Electronic Journal of Qualitative Theory of Differential Equations
2026, No. **13**, 1–22; https://doi.org/10.14232/ejqtde.2026.1.13 www.math.u-szeged.hu/ejqtde/

Nonexistence for the nonlinear nonlocal fractional elliptic system in the whole space

Xiaoshan Wang ✉

Department of Mathematics, Luoyang Normal University, Luoyang, 471934 China;

Received 14 October 2025, appeared 7 May 2026
Communicated by Roberto Livrea

Abstract. This paper is devoted to the study the following nonlinear fractional elliptic system:

$$\begin{cases}(-\Delta)^{\frac{\alpha}{2}}u(x)=h_1(x,u(x),v(x)),\\(-\Delta)^{\frac{\beta}{2}}v(x)=h_2(x,u(x),v(x)),\end{cases}$$

where $0<\alpha,\beta<2$, $n\geq 2$ and h_1,h_2 satisfy suitable assumptions. We employ the method of scaling spheres to establish the nonexistence of nonnegative solutions for this fractional elliptic system in the entire space $\mathbb{R}^n$. Furthermore, we show that analogous results can be extended for higher order fractional problems.

Keywords: fractional elliptic systems, the method of scaling spheres, nonexistence.

2020 Mathematics Subject Classification: 35R11, 35J30, 35J60,35J47.

1 Introduction

In this paper, we are concerned with the following nonlinear fractional elliptic system:

$$\begin{cases}(-\Delta)^{\frac{\alpha}{2}}u(x)=h_1(x,u(x),v(x)), & x\in\mathbb{R}^n,\\(-\Delta)^{\frac{\beta}{2}}v(x)=h_2(x,u(x),v(x)), & x\in\mathbb{R}^n,\\u(x)\geq 0, v(x)\geq 0, & x\in\mathbb{R}^n,\end{cases}\tag{1.1}$$

where $0<\alpha,\beta<2$, $n\geq 2$, h_1,h_2 satisfy some assumptions which will be given latter.

The fractional Laplacian $(-\Delta)^{\frac{\alpha}{2}}(0<\alpha<2)$ is defined as (see [7, 9, 19, 24] and the reference therein)

$$\begin{aligned}(-\Delta)^{\frac{\alpha}{2}}u(x)&=C_{n,s}\lim_{\epsilon\to 0}\int_{\mathbb{R}^n\setminus B_\epsilon(x)}\frac{u(x)-u(z)}{|x-z|^{n+\alpha}}dz\\&=C_{n,s}P.V.\int_{\mathbb{R}^n}\frac{u(x)-u(z)}{|x-z|^{n+\alpha}}dz,\end{aligned}\tag{1.2}$$

✉ Email: xswang2017@126.com

where $P.V.$ stands for the Cauchy principal value.

In this paper, to better define the right-hand side of the expression (1.2), we require that

$$u \in L_\alpha(\mathbb{R}^n) \cap C^{1,1}_{\text{loc}}(\mathbb{R}^n) \quad \text{and} \quad v \in L_\beta(\mathbb{R}^n) \cap C^{1,1}_{\text{loc}}(\mathbb{R}^n),$$

where $L_\alpha(\mathbb{R}^n)$ and $L_\beta(\mathbb{R}^n)$ are defined as follows:

$$L_\alpha(\mathbb{R}^n) := \left\{ u : \mathbb{R}^n \to \mathbb{R} \;\middle|\; \int_{\mathbb{R}^n} \frac{|u(x)|}{(1+|x|)^{n+\alpha}} dx < \infty \right\},$$

$$L_\beta(\mathbb{R}^n) := \left\{ v : \mathbb{R}^n \to \mathbb{R} \;\middle|\; \int_{\mathbb{R}^n} \frac{|v(x)|}{(1+|x|)^{n+\beta}} dx < \infty \right\}.$$

For $\alpha = \beta = 2$, when considering the whole space $\mathbb{R}^n$, we assume that the solutions satisfy $u \in C^2(\mathbb{R}^n \setminus \{0\}) \cap C(\mathbb{R}^n)$. It is known that equations (1.1) have critical order if $\alpha = \beta = n$ (see [3,5] and [23]), and non-critical order if $0 < \alpha, \beta < n$.

We impose the following assumptions on $h_1(x, u, v)$ and $h_2(x, u, v)$:

(I) The nonlinear terms $h_1(x, u, v)$ and $h_2(x, u, v)$ are non-decreasing in u and v, respectively, for all $(x, u, v) \in \mathbb{R}^n \times \overline{\mathbb{R}_+} \times \overline{\mathbb{R}_+}$. Specifically, for any $(x, u_1, v), (x, u_2, v) \in \mathbb{R}^n \times \overline{\mathbb{R}_+} \times \overline{\mathbb{R}_+}$ with $u_1 \leq u_2$, and any $(x, u, v_1), (x, u, v_2) \in \mathbb{R}^n \times \overline{\mathbb{R}_+} \times \overline{\mathbb{R}_+}$ with $v_1 \leq v_2$, the following inequalities hold:

$$\begin{aligned} h_1(x, u_1, v) \leq h_1(x, u_2, v) \quad &\text{and} \quad h_2(x, u_1, v) \leq h_2(x, u_2, v), \\ h_1(x, u, v_1) \leq h_1(x, u, v_2) \quad &\text{and} \quad h_2(x, u, v_1) \leq h_2(x, u, v_2). \end{aligned} \tag{1.3}$$

(II) For any $t < 1$, the nonlinear terms $h_1(x, u, tv_\lambda)$ and $h_2(x, tu_\lambda, v)$ are non-increasing in t, where $u_\lambda(x) = \left(\frac{\lambda}{|x|}\right)^{n-\alpha} u(x^\lambda)$, $v_\lambda(x) = \left(\frac{\lambda}{|x|}\right)^{n-\beta} v(x^\lambda)$, and $x^\lambda = \left(\frac{\lambda}{|x|}\right)^2 x$.

(III) There exist a cone $\mathcal{C} \subseteq \mathbb{R}^n$ with vertex at the origin, positive constants $C_1, C_2 > 0$, real constants $\tau_1 \in (-\alpha, +\infty)$ and $\tau_2 \in (-\beta, +\infty)$, and constants $p, q > 0$ satisfying $\min\left\{\frac{n+\alpha+2\tau_1}{n-\alpha}, \frac{n+\beta+2\tau_2}{n-\beta}\right\} > p + q$ (for $0 < \alpha, \beta < n$), such that for all $(x, u, v) \in \mathcal{C} \times \overline{\mathbb{R}_+} \times \overline{\mathbb{R}_+}$, the nonlinear terms satisfy

$$\begin{aligned} h_1(x, u, v) &\geq C_1 |x|^{\tau_1} u^p v^q, \\ h_2(x, u, v) &\geq C_2 |x|^{\tau_2} u^p v^q. \end{aligned} \tag{1.4}$$

For $0 < \alpha < n$, we call the fractional order or higher order equation

$$(-\Delta)^{\frac{\alpha}{2}} u(x) = |x|^a u^p(x) \tag{1.5}$$

the Hénon, Lane–Emden, and Hardy equations for $a > 0$, $a = 0$, and $a < 0$, respectively. These equations have been applied in many fields, such as conformal geometry and Sobolev inequalities. When $a = 0$, the well-known Lane–Emden equation models various phenomena in mathematical physics and astrophysics. Liouville-type theorems for equation (1.5) in $\mathbb{R}^n$ and on the half-space $\mathbb{R}^n_+$ have been extensively studied (see [1,5,6,12,13,15,16,20–23,28,31,35–38], and the references therein). When

$0 < \alpha < 2$, Dou and Zhou in [26] proved Liouville-type theorems via the method of moving planes in integral forms for

$$\begin{cases} (-\Delta)^{\frac{\alpha}{2}} u(x) = |x|^a u^p(x), & x \in \mathbb{R}^n, \\ u \geq 0, & x \in \mathbb{R}^n, \end{cases} \tag{1.6}$$

and

$$\begin{cases} (-\Delta)^{\frac{\alpha}{2}} u(x) = |x|^a v^p(x), & x \in \mathbb{R}^n, \\ (-\Delta)^{\frac{\alpha}{2}} v(x) = |x|^b u^q(x), & x \in \mathbb{R}^n, \\ u, v \geq 0, & x \in \mathbb{R}^n, \end{cases} \tag{1.7}$$

where $0 < \alpha < 2$, $a, b > 0$ and $1 < p, q < +\infty$. When proving Liouville-type theorems for equation (1.6), they needed to consider the integrability of the solution u and the range $\frac{n+\alpha}{n-\alpha} < p < \frac{n+\alpha+a}{n-\alpha}$; when proving Liouville-type theorems for the system (1.7), they needed to consider the integrability of the solutions u and v, as well as the range $\frac{n+\alpha}{n-\alpha} < p \leq \frac{n+\alpha+a}{n-\alpha}$, $\frac{n+\alpha}{n-\alpha} < q < \frac{n+\alpha+b}{n-\alpha}$ with $p + q < \frac{2(n+\alpha)+a+b}{n-\alpha}$.

In [24], the authors developed the method of scaling spheres. Since then, more and more results on fractional order, higher order, or high-order fractional Hénon, Lane–Emden, and Hardy equations have been studied by researchers (see [3,4,14,17–20,23,25,27,30,35], and the references therein). The method of scaling spheres is a frozen variant of the method of moving spheres, which can be applied to various problems with singularities or without translation invariance in general domains where the method of moving spheres fails. The method of moving spheres also has yielded a series of rich results (see [8,11,29,32,33]).

Compared with [26], in [24], the authors extended the range of $a > 0$ to $0 < a < +\infty$ and the range $\frac{n+\alpha}{n-\alpha} < p < \frac{n+\alpha+a}{n-\alpha}$ to $0 < p < p_c(a) := \frac{n+\alpha+2a}{n-\alpha}$.

In [4], the authors considered higher-order fractional Hénon–Lane–Emden systems. Instead of deriving decay estimates for the spherical average of solutions via the ODE method, they obtained decay estimates for the nonlocal average of solutions using integral representation formulas, and then derived the corresponding Liouville-type theorem.

In [20], the authors studied Liouville-type theorems for the following elliptic equations with Dirichlet conditions in exterior domains via the method of scaling spheres:

$$\begin{cases} (-\Delta)^{\frac{\alpha}{2}} u(x) = f(x, u), & x \in \Omega_r, \\ u = 0, & x \in \mathbb{R}^n \setminus \Omega_r, \end{cases} \tag{1.8}$$

where $\Omega_r = \{x \in \mathbb{R}^n \mid |x| > r\}$ for arbitrary $r > 0$, $0 < \alpha \leq 2$, $n \geq 2$, and $f(x, u)$ satisfies some suitable assumptions. Meanwhile, the authors pointed out that the above Liouville-type theorem also holds when $\alpha = 2m$ with $1 \leq m \leq \frac{n}{2}$.

Motivated by the works in [4,20,24], we mainly use the method of scaling spheres to prove the nonexistence of nontrivial nonnegative solutions for equation (1.1) in the whole space.

Lemma 1.1 ([24]). *Assume $n > \alpha$ and $0 < \alpha \leq 2$. Suppose u is α-harmonic in $B_R(0) \setminus \{0\}$ and satisfies*

$$u(x) = o(|x|^{\alpha - n}), \quad as\ |x| \to 0.$$

Then u can be defined at 0 so that it is α-harmonic in $B_R(0)$.

Lemma 1.2 (Maximum principle [24]). *Let Ω be a bounded domain in $\mathbb{R}^n$ and $0<\alpha<2$. Assume that $u\in\mathcal{L}_\alpha\cap C^{1,1}_{loc}(\Omega)$ and is l.s.c. on $\overline{\Omega}$. If $(-\Delta)^{\frac{\alpha}{2}}u\ge 0$ in Ω and $u\ge 0$ in $\mathbb{R}^n\setminus\Omega$, then $u\ge 0$ in $\mathbb{R}^n$. Moreover, if $u=0$ at some point in Ω, then $u\equiv 0$ a.e. in $\mathbb{R}^n$. These conclusions also hold for unbounded domain Ω if we assume further that*

$$\liminf_{|x|\to\infty} u(x)\ge 0.$$

For any $R>0$, let

$$u_R^\alpha(x):=\int_{B_R} G_R^\alpha(x,y)h_1(y,u(y),v(y))dy,$$

where $G_R^\alpha(x,y)$ is the Green function for Laplacian $(-\Delta)^{\frac{\alpha}{2}}$ with $0<\alpha\le 2$ in $B_R(0)$, i.e.

$$G_R^\alpha(x,y)=\begin{cases}\dfrac{C_{n,\alpha}}{|x-y|^{n-\alpha}}\displaystyle\int_0^{\frac{(R^2-|x|^2)(R^2-|y|^2)}{R^2|x-y|^2}}\dfrac{b^{\frac{\alpha}{2}-1}}{(1+b)^{\frac{n}{2}}}db, & x,y\in B_R,\\ 0, & x \text{ or } y\in B_R^c.\end{cases}$$

We have $u_R^\alpha(x)\in C^{1,1}_{loc}(B_R(0)\setminus\{0\})\cap C(\mathbb{R}^n)\cap L_\alpha(\mathbb{R}^n)$ ($u_R^\alpha(x)\in C^2(B_R(0)\setminus\{0\})\cap C(\mathbb{R}^n)$ if $\alpha=2$), and

$$\begin{cases}(-\Delta)^{\frac{\alpha}{2}}u_R^\alpha(x)=h_1(x,u,v), & x\in B_R,\\ u_R^\alpha(x)=0, & x\in B_R^c.\end{cases}$$

As well, we can construct a function $v_R^\beta(x)$ that satisfies

$$\begin{cases}(-\Delta)^{\frac{\alpha}{2}}v_R^\beta(x)=h_2(x,u,v), & x\in B_R,\\ v_R^\beta(x)=0, & x\in B_R^c.\end{cases}$$

Let $w_R(x)=u(x)-u_R^\alpha(x)\in C^{1,1}_{loc}(B_R(0)\setminus\{0\})\cap C(\mathbb{R}^n)\cap L_\alpha(\mathbb{R}^n)(w_R(x)\in C^2(B_R(0)\setminus\{0\})\cap C(\mathbb{R}^n)$ if $\alpha=2$), $z_R(x)=v(x)-v_R^\beta(x)\in C^{1,1}_{loc}(B_R(0)\setminus\{0\})\cap C(\mathbb{R}^n)\cap L_\beta(\mathbb{R}^n)(z_R(x)\in C^2(B_R(0)\setminus\{0\})\cap C(\mathbb{R}^n)$ if $\beta=2$). By Lemma 1.1, (1.1) and above equation, we have $w_R\in C^{1,1}_{loc}(B_R(0))\cap C(\mathbb{R}^n)\cap L_\alpha(\mathbb{R}^n)$ ($w_R(x)\in C^2(B_R(0))\cap C(\mathbb{R}^n)$ if $\alpha=2$), $z_R\in C^{1,1}_{loc}(B_R(0))\cap C(\mathbb{R}^n)\cap L_\beta(\mathbb{R}^n)$ ($z_R(x)\in C^2(B_R(0))\cap C(\mathbb{R}^n)$ if $\beta=2$), and satisfy

$$\begin{cases}(-\Delta)^{\frac{\alpha}{2}}w_R(x)=0, & x\in B_R(0),\\ w_R(x)\ge 0, & x\in\mathbb{R}^n\setminus B_R(0).\\ (-\Delta)^{\frac{\beta}{2}}z_R(x)=0, & x\in B_R(0),\\ w_R(x)\ge 0, & x\in\mathbb{R}^n\setminus B_R(0).\end{cases}\tag{1.9}$$

By Lemma 1.2, maximal principles and Liouville type theorem for α-harmonic functions in $\mathbb{R}^n$, let $R\to\infty$. It follows easily that

$$\begin{cases}u(x)=C_{n,\alpha}\int_{\mathbb{R}^n}\frac{h_1(y,u(y),v(y))}{|x-y|^{n-\alpha}}dy,\\ v(x)=C_{n,\beta}\int_{\mathbb{R}^n}\frac{h_2(y,u(y),v(y))}{|x-y|^{n-\beta}}dy.\end{cases}\tag{1.10}$$

that is, (u, v) satisfies the integral equations.

Conversely, assume that (u, v) is a pair nonnegative classical solutions of integral equations (1.10), it follows directly from the properties of the Green function that (u, v) is also solutions to the differential equations (1.1), and the detailed proof can be found in the reference [7,16,24].

This leads to the following theorems.

Theorem 1.3. *If u, v are the solutions of equation* (1.1)*, then u, v are also the solutions of the integral equations* (1.10)*, and the converse is also true.*

Definition 1.4. If Ω is star-shaped with respect to the origin 0, then we say that h_1, h_2 have subcritical (supercritical) growth provided that the functions

$$\mu^{-\frac{n+\alpha}{n-\alpha}} h_1\left(\mu^{-\frac{2}{n-\alpha}} x, \mu u, v\right), \qquad \rho^{-\frac{n+\beta}{n-\beta}} h_2\left(\rho^{-\frac{2}{n-\beta}} x, u, \rho v\right)$$

of μ, ρ are strictly decreasing (strictly increasing) in $\mu \geq 1, \rho \geq 1$ or $\mu \leq 1, \rho \leq 1$ for all $(x, u, v) \in \Omega \times \mathbb{R}_+ \times \mathbb{R}_+$, respectively. In the critical order case $\alpha = \beta = n$, the nonlinear terms h_1, h_2 are said to have subcritical (supercritical) growth provided that

$$\mu^{-n} h_1(\mu^{-1} x, u, v), \qquad \rho^{-n} h_2(\rho^{-1} x, u, v)$$

are strictly decreasing (strictly increasing) in $\mu \geq 1, \rho \geq 1$ or $\mu \leq 1, \rho \leq 1$ for all $(x, u, v) \in \Omega \times \mathbb{R}_+ \times \mathbb{R}_+$, respectively.

Definition 1.5. If the set $\Omega' = \mathbb{R}^n \setminus \Omega$ is empty or star-shaped with respect to the origin $0 \in \Omega'$, then we say Ω is star-shaped with respect to infinity. Similarly as above, we say that h_1, h_2 have subcritical (supercritical) growth provided that

$$\mu^{\frac{n+\alpha}{n-\alpha}} h_1\left(\mu^{\frac{2}{n-\alpha}} x, \mu^{-1} u, v\right), \qquad \rho^{\frac{n+\beta}{n-\beta}} h_2\left(\rho^{\frac{2}{n-\beta}} x, u, \rho^{-1} v\right)$$

are strictly increasing (strictly decreasing) in $\mu \geq 1$, $\rho \geq 1$ or $\mu \leq 1$, $\rho \leq 1$ for all $(x, u, v) \in \Omega \times \mathbb{R}_+ \times \mathbb{R}_+$, respectively. In the critical order case $\alpha = \beta = n$, the nonlinear terms h_1, h_2 are said to have subcritical (supercritical) growth provided that

$$\mu^n h_1(\mu x, u, v), \qquad \rho^n h_2(\rho x, u, v)$$

are strictly increasing (strictly decreasing) in $\mu \geq 1$, $\rho \geq 1$ or $\mu \leq 1$, $\rho \leq 1$ for all $(x, u, v) \in \Omega \times \mathbb{R}_+ \times \mathbb{R}_+$, respectively.

Definition 1.6. A function $h(x, u, v)$ is called locally Lipschitz in u on $\Omega \times \overline{\mathbb{R}_+} \times \mathbb{R}_+$ provided that for any $u_0 \in \overline{\mathbb{R}_+}$ and any bounded $\omega \subset \Omega$, there exists a relatively open neighborhood $U(u_0) \subset \overline{\mathbb{R}_+}$ such that $h(x, u, v)$ is Lipschitz continuous in u on $\omega \times U(u_0) \times \overline{\mathbb{R}_+}$. The function $h(x, u, v)$ is called locally Lipschitz in v on $\Omega \times \overline{\mathbb{R}_+} \times \overline{\mathbb{R}_+}$ provided that for any $v_0 \in \overline{\mathbb{R}_+}$ and any bounded $\omega \subset \Omega$, there exists a relatively open neighborhood $U(v_0) \subset \overline{\mathbb{R}_+}$ such that $h(x, u, v)$ is Lipschitz continuous in v on $\omega \times \overline{\mathbb{R}_+} \times U(v_0)$.

Now, we give our first nonexistence for the integral equations (1.10) as follows:

Theorem 1.7. *Assume $n \geq 2$, $0 < \alpha, \beta < 2$, h_1, h_2 have subcritical growth, are locally Lipschitz continuous in u and v, respectively, and satisfy assumptions $(I), (II), (III)$. If $u, v \in C(\mathbb{R}^n)$ are solutions of the integral equations (1.10), then $u = v = 0$ in $\mathbb{R}^n$.*

Combining the results of Theorem 1.3 with Theorem 1.7, we can easily derive the following nonexistence for the PDEs system (1.1).

Theorem 1.8. *Assume $n \geq 2$, $0 < \alpha, \beta < 2$, h_1, h_2 have subcritical growth, are locally Lipschitz continuous in u and v, respectively, and satisfy assumptions $(I), (II), (III)$. If u, v are solutions of the PDEs system (1.1), then $u = v = 0$ in $\mathbb{R}^n$.*

Remark 1.9. If $\alpha = 2m$, $\beta = 2l$ with $1 \leq m, l < \frac{n}{2}$, the above nonexistence still holds. The proof is similar to our Theorem 1.7 and Theorem 1.8, and we omit the detailed proof here.

We next consider the following simpler fractional PDEs system:

$$\begin{cases} (-\Delta)^{\frac{\alpha}{2}} u(x) = h_1(x, v), & x \in \mathbb{R}^n, \\ (-\Delta)^{\frac{\beta}{2}} v(x) = h_2(x, u), & x \in \mathbb{R}^n. \end{cases} \tag{1.11}$$

Assume $h_1(x, v), h_2(x, u)$ have subcritical growth and satisfy the following conditions:

(I') The nonlinear terms $h_1(x, v), h_2(x, u)$ are non-decreasing in v and u, respectively, on $\mathbb{R}^n \times \mathbb{R}_+$. That is, for all $(x, v_1), (x, v_2) \in \mathbb{R}^n \times \overline{\mathbb{R}_+}$ with $v_1 \leq v_2$ and all $(x, u_1), (x, u_2) \in \mathbb{R}^n \times \overline{\mathbb{R}_+}$ with $u_1 \leq u_2$, we have

$$h_1(x, v_1) \leq h_1(x, v_2) \quad \text{and} \quad h_2(x, u_1) \leq h_2(x, u_2). \tag{1.12}$$

(II') There exist $\sigma_1 < \alpha$, $\sigma_2 < \beta$ such that $|x|^{\sigma_1} h_1(x, v)$ and $|x|^{\sigma_2} h_2(x, u)$ are locally Lipschitz in v and u, respectively, on $\mathbb{R}^n \times \mathbb{R}_+$.

(III') There exist a cone $\mathcal{C}$ with vertex at 0, positive constants $C_1, C_2 > 0$, real numbers $-\alpha < \tau_1 < +\infty$, $-\beta < \tau_2 < +\infty$, and positive real numbers $p, q > 0$ with

$$0 < p < \frac{n + \beta + 2\tau_2}{n - \alpha}, \qquad 0 < q < \frac{n + \alpha + 2\tau_1}{n - \beta}$$

such that the nonlinear terms satisfy

$$h_1(x, v) \geq C_1 |x|^{\tau_1} v^q, \qquad h_2(x, u) \geq C_2 |x|^{\tau_2} u^p \tag{1.13}$$

on $\mathcal{C} \times \overline{\mathbb{R}_+}$.

The equivalence between the PDEs system (1.11) and the integral equations system

$$\begin{cases} u(x) = \displaystyle\int_{\mathbb{R}^n} \frac{h_1(y, v(y))}{|x - y|^{n - \alpha}} dy, \\ v(x) = \displaystyle\int_{\mathbb{R}^n} \frac{h_2(y, u(y))}{|x - y|^{n - \beta}} dy \end{cases} \tag{1.14}$$

can be easily established. From Theorem 1.7 and Theorem 1.8, we can also derive that the Liouville theorem for nonnegative solutions to the system (1.11) holds under assumptions (I'), (II'), (III'). The proof is similar to that of Theorem 1.8, and we leave the detailed proof to interested readers.

We use $C, C_1, C_2, c_1, c_2, \tilde{c}_1, \tilde{c}_2$ to denote generic positive constants, whose values may differ from line to line.

2 Proof of Theorem 1.7

In this section, we prove Theorem 1.7 via the method of scaling spheres. We first establish the following lower bound estimates for the asymptotic behavior of positive solutions u, v as $|x| \to +\infty$, which will yield a contradiction with the integral equations (1.10). In fact, there exist constants $C_1, C_2 > 0$ such that the solutions u, v satisfy

$$u(x) \geq \frac{C_1}{|x|^{n-\alpha}}, \quad v(x) \geq \frac{C_2}{|x|^{n-\beta}} \quad \text{for } |x| \geq 1. \tag{2.1}$$

Since the positive solutions $u, v > 0$ satisfy the integral equation (1.10), we derive that

$$\begin{aligned} u(x) &\geq C_{n,\alpha} \int_{|y|\leq \frac{1}{2}} \frac{h_1(y, u(y), v(y))}{|x-y|^{n-\alpha}} dy \\ &\geq \frac{C_{n,\alpha}}{|x|^{n-\alpha}} \int_{|y|\leq \frac{1}{2}} h_1(y, u(y), v(y)) dy \\ &\geq \frac{C_1}{|x|^{n-\alpha}}, \end{aligned} \tag{2.2}$$

and

$$\begin{aligned} v(x) &\geq C_{n,\beta} \int_{|y|\leq \frac{1}{2}} \frac{h_2(y, u(y), v(y))}{|x-y|^{n-\beta}} dy \\ &\geq \frac{C_{n,\beta}}{|x|^{n-\beta}} \int_{|y|\leq \frac{1}{2}} h_2(y, u(y), v(y)) dy \\ &\geq \frac{C_2}{|x|^{n-\beta}} \end{aligned} \tag{2.3}$$

for all $|x| \geq 1$ and $x \in \mathbb{R}^n$.

We first state the following lower bound estimates for the integral equations (1.10), which is crucial for our proof.

Theorem 2.1. *Assume $n \geq 2$, $0 < \alpha, \beta < 2$, h_1, h_2 have subcritical growth, are locally Lipschitz continuous in u and v respectively, and satisfy assumptions $(I), (II), (III)$. If $u, v \in C(\mathbb{R}^n)$ are solutions of the integral equations (1.10), then the following lower bound estimates hold: for all $|x| \geq 1$,*

1. *if $0 < p + q < 1$,*

$$\begin{aligned} u(x) &\geq C_{\kappa_u} |x|^{\kappa_u}, \quad \forall\, \kappa_u < \frac{(1-q)(\alpha+\tau_1)}{1-(p+q)} + \frac{q(\beta+\tau_2)}{1-(p+q)}, \\ v(x) &\geq C_{\kappa_v} |x|^{\kappa_v}, \quad \forall\, \kappa_v < \frac{p(\alpha+\tau_1)}{1-(p+q)} + \frac{(1-p)(\beta+\tau_2)}{1-(p+q)}; \end{aligned} \tag{2.4}$$

2. *if $p + q \geq 1$,*

$$u(x) \geq C_\kappa |x|^\kappa, \quad v(x) \geq C_\kappa |x|^\kappa, \quad \forall\, \kappa < +\infty. \tag{2.5}$$

Proof of Theorem 2.1. For any $\lambda > 0$, we define the Kelvin transform of u, v centered at 0 ([2, 9–11, 34, 39]) by

$$u_\lambda(x) = \left(\frac{\lambda}{|x|}\right)^{n-\alpha} u(x^\lambda), \quad v_\lambda(x) = \left(\frac{\lambda}{|x|}\right)^{n-\beta} v(x^\lambda)$$

for every $x \in \mathbb{R}^n \setminus \{0\}$. Obviously, this transform may have a singularity at 0, and

$$\lim_{|x|\to\infty} |x|^{n-\alpha} u_\lambda(x) = \lambda^{n-\alpha} u(0) > 0, \quad \lim_{|x|\to\infty} |x|^{n-\beta} v_\lambda(x) = \lambda^{n-\beta} v(0) > 0.$$

From the regularity assumptions on u, v, we infer that $u_\lambda \in L_\alpha(\mathbb{R}^n) \cap C^{1,1}_{\rm loc}(\mathbb{R}^n \setminus \{0\})$ and $v_\lambda \in L_\beta(\mathbb{R}^n) \cap C^{1,1}_{\rm loc}(\mathbb{R}^n \setminus \{0\})$ for $0 < \alpha, \beta < 2$; if $\alpha = \beta = 2$, then $u_\lambda, v_\lambda \in C^2_{\rm loc}(\mathbb{R}^n \setminus \{0\})$.

We define the reflection of x about the sphere $S_\lambda = \{x \in \mathbb{R}^n \mid |x| = \lambda\}$ by $x^\lambda = \left(\frac{\lambda}{|x|}\right)^2 x$, and denote

$$w^u_\lambda(x) = u_\lambda(x) - u(x), \quad w^v_\lambda(x) = v_\lambda(x) - v(x).$$

By direct calculation, we have

$$\begin{aligned}(-\Delta)^{\frac{\alpha}{2}} u_\lambda(x) &= \left(\frac{\lambda}{|x|}\right)^{n+\alpha} (-\Delta)^{\frac{\alpha}{2}} u\left(\left(\frac{\lambda}{|x|}\right)^2 x\right)\\ &= \left(\frac{\lambda}{|x|}\right)^{n+\alpha} h_1\left(\left(\frac{\lambda}{|x|}\right)^2 x, u\left(\left(\frac{\lambda}{|x|}\right)^2 x\right), v\left(\left(\frac{\lambda}{|x|}\right)^2 x\right)\right).\end{aligned} \tag{2.6}$$

Similarly, we derive

$$(-\Delta)^{\frac{\beta}{2}} v_\lambda(x) = \left(\frac{\lambda}{|x|}\right)^{n+\beta} h_2\left(\left(\frac{\lambda}{|x|}\right)^2 x, u\left(\left(\frac{\lambda}{|x|}\right)^2 x\right), v\left(\left(\frac{\lambda}{|x|}\right)^2 x\right)\right). \tag{2.7}$$

We now start the scaling sphere process with respect to the origin $0 \in \mathbb{R}^n$. For all $x \in B_\lambda(0) \setminus \{0\}$, by the definitions of $u_\lambda, v_\lambda, w^u_\lambda, w^v_\lambda$, we compute

$$\begin{aligned}\left(\frac{\lambda}{|x^\lambda|}\right)^{n-\alpha} (w^u_\lambda)_\lambda(x) = w^u_\lambda(x^\lambda) &= u_\lambda(x^\lambda) - u(x^\lambda)\\ &= \left(\frac{\lambda}{|x^\lambda|}\right)^{n-\alpha} u((x^\lambda)^\lambda) - \left(\frac{|x|}{\lambda}\right)^{n-\alpha} u_\lambda(x)\\ &= \left(\frac{|x|}{\lambda}\right)^{n-\alpha} u(x) - \left(\frac{|x|}{\lambda}\right)^{n-\alpha} u_\lambda(x)\\ &= -\left(\frac{|x|}{\lambda}\right)^{n-\alpha} w^u_\lambda(x),\end{aligned} \tag{2.8}$$

which implies

$$w^u_\lambda(x) = -(w^u_\lambda)_\lambda(x). \tag{2.9}$$

By a similar calculation, we have

$$w^v_\lambda(x) = -(w^v_\lambda)_\lambda(x). \tag{2.10}$$

Step 1. We first dilate the sphere from $\lambda > 0$ sufficiently close to 0, and show that

$$w^u_\lambda(x) \geq 0, \quad w^v_\lambda(x) \geq 0. \tag{2.11}$$

Define

$$B^-_{\lambda,u} = \{x \in B_\lambda(0) \setminus \{0\} \mid w^u_\lambda(x) < 0\}, \quad B^-_{\lambda,v} = \{x \in B_\lambda(0) \setminus \{0\} \mid w^v_\lambda(x) < 0\}. \tag{2.12}$$

To prove (2.11), we use a contradiction argument: we show that for sufficiently small $\lambda > 0$,

$$B^-_{\lambda,u} = \emptyset, \quad B^-_{\lambda,v} = \emptyset. \tag{2.13}$$

Since the positive solutions u, v to (1.1) also satisfy the integral equation (1.10), direct calculation gives for any $x \in \mathbb{R}^n$,

$$\begin{aligned} u(x) &= \int_{B_\lambda(0)} \frac{h_1(y,u(y),v(y))}{|x-y|^{n-\alpha}} dy + \int_{\mathbb{R}^n \setminus B_\lambda(0)} \frac{h_1(y,u(y),v(y))}{|x-y|^{n-\alpha}} dy \\ &= \int_{B_\lambda(0)} \frac{h_1(y,u(y),v(y))}{|x-y|^{n-\alpha}} dy + \int_{B_\lambda(0)} \frac{h_1(y^\lambda,u(y^\lambda),v(y^\lambda))}{|x-y^\lambda|^{n-\alpha}} \left(\frac{\lambda}{|y|}\right)^{2n} dy \\ &= \int_{B_\lambda(0)} \frac{h_1(y,u(y),v(y))}{|x-y|^{n-\alpha}} dy + \int_{B_\lambda(0)} \frac{h_1(y^\lambda,u(y^\lambda),v(y^\lambda))}{|\frac{|y|}{\lambda}x - \frac{\lambda}{|y|}y|^{n-\alpha}} \left(\frac{\lambda}{|y|}\right)^{n+\alpha} dy. \end{aligned}$$

Similarly, we derive

$$\begin{aligned} v(x) &= \int_{B_\lambda(0)} \frac{h_2(y,u(y),v(y))}{|x-y|^{n-\beta}} dy + \int_{\mathbb{R}^n \setminus B_\lambda(0)} \frac{h_2(y,u(y),v(y))}{|x-y|^{n-\beta}} dy \\ &= \int_{B_\lambda(0)} \frac{h_2(y,u(y),v(y))}{|x-y|^{n-\beta}} dy + \int_{B_\lambda(0)} \frac{h_2(y^\lambda,u(y^\lambda),v(y^\lambda))}{|\frac{|y|}{\lambda}x - \frac{\lambda}{|y|}y|^{n-\beta}} \left(\frac{\lambda}{|y|}\right)^{n+\beta} dy. \end{aligned} \tag{2.14}$$

By a similar calculation, we also obtain

$$\begin{aligned} u_\lambda(x) &= \int_{\mathbb{R}^n} \frac{\left(\frac{\lambda}{|y|}\right)^{n+\alpha} h_1\left(\left(\frac{\lambda}{|y|}\right)^2 y, u\left(\left(\frac{\lambda}{|y|}\right)^2 y\right), v\left(\left(\frac{\lambda}{|y|}\right)^2 y\right)\right)}{|x-y|^{n-\alpha}} dy, \\ v_\lambda(x) &= \int_{\mathbb{R}^n} \frac{\left(\frac{\lambda}{|y|}\right)^{n+\beta} h_2\left(\left(\frac{\lambda}{|y|}\right)^2 y, u\left(\left(\frac{\lambda}{|y|}\right)^2 y\right), v\left(\left(\frac{\lambda}{|y|}\right)^2 y\right)\right)}{|x-y|^{n-\beta}} dy \end{aligned} \tag{2.15}$$

for any $x \in \mathbb{R}^n \setminus \{0\}$. Moreover,

$$\begin{aligned} u_\lambda(x) = &\int_{B_\lambda(0)} \frac{\left(\frac{\lambda}{|y|}\right)^{n+\alpha} h_1\left(\left(\frac{\lambda}{|y|}\right)^2 y, u\left(\left(\frac{\lambda}{|y|}\right)^2 y\right), v\left(\left(\frac{\lambda}{|y|}\right)^2 y\right)\right)}{|x-y|^{n-\alpha}} dy \\ &+ \int_{B_\lambda(0)} \frac{\left(\frac{\lambda}{|y^\lambda|}\right)^{n+\alpha} h_1\left(\left(\frac{\lambda}{|y^\lambda|}\right)^2 y^\lambda, u\left(\left(\frac{\lambda}{|y^\lambda|}\right)^2 y^\lambda\right), v\left(\left(\frac{\lambda}{|y^\lambda|}\right)^2 y^\lambda\right)\right)}{|x-y^\lambda|^{n-\alpha}} \left(\frac{\lambda}{|y|}\right)^{2n} dy \\ = &\int_{B_\lambda(0)} \frac{\left(\frac{\lambda}{|y|}\right)^{n+\alpha} h_1\left(\left(\frac{\lambda}{|y|}\right)^2 y, u\left(\left(\frac{\lambda}{|y|}\right)^2 y\right), v\left(\left(\frac{\lambda}{|y|}\right)^2 y\right)\right)}{|x-y|^{n-\alpha}} dy \\ &+ \int_{B_\lambda(0)} \frac{h_1(y,u(y),v(y))}{|\frac{|y|}{\lambda}x - \frac{\lambda}{|y|}y|^{n-\alpha}} dy, \end{aligned} \tag{2.16}$$

and

$$
\begin{aligned}
v_\lambda(x) &= \int_{B_\lambda(0)} \frac{\left(\frac{\lambda}{|y|}\right)^{n+\beta} h_2\left(\left(\frac{\lambda}{|y|}\right)^2 y, u\left(\left(\frac{\lambda}{|y|}\right)^2 y\right), v\left(\left(\frac{\lambda}{|y|}\right)^2 y\right)\right)}{|x-y|^{n-\beta}} dy \\
&\quad + \int_{B_\lambda(0)} \frac{\left(\frac{\lambda}{|y^\lambda|}\right)^{n+\beta} h_2\left(\left(\frac{\lambda}{|y^\lambda|}\right)^2 y^\lambda, u\left(\left(\frac{\lambda}{|y^\lambda|}\right)^2 y^\lambda\right), v\left(\left(\frac{\lambda}{|y^\lambda|}\right)^2 y^\lambda\right)\right)}{|x-y^\lambda|^{n-\beta}} \left(\frac{\lambda}{|y|}\right)^{2n} dy \\
&= \int_{B_\lambda(0)} \frac{\left(\frac{\lambda}{|y|}\right)^{n+\beta} h_2\left(\left(\frac{\lambda}{|y|}\right)^2 y, u\left(\left(\frac{\lambda}{|y|}\right)^2 y\right), v\left(\left(\frac{\lambda}{|y|}\right)^2 y\right)\right)}{|x-y|^{n-\beta}} dy \\
&\quad + \int_{B_\lambda(0)} \frac{h_2(y,u(y),v(y))}{\left|\frac{|y|}{\lambda}x - \frac{\lambda}{|y|}y\right|^{n-\beta}} dy.
\end{aligned} \tag{2.17}
$$

From (2.14) and (2.16), for any $x \in B^-_{\lambda,u}$, we have

$$
\begin{aligned}
0 > w^u_\lambda(x) &= u_\lambda(x) - u(x) \\
&= \int_{B_\lambda(0)} \left[\frac{1}{|x-y|^{n-\alpha}} - \frac{1}{\left|\frac{|y|}{\lambda}x - \frac{\lambda}{|y|}y\right|^{n-\alpha}}\right] \\
&\qquad \times \left[\left(\frac{\lambda}{|y|}\right)^{n+\alpha} h_1(y^\lambda, u(y^\lambda), v(y^\lambda)) - h_1(y,u(y),v(y))\right] dy \\
&\geq \int_{B_\lambda(0)} \left[\frac{1}{|x-y|^{n-\alpha}} - \frac{1}{\left|\frac{|y|}{\lambda}x - \frac{\lambda}{|y|}y\right|^{n-\alpha}}\right] \\
&\qquad \times \Big[h_1(y,u_\lambda(y),v(y^\lambda)) - h_1(y,u(y),v(y^\lambda)) \\
&\qquad\quad + h_1(y,u(y),v(y^\lambda)) - h_1(y,u(y),v(y))\Big] dy \\
&\geq \int_{B_\lambda(0)} \left[\frac{1}{|x-y|^{n-\alpha}} - \frac{1}{\left|\frac{|y|}{\lambda}x - \frac{\lambda}{|y|}y\right|^{n-\alpha}}\right] \\
&\qquad \times \Big[h_1(y,u_\lambda(y),v(y^\lambda)) - h_1(y,u(y),v(y^\lambda)) \\
&\qquad\quad + h_1(y,u(y),v_\lambda(y)) - h_1(y,u(y),v(y))\Big] dy \\
&\geq \int_{B^-_{\lambda,u}} \left[\frac{1}{|x-y|^{n-\alpha}} - \frac{1}{\left|\frac{|y|}{\lambda}x - \frac{\lambda}{|y|}y\right|^{n-\alpha}}\right] \left[h_1(y,u_\lambda(y),v(y^\lambda)) - h_1(y,u(y),v(y^\lambda))\right] dy \\
&\quad + \int_{B^-_{\lambda,v}} \left[\frac{1}{|x-y|^{n-\alpha}} - \frac{1}{\left|\frac{|y|}{\lambda}x - \frac{\lambda}{|y|}y\right|^{n-\alpha}}\right] \left[h_1(y,u(y),v_\lambda(y)) - h_1(y,u(y),v(y))\right] dy
\end{aligned}
$$

$$
\begin{aligned}
&\geq \int_{B_{\lambda,u}^-} \frac{1}{|x-y|^{n-\alpha}} \left[h_1(y,u_\lambda(y),v(y^\lambda)) - h_1(y,u(y),v(y^\lambda))\right] dy \\
&\quad + \int_{B_{\lambda,v}^-} \frac{1}{|x-y|^{n-\alpha}} \left[h_1(y,u(y),v_\lambda(y)) - h_1(y,u(y),v(y))\right] dy \\
&\geq \int_{B_{\lambda,u}^-} \frac{1}{|x-y|^{n-\alpha}} \frac{h_1(y,u_\lambda(y),v(y^\lambda)) - h_1(y,u(y),v(y^\lambda))}{u_\lambda(y)-u(y)} w_\lambda^u(y)dy \\
&\quad + \int_{B_{\lambda,v}^-} \frac{1}{|x-y|^{n-\alpha}} \frac{h_1(y,u(y),v_\lambda(y)) - h_1(y,u(y),v(y))}{v_\lambda(y)-v(y)} w_\lambda^v(y)dy,
\end{aligned}
$$

where we use the subcritical growth condition of $h_1(x,u,v)$ for $\mu = \left(\frac{\lambda}{|x|}\right)^{n-\alpha} > 1$ to derive the second inequality, and assumption (*II*) on $h_1(x,u,v)$ to derive the third inequality.

Similarly, for any $x \in B_{\lambda,v}^-$, we deduce

$$
\begin{aligned}
0 &> w_\lambda^v(x) = v_\lambda(x) - v(x) \\
&\geq \int_{B_{\lambda,v}^-} \frac{1}{|x-y|^{n-\beta}} \frac{h_2(y,u(y^\lambda),v_\lambda(y)) - h_2(y,u(y^\lambda),v(y))}{v_\lambda(y)-v(y)} w_\lambda^v(y)dy \\
&\quad + \int_{B_{\lambda,u}^-} \frac{1}{|x-y|^{n-\beta}} \frac{h_2(y,u_\lambda(y),v(y)) - h_2(y,u(y),v(y))}{u_\lambda(y)-u(y)} w_\lambda^u(y)dy.
\end{aligned} \tag{2.18}
$$

We now invoke the Hardy–Littlewood–Sobolev inequality:

Lemma 2.2 (Hardy–Littlewood–Sobolev (HLS) inequality)**.** *Let $n \geq 1$, $0 < \alpha < n$, and $1 < p < q < +\infty$ satisfy $\frac{n}{q} = \frac{n}{p} - \alpha$. Then*

$$
\left\| \int_{\mathbb{R}^n} \frac{f(y)}{|x-y|^{n-\alpha}} dy \right\|_{L^q(\mathbb{R}^n)} \leq C_{n,\alpha,p,q} \|f\|_{L^p(\mathbb{R}^n)} \tag{2.19}
$$

for all $f \in L^p(\mathbb{R}^n)$.

By the HLS inequality, the Minkowski inequality, and the Hölder inequality, combining (2.18) with (2.19), for any $q > \max\{\frac{n}{n-\alpha}, \frac{n}{n-\beta}\}$ we have

$$
\begin{aligned}
&\|w_\lambda^u\|_{L^q(B_{\lambda,u}^-)} \\
&\quad \leq \left\| \int_{B_{\lambda,u}^-} \frac{1}{|x-y|^{n-\alpha}} \frac{h_1(y,u_\lambda(y),v(y^\lambda)) - h_1(y,u(y),v(y^\lambda))}{u_\lambda(y)-u(y)} w_\lambda^u(y)dy \right\|_{L^q(B_{\lambda,u}^-)} \\
&\qquad + \left\| \int_{B_{\lambda,v}^-} \frac{1}{|x-y|^{n-\alpha}} \frac{h_1(y,u(y),v_\lambda(y)) - h_1(y,u(y),v(y))}{v_\lambda(y)-v(y)} w_\lambda^v(y)dy \right\|_{L^q(B_{\lambda,u}^-)} \\
&\quad \leq C \left\| \frac{h_1(y,u_\lambda(y),v(y^\lambda)) - h_1(y,u(y),v(y^\lambda))}{u_\lambda(y)-u(y)} w_\lambda^u(y) \right\|_{L^{\frac{nq}{n+\alpha q}}(B_{\lambda,u}^-)} \\
&\qquad + C \left\| \frac{h_1(y,u(y),v_\lambda(y)) - h_1(y,u(y),v(y))}{v_\lambda(y)-v(y)} w_\lambda^v(y) \right\|_{L^{\frac{nq}{n+\alpha q}}(B_{\lambda,u}^- \cap B_{\lambda,v}^-)} \\
&\quad \leq C \left\| \frac{h_1(y,u_\lambda(y),v(y^\lambda)) - h_1(y,u(y),v(y^\lambda))}{u_\lambda(y)-u(y)} \right\|_{L^{\frac{n}{\alpha}}(B_{\lambda,u}^-)} \|w_\lambda^u\|_{L^q(B_{\lambda,u}^-)}
\end{aligned}
$$

$$+ C\left\|\frac{h_1(y,u(y),v_\lambda(y)) - h_1(y,u(y),v(y))}{v_\lambda(y)-v(y)}\right\|_{L^{\frac{n}{\alpha}}(B^-_{\lambda,u}\cap B^-_{\lambda,v})}\|w^v_\lambda\|_{L^q(B^-_{\lambda,u}\cap B^-_{\lambda,v})}$$
$$\leq C\|\phi_u\|_{L^{\frac{n}{\alpha}}(B^-_{\lambda,u})}\|w^u_\lambda\|_{L^q(B^-_{\lambda,u})} + C\|\phi_v\|_{L^{\frac{n}{\alpha}}(B^-_{\lambda,v})}\|w^v_\lambda\|_{L^q(B^-_{\lambda,v})}, \tag{2.20}$$

Similarly,

$$\|w^v_\lambda\|_{L^q(B^-_{\lambda,v})} \leq C\|\psi_v\|_{L^{\frac{n}{\beta}}(B^-_{\lambda,v})}\|w^v_\lambda\|_{L^q(B^-_{\lambda,v})} + C\|\psi_u\|_{L^{\frac{n}{\beta}}(B^-_{\lambda,u})}\|w^u_\lambda\|_{L^q(B^-_{\lambda,u})}. \tag{2.21}$$

We denote

$$\phi_u = \frac{h_1(y,u_\lambda(y),v(y^\lambda)) - h_1(y,u(y),v(y^\lambda))}{u_\lambda(y)-u(y)},$$
$$\phi_v = \frac{h_1(y,u(y),v_\lambda(y)) - h_1(y,u(y),v(y))}{v_\lambda(y)-v(y)},$$
$$\psi_v = \frac{h_2(y,u(y^\lambda),v_\lambda(y)) - h_2(y,u(y^\lambda),v(y))}{v_\lambda(y)-v(y)},$$
$$\psi_u = \frac{h_2(y,u_\lambda(y),v(y)) - h_2(y,u(y),v(y))}{u_\lambda(y)-u(y)}.$$

Since $u,v\in C(\mathbb{R}^n)$ and $h_1(x,u,v), h_2(x,u,v)$ are locally Lipschitz continuous in u and v, respectively, there exists a sufficiently small $\varepsilon>0$ such that

$$C\|\phi_u\|_{L^{\frac{n}{\alpha}}(B^-_{\lambda,u})},\ C\|\phi_v\|_{L^{\frac{n}{\alpha}}(B^-_{\lambda,v})},\ C\|\psi_v\|_{L^{\frac{n}{\beta}}(B^-_{\lambda,v})},\ C\|\psi_u\|_{L^{\frac{n}{\beta}}(B^-_{\lambda,u})} \leq \frac{1}{4} \tag{2.22}$$

for all $0<\lambda<\varepsilon$.

From (2.20) and (2.21), we obtain

$$\left(1 - C\|\phi_u\|_{L^{\frac{n}{\alpha}}(B^-_{\lambda,u})}\right)\|w^u_\lambda\|_{L^q(B^-_{\lambda,u})} \leq C\|\phi_v\|_{L^{\frac{n}{\alpha}}(B^-_{\lambda,v})}\|w^v_\lambda\|_{L^q(B^-_{\lambda,v})}, \tag{2.23}$$

and

$$\left(1 - C\|\psi_v\|_{L^{\frac{n}{\beta}}(B^-_{\lambda,v})}\right)\|w^v_\lambda\|_{L^q(B^-_{\lambda,v})} \leq C\|\psi_u\|_{L^{\frac{n}{\beta}}(B^-_{\lambda,u})}\|w^u_\lambda\|_{L^q(B^-_{\lambda,u})}. \tag{2.24}$$

Combining these two inequalities, we deduce

$$\begin{aligned}
\|w^u_\lambda\|_{L^q(B^-_{\lambda,u})} &\leq \frac{C\|\phi_v\|_{L^{\frac{n}{\alpha}}(B^-_{\lambda,v})}}{\left(1 - C\|\phi_u\|_{L^{\frac{n}{\alpha}}(B^-_{\lambda,u})}\right)}\|w^v_\lambda\|_{L^q(B^-_{\lambda,v})}\\
&\leq \frac{C\|\psi_u\|_{L^{\frac{n}{\beta}}(B^-_{\lambda,u})}C\|\phi_v\|_{L^{\frac{n}{\alpha}}(B^-_{\lambda,v})}}{\left(1 - C\|\psi_v\|_{L^{\frac{n}{\beta}}(B^-_{\lambda,v})}\right)\left(1 - C\|\phi_u\|_{L^{\frac{n}{\alpha}}(B^-_{\lambda,u})}\right)}\|w^u_\lambda\|_{L^q(B^-_{\lambda,u})}\\
&=: C_{\lambda,u,v}\|w^u_\lambda\|_{L^q(B^-_{\lambda,u})},\\
\|w^v_\lambda\|_{L^q(B^-_{\lambda,v})} &\leq \frac{C\|\psi_u\|_{L^{\frac{n}{\beta}}(B^-_{\lambda,u})}}{\left(1 - C\|\psi_v\|_{L^{\frac{n}{\beta}}(B^-_{\lambda,v})}\right)}\|w^u_\lambda\|_{L^q(B^-_{\lambda,u})}\\
&\leq C_{\lambda,u,v}\|w^v_\lambda\|_{L^q(B^-_{\lambda,v})}.
\end{aligned} \tag{2.25}$$

From (2.22), we have

$$C_{\lambda,u,v} \le \frac{1}{9} < \frac{1}{2} \tag{2.26}$$

for all $0 < \lambda < \varepsilon$ with sufficiently small ε. Combining (2.25) with (2.26) implies

$$\|w_\lambda^u\|_{L^q(B_{\lambda,u}^-)} = 0, \quad \|w_\lambda^v\|_{L^q(B_{\lambda,v}^-)} = 0, \tag{2.27}$$

which yields $B_{\lambda,u}^- = B_{\lambda,v}^- = \emptyset$. Therefore, for all $0 < \lambda < \varepsilon$,

$$w_\lambda^u(x) \ge 0, \quad w_\lambda^v(x) \ge 0, \quad \forall\, x \in B_\lambda(0) \setminus \{0\}. \tag{2.28}$$

This completes Step 1.

Step 2. Step 1 gives a starting point to dilate the sphere S_λ from near $\lambda = 0$. We now dilate S_λ outward as long as (2.11) holds, and define

$$\lambda_0 = \sup\{\lambda > 0 \mid w_\mu^u(x) \ge 0,\ w_\mu^v(x) \ge 0,\ \forall\, x \in B_\mu(0) \setminus \{0\},\ \forall\, 0 < \mu < \lambda\}. \tag{2.29}$$

We prove $\lambda_0 = +\infty$ by contradiction. By the definition of λ_0,

$$w_{\lambda_0}^u(x) \ge 0, \quad w_{\lambda_0}^v(x) \ge 0, \quad \forall\, x \in B_{\lambda_0}(0) \setminus \{0\}. \tag{2.30}$$

Suppose for contradiction that $\lambda_0 < +\infty$; we first show

$$w_{\lambda_0}^u(x) > 0, \quad w_{\lambda_0}^v(x) > 0, \quad \forall\, x \in B_{\lambda_0}(0) \setminus \{0\}. \tag{2.31}$$

If there exists $x \in B_{\lambda_0}(0) \setminus \{0\}$ such that $w_{\lambda_0}^u(x) = w_{\lambda_0}^v(x) = 0$, then

$$\begin{aligned}
0 &= w_{\lambda_0}^u(x) \\
&= \int_{B_{\lambda_0}(0)} \left[\frac{1}{|x-y|^{n-\alpha}} - \frac{1}{\left|\frac{|y|}{\lambda_0}x - \frac{\lambda_0}{|y|}y\right|^{n-\alpha}}\right] \\
&\qquad \times \left[\left(\frac{\lambda_0}{|y|}\right)^{n+\alpha} h_1(y^{\lambda_0}, u(y^{\lambda_0}), v(y^{\lambda_0})) - h_1(y, u(y), v(y))\right] dy \\
&> \int_{B_{\lambda_0}(0)} \left[\frac{1}{|x-y|^{n-\alpha}} - \frac{1}{\left|\frac{|y|}{\lambda_0}x - \frac{\lambda_0}{|y|}y\right|^{n-\alpha}}\right] \\
&\qquad \times \Big[h_1(y, u_{\lambda_0}(y), v(y^{\lambda_0})) - h_1(y, u(y), v(y^{\lambda_0})) \\
&\qquad\quad + h_1(y, u(y), v(y^{\lambda_0})) - h_1(y, u(y), v(y))\Big] dy \\
&\ge \int_{B_{\lambda_0}(0)} \left[\frac{1}{|x-y|^{n-\alpha}} - \frac{1}{\left|\frac{|y|}{\lambda_0}x - \frac{\lambda_0}{|y|}y\right|^{n-\alpha}}\right] \\
&\qquad \times \Big[h_1(y, u_{\lambda_0}(y), v(y^{\lambda_0})) - h_1(y, u(y), v(y^{\lambda_0})) \\
&\qquad\quad + h_1(y, u(y), v_{\lambda_0}(y)) - h_1(y, u(y), v(y))\Big] dy \\
&\ge 0,
\end{aligned} \tag{2.32}$$

and

$$
\begin{aligned}
0 &= w^v_{\lambda_0}(x) \\
&\geq \int_{B_{\lambda_0}(0)} \left[\frac{1}{|x-y|^{n-\beta}} - \frac{1}{\left|\frac{|y|}{\lambda_0}x - \frac{\lambda_0}{|y|}y\right|^{n-\beta}} \right] \\
&\quad \times \left[h_2(y, u(y^{\lambda_0}), v_{\lambda_0}(y)) - h_2(y, u(y^{\lambda_0}), v(y)) \right] dy \\
&\quad + \int_{B_{\lambda_0}(0)} \left[\frac{1}{|x-y|^{n-\beta}} - \frac{1}{\left|\frac{|y|}{\lambda_0}x - \frac{\lambda_0}{|y|}y\right|^{n-\beta}} \right] \\
&\quad \times \left[h_2(y, u_{\lambda_0}(y), v(y)) - h_2(y, u(y), v(y)) \right] dy \\
&\geq 0.
\end{aligned} \tag{2.33}
$$

This contradiction implies there exists $x^0 \in B_{\lambda_0}(0) \setminus \{0\}$ such that $w^u_{\lambda_0}(x^0) > 0$ or $w^v_{\lambda_0}(x^0) > 0$. Without loss of generality, assume $w^u_{\lambda_0}(x^0) > 0$; then (2.33) forces $w^v_{\lambda_0}(x^0) > 0$, and vice versa. Since $u, v \in C(\mathbb{R}^n)$, there exist a small $\delta > 0$ and constants $c_1, c_2 > 0$ such that

$$
B_\delta(x^0) \subset B_{\lambda_0}(0) \setminus \{0\}, \quad w^u_{\lambda_0}(x) > c_1, \quad w^v_{\lambda_0}(x) > c_2, \quad \forall\, x \in B_\delta(x^0). \tag{2.34}
$$

From (2.34), the integral equation (1.10), and arguments similar to those for (2.32) and (2.33), for all $x \in B_{\lambda_0}(0) \setminus \{0\}$ we have

$$
\begin{aligned}
w^u_{\lambda_0}(x) &> \int_{B_\delta(x^0)} \left[\frac{1}{|x-y|^{n-\alpha}} - \frac{1}{\left|\frac{|y|}{\lambda_0}x - \frac{\lambda_0}{|y|}y\right|^{n-\alpha}} \right] \\
&\quad \times \left[h_1(y, u_{\lambda_0}(y), v(y^{\lambda_0})) - h_1(y, u(y), v(y^{\lambda_0})) \right] dy \\
&\quad + \int_{B_\delta(x^0)} \left[\frac{1}{|x-y|^{n-\alpha}} - \frac{1}{\left|\frac{|y|}{\lambda_0}x - \frac{\lambda_0}{|y|}y\right|^{n-\alpha}} \right] \\
&\quad \times \left[h_1(y, u(y), v_{\lambda_0}(y)) - h_1(y, u(y), v(y)) \right] dy \\
&> 0,
\end{aligned} \tag{2.35}
$$

and

$$
\begin{aligned}
w^v_{\lambda_0}(x) &> \int_{B_\delta(x^0)} \left[\frac{1}{|x-y|^{n-\beta}} - \frac{1}{\left|\frac{|y|}{\lambda_0}x - \frac{\lambda_0}{|y|}y\right|^{n-\beta}} \right] \\
&\quad \times \left[h_2(y, u(y^{\lambda_0}), v_{\lambda_0}(y)) - h_2(y, u(y^{\lambda_0}), v(y)) \right] dy \\
&\quad + \int_{B_\delta(x^0)} \left[\frac{1}{|x-y|^{n-\beta}} - \frac{1}{\left|\frac{|y|}{\lambda_0}x - \frac{\lambda_0}{|y|}y\right|^{n-\beta}} \right] \\
&\quad \times \left[h_2(y, u_{\lambda_0}(y), v(y)) - h_2(y, u(y), v(y)) \right] dy \\
&> 0.
\end{aligned} \tag{2.36}
$$

This proves (2.31).

We now reach a contradiction with the definition of λ_0 by showing that S_λ can be dilated outward slightly further. By (2.35) and (2.36), there exists a sufficiently small $0 < \eta < \lambda_0$ such that for all $x \in \overline{B_\eta(0)} \setminus \{0\}$,

$$\begin{aligned} w^u_{\lambda_0}(x) &\geq C_1 \int_{B_{\frac{\eta}{2}}(x^0)} c c_1^p c_2^q dy = \widetilde{c}_1 > 0, \\ w^v_{\lambda_0}(x) &\geq C_2 \int_{B_{\frac{\eta}{2}}(x^0)} c c_1^p c_2^q dy = \widetilde{c}_2 > 0. \end{aligned} \tag{2.37}$$

Fix a sufficiently small $0 < r_0 < \frac{\lambda_0}{2}$; then

$$C_{\lambda,u,v} := \frac{C\|\psi_u\|_{L^{\frac{n}{\beta}}(A_{\lambda_0+r_0,2r_0})} C\|\phi_v\|_{L^{\frac{n}{\alpha}}(A_{\lambda_0+r_0,2r_0})}}{\left(1 - C\|\psi_v\|_{L^{\frac{n}{\beta}}(A_{\lambda_0+r_0,2r_0})}\right)\left(1 - C\|\phi_u\|_{L^{\frac{n}{\alpha}}(A_{\lambda_0+r_0,2r_0})}\right)} < \frac{1}{2}, \tag{2.38}$$

where the annular region is defined by

$$A_{\lambda_0+r_0,2r_0} := \left\{x \in B_{\lambda_0+r_0}(0) \mid |x| > \lambda_0 - r_0\right\}. \tag{2.39}$$

Arguments similar to those for (2.20) and (2.21) show that for all $\lambda \in [\lambda_0, \lambda_0 + \epsilon_0)$ and all $r > \max\{\frac{n}{n-\alpha}, \frac{n}{n-\beta}\}$,

$$\left(1 - C\|\phi_u\|_{L^{\frac{n}{\alpha}}(B^-_{\lambda,u})}\right)\|w^u_\lambda\|_{L^r(B^-_{\lambda,u})} \leq C\|\phi_v\|_{L^{\frac{n}{\alpha}}(B^-_{\lambda,v})}\|w^v_\lambda\|_{L^r(B^-_{\lambda,v})}, \tag{2.40}$$

and

$$\left(1 - C\|\psi_v\|_{L^{\frac{n}{\beta}}(B^-_{\lambda,v})}\right)\|w^v_\lambda\|_{L^r(B^-_{\lambda,v})} \leq C\|\psi_u\|_{L^{\frac{n}{\beta}}(B^-_{\lambda,u})}\|w^u_\lambda\|_{L^r(B^-_{\lambda,u})}. \tag{2.41}$$

It follows immediately that

$$\begin{aligned} \|w^u_\lambda\|_{L^r(B^-_{\lambda,u})} &\leq C_{\lambda,u,v}\|w^u_\lambda\|_{L^r(B^-_{\lambda,u})}, \\ \|w^v_\lambda\|_{L^r(B^-_{\lambda,v})} &\leq C_{\lambda,u,v}\|w^v_\lambda\|_{L^r(B^-_{\lambda,v})}. \end{aligned} \tag{2.42}$$

From (2.31) and (2.37), we deduce

$$m_1 := \inf_{\overline{B_{\lambda_0-r_0}}\setminus\{0\}} w^u_{\lambda_0}(x) > 0, \quad m_2 := \inf_{\overline{B_{\lambda_0-r_0}}\setminus\{0\}} w^v_{\lambda_0}(x) > 0. \tag{2.43}$$

This is equivalent to

$$\begin{aligned} |x|^{n-\alpha}u(x) - \lambda_0^{n-\alpha}u(x^{\lambda_0}) &\geq m_1\lambda_0^{n-\alpha}, \quad \forall\, |x| \geq \frac{\lambda_0^2}{\lambda_0 - r_0}, \\ |x|^{n-\beta}v(x) - \lambda_0^{n-\beta}v(x^{\lambda_0}) &\geq m_2\lambda_0^{n-\beta}, \quad \forall\, |x| \geq \frac{\lambda_0^2}{\lambda_0 - r_0}. \end{aligned} \tag{2.44}$$

Since u and v are uniformly continuous on any compact set $K \subset \mathbb{R}^n$ (e.g., $K = \overline{B_{4\lambda_0}(0)}$), from (2.44) there exists a sufficiently small $0 < \epsilon_1 < r_0$ such that for all $\lambda \in [\lambda_0, \lambda_0 + \epsilon_0)$,

$$\begin{aligned} |x|^{n-\alpha}u(x) - \lambda^{n-\alpha}u(x^\lambda) &\geq \frac{m_1}{2}\lambda^{n-\alpha}, \quad \forall\, |x| \geq \frac{\lambda^2}{\lambda_0 - r_0}, \\ |x|^{n-\beta}v(x) - \lambda^{n-\beta}v(x^\lambda) &\geq \frac{m_2}{2}\lambda^{n-\beta}, \quad \forall\, |x| \geq \frac{\lambda^2}{\lambda_0 - r_0}. \end{aligned} \tag{2.45}$$

In other words, for any $\lambda \in [\lambda_0, \lambda_0 + \epsilon_0)$,

$$w_\lambda^u(x) \geq \frac{m_1}{2}, \quad w_\lambda^v(x) \geq \frac{m_2}{2}, \quad \forall\, x \in \overline{B_{\lambda_0 - r_0}} \setminus \{0\}. \tag{2.46}$$

By (2.45), for all $\lambda \in [\lambda_0, \lambda_0 + \epsilon_0)$,

$$B_{\lambda,u}^- \subset A_{\lambda_0 + r_0, 2r_0}, \quad B_{\lambda,v}^- \subset A_{\lambda_0 + r_0, 2r_0}. \tag{2.47}$$

Combining (2.38) with (2.43), we obtain

$$\|w_\lambda^u\|_{L^r(B_{\lambda,u}^-)} = 0, \quad \|w_\lambda^v\|_{L^r(B_{\lambda,v}^-)} = 0. \tag{2.48}$$

Thus for all $\lambda \in [\lambda_0, \lambda_0 + \epsilon_0)$, $B_{\lambda,u}^- = B_{\lambda,v}^- = \emptyset$, i.e.,

$$w_\lambda^u(x) \geq 0, \quad w_\lambda^v(x) \geq 0, \quad \forall\, x \in B_\lambda(0) \setminus \{0\}. \tag{2.49}$$

This contradicts the definition of λ_0, so we must have $\lambda_0 = +\infty$. Therefore,

$$\begin{aligned} u(x) &\geq \left(\frac{\lambda}{|x|}\right)^{n-\alpha} u\left(\frac{\lambda^2 x}{|x|^2}\right), \quad \forall\, |x| \geq \lambda,\ \forall\, 0 < \lambda < +\infty, \\ v(x) &\geq \left(\frac{\lambda}{|x|}\right)^{n-\beta} v\left(\frac{\lambda^2 x}{|x|^2}\right), \quad \forall\, |x| \geq \lambda,\ \forall\, 0 < \lambda < +\infty. \end{aligned} \tag{2.50}$$

For arbitrary $|x| \geq 1$, choose $\lambda = 1$ (note: original $\lambda = \sqrt{x}$ is a typo); then (2.50) implies

$$\begin{aligned} u(x) &\geq \left(\frac{1}{|x|}\right)^{n-\alpha} u\left(\frac{x}{|x|}\right) \geq \min_{S^{n-1}} u \cdot \frac{1}{|x|^{n-\alpha}} =: \frac{C_1}{|x|^{\mu_{u,0}}}, \quad \forall\, |x| \geq 1, \\ v(x) &\geq \left(\frac{1}{|x|}\right)^{n-\beta} v\left(\frac{x}{|x|}\right) \geq \min_{S^{n-1}} v \cdot \frac{1}{|x|^{n-\beta}} =: \frac{C_2}{|x|^{\mu_{v,0}}}, \quad \forall\, |x| \geq 1, \end{aligned} \tag{2.51}$$

where $\mu_{u,0} = n - \alpha$, $\mu_{v,0} = n - \beta$ (corrected to match subsequent iteration). From

(1.10) and (2.51), for $|x| \geq 1$, we have

$$
\begin{aligned}
u(x) &\geq C \int_{2|x|\leq|y|\leq4|x|} \frac{h_1(y,u(y),v(y))}{|x-y|^{n-\alpha}}dy \\
&\geq C \int_{2|x|\leq|y|\leq4|x|} \frac{|y|^{\tau_1}u^p(y)v^q(y)}{|x-y|^{n-\alpha}}dy \\
&\geq \frac{C}{|x|^{n-\alpha}} \int_{2|x|\leq|y|\leq4|x|} \frac{|y|^{\tau_1}}{|y|^{p\mu_{u,0}}|y|^{q\mu_{v,0}}}dy \\
&\geq \frac{C}{|x|^{n-\alpha}} \int_{2|x|}^{4|x|} r^{n-1+\tau_1-p\mu_{u,0}-q\mu_{v,0}}dr \\
&= \frac{C}{|x|^{p\mu_{u,0}+q\mu_{v,0}-(\alpha+\tau_1)}} \\
&=: \frac{C_1}{|x|^{\mu_{u,1}}},
\end{aligned}
\tag{2.52}
$$

and

$$
\begin{aligned}
v(x) &\geq C \int_{2|x|\leq|y|\leq4|x|} \frac{h_2(y,u(y),v(y))}{|x-y|^{n-\beta}}dy \\
&\geq \frac{C_2}{|x|^{\mu_{v,1}}},
\end{aligned}
\tag{2.53}
$$

where $\mu_{v,1} = p\mu_{u,0} + q\mu_{v,0} - (\beta + \tau_2)$. This gives a better lower bound than (2.51). For $k = 0, 1, 2, \ldots$, define the iteration

$$
\mu_{u,k+1} = p\mu_{u,k} + q\mu_{v,k} - (\alpha+\tau_1), \quad \mu_{v,k+1} = p\mu_{u,k} + q\mu_{v,k} - (\beta+\tau_2). \tag{2.54}
$$

Continuing this iteration with (1.10), for every $k = 0, 1, 2, \ldots$, we obtain the lower bound estimates

$$
u(x) \geq \frac{C_1}{|x|^{\mu_{u,k}}}, \quad v(x) \geq \frac{C_2}{|x|^{\mu_{v,k}}}, \quad \forall\ |x| \geq 1. \tag{2.55}
$$

Moreover, the iterates satisfy

$$
\begin{aligned}
\mu_{u,k+2} &= (p^2+pq)\mu_{u,k} + (pq+q^2)\mu_{v,k} - \big((p+1)(\alpha+\tau_1) + q(\beta+\tau_2)\big), \\
\mu_{v,k+2} &= (p^2+pq)\mu_{u,k} + (pq+q^2)\mu_{v,k} - \big(p(\alpha+\tau_1) + (q+1)(\beta+\tau_2)\big).
\end{aligned}
\tag{2.56}
$$

One verifies that for $p + q < 1$, as $k \to +\infty$,

$$
\begin{aligned}
\mu_{u,2k} &\to -\frac{(1-q)(\alpha+\tau_1) + q(\beta+\tau_2)}{1-(p+q)}, \\
\mu_{v,2k} &\to -\frac{p(\alpha+\tau_1) + (1-p)(\beta+\tau_2)}{1-(p+q)},
\end{aligned}
\tag{2.57}
$$

and for $p + q \geq 1$, if $\max\{\frac{n+\alpha+2\tau_1}{n-\alpha}, \frac{n+\beta+2\tau_2}{n-\beta}\} > p + q$, then $\mu_{u,2k} \to -\infty$ and $\mu_{v,2k} \to -\infty$ as $k \to +\infty$. This completes the proof of Theorem 2.1. □

We first show that the nontrivial nonnegative solutions (u, v) to (1.1) are in fact positive solutions, and they satisfy the integral equation (1.10). We then use the lower bound estimates established in Theorem 2.1 to derive a contradiction with the finiteness of $u(0)$ and $v(0)$. Specifically, we have

$$\begin{aligned} +\infty > u(0) &= C\int_{\mathbb{R}^n} \frac{h_1(y,u(y),v(y))}{|y|^{n-\alpha}}dy \\ &\geq C\int_{\mathbb{R}^n} \frac{|y|^{\tau_1}u^p(y)v^q(y)}{|y|^{n-\alpha}}dy \\ &\geq C\int_{\mathbb{R}^n} \frac{1}{|y|^{n-(\alpha+\tau_1-p\mu_{u,k}-q\mu_{v,k})}}dy \\ &= +\infty, \end{aligned} \tag{2.58}$$

and similarly,

$$\begin{aligned} +\infty > v(0) &= C\int_{\mathbb{R}^n} \frac{h_2(y,u(y),v(y))}{|y|^{n-\beta}}dy \\ &\geq C\int_{\mathbb{R}^n} \frac{|y|^{\tau_2}u^p(y)v^q(y)}{|y|^{n-\beta}}dy \\ &\geq C\int_{\mathbb{R}^n} \frac{1}{|y|^{n-(\beta+\tau_2-p\mu_{u,k}-q\mu_{v,k})}}dy \\ &= +\infty. \end{aligned} \tag{2.59}$$

This is a clear contradiction. Thus our initial assumption of the existence of a nontrivial nonnegative solution is false, and we conclude that $u \equiv 0$ and $v \equiv 0$ in $\mathbb{R}^n$. In other words, the only nonnegative solution to the integral equation (1.10) (and hence to the original equation (1.1)) is the trivial solution $(u,v) \equiv (0,0)$ in $\mathbb{R}^n$.

This completes the proof of Theorem 1.7.

Acknowledgements

This work was supported by the Natural Science Foundation of Henan Province (Grant No. 242300421691), and the National Natural Science Foundation of China (Grant No. 12301251,12271232) and the Scientific Research Foundation of Linyi University, China (Grant No. LYDX2020BS014).

References

[1] M. F. Bidaut-Véron, H. Giacomini, A new dynamical approach of Emden–Fowler equations and systems, *Adv. Differ. Equ.* **15**(2010), No. 11–12, 1033–1082. https://doi.org/10.48550/arXiv.1001.0562; MR2743494; Zbl 1230.34021

[2] L. Caffarelli, B. Gidas, J. Spruck, Asymptotic symmetry and local behavior of semilinear elliptic equation with critical Sobolev growth, *Comm. Pure Appl. Math.* **42**(1989), No. 3, 271–297. https://doi.org/10.1002/cpa.3160420304; MR0982351; Zbl 0702.35085

[3] D. Cao, W. Dai, G. Qin, Super poly-harmonic properties, Liouville theorems and classification of nonnegative solutions to equations involving higher-order fractional Laplacians, *Trans. Amer. Math. Soc.* **374**(2021), No. 7, 4781–4813. https://doi.org/10.1090/tran/8389; MR4273176; Zbl 1465.35388

[4] D. Cao, G. Qin, Liouville type theorems for fractional and higher-order fractional systems, *Discrete Contin. Dyn. Syst.* **41**(2021), No. 5, 2269–2283. https://doi.org/10.3934/dcds.2020361; MR4225913; Zbl 1466.35118

[5] W. Chen, W. Dai, G. Qin, Liouville type theorems, a priori estimates and existence of solutions for critical and super-critical order Hardy–Hénon type equations in $\mathbb{R}^n$, *Math. Z.* **303**(2023), No. 4, 1–36. https://doi.org/10.1007/s00209-023-03265-y; MR4565803; Zbl 1519.35043

[6] W. Chen, Y. Fang, C. Li, Super poly-harmonic property of solutions for Navier boundary problems on a half space, *J. Funct. Anal.* **265**(2013), No. 8, 1522–1555. https://doi.org/10.1016/j.jfa.2013.06.010; MR3079228; Zbl 1288.35230

[7] W. Chen, Y. Fang, R. Yang, Liouville theorems involving the fractional Laplacian on a half space, *Adv. Math.* **274**(2015), 167–198. https://doi.org/10.1016/j.aim.2014.12.013; MR3318148; Zbl 1372.35332

[8] W. Chen, C. Li, Moving planes, moving spheres, and a priori estimates, *J. Differential Equations* **195**(2003), No. 1, 1–13. https://doi.org/10.1016/j.jde.2003.06.004; MR2019239; Zbl 1134.35331

[9] W. Chen, C. Li, Y. Li, A direct method of moving planes for the fractional Laplacian, *Adv. Math.* **308**(2017), 404–437. https://doi.org/10.1016/j.aim.2016.11.038; MR3600062; Zbl 1362.35320

[10] W. Chen, C. Li, B. Ou, Classification of solutions for an integral equation, *Comm. Pure Appl. Math.* **59**(2006), No. 3, 330–343. https://doi.org/10.1002/cpa.20116; MR2200258; Zbl 1093.45001

[11] W. Chen, Y. Li, R. Zhang, A direct method of moving spheres on fractional order equations, *J. Funct. Anal.* **272**(2017), No. 10, 4131–4157. https://doi.org/10.1016/j.jfa.2017.02.022; MR3626036; Zbl 1431.35225

[12] T. Cheng, S. Liu, A Liouville type theorem for higher order Hardy–Hénon equation in $\mathbb{R}^n$, *J. Math. Anal. Appl.* **444**(2016), No. 1, 370–389. https://doi.org/10.1016/j.jmaa.2016.05.035; MR3523382; Zbl 1345.35025

[13] C. Cowan, A Liouville theorem for a fourth order Hénon equation, *Adv. Nonlinear Stud.* **14**(2014), No. 3, 767–776. https://doi.org/10.1515/ans-2014-0313; MR3244358; Zbl 1301.35023

[14] X. Cui, M. Yu, Non-existence of positive solutions for a higher order fractional equation, *Discrete Contin. Dyn. Syst.* **39**(2019), No. 3, 1379–1387. https://doi.org/10.3934/dcds.2019059; MR3918222; Zbl 1407.35203

[15] W. Dai, Liouville type theorems for poly-harmonic Dirichlet problems of Hénon-Hardy type equations on a half space or a ball, *Collect. Math.* **74**(2023), No. 3, 729–751. https://doi.org/10.1007/s13348-022-00371-8; MR4614641; Zbl 1519.35044

[16] W. Dai, Y. Fang, G. Qin, Classification of positive solutions to fractional order Hartree equations via a direct method of moving planes, *J. Differential Equations* **265**(2018), No. 3, 2044–2063. https://doi.org/10.1016/j.jde.2018.04.026; MR3800111; Zbl 1392.35329

[17] W. Dai, S. Peng, Liouville theorems for nonnegative solutions to static weighted Schrödinger–Hartree–Maxwell type equations with combined nonlinearities, *Anal. Math. Physics* **11**(2021), No. 2, 1–21. https://doi.org/10.1007/s13324-021-00479-3; MR4213755; Zbl 1458.35088

[18] W. Dai, S. Peng, G. Qin, Liouville type theorems, a priori estimates and existence of solutions for sub-critical order Lane–Emden–Hardy equations, *J. Anal. Math.* **146**(2022), No. 2, 673–718. https://doi.org/10.1007/s11854-022-0207-6; MR4468010; Zbl 1497.35255

[19] W. Dai, G. Qin, Classification of nonnegative classical solutions to third-order equations, *Adv. Math.* **328**(2018), 822-857. https://doi.org/10.1016/j.aim.2018.02.016; MR3771143; Zbl 1429.35198

[20] W. Dai, G. Qin, Liouville type theorems for elliptic equations with Dirichlet conditions in exterior domains, *J. Differential Equations* **269**(2020), No. 9, 7231–7252. https://doi.org/10.1016/j.jde.2020.05.026; MR4108363; Zbl 1441.35083

[21] W. Dai, G. Qin, Liouville type theorems for Hardy-Hénon equations with concave nonlinearities, *Math. Nachr.* **293**(2020), No. 6, 1084–1093. https://doi.org/10.1002/mana.201800532; MR4107983; Zbl 1475.35161

[22] W. Dai, G. Qin, Liouville theorems for poly-harmonic functions on $\mathbb{R}^n_+$, *Arch. Math.* **115**(2020), No. 3, 317–327. https://doi.org/10.1007/s00013-020-01464-1; MR4134926; Zbl 1445.35096

[23] W. Dai, G. Qin, Liouville type theorem for critical order Hénon–Lane–Emden type equations on a half space and its applications, *J. Funct. Anal.* **281**(2021), No. 10, 1–37. https://doi.org/10.1016/j.jfa.2021.109227; MR4308059; Zbl 1473.35082

[24] W. Dai, G. Qin, Liouville-type theorems for fractional and higher-order Hénon–Hardy type equations via the method of scaling spheres, *Int. Math. Res. Not. IMRN* **2023**, No. 11, 9001–9070. https://doi.org/10.1093/imrn/rnac079; MR4597203; Zbl 1516.35142

[25] W. Dai, G. Qin, Y. Zhang, Liouville type theorem for higher order Hénon equations on a half space, *Nonlinear Anal.* **183**(2019), 284–302. https://doi.org/10.1016/j.na.2019.01.033; MR3914212; Zbl 1418.35053

[26] J. Dou, H. Zhou, Liouville theorems for fractional Hénon equation and system on $\mathbb{R}^n$, *Comm. Pure Appl. Anal.* **14**(2015), No. 5, 1915–1927. https://doi.org/10.3934/cpaa.2015.14.1915; MR3359551; Zbl 1320.35113

[27] A. T. Duong, P. Le, N. T. Nguyen, Symmetry and nonexistence results for a fractional choquard equation with weights, *Discrete Contin. Dyn. Syst.* **41**(2021), No. 2, 489–505. https://doi.org/10.3934/dcds.2020265; MR4191515; Zbl 1458.35449

[28] B. Gidas, J. Spruck, Global and local behavior of positive solutions of nonlinear elliptic equations, *Comm. Pure Appl. Math.* **34**(1981), No. 4, 525–598. https://doi.org/10.1002/cpa.3160340406; MR0615628; Zbl 0465.35003

[29] Q. Jin, Y. Li, H. Xu, Symmetry and asymmetry: the method of moving spheres, *Adv. Differ. Equ.* **13**(2008), No. 7–8, 601–640. https://doi.org/10.48550/arXiv.math/0703808; MR2479025; Zbl 1201.35099

[30] P. Le, Liouville theorems for fractional Hénon–Lane–Emden systems on a half space, *Proc. R. Soc. Edinburgh* **150**(2020), No. 6, 3060–3073. https://doi.org/10.1017/prm.2019.58; MR4190101; Zbl 1459.35058

[31] Y. Lei, Asymptotic properties of positive solutions of the Hardy–Sobolev type equations, *J. Differential Equations* **254**(2013), No. 4, 1774–1799. https://doi.org/10.1016/j.jde.2012.11.008; MR3003292; Zbl 1261.35023

[32] Y. Li, Remark on some conformally invariant integral equations: the method of moving spheres, *J. Eur. Math. Soc.* **6**(2004), No. 2, 153–180. https://doi.org/10.4171/JEMS/6; MR2055032; Zbl 1075.45006

[33] Y. Li, M. Zhu, Uniqueness theorems through the method of moving spheres, *Duke Math. J.* **80**(1995), No. 2, 383–417. https://doi.org/10.1215/S0012-7094-95-08016-8; MR1369398; Zbl 0846.35050

[34] C. Lin, A classification of solutions of a conformally invariant fourth order equation in $\mathbb{R}^n$, *Comment. Math. Helv.* **73**(1998), No. 6, 206-231. https://doi.org/10.1007/s000140050052; MR1611691; Zbl 0933.35057

[35] S. Peng, Liouville theorems for fractional and higher order Hénon–Hardy systems on $\mathbb{R}^n$, *Complex Var. Elliptic Equ.* **66**(2021), No. 11, 1839–1863. https://doi.org/10.1080/17476933.2020.1783661; MR4334040; Zbl 1483.35325

[36] Q. H. Phan, Liouville-type theorems for polyharmonic Hénon–Lane–Emden system, *Adv. Nonlinear Stud.* **15**(2015), No. 2, 415–432. https://doi.org/10.48550/arXiv.1405.4584; MR3337881; Zbl 1330.35054

[37] Q. H. Phan, P. Souplet, Liouville-type theorems and bounds of solutions of Hardy–Hénon equations, *J. Differential Equations* **252**(2012), No. 3, 2544–2562. https://doi.org/10.1016/j.jde.2011.09.022; MR2860629; Zbl 1233.35093

[38] W. Reichel, T. Weth, A priori bounds and a Liouville theorem on a half-space for higher-order elliptic Dirichlet problems, *Math. Z.* **261**(2009), No. 4, 805–827. https://doi.org/10.1007/s00209-008-0352-3; MR2480759; Zbl 1167.35014

[39] J. Wei, X. Xu, Classification of solutions of higher order conformally invariant equations, *Math. Ann.* **313**(1999), No. 2, 207–228. https://doi.org/10.1007/s002080050258; MR1679783; Zbl 0940.35082