin{aligned}
\|\mathcal{T}_{\lambda,0}u\|_\infty &= \lambda \sup_{t\in[-1,1]} \int_{-1}^{1} G(t,s)f(s,u(s))ds \\
&\le \lambda \sup_{t\in[-1,1]} \int_{-1}^{1} G(t,s)\left(\frac{u(s)}{2\lambda K_0} + \beta\right)ds \\
&\le \lambda \sup_{t\in[-1,1]} \int_{-1}^{1} G(t,s)ds\left(\frac{\|u\|_\infty}{2\lambda K_0} + \beta\right) \\
&< \|u\|_\infty.
\end{aligned} \tag{5.9}$$

Thus,

$$i(\mathcal{T}_{\lambda,0}, K_{R_0}, K) = 1. \tag{5.10}$$

Finally, combining (5.3), (5.8) and (5.10), we have

$$i(\mathcal{T}_{\lambda,0}, K_{\rho_0} \setminus \overline{K_{r_0}}, K) = -1, \qquad i(\mathcal{T}_{\lambda,0}, K_{R_0} \setminus \overline{K_{\rho_0}}, K) = 1.$$

Therefore, $\mathcal{T}_{\lambda,0}$ has at least two fixed points $u_1 \in K_{\rho_0} \setminus \overline{K_{r_0}}$, $u_2 \in K_{R_0} \setminus \overline{K_{\rho_0}}$. Obviously, $u_1 \ne 0$ and $u_2 \ne 0$. This proof of Lemma 5.5 is now complete. □

Proof of Theorem 5.1(i). It is an immediate consequence of Lemma 5.5. □

Proof of Theorem 5.1(ii). Fixed $\lambda \in (\lambda_0, \infty)$. Let

$$\mathcal{W}_1 := \left\{(\lambda, \mu, u) \in \mathbb{R}^+ \times \mathbb{R} \times K : \ (\lambda, 0, u) \text{ solves } (Q_{\lambda,\mu}) \text{ and } u \in K_{\rho_0} \setminus \overline{K_{r_0}}\right\};$$

$$\mathcal{W}_2 := \left\{(\lambda, \mu, u) \in \mathbb{R}^+ \times \mathbb{R} \times K : \ (\lambda, 0, u) \text{ solves } (Q_{\lambda,\mu}) \text{ and } u \in K_{R_0} \setminus \overline{K_{\rho_0}}\right\}.$$

Using Lemma 5.5 and Lemma 2.4, and the standard continuation method, see for example, Ambrosetti, Arcoya and Buffoni [2, p. 663], Mawhin and Schmitt [26, p. 142–143], there exist two components $\mathcal{L}$ and $\mathcal{U}$ of positive solutions of $(Q_{\lambda,\mu})$, and four functions $Q_+ : (0, \infty) \to (0, \infty]$, $Q_- : (0, \infty) \to [-\infty, 0)$, $M_+ : (0, \infty) \to (0, \infty]$, $M_- : (0, \infty) \to [-\infty, 0)$ satisfying

(i) $\mathcal{W}_1 \subset \mathcal{L}$, $\mathcal{W}_2 \subset \mathcal{U}$;

(ii) $\mathrm{Proj}_{\mathbb{R}}\mathcal{L} \supseteq (M_-(\lambda), M_+(\lambda))$; $\mathrm{Proj}_{\mathbb{R}}\mathcal{U} \supseteq (Q_-(\lambda), Q_+(\lambda))$.

Therefore, $(Q_{\lambda,\mu})$ has at least two positive solutions for

$$\min\{M_-(\lambda), Q_-(\lambda)\} < \mu < \min\{M_+(\lambda), Q_+(\lambda)\}.$$

For fixed $\lambda > \lambda_0$, it is easy to check that $\mu = 0$ is a bifurcation point from infinity. Using the same method to prove Theorem 1.1(ii), with obvious changes, we may get the third positive solution if $\mu \in (0, M(\lambda))$ for some constant $M(\lambda) > 0$. This completes the proof of Theorem 5.1. □

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