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Multiplicity of solutions for nonautonomous (p, q)-equations

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Abstract. We consider a nonlinear Dirichlet problem driven by a nonautonomous (p, q)-Laplacian and a Carathéodory reaction which is $(p-1)$-linear near $\pm\infty$. Using variational tools, comparison principles and critical groups, we prove two multiplicity theorems producing six nontrivial smooth solutions, all with sign information and ordered.

Keywords: comparison principles, nonlinear regularity theory, tangency principle, constant sign solutions, nodal solutions, critical groups.

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1 Introduction

Let $\Omega \subseteq \mathbb{R}^N$ be a bounded domain with a C^2-boundary $\partial\Omega$. In this paper we study the following nonhomogeneous Dirichlet (p, q)-equation:

$$\begin{cases} -\Delta_p^{a_1} u(z) - \Delta_q^{a_2} u(z) = f(z, u(z)) \text{ in } \Omega, \\ u|_{\partial\Omega} = 0, \ 1 < q < p. \end{cases} \tag{1.1}$$

If $a \in L^\infty(\Omega)$, $0 < \hat{c} \leq \text{essinf}_\Omega a$ and $r \in (1, \infty)$, then by Δ_r^a we denote the weighted r-Laplace differential operator defined by

$$\Delta_r^a u = \text{div}(a(z)|Du|^{r-2}|Du), \quad \text{for all } u \in W_0^{1,r}(\Omega).$$

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In (1.1) the equation is driven by the sum of two such operators with different exponents $(1 < q < p)$ and in general different weights $(a_1(\cdot), a_2(\cdot))$. So in (1.1) the differential operator is nonautonomous and nonhomogeneous. The reaction (right-hand side) $f(z,x)$ is a Carathéodory function (that is, for all $x \in \mathbb{R}$, $z \mapsto f(z,x)$ is measurable, and for a.a. $z \in \Omega$, $x \mapsto f(z,x)$ is continuous). Such a function is jointly measurable and so superpositionally measurable (that is, for every measurable function $u : \Omega \to \mathbb{R}$, the function $z \mapsto f(z,u(z))$ is measurable). We assume that $f(z,\cdot)$ exhibits $(p-1)$-linear growth as $x \to \pm\infty$.

Using variational tools from the critical point theory and the spectral properties of the nonhomogeneous p and q Laplacians (see Liu–Papageorgiou [9]), we prove the existence of at least six nontrivial smooth solutions, all with sign information and ordered. Our work here extends that of Liu–Papageorgiou [8,9] where the authors obtain three nontrivial smooth solutions. In the process of the proof, we prove two strong comparison theorems, which provide powerful tools for the analysis of nonautonomous equations.

2 Mathematical background – hypotheses

The main spaces in the analysis of problem (1.1) are the Sobolev spaces $W_0^{1,p}(\Omega)$ and the Banach space $C_0^1(\overline{\Omega}) = \{u \in C^1(\overline{\Omega}) : u|_{\partial\Omega} = 0\}$. On account of the Poincaré inequality, on $W_0^{1,p}(\Omega)$ we can use the norm

$$\|u\| = \|Du\|_p \quad \text{for all } u \in W_0^{1,p}(\Omega).$$

The Banach space $C_0^1(\overline{\Omega})$ is an ordered Banach space with positive (order) cone

$$C_+ = \left\{ u \in C_0^1(\overline{\Omega}) : u(z) \geq 0 \text{ for all } z \in \overline{\Omega} \right\}.$$

This cone has a nonempty interior given by

$$\operatorname{int} C_+ = \left\{ u \in C_+ : u(z) > 0 \text{ for all } z \in \Omega,\ \left.\frac{\partial u}{\partial n}\right|_{\partial\Omega} < 0 \right\},$$

where $\frac{\partial u}{\partial n} = (Du, n)_{\mathbb{R}^N}$ with $n(\cdot)$ being the outward unit normal on $\partial\Omega$.

Let $C^{0,1}(\overline{\Omega})$ be the space of all Lipschitz continuous functions on $\overline{\Omega}$. Suppose $a \in C^{0,1}(\overline{\Omega})$, $0 < \hat{c} \leq a(z)$ for all $z \in \overline{\Omega}$ and $1 < r < \infty$. We consider the following nonlinear eigenvalue problem

$$-\Delta_r^a u(z) = \hat{\lambda} |u(z)|^{r-2} u(z) \quad \text{in } \Omega,\ u|_{\partial\Omega} = 0. \tag{2.1}$$

From Liu–Papageorgiou [9], we know that

(a) Problem (2.1) has a smallest eigenvalue $\hat{\lambda}_1^a(r) > 0$ which has the following variational characterization

$$\hat{\lambda}_1^a(r) = \inf \left\{ \frac{\int_\Omega a(z)|Du|^r \, dz}{\|u\|_r^r} : u \in W_0^{1,r}(\Omega), u \neq 0 \right\}. \tag{2.2}$$

(b) $\hat{\lambda}_1^a(r) > 0$ is isolated, that is, if $\hat{\sigma}(r) > 0$ denotes the spectrum of (2.1), then there exists $\varepsilon > 0$ such that

$$\hat{\sigma}(r) \cap (\hat{\lambda}_1^a(r), \hat{\lambda}_1^a(r) + \varepsilon) = \emptyset.$$

(c) $\hat{\lambda}_1^a(r) > 0$ is simple, that is, if $\hat{u}, \hat{v} \in W_0^{1,p}(\Omega)$ are eigenfunctions corresponding to $\hat{\lambda}_1^a(r)$, then $\hat{u} = \theta\hat{v}$ for some $\theta \in \mathbb{R}\backslash\{0\}$.

(d) All eigenfunctions of (2.1) belong in $C_0^1(\bar{\Omega})$ (nonlinear regularity theory of Lieberman [7]), if $\lambda > \hat{\lambda}_1^a(r)$ is an eigenvalue of (2.1), then the eigenfunctions corresponding to λ are nodal (sign-changing) and the eigenfunctions of $\hat{\lambda}_1^a(r)$ have fixed sign (see (2.2)) and by $\hat{u}_1^a(r)$ we denote the positive, r-normalized (that is, $\|\hat{u}_1^a(r)\|_r = 1$) eigenfunction for $\hat{\lambda}_1^a(r)$. Using the nonlinear maximum principle (see Pacci–Serrin [16], p. 120), we have

$$\hat{u}_1^a(r) \in \operatorname{int} C_+.$$

(e) We do not have a full knowledge of $\hat{\sigma}(r)$, $r \neq 2$. Using the Ljusternik–Schnirelmann minimax scheme, we know that (2.1) has a sequence $\{\hat{\lambda}_k^a(r)\}_{k\in\mathbb{N}}$ of eigenvalues (known as variational eigenvalues) such that $\hat{\lambda}_k^a(r) \to +\infty$ as $k \to \infty$. We do not know if this sequence exhausts $\hat{\sigma}(r)$.

If $r = 2$ (linear eigenvalue problem), then using the spectral theorem for compact operators, we have a full description of the spectrum $\hat{\sigma}(2)$ which consists of a sequence $\{\hat{\lambda}_k(2)\}_{k\in\mathbb{N}}$ of distinct eigenvalues with $\hat{\lambda}_k(2) \to +\infty$. The eigenspaces $E(\hat{\lambda}_k(2))$ are finite dimensional linear spaces and we have $E(\hat{\lambda}_k(2)) \subseteq C_0^1(\overline{\Omega})$ for all $k \in \mathbb{N}$. We have variational characterizations for all the eigenvalues:

$$\hat{\lambda}_1^a(2) = \inf\left\{\frac{\int_\Omega a(z)|Du|^2dz}{\|u\|_2^2} : u \in H_0^1(\Omega), u \neq 0\right\}, \tag{2.3}$$

$$\begin{aligned} m \geq 2 \;\; \hat{\lambda}_m^a(2) &= \inf\left\{\frac{\int_\Omega a(z)|Du|^2dz}{\|u\|_2^2} : u \in V = \overline{\bigoplus_{k\geq m} E(\hat{\lambda}_k(2))}, u \neq 0\right\} \\ &= \sup\left\{\frac{\int_\Omega a(z)|Du|^2dz}{\|u\|_2^2} : u \in V = \bigoplus_{k=1}^{m} E(\hat{\lambda}_k(2)), u \neq 0\right\}. \end{aligned} \tag{2.4}$$

The extrema in (2.3) and (2.4) are realized on the corresponding eigenspaces.

The next inequalities are easy consequences of the aforementioned properties (see Hu–Papageorgiou [5]).

Proposition 2.1. *If $\hat{\eta} \in L^\infty(\Omega)$, $\hat{\eta}(z) \leq \hat{\lambda}_1^a(r)$ for a.a. $z \in \Omega$ and $\hat{\eta} \not\equiv \hat{\lambda}_1^a(r)$, then there exists $c_1 > 0$ such that*

$$c_1\|u\|^r \leq \|Du\|_r^r - \int_\Omega \hat{\eta}(z)|u|^r dz \quad \textit{for all } u \in W_0^{1,r}(\Omega).$$

Proposition 2.2.

(a) If $\hat{\eta} \in L^\infty(\Omega)$, $\hat{\eta}(z) \leq \hat{\lambda}_k^a(2)$ for a.a. $z \in \Omega$ and $\hat{\eta} \not\equiv \hat{\lambda}_k^a(2)$, then there exists $c_2 > 0$ such that

$$c_2\|u\|^2 \leq \|Du\|_2^2 - \int_\Omega \hat{\eta}(z)u^2dz \quad \textit{for all } u \in V = \overline{\bigoplus_{m\geq k} E(\hat{\lambda}_m(2))}.$$

(b) If $\hat{\eta} \in L^\infty(\Omega)$, $\hat{\eta}(z) \geq \hat{\lambda}_k^a(2)$ for a.a. $z \in \Omega$ and $\hat{\eta} \not\equiv \hat{\lambda}_k^a(2)$, then there exists $c_3 > 0$ such that

$$\|Du\|_2^2 - \int_\Omega \hat{\eta}(z)u^2\,dz \leq -c_3\|u\|_2^2 \quad \textit{for all } u \in V = \bigoplus_{m=1}^{k} E(\hat{\lambda}_m(2)).$$

Let $A_p^{a_1}: W_0^{1,p}(\Omega) \to W^{-1,p'}(\Omega) = W_0^{1,p}(\Omega)^* \left(p' = \frac{p}{p-1}\right)$ and $A_q^{a_2}: W_0^{1,q}(\Omega) \to W^{-1,q'}(\Omega) = W_0^{1,q}(\Omega)^*$ be defined by

$$\langle A_p^{a_1}(u), h\rangle = \int_\Omega a_1(z)|Du|^{p-2}(Du, Dh)_{\mathbb{R}^N}\, dz \quad \text{for all } u, h \in W_0^{1,p}(\Omega),$$
$$\langle A_q^{a_2}(u), h\rangle = \int_\Omega a_2(z)|Du|^{q-2}(Du, Dh)_{\mathbb{R}^N}\, dz \quad \text{for all } u, h \in W_0^{1,q}(\Omega).$$

We set $V = A_p^{a_1} + A_q^{a_2}$ (recall $W_0^{1,p}(\Omega) \hookrightarrow W_0^{1,q}(\Omega)$).

Proposition 2.3. *The operators $A_p^{a_1}$, $A_q^{a_2}$, V are bounded (map bounded sets to bounded ones), continuous, strictly monotone (thus maximal monotone too), coercive and of type $(S)_+$, that is,*

$$\textit{if } u_n \rightharpoonup u \textit{ in } X \quad (\textit{with } X = W_0^{1,p}(\Omega) \textit{ or } X = W_0^{1,q}(\Omega))$$

and $\limsup_{n\to\infty}\langle G(u_n), u_n - u\rangle \le 0$ *(with $G = A_p^{a_1}$ or $G = A_q^{a_2}$ or $G = V$), then $u_n \to u$ in X.*

Let X be a Banach space and $\varphi \in C^1(X)$. We say that $\varphi(\cdot)$ satisfies the "C-condition" if it has the following property:

"every sequence $\{u_n\}_{n\in\mathbb{N}} \subseteq X$ such that $|\varphi(u_n)| \le M$ for some $M > 0$, all $n \in \mathbb{N}$, $(1 + \|u_n\|_X)\varphi'(u_n) \to 0$ in X^* as $n \to \infty$, admits a strongly convergent subsequence".

We introduce the following sets:

$$K_\varphi = \{u \in X : \varphi'(u) = 0\} \quad \text{(critical set of } \varphi(\cdot)),$$
$$\varphi^c = \{u \in X : \varphi(u) \le c\} \quad \text{for every } c \in \mathbb{R}.$$

If $Y_2 \subseteq Y_1 \subseteq X$ and $k \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$, then by $H_k(Y_1, Y_2)$ we denote the k^{th}-singular homology group with real coefficients (so the critical groups are vector spaces). Let $u \in K_\varphi$ be isolated and $c = \varphi(u)$. The critical groups of $\varphi(\cdot)$ at u are defined by

$$C_k(\varphi, u) = H_k(\varphi^c \cap U, \varphi^c \cap U \setminus \{u\}) \quad \text{for all } k \in \mathbb{N}_0,$$

where U is a neighborhood of u such that $K_\varphi \cap U \cap \varphi^c = \{u\}$. The excision property of singular homology implies that this definition is independent of the isolating neighborhood U.

If $u, v : \Omega \to \mathbb{R}$ are measurable functions such that $u(z) \le v(z)$ for a.a. $z \in \Omega$". We define

$$[u, v] = \{h \in W_0^{1,p}(\Omega) : u(z) \le h(z) \le v(z) \text{ for a.a. } z \in \Omega\}$$
$$\text{int}_{C_0^1(\overline{\Omega})}[u, v] = \text{ interior in } C_0^1(\overline{\Omega}) \text{ of } [u, v] \cap C_0^1(\overline{\Omega}).$$

Also given $u : \Omega \to \mathbb{R}$ measurable, we write $0 \prec u$ if for every $K \subseteq \Omega$ compact we have $0 < c_K \le u(z)$ for a.a $z \in K$. Also, we set $u^\pm = \max\{\pm u, 0\}$ and we have $u = u^+ - u^-$, $|u| = u^+ + u^-$ and if $u \in W_0^{1,p}(\Omega)$, then $u^\pm \in W_0^{1,p}(\Omega)$. By $|\cdot|_N$ we denote the Lebesgue measure on $\mathbb{R}^N$.

The hypotheses on the data of (1.1), are the following:

H_0 $a_1, a_2 \in C^{0,1}(\overline{\Omega})$, $0 < \hat{c} \le a_1(z), a_2(z)$ for all $z \in \overline{\Omega}$, $1 < q < p < N$.

H_1 $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function such that

(i) $|f(z, x)| \le \hat{a}(z)[1 + |x|^{p-1}]$ for a.a. $z \in \Omega$, all $x \in \mathbb{R}$, with $\hat{a} \in L^\infty(\Omega)$;

(ii) there exists $\hat{\eta} \in L^\infty(\Omega)$ such that

$$\hat{\lambda}_1^{a_1}(p) \le \hat{\eta}(z) \text{ for a.a. } z \in \Omega, \quad \hat{\eta} \not\equiv \hat{\lambda}_1^{a_1}(p),$$

$$\hat{\eta}(z) \le \liminf_{x \to \pm\infty} \frac{f(z,x)}{|x|^{p-2}x} \text{ uniformly for a.a. } z \in \Omega;$$

(iii) there exists $\eta \in L^\infty(\Omega)$ such that

$$\hat{\lambda}_1^{a_2}(q) \le \eta(z) \text{ for a.a. } z \in \Omega, \quad \eta \not\equiv \hat{\lambda}_1^{a_2}(q),$$

$$\eta(z) \le \liminf_{x \to 0} \frac{f(z,x)}{|x|^{q-2}x} \text{ uniformly for a.a. } z \in \Omega;$$

(iv) there exist real numbers $\theta_- < 0 < \theta_+$ such that

$$f(z,\theta_+) \le -l < 0 < l \le f(z,\theta_-) \text{ for a.a. } z \in \Omega;$$

(v) for every $\rho > 0$, there exists $\hat{\xi}_\rho > 0$ such that for a.a. $z \in \Omega$, the function $x \to f(z,x) + \hat{\xi}_\rho |x|^{p-2}x$ is nondecreasing on $[-\rho,\rho]$.

Remark 2.4. Hypotheses H_1(i),(ii) imply that $f(z,\cdot)$ is $(p-1)$-linear as $x \to \pm\infty$. We can have partial interaction with $\hat{\lambda}_1^{a_1}(p) > 0$ from the right (nonuniform nonresonance). Similarly as $x \to 0$ with respect to $\hat{\lambda}_1^{a_2}(q) > 0$, Hypothesis H_1(i),(iii) imply that $f(z,0) = 0$ for a.a. $z \in \Omega$. Hypothesis H_1(iv) implies that $f(z,\cdot)$ is necessarily sign changing (it exhibits an oscillatory behavior near zero). Later (see hypotheses H_2 in Section 4), we will see that by requiring that $q = 2$, we can relax hypothesis H_1(iii) and incorporate in our framework reactions which satisfy the sign condition (that is, $f(z,x)x \ge 0$ for a.a. $z \in \Omega$, all $x \in \mathbb{R}$).

In what follows, for notational economy, we write

$$\rho_{a_1,p}(Du) = \int_\Omega a_1(z)|Du|^p\,dz,$$
$$\rho_{a_2,q}(Du) = \int_\Omega a_2(z)|Du|^q\,dz \quad \text{for all } u \in W_0^{1,p}(\Omega).$$

Let $\varphi : W_0^{1,p}(\Omega) \to \mathbb{R}$ be the energy functional for problem (1.1) defined by

$$\varphi(u) = \frac{1}{p}\rho_{a_1,p}(Du) + \frac{1}{q}\rho_{a_2,q}(Du) - \int_\Omega F(z,u)\,dz \quad \text{for all } u \in W_0^{1,p}(\Omega),$$

where $F(z,x) = \int_0^x f(z,s)ds$. Evidently $\varphi \in C^1(W_0^{1,p}(\Omega))$.

To produce solutions of constant sign, we introduce the positive and negative truncations of $\varphi(\cdot)$, namely the C^1-functionals $\varphi_\pm : W_0^{1,p}(\Omega) \to \mathbb{R}$ defined by

$$\varphi_\pm(u) = \frac{1}{p}\rho_{a_1,p}(Du) + \frac{1}{q}\rho_{a_2,q}(Du) - \int_\Omega F(z,\pm u^\pm)dz \quad \text{for all } u \in W_0^{1,p}(\Omega).$$

3 Comparison theorems

In this section, we prove two strong comparison theorems which we will use in the sequel and which are of independent interest.

Given two measurable functions $u, v : \Omega \to \mathbb{R}$, we write

$$u \prec v \quad \text{if } 0 \prec u - v \quad \text{(see Section 2).}$$

Proposition 3.1. *If hypotheses H_0 hold, $\hat{\xi} \geq 0$, $h_1, h_2 \in L^\infty(\Omega)$ and $h_1 \prec h_2$ and $u \in C_0^1(\overline{\Omega}) \setminus \{0\}$, $v \in \mathrm{int}C_+$ satisfy the equations*

$$\begin{aligned} &-\Delta_p^{a_1} u - \Delta_q^{a_2} u + \hat{\xi}|u|^{p-2}u = h_1 \quad \text{in } \Omega, \\ &-\Delta_p^{a_1} v - \Delta_q^{a_2} v + \hat{\xi} v^{p-1} = h_2 \quad \text{in } \Omega, \end{aligned}$$

then $v - u \in \mathrm{int}\, C_+$.

Proof. From Theorem 3.4.1, p. 61 of Pucci–Serrin [16] (the weak comparison principle), we have

$$u(z) \leq v(z) \quad \text{for all } z \in \overline{\Omega}. \tag{3.1}$$

We introduce the following closed subsets of Ω:

$$E_0 = \{z \in \Omega : u(z) = v(z)\}, \quad E_1 = \{z \in \Omega : Du(z) = Dv(z) = 0\}.$$

First we show that $E_0 \subseteq E_1$. To this end let $\hat{z} \in E_0$ and set $y = v - u$. Then $y \in C_+ \setminus \{0\}$ and $y(\hat{z}) = \min_{\overline{\Omega}} y$. Hence

$$\begin{aligned} &Dy(\hat{z}) = 0, \\ \Rightarrow\ &Dv(\hat{z}) = Du(\hat{z}). \end{aligned}$$

If $\hat{z} \notin E_1$, then $Dv(\hat{z}) = Du(\hat{z}) \neq 0$ and so we can find an open ball B centered at $\hat{z}$ such that $Dv(z) \neq 0$, $Du(z) \neq 0$ for all $z \in B$. As in the proof of Proposition 2.1 of Guedda–Véron [3], we show that there exists $K \in W^{1,\infty}(\Omega, \mathbb{R}^{N\times N})$ such that

$$-\mathrm{div}(K(z)Dy) = h_2(z) - h_1(z) - \hat{\xi}(v^{p-1} - |u|^{p-2}u) \text{ in } B. \tag{3.2}$$

By taking B even smaller if necessary, we can apply the strong maximum principle of Pucci–Serrin [16] (p. 111) and obtain

$$y(z) > 0 \quad \text{for all } z \in B,$$

$$\Rightarrow u(z) < v(z) \quad \text{for all } z \in B,$$

a contradiction since $\hat{z} \in E_0 \cap B$. This proves that

$$E_0 \subseteq E_1. \tag{3.3}$$

Since $v \in \mathrm{int}\, C_+$, it follows that $E_1 \subseteq \Omega$ is compact. Hence E_0 is compact too (see (3.3)). So, we can find $\Omega' \subset \Omega$ open such that

$$E_0 \subseteq \Omega' \subseteq \overline{\Omega'} \subseteq \Omega. \tag{3.4}$$

From (3.4) and since $\overline{\Omega'}$ is compact, we can find $\varepsilon > 0$ such that

$$u(z) + \varepsilon \le v(z) \quad \text{for all } z \in \partial\Omega', \tag{3.5}$$

$$h_1(z) + \varepsilon \le h_2(z) \quad \text{for all } z \in \overline{\Omega'} \quad (\text{recall } h_1 \prec h_2). \tag{3.6}$$

We choose $\delta \in (0, \varepsilon)$ small such that

$$\hat{\xi}\left||s|^{p-2}s - |t|^{p-2}t\right| \le \varepsilon \quad \text{for all } s,t \in [\min_{\overline{\Omega}} u, \|v\|_\infty],\ |t-s| \le \delta. \tag{3.7}$$

Then we have

$$\begin{aligned}
&- \Delta_p^{a_1}(u+\delta) - \Delta_q^{a_2}(u+\delta) + \hat{\xi}|u+\delta|^{p-2}(u+\delta)\\
&\qquad = -\Delta_p^{a_1}u - \Delta_q^{a_2}u + \hat{\xi}|u+\delta|^{p-2}(u+\delta)\\
&\qquad = h_1 - \hat{\xi}|u|^{p-2}u + \hat{\xi}|u+\delta|^{p-2}(u+\delta)\\
&\qquad \le h_2 - \varepsilon + \varepsilon \quad (\text{see (3.6), (3.7)})\\
&\qquad = -\Delta_p^{a_1}v - \Delta_q^{a_2}v + \hat{\xi}v^{p-1} \quad \text{in } \Omega',\\
&\Rightarrow u + \delta \le v \quad \text{in } \Omega' \quad (\text{by the weak comparison principle, see [16], p. 61})\\
&\Rightarrow E_0 = \emptyset \quad (\text{see (2.4)}),\\
&\Rightarrow u(z) < v(z) \quad \text{for all } z \in \Omega.
\end{aligned}$$

Now let $z_0 \in \partial\Omega$ and let $z_1 = z_0 - 2rn$ with $n = n(z_0)$ the outward unit normal at $z_0 \in \partial\Omega$. For $r \in (0,1)$ small, we consider the annulus

$$R = \{z \in \Omega : r \le |z - z_1| \le 2r\}$$

and let $m = \min\{y(z) : z \in \partial B_r(z_1)\} > 0$. From the proof of Proposition 2.3 of Papageorgiou–Vetro–Vetro [13], we can find $\widetilde{w} \in C^1(\overline{R}) \cap C^2(R)$ satisfying

$$-\Delta_p^{a_1}\widetilde{w} - \Delta_q^{a_2}\widetilde{w} + \hat{\xi}|\widetilde{w}|^{p-2}\widetilde{w} \le 0 \quad \text{in } R$$

with $\widetilde{w} \le u$ on ∂R. Then by the weak comparison principle, we have

$$\widetilde{w} \le u \quad \text{in } R.$$

Moreover, it follows $\widetilde{w}(z_0) = 0$, $\frac{\partial y}{\partial n}(z_0) \le \frac{\partial \widetilde{w}}{\partial n}(z_0)$ and so $y \in \operatorname{int} C_+$,

$$\Longrightarrow \ v - u \in \operatorname{int} C_+. \qquad \square$$

In the second comparison result, we strengthen the relation between the two forcing terms h_1, h_2 but relax the conditions on u, v.

Proposition 3.2. *If hypothesis H_0 holds, $\hat{\xi} \ge 0$, $h_1, h_2 \in L^1(\Omega)$ satisfy*

$$\alpha \le \hat{c}_0 \le h_2(z) - h_1(z) \quad \textit{for a.a. } z \in \Omega$$

and $u, v \in C^1(\overline{\Omega}) \setminus \{0\}$ with $0 < v(z)$ for all $z \in \Omega$ satisfy

$$\begin{aligned}
&- \Delta_p^{a_1}u - \Delta_q^{a_2}u + \hat{\xi}|u|^{p-2}u = h_1 \quad \textit{in } \Omega,\\
&- \Delta_p^{a_1}v - \Delta_q^{a_2}v + \hat{\xi}|v|^{p-1} = h_2 \quad \textit{in } \Omega, \quad u \le v \textit{ on } \partial\Omega,
\end{aligned}$$

then $u(z) < v(z)$ for all $z \in \Omega$.

Proof. Let $\eta(z,y) = a_1(z)|y|^{p-2}y + a_2(z)|y|^{q-2}y$ for all $z \in \Omega$, all $y \in \mathbb{R}^N$. We have

$$-\operatorname{div}(\eta(z,Dv) - \eta(z,Du)) = h_2 - h_1 - \hat{\xi}(v^{p-1} - |u|^{p-2}u) \quad \text{in } \Omega. \tag{3.8}$$

As before the weak comparison principle implies $u \leq v$. Let $y = v - u$. Then $y \in C^1(\overline{\Omega})$, $y(z) \geq 0$ for all $z \in \Omega$ and as in Guedda–Véron [3], we have

$$-\operatorname{div}(K(z)Dy(z)) = h_2 - h_1 - \hat{\xi}(v^{p-1} - |u|^{p-2}u) \quad \text{in } \Omega,$$

with $K \in W^{1,\infty}(\Omega, \mathbb{R}^{N\times N})$.

Suppose that for some $\hat{z} \in \Omega$, we have $y(\hat{z}) = 0$, hence $v(\hat{z}) = u(\hat{z})$. Then from (3.8) and the hypothesis on h_1, h_2 we can find $r \in (0,1)$ small such that

$$-\operatorname{div}(K(z)Dy(z)) \geq \hat{c}_1 > 0 \quad \text{in } B_r(\hat{z}) = \{z \in \mathbb{R}^N : |z - \hat{z}| \leq r\} \subseteq \Omega.$$

Invoking Theorem 4 of Vázquez [18], we obtain

$$0 < y(z) \quad \text{for all } z \in B_r(\hat{z}),$$

a contradiction since $y(\hat{z}) = 0$. It follows that

$$u(z) < v(z) \quad \text{for all } z \in \Omega. \qquad \square$$

In the next section we will use these comparison results to produce multiple constant sign solutions of (1.1).

4 Constant sign solutions

We start by obtaining constant sign solutions which are local minimizers of the energy functional $\varphi(\cdot)$.

Proposition 4.1. *If hypotheses H_0, H_1 (i), (iii), (iv), (v) hold, then problem (1.1) has two constant sign solutions*

$$u_0 \in \operatorname{int} C_+ \quad \textit{with} \quad u(z) < \theta_+ \quad \textit{for all } z \in \overline{\Omega},$$

$$v_0 \in -\operatorname{int} C_+ \quad \textit{with} \quad \theta_- < v(z) \quad \textit{for all } z \in \overline{\Omega}$$

and u_0 is a local minimizer of the functionals φ, φ_+, v_0 is a local minimizer of the functionals φ, φ_-.

Proof. First we produce the positive solution. To this end we introduce the Carathéodory function $\hat{f}_+ : \Omega \times \mathbb{R} \to \mathbb{R}$ defined by

$$\hat{f}_+(z,x) = \begin{cases} f(z,x^+) & \text{if } x \leq \theta_+, \\ f(z,\theta_+) & \text{if } \theta_+ < x. \end{cases} \tag{4.1}$$

We set $\hat{F}_+(z,x) = \int_0^x \hat{f}_+(z,s)ds$ and consider the C^1-functional $\hat{\varphi}_+ : W_0^{1,p}(\Omega) \to \mathbb{R}$ defined by

$$\hat{\varphi}_+(u) = \frac{1}{p}\rho_{a_1,p}(Du) + \frac{1}{q}\rho_{a_2,q}(Du) - \int_\Omega \hat{F}_+(z,u)\,dz \quad \text{for all } u \in W_0^{1,p}(\Omega).$$

From (4.1) and hypotheses H_0, we see that $\hat{\varphi}_+(\cdot)$ is coercive. Also, using the Sobolev embedding theorem, we have that $\hat{\varphi}_+(\cdot)$ is sequentially weakly lower semicontinuous, so by the Weierstrass–Tonelli theorem, we can find $u_0 \in W_0^{1,p}(\Omega)$ such that

$$\hat{\varphi}_+(u_0) = \inf\left\{\hat{\varphi}_+(u) : u \in W_0^{1,p}(\Omega)\right\}. \tag{4.2}$$

On account of hypothesis H_1(iii), given $\varepsilon > 0$, we can find $\delta \in (0, \theta_+)$ such that

$$\frac{1}{q}[\eta(z) - \varepsilon]x^q \le F(z,x) \quad \text{for a.a. } z \in \Omega, \text{ all } 0 \le x \le \delta. \tag{4.3}$$

From Section 2, we know that $\hat{u}_1^{a_2}(q) \in \operatorname{int} C_+$. So, we can find $t \in (0,1)$ small such that

$$0 \le t\hat{u}_1^{a_2}(q)(z) \le \delta \quad \text{for all } z \in \overline{\Omega}. \tag{4.4}$$

Using (4.3), (4.4) and (4.1), we have

$$\begin{aligned}\hat{\varphi}_+(t\hat{u}_1^{a_2}(q)) \le{}& \frac{t^p}{p}\rho_{a_1,p}(D\hat{u}_1^{a_2}(q)) \\ &+ \frac{t^q}{q}\left[\int_\Omega (\hat{\lambda}_1^{a_2}(q) - \eta(z))(\hat{u}_1^{a_2}(q)^q)\,dz + \varepsilon\right] \text{ (recall that } \|\hat{u}_1^{a_2}(q)\|_q = 1). \end{aligned} \tag{4.5}$$

Since by hypothesis the inequality $\hat{\lambda}_1^{a_2}(q) \le \eta(z)$ is strict on a set of positive Lebesgue measure, we have

$$\beta = \int_\Omega (\eta(z) - \hat{\lambda}_1^{a_2}(q))\hat{u}_1^{a_2}(q)^q\,dz > 0.$$

So, choosing $\varepsilon \in (0,\beta)$ small, from (4.5), we obtain

$$\hat{\varphi}_+(t\hat{u}_1^{a_2}(q)) \le c_4 t^p - c_5 t^q \quad \text{for some } c_4, c_5 > 0.$$

Since $q < p$, choosing $t \in (0,1)$ even smaller if necessary, we have

$$\begin{aligned}&\hat{\varphi}_+(t\hat{u}_1^{a_2}(q)) < 0,\\ \Rightarrow\ &\hat{\varphi}_+(u_0) < 0 = \hat{\varphi}_+(0) \quad (\text{see } (4.2)),\\ \Rightarrow\ &u_0 \ne 0.\end{aligned}$$

From (4.2) we have

$$\begin{aligned}&\langle \hat{\varphi}_+'(u_0), h\rangle = 0 \quad \text{for all } h \in W_0^{1,p}(\Omega),\\ \Rightarrow\ &\langle V(u_0), h\rangle = \int_\Omega \hat{f}_+(z,u_0)h\,dz \quad \text{for all } h \in W_0^{1,p}(\Omega).\end{aligned} \tag{4.6}$$

In (4.6) first we choose the test function $h = -u_0^- \in W_0^{1,p}(\Omega)$. Then from (4.1) we have

$$\hat{c}\|Du_0^-\|_p^p \le 0 \Rightarrow u_0 \ge 0, \quad u_0 \ne 0.$$

Next in (4.6) we choose the test function $(u_0 - \theta_+)^+ \in W_0^{1,p}(\Omega)$. Then

$$\begin{aligned}\langle V(u_0), (u_0-\theta_+)^+\rangle &= \int_\Omega f(z,\theta_+)(u_0-\theta_+)^+\,dz \quad (\text{see } (4.1))\\ &\le 0 \quad (\text{see hypothesis } H_1(\text{iv}))\\ &= \langle V(\theta_+), (u_0-\theta_+)^+\rangle\\ &\Rightarrow u_0 \le \theta_+.\end{aligned}$$

So, we have proved that

$$u_0 \in [0, \theta_+], \quad u_0 \neq 0. \tag{4.7}$$

From (4.7), (4.1) and (4.6), we infer that u_0 is a nontrivial nonnegative solution of (1.1). Evidently $u_0 \in L^\infty(\Omega)$ and then the nonlinear regularity theory of Lieberman [7] implies $u_0 \in C_+ \setminus \{0\}$. Let $\rho = \|u_0\|_\infty$ and let $\hat{\xi}_\rho > 0$ be as postulated by hypothesis H_1(v). We have

$$\begin{aligned} & -\Delta_p^{a_1} u_0 - \Delta_q^{a_2} u_0 + \hat{\xi}_\rho u_0^{p-1} \geq 0 \quad \text{in } \Omega, \\ \Rightarrow\ & u_0 \in \operatorname{int} C_+ \quad \text{(see Pucci–Serrin [16], pp. 111, 120).} \end{aligned} \tag{4.8}$$

We have

$$\begin{aligned} -\Delta_p^{a_1} u_0 - \Delta_q^{a_2} u_0 + \hat{\xi}_\rho u_0^{p-1} &= f(z, u_0) + \hat{\xi}_\rho u_0^{p-1} \quad \text{(see (4.7), (4.1))} \\ &\leq f(z, \theta_+) + \hat{\xi}_\rho \theta_+^{p-1} \quad \text{(see (4.7) and hypothesis } H_1\text{(v))} \\ &\leq -l + \hat{\xi} \theta_+^{p-1} \quad \text{(see hypothesis } H_1\text{(iv))} \\ &\leq -\Delta_p^{a_1} \theta_+ - \Delta_q^{a_2} \theta_+ + \hat{\xi}_\rho \theta_+^{p-1} \quad \text{in } \Omega. \end{aligned}$$

Invoking Proposition 3.2, we infer that

$$u_0(z) < \theta_+ \quad \text{for all } z \in \overline{\Omega}. \tag{4.9}$$

Then (4.8) and (4.9) imply that

$$u_0 \in \operatorname{int}_{C_0^1(\overline{\Omega})} [0, \theta_+]. \tag{4.10}$$

Note that

$$\hat{\varphi}_+ \big|_{[0,\theta_+]} = \varphi_+ \big|_{[0,\theta_+]} \quad \text{(see (4.1)).} \tag{4.11}$$

Recall that u_0 is a minimizer of $\hat{\varphi}_+(\cdot)$. So, from (4.10) and (4.11) it follows that

$$\begin{aligned} & u_0 \text{ is a local } C_0^1(\overline{\Omega})\text{-minimizer of } \varphi_+(\cdot), \\ \Rightarrow\ & u_0 \text{ is a local } W_0^{1,p}(\Omega)\text{-minimizer of } \varphi_+(\cdot). \\ & \text{(see Hu–Papageorgiou [5, p. 339]).} \end{aligned} \tag{4.12}$$

Let $W_+ = \{u \in W_0^{1,p}(\Omega) : u(z) \geq 0 \quad \text{for a.a. } z \in \Omega\}$. We have

$$\varphi\big|_{W_+} = \varphi_+\big|_{W_+}, \quad u_0 \in \operatorname{int} W_+. \tag{4.13}$$

So, from (4.12) and (4.13) it follows that

$$u_0 \text{ is a local } W_0^{1,p}(\Omega)\text{-minimizer of } \varphi(\cdot).$$

For the negative solution, we start with the Carathéodory function $\hat{f}_-(z, x)$ defined by

$$\hat{f}_-(z, x) = \begin{cases} f(z, \theta_-) & \text{if } x < \theta_-, \\ f(z, -x^-) & \text{if } \theta_- \leq x. \end{cases}$$

We set $\hat{F}_-(z, x) = \int_0^x \hat{f}_-(z, s)\, ds$ and consider the C^1-functional $\hat{\varphi}_- : W_0^{1,p}(\Omega) \to \mathbb{R}$ defined by

$$\hat{\varphi}_-(u) = \frac{1}{p} \rho_{a_1,p}(Du) + \frac{1}{q} \rho_{a_2,q}(Du) - \int_\Omega \hat{F}_-(z, u)\, dz \quad \text{for all } u \in W_0^{1,p}(\Omega).$$

Working as above, this time with the functional $\hat{\varphi}_-(\cdot)$, we produce a negative solution $v_0 \in -\operatorname{int} C_+$, such that

$$v_0 \in \operatorname{int}_{C^1_0(\overline{\Omega})}[\theta_-, 0],$$

and v_0 is a local $W^{1,p}_0(\Omega)$ minimizer of both $\varphi_-(\cdot)$, $\varphi(\cdot)$. □

As we already mentioned, hypothesis H_1(iv) forces $f(z,\cdot)$ to be sign-changing. We can modify hypothesis H_1(iv) so as to include in our framework reactions which satisfy the sign condition. We can do this if we restrict to $q = 2$. The new hypotheses on $f(z,x)$ are the following:

$H_2 : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function such that

(i) same as hypothesis H_1(i);

(ii) same as hypothesis H_1(ii);

(iii) same as hypothesis H_1(iii) with $q = 2$;

(iv) there exist $\theta_- < 0 < \theta_+$ such that $f(z,\theta_+) \leq 0 \leq f(z,\theta_-)$ for a.a. $z \in \Omega$;

(v) same as hypothesis H_1(v).

Remark 4.2. We see that this set of hypotheses includes also reactions which satisfy the sign condition, that is, $f(z,x)x \geq 0$ for a.a. $z \in \Omega$, all $x \in \mathbb{R}$. Such functions were excluded under hypotheses H_1.

Proposition 4.3. *If hypotheses H_0, H_2(i), (iii), (iv), (v) hold, then problem (1.1) has at least two constant sign solutions*

$$u_0 \in \operatorname{int} C_+ \quad \textit{with} \quad u_0(z) < \theta_+ \quad \textit{for all } z \in \overline{\Omega},$$

$$v_0 \in -\operatorname{int} C_+ \quad \textit{with} \quad \theta_- < v_0(z) \quad \textit{for all } z \in \overline{\Omega},$$

u_0 is a local minimizer of the functionals φ, φ_+, v_0 is a local minimizer of the functionals φ, φ_-.

Proof. The proof is the same as that of Proposition 4.1 and it differs only at the point where we show that $u_0(z) < \theta_+$ (resp. $\theta_- < v_0(z)$) for all $z \in \overline{\Omega}$. In this case we argue as follows: let $\eta(z,y) = a_1(z)|y|^{p-2}y + a_2(z)|y|^{q-2}y$ for all $z \in \Omega$, all $y \in \mathbb{R}^N$. Then

$$\operatorname{div} \eta(z, Du) = \Delta^{a_1}_p u + \Delta^{a_2}_q u \quad \text{for all } u \in W^{1,p}_0(\Omega).$$

Since $2 < p$ (see hypothesis H_2(iii)), we see that $\eta(z,\cdot) \in C^1(\mathbb{R}^N, \mathbb{R}^N)$ and we have

$$\nabla_y \eta(z,y) = a_1(z)|y|^{p-2}\left[\mathrm{id} + (p-2)\frac{y \otimes y}{|y|^2}\right] + a_2(z)\mathrm{id}.$$

Therefore for any $h \in \mathbb{R}^N \setminus \{0\}$, we have

$$\begin{aligned} &(\nabla_y \eta(z,y)h, h)_{\mathbb{R}^N} \geq \hat{c}|h|^2, \\ \Rightarrow\ &\nabla_y \eta(z,\cdot) \text{ is positive definite.} \end{aligned} \tag{4.14}$$

Note that

$$\operatorname{div} \eta(z, Du_0) + f(z,u_0) \geq -\operatorname{div} \eta(z,\theta_+) + f(z,\theta_+) \quad \text{in } \Omega. \tag{4.15}$$

(see hypothesis H_2(iv)).

From (4.14), (4.15) and the tangency principle of Serrin [17] (see also Pucci–Serrin [16], p. 35), we have

$$u_0(z) < \theta_+ \quad \text{for all } z \in \overline{\Omega}.$$

Similarly, we show that $\theta_- < v_0(z)$ for all $z \in \overline{\Omega}$. □

Next we will show the existence of extremal constant sign solutions, that is, a minimal positive solution and a maximal negative solution. In Section 4 we will use these extremal constant sign solutions in order to produce a nodal (sign-changing) solutions.

On account of hypotheses H_1(i), (iii) (they are also valid when H_2 hold, with $q = 2$), given $\varepsilon > 0$, we can find $c_6 = c_6(\varepsilon) \geq 0$ such that

$$f(z, x)x \geq [\eta(z) - \varepsilon]|x|^q - c_6|x|^p \quad \text{for a.a. } z \in \Omega, \text{ all } x \in \mathbb{R}. \tag{4.16}$$

This unilateral growth condition suggests that we consider the following auxiliary Dirichlet problem

$$\begin{cases} -\Delta_p^{a_1} u(z) - \Delta_q^{a_2} u(z) = [\eta(z) - \varepsilon]|u(z)|^{q-2}u(z) - c_6|u(z)|^{p-2}u(z) & \text{in } \Omega, \\ u|_{\partial\Omega} = 0. \end{cases} \tag{4.17}$$

For this problem we have the following existence and uniqueness result.

Proposition 4.4. *If hypothesis H_0 holds, then for all $\varepsilon \in (0, 1)$ small problem has a unique positive solution*

$$\bar{u} \in \operatorname{int} C_+,$$

and since the equation is odd $\bar{v} = -\bar{u} \in -\operatorname{int} C_+$ is the unique negative solution of (4.17).

Proof. Let $\sigma_+ : W_0^{1,p}(\Omega) \to \mathbb{R}$ be the C^1-functional defined by

$$\sigma_+(u) = \frac{1}{p}\rho_{a_1,p}(Du) + \frac{1}{q}\rho_{a_2,q}(Du) - \frac{1}{q}\int_\Omega [\eta(z) - \varepsilon](u^+)^q \, dz + \frac{c_6}{p}\|u^+\|_p^p$$

for all $u \in W_0^{1,p}(\Omega)$.

Since $q < p$, we see that $\sigma_+(\cdot)$ is coercive. Also using the Sobolev embedding theorem, we show that $\sigma_+(\cdot)$ is sequentially weakly lower semicontinuous. So, we can find $\bar{u} \in W_0^{1,p}(\Omega)$ such that

$$\sigma_+(\bar{u}) = \inf\left\{\sigma_+(u) : u \in W_0^{1,p}(\Omega)\right\}. \tag{4.18}$$

As in the proof of Proposition 4.1, for $t \in (0, 1)$ small, we have

$$\begin{aligned} &\sigma_+(t\hat{u}_1^{a_2}(q)) < 0, \\ \Rightarrow\ &\sigma_+(\bar{u}) < 0 = \sigma_+(0) \quad (\text{see } (4.18)), \\ \Rightarrow\ &\bar{u} \neq 0. \end{aligned}$$

From (4.18) we have

$$\begin{aligned} &\langle \sigma_+'(\bar{u}), h\rangle = 0 \quad \text{for all } h \in W_0^{1,p}(\Omega), \\ \Rightarrow\ &\langle V(\bar{u}), h\rangle = \int_\Omega [\eta(z) - \varepsilon](\bar{u}^+)^{q-1} h \, dz - c_6 \int_\Omega (\bar{u}^+)^{p-1} h \, dz \end{aligned} \tag{4.19}$$

for all $h \in W_0^{1,p}(\Omega)$.

In (4.19) we choose the test function $h = \bar{u}^- \in W_0^{1,p}(\Omega)$ and obtain

$$\hat{c}\|D\bar{u}\|_p^p \leq 0, \quad \Rightarrow \quad \bar{u} \geq 0, \quad \bar{u} \neq 0.$$

Therefore $\bar{u}$ is a nontrivial positive solution of (4.17). Using Theorem 7.1, p. 286, of Ladyzhenskaya–Uraltseva [6], we have that $\bar{u} \in L^\infty(\Omega)$ and then the nonlinear regularity theory of Lieberman [7] implies that $\bar{u} \in C_+ \setminus \{0\}$. We have

$$\Delta_p^{a_1}\bar{u} + \Delta_q^{a_2}\bar{u} \leq c_6\bar{u}^{p-1} \quad \text{in } \Omega,$$

$$\Rightarrow \bar{u} \in \operatorname{int} C_+ \quad \text{(see Pucci–Serrin [16], p. 120).}$$

Now we show the uniqueness of this positive solution of (4.17). To this end, we introduce the integral functional $j : L^1(\Omega) \to \overline{\mathbb{R}} = \mathbb{R} \cup \{+\infty\}$ defined by

$$j(u) = \begin{cases} \frac{1}{p}\rho_{a_1,p}(Du^{1/q}) + \frac{1}{q}\rho_{a_2,q}(Du^{1/q}) & \text{if } u \geq 0,\ u^{1/q} \in W_0^{1,p}(\Omega), \\ +\infty & \text{otherwise.} \end{cases}$$

From Diaz-Saa [2], we know that $j(\cdot)$ is convex. Suppose that $\bar{y}$ is another positive solution of (4.17). Again we have $\bar{y} \in \operatorname{int} C_+$. On account of Proposition 4.1.22, p. 274, of Papageorgiou–Rădulescu–Repovš [12], implies that

$$\frac{\bar{u}}{\bar{y}} \in L^\infty(\Omega), \quad \frac{\bar{y}}{\bar{u}} \in L^\infty(\Omega). \tag{4.20}$$

Set $h = \bar{u}^q - \bar{y}^q \in C_0^1(\overline{\Omega})$. Then (4.20) implies that for $t > 0$ small, we have

$$\bar{u} + th \in \operatorname{dom} j, \quad \bar{y} + th \in \operatorname{dom} j,$$

where $\operatorname{dom} j = \{u \in L^1(\Omega) : j(u) < \infty\}$ (effective domain of $j(\cdot)$). Then we can compute the directional derivatives of $j(\cdot)$ at $\bar{u}^q$ and at $\bar{y}^q$ in the direction h. A simple computation using the convexity of $j(\cdot)$ and the nonlinear Green's theorem (see [12], p. 35), we obtain

$$\begin{aligned} j'(\bar{u}^q)(h) &= \frac{1}{q}\int_\Omega \frac{-\Delta_p^{a_1}\bar{u} - \Delta_q^{a_2}\bar{u}}{\bar{u}^{q-1}}(\bar{u}^q - \bar{y}^q)\,dz \\ &= \frac{1}{q}\int_\Omega \left[(\eta(z) - \varepsilon) - c_4\bar{u}^{p-q}\right](\bar{u}^q - \bar{y}^q)\,dz, \\ j'(\bar{y}^q)(h) &= \frac{1}{q}\int_\Omega \frac{-\Delta_p^{a_1}\bar{y} - \Delta_q^{a_2}\bar{y}}{\bar{y}^{q-1}}(\bar{u}^q - \bar{y}^q)\,dz \\ &= \frac{1}{q}\int_\Omega \left[(\eta(z) - \varepsilon) - c_4\bar{y}^{p-q}\right](\bar{u}^q - \bar{y}^q)\,dz. \end{aligned}$$

The convexity of $j(\cdot)$ implies the monotonicity of $j'(\cdot)$. So, we have

$$0 \leq \frac{c_6}{q}\int_\Omega \left(\bar{y}^{p-q} - \bar{u}^{p-q}\right)(\bar{u}^q - \bar{y}^q)\,dz,$$

$$\Rightarrow \bar{u} = \bar{y} \quad \text{(since } q < p).$$

Therefore $\bar{u} \in \operatorname{int} C_+$ is the unique positive solution of (4.17). The equation is odd, so $\bar{v} = -\bar{u} \in -\operatorname{int} C_+$ is the unique negative solution of (4.17). □

Let S_+ (resp. S_-) be the set of positive (resp. negative) solutions of (1.1). From Propositions 4.1 and 4.3 and the nonlinear regularity theory, we have

$$\emptyset \neq S_+ \subseteq \operatorname{int} C_+ \quad \text{and} \quad \emptyset \neq S_- \subseteq -\operatorname{int} C_+.$$

Next we show that the solutions $\bar{u} \in \operatorname{int} C_+$ (resp. $\bar{v} \in -\operatorname{int} C_+$) from Proposition 4.4, is a lower bound (resp. upper bound) for the elements of S_+ (resp. of S_-).

Proposition 4.5. *If hypotheses H_0 and H_1 or H_2 ($q = 2$) hold, then $\bar{u} \leq u$ for all $u \in S_+$, and $v \leq \bar{v}$ for all $v \in S_-$.*

Proof. For $u \in S_+ \subseteq \operatorname{int} C_+$ and consider the Carathéodory function $k_+ : \Omega \times \mathbb{R} \to \mathbb{R}$ defined by

$$k_+(z,x) = \begin{cases} (\eta(z) - \varepsilon)(x^+)^{q-1} - c_6(x^+)^{p-1} & \text{if } x \leq u(z), \\ (\eta(z) - \varepsilon)(u(z))^{q-1} - c_6(u(z))^{p-1} & \text{if } u(z) < x. \end{cases} \tag{4.21}$$

We set $K_+(z,x) = \int_0^x k_+(z,s)\,ds$ and consider the C^1-functional $\gamma_+ : W_0^{1,p}(\Omega) \to \mathbb{R}$ defined by

$$\gamma_+(u) = \frac{1}{p}\rho_{a_1,p}(Du) + \frac{1}{q}\rho_{a_2,q}(Du) - \int_\Omega K_+(z,u)\,dz \quad \text{for all } u \in W_0^{1,p}(\Omega).$$

Clearly $\gamma_+(\cdot)$ is coercive (see (4.21)). Also using the Sobolev embedding theorem, we see that $\gamma_+(\cdot)$ is sequentially weakly lower semicontinuous. So, we can find $\tilde{u} \in W_0^{1,p}(\Omega)$ such that

$$\gamma_+(\tilde{u}) = \inf\left\{\gamma_+(u) : u \in W_0^{1,p}(\Omega)\right\}. \tag{4.22}$$

As in the proof of Proposition 4.1, we choose $t \in (0,1)$ small so that $t\hat{u}_1^{a_2}(q) \leq u$ (recall $u \in \operatorname{int} C_+$ and use Proposition 4.1.22, p. 274 of [12]) and using (4.21) and since $q < p$, we have

$$\begin{aligned} &\gamma_+(t\hat{u}_1^{a_2}(q)) < 0 \ \ \text{(choosing } t \in (0,1) \text{ even smaller if necessary)}, \\ \Rightarrow\ &\gamma_+(\tilde{u}) < 0 = \gamma_+(0) \quad \text{(see (4.22))}, \\ \Rightarrow\ &\tilde{u} \neq 0. \end{aligned}$$

From (4.22) we have

$$\begin{aligned} &\langle \gamma_+'(\tilde{u}), h\rangle = 0 \quad \text{for all } h \in W_0^{1,p}(\Omega), \\ \Rightarrow\ &\langle V(\tilde{u}), h\rangle = \int_\Omega k_+(z,\tilde{u})h\,dz \quad \text{for all } h \in W_0^{1,p}(\Omega). \end{aligned} \tag{4.23}$$

In (4.23) we choose the test function $h = -\tilde{u}^- \in W_0^{1,p}(\Omega)$ and obtain

$$\hat{c}\|D\tilde{u}^-\|_p^p \leq 0 \quad \text{(see (4.21))},$$

$$\Rightarrow \tilde{u} \geq 0, \quad \tilde{u} \neq 0.$$

Next in (4.23), we choose the test function $h = (\tilde{u} - u)^+ \in W_0^{1,p}(\Omega)$. Then

$$\begin{aligned} \langle V(\tilde{u}), (\tilde{u}-u)^+\rangle &= \int_\Omega \left[(\eta(z) - \varepsilon)u^{q-1} - c_6 u^{p-1}\right](\tilde{u}-u)^+\,dz \quad \text{(see (4.21))} \\ &\leq \int_\Omega f(z,u)(\tilde{u}-u)^+\,dz \quad \text{(see (4.16))} \\ &= \langle V(u), (\tilde{u}-u)^+\rangle \quad \text{(since } u \in S_+). \\ &\Rightarrow \tilde{u} \leq u. \end{aligned}$$

So, we have proved that

$$\tilde{u} \in [0,u], \quad \tilde{u} \neq 0. \tag{4.24}$$

From (4.24), (4.21), (4.23) and Proposition 4.5, it follows that

$$\tilde{u} = \bar{u} \leq u \quad \text{for all } u \in S_+ \ (u \in S_+ \text{ was arbitrary}).$$

Similarly, we show that

$$v \leq \bar{v} \quad \text{for all } v \in S_-. \qquad \square$$

Using these bounds, we can produce the extremal constant sign solutions for problem (1.1).

Proposition 4.6. *If hypotheses H_0 and H_1 or H_2 $(q=2)$ hold, then there exist $u^* \in S_+$ and $v^* \in S_-$ such that $u^* \leq u$ for all $u \in S_+$, $v \leq v^*$ for all $v \in S_-$.*

Proof. We know that S_+ is downward directed, that is, if $u_1, u_2 \in S_+$, then there exists $u \in S_+$ such that $u \leq u_1$, $u \leq u_2$ (see Papageorgiou–Rădulescu–Repovš [11]). Then using Theorem 5.109, p. 308, of Papageorgiou–Rădulescu [4], we can find $\{u_n\}_{n\in\mathbb{N}} \subseteq S_+$ decreasing such that

$$\inf S_+ = \inf_{n\in\mathbb{N}} u_n.$$

From Proposition 4.5, we have

$$\bar{u} \leq u_n \leq u_1 \quad \text{for all } n \in \mathbb{N}. \tag{4.25}$$

Also, since $u_n \in S_+$ for all $n \in \mathbb{N}$, we have

$$\langle V(u_n), h\rangle = \int_\Omega f(z,u_n)h\,dz \quad \text{for all } h \in W_0^{1,p}(\Omega). \tag{4.26}$$

In (4.26) we choose the test function $h = u_n \in W_0^{1,p}(\Omega)$. Then using (4.25) and hypothesis $H_1(\mathrm{i}) = H_2(\mathrm{i})$, we infer that

$$\{u_n\}_{n\in\mathbb{N}} \subseteq W_0^{1,p}(\Omega) \quad \text{is bounded}.$$

So, we may assume that

$$u_n \xrightarrow{w} u^* \quad \text{in } W_0^{1,p}(\Omega), \quad u_n \to u^* \quad \text{in } L^p(\Omega). \tag{4.27}$$

In (4.26) we choose the test function $h = u_n - u^* \in W_0^{1,p}(\Omega)$, pass to the limit as $n \to \infty$ and use (4.27). Then

$$\lim_{n\to\infty} \langle V(u_n), u_n - u^*\rangle = 0,$$

$$\Rightarrow u_n \to u^* \quad \text{in } W_0^{1,p}(\Omega) \quad (\text{see Proposition 2.3}).$$

Passing to the limit as $n \to \infty$ in (4.26), we obtain

$$\langle V(u^*), h\rangle = \int_\Omega f(z,u^*)h\,dz \quad \text{for all } h \in W_0^{1,p}(\Omega),$$

$$\bar{u} \leq u^* \quad (\text{see } (4.25)).$$

Therefore, $u^* \in S_+$, $u^* = \inf S_+$.

To produce the maximal negative solution, we note that S_- is upward directed (that is, if $v_1, v_2 \in S_-$, there exists $v \in S_-$ such that $v_1 \le v$, $v_2 \le v$). Therefore, we can find $\{v_n\}_{n\in\mathbb{N}}$ increasing such that

$$\sup S_- = \sup_{n\in\mathbb{N}} v_n.$$

Reasoning as above, we produce $v^* \in S_-$ such that

$$v \le v^* \quad \text{for all } v \in S_-. \qquad \square$$

In Proposition 4.1, to obtain the solutions $u_0 \in \operatorname{int} C_+$ and $v_0 \in -\operatorname{int} C_+$, we did not use the asymptotic condition as $x \to \pm\infty$ (hypothesis H_1(ii) = H_2(ii)). If we bring this condition in the picture and strengthen hypotheses H_1(iii), H_2(iii), we can generate additional constant sign solutions.

The new hypotheses on $f(z,x)$ are the following:

H_1' : $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function such that H_1'(i), (ii), (iv), (v) are the same as the corresponding hypotheses H_1(i), (ii), (iv), (v) and

(iii) there exist $\tau \in (1,q)$, $\delta > 0$ and $c_0^* > 0$ such that

$$c_0^* |x|^\tau \le f(z,x)x \le \tau F(z,x) \quad \text{for a.a. } z \in \Omega, \text{ all } |x| \le \delta.$$

H_2' : $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function such that H_2'(i), (ii), (iv), (v) are the same as the corresponding hypotheses H_2(i), (ii), (iv), (v) and

(iii) $q = 2$ and there exist $\eta_1, \eta_2 \in L^\infty(\Omega)$ and $k \ge 2$ such that

$$\begin{aligned} &\hat{\lambda}_k^{a_2}(2) \le \eta_1(z) \le \eta_2(z) \le \hat{\lambda}_{k+1}^{a_2}(2) \quad \text{for a.a. } z \in \Omega, \\ &\eta_1 \not\equiv \hat{\lambda}_k^{a_2}(2), \quad \eta_2 \not\equiv \hat{\lambda}_{k+1}^{a_2}(2), \\ &\eta_1(z) \le \liminf_{x\to 0} \frac{f(z,x)}{x} \le \limsup_{x\to 0} \frac{f(z,x)}{x} \le \eta_2(z) \\ &\qquad\qquad\qquad\qquad \text{uniformly for a.a. } z \in \Omega. \end{aligned}$$

Proposition 4.7. *If hypotheses H_0 and H_1' or H_2' ($q = 2$) hold, then we can find two more constant sign solutions:*

$$\hat{u} \in \operatorname{int} C_+, \quad u_0 \le \hat{u}, \quad \hat{u} \ne u_0,$$
$$\hat{v} \in -\operatorname{int} C_+, \quad \hat{v} \le v_0, \quad \hat{v} \ne v_0.$$

Proof. From Proposition 4.1, we know that there is a positive solution $u_0 \in \operatorname{int} C_+$ such that

$$u_0 \in \operatorname{int}_{C_0^1(\overline{\Omega})} [0, \theta_+]$$

and u_0 is a local minimizer of $\varphi_+(\cdot)$. We assume that K_{φ_+} is finite, otherwise we already have an infinite number of positive smooth solutions and so we are done.

Claim: $\varphi_+(\cdot)$ satisfies the C-condition.

Consider a sequence $\{u_n\}_{n\in\mathbb{N}} \subseteq W_0^{1,p}(\Omega)$ such that $\{\varphi_+(u_n)\}_{n\in\mathbb{N}} \subseteq \mathbb{R}$ is bounded and

$$\left| \langle V(u_n), h \rangle - \int_\Omega f(z, u_n^+) h \, dz \right| \le \frac{\varepsilon_n \|h\|}{1 + \|u_n\|} \tag{4.28}$$

for all $h \in W_0^{1,p}(\Omega)$, all $n \in \mathbb{N}$ with $\varepsilon_n \to 0$.

In (4.28), we choose the test function $h = -u_n^- \in W_0^{1,p}(\Omega)$ and obtain

$$\hat{c}\|Du_n^-\|_p^p \le \varepsilon_n \quad \text{for all } n \in \mathbb{N},$$

$$\Rightarrow u_n^- \to 0 \quad \text{in } W_0^{1,p}(\Omega). \tag{4.29}$$

Suppose that $\|u_n^+\| \to \infty$ and let $y_n = \frac{u_n^+}{\|u_n^+\|}$, $n \in \mathbb{N}$. Then

$$\|y_n\| = 1,\ y_n \ge 0 \quad \text{for all } n \in \mathbb{N}.$$

So, we may assume that

$$y_n \xrightarrow{w} y \quad \text{in } W_0^{1,p}(\Omega), \quad y_n \to y \quad \text{in } L^p(\Omega), \quad y \ge 0. \tag{4.30}$$

From (4.28) and (4.29) it follows that

$$|\langle V(u_n^+), h\rangle| \le \varepsilon_n' \|h\| + \int_\Omega f(z, u_n^+) h \, dz \tag{4.31}$$
$$\text{for all } h \in W_0^{1,p}(\Omega), \text{ all } n \in \mathbb{N}, \text{ with } \varepsilon_n' \to 0^+.$$

$$\Rightarrow \int_\Omega a_1(z)|Dy_n|^{p-2}(Dy_n, Dh)_{\mathbb{R}^N} dz + \frac{1}{\|u_n\|^{p-q}} \int_\Omega a_2 |Dy_n|^{q-2}(Dy_n, Dh)_{\mathbb{R}^N} \, dz$$
$$\le \frac{\varepsilon_n'}{\|u_n^+\|^{p-1}}\|h\| + \int_\Omega \frac{f(z, u_n^+)}{\|u_n^+\|^{p-1}} h \, dz \quad \text{for all } h \in W_0^{1,p}(\Omega), \text{ all } n \in \mathbb{N}. \tag{4.32}$$

On account of hypothesis H_1(i)=H_2(i), we have that

$$\left\{ \frac{f(\cdot, u_n^+(\cdot))}{\|u_n^+\|^{p-1}} \right\}_{n \in \mathbb{N}} \subseteq L^{p'}(\Omega) \text{ is bounded.} \tag{4.33}$$

So using hypothesis H_1(ii) = H_2(ii), we may assume that

$$\frac{f(\cdot, u_n^+(\cdot))}{\|u_n^+\|^{p-1}} \xrightarrow{w} \hat{\eta}_0(\cdot) y^{p-1} \tag{4.34}$$

with $\hat{\eta}_0 \in L^\infty(\Omega)$, $\hat{\eta}(z) \le \hat{\eta}_0(z)$ for a.a. $z \in \Omega$. (see Aizicovici–Papageorgiou–Staicu [1], proof of Proposition 16). In (4.32), we choose the test function $h = y_n - y \in W_0^{1,p}(\Omega)$, pass to the limit as $n \to \infty$ and use (4.30) and (4.33). Then

$$\limsup_{n \to \infty} \int_\Omega a_1(z)|Dy_n|^{p-2}(Dy_n, Dy_n - Dy)_{\mathbb{R}^N} \, dz \le 0,$$
$$\Rightarrow y_n \to y \quad \text{in } W_0^{1,p}(\Omega), \quad \|y\| = 1, \quad y \ge 0 \tag{4.35}$$
(see Papageorgiou–Winkert [14], p. 665).

If in (4.32), we pass to the limit as $n \to \infty$ and use (4.34), (4.35), we obtain

$$\int_\Omega a_1(z)|Dy|^{p-2}(Dy, Dh)_{\mathbb{R}^N} \, dz = \int_\Omega \hat{\eta}_0(z) y^{p-1} h \, dz \quad \text{for all } h \in W_0^{1,p}(\Omega),$$
$$\Rightarrow -\Delta_p^{a_1} y = \hat{\eta}_0(z) y^{p-1} \quad \text{in } \Omega, \quad y|_{\partial\Omega} = 0. \tag{4.36}$$

Recall that $\hat{\lambda}_1^{a_1}(p) \le \hat{\eta}(z) \le \hat{\eta}_0(z)$ for a.a. $z \in \Omega$. Let $\tilde{\lambda}_1^{a_1}(p, \hat{\eta}_0)$ be the principal eigenvalue of $-\Delta_p^{a_1} y = \tilde{\lambda}\hat{\eta}_0(z)|y|^{p-2}y$ in Ω, $y|_{\partial\Omega} = 0$ (weighted version of the eigenvalue problem (2.1). From Proposition 4.133, p. 271, of Hu–Papageorgiou [5] we have

$$\tilde{\lambda}_1^{a_1}(p, \hat{\eta}_0) < \tilde{\lambda}_1^{a_1}(p, \hat{\lambda}_1^{a_1}(p)) = 1.$$

Then from (4.36) it follows that

$y(\cdot)$ is nodal or zero (see Hu–Papageorgiou [5, Proposition 4.127, p. 268]).

Both possibilities contradict (4.35). Therefore

$$\begin{aligned} &\{u_n^+\}_{n\in\mathbb{N}} \subseteq W_0^{1,p}(\Omega) \text{ is bounded,} \\ \Rightarrow\ &\{u_n\}_{n\in\mathbb{N}} \subseteq W_0^{1,p}(\Omega) \text{ is bounded (see (4.29)).} \end{aligned}$$

So, we may assume that

$$u_n \xrightarrow{w} u \quad \text{in } W_0^{1,p}(\Omega), \quad u_n \to u \quad \text{in } L^p(\Omega). \tag{4.37}$$

In (4.28) we choose the test function $h = u_n - u \in W_0^{1,p}(\Omega)$, pass to the limit as $n \to \infty$ and use (4.37). Then

$$\begin{aligned} &\lim_{n\to\infty} \langle V(u_n), u_n - u\rangle = 0 \\ \Rightarrow\ & u_n \to u \quad \text{in } W_0^{1,p}(\Omega) \quad \text{(see Proposition 2.3),} \\ \Rightarrow\ & \varphi_+(\cdot) \text{ satisfies the C-condition.} \end{aligned}$$

This proves the claim.

Recall that u_0 is a local minimizer of $\varphi_+(\cdot)$ and that K_{φ_+} is finite (see the beginning of the proof). These facts and Proposition 3.132, p. 179, of Hu–Papageorgiou [5], imply that we can find $\rho \in (0,1)$ small such that

$$\varphi_+(u_0) < \inf\{\varphi_+(u) : \|u - u_0\| = \rho\} = m_+. \tag{4.38}$$

Hypotheses H_1(i),(ii) = H_2(i),(ii) imply that given $\varepsilon > 0$, we can find $\delta = \delta(\varepsilon) > 0$ such that

$$F(z,x) \ge \frac{1}{p}[\hat{\eta}(z) - \varepsilon]|x|^p \quad \text{for a.a. } z \in \Omega, \text{ all } |x| \le \delta.$$

For $t \in (0,1)$, we have

$$\begin{aligned} \varphi_+(t\hat{u}_1^{a_1}(p)) \le \frac{t^p}{p}\int_\Omega [\hat{\lambda}_1^{a_1}(p) - \eta(z)]\hat{u}_1^{a_1}(p)^p\,dz + \frac{\varepsilon t^p}{p} + \frac{t^q}{t^q}\rho_{a_2,q}(D\hat{u}_1^{a_1}(p)) \\ \text{(since } \rho_{a_1,p}(D\hat{u}_1^{a_1}(p)) = \hat{\lambda}_1^{a_1}(p)\|\hat{u}_1^{a_1}(p)\|_p^p,\ \|\hat{u}_1^{a_1}(p)\|_p = 1). \end{aligned}$$

Since $\hat{u}_1^{a_1} \in \operatorname{int} C_+$ and the inequality $\hat{\lambda}_1^{a_1}(p) \le \eta(z)$ for a.a. $z \in \Omega$ is strict on a set of positive Lebesgue measure, it follows that

$$\hat{\beta} = \int_\Omega [\hat{\eta}(z) - \hat{\lambda}_1^{a_1}(p)]\hat{u}_1(p)^p\,dz > 0.$$

Choosing $\varepsilon \in (0, \hat{\beta})$ small, we obtain

$$\varphi_+(t\hat{u}_1^{a_1}(p)) \le c_7 t^q - c_8 t^p \quad \text{for some } c_7, c_8 > 0.$$

Since $q < p$, we infer that

$$\varphi_+(t\hat{u}_1(p)) \to -\infty \quad \text{as } t \to +\infty. \tag{4.39}$$

Then (4.38), (4.39) and the Claim, permit the use of the mountain pass theorem and obtain $\hat{u} \in W_0^{1,p}(\Omega)$ such that

$$\begin{aligned} &\hat{u} \in K_{\varphi_+} \subseteq \operatorname{int} C_+ \cup \{0\}, \quad m_+ \leq \varphi_+(\hat{u}), \\ \Rightarrow\ &\hat{u} \neq u_0. \end{aligned}$$

If we show that $\hat{u} \neq 0$, then $\hat{u} \in \operatorname{int} C_+$ is the second positive solution of (1.1). First assume that hypotheses H_1' hold. On account of hypothesis H_1'(iii) and of Lemma 5.125, p. 459, of Hu–Papageorgiou [5] we have

$$C_k(\varphi_+, 0) = 0 \quad \text{for all } k \in \mathbb{N}_0. \tag{4.40}$$

On the other hand since $\hat{u} \in K_{\varphi_+}$ is a critical point of mountain pass type, using Corollary 3.125, p. 178, of Hu–Papageorgiou [5], we have

$$C_1(\varphi_+, \hat{u}) \neq 0. \tag{4.41}$$

Comparing (4.40) and (4.41), we conclude that $\hat{u} \neq 0$. So $\hat{u} \in \operatorname{int} C_+$ is the second positive solution of (1.1).

Similarly working this time with the functional $\varphi_-(\cdot)$, we produce a second negative solution $\hat{v} \in -\operatorname{int} C_+$, $\hat{v} \neq v_0$. □

5 Nodal solutions

In this section we prove the existence of nodal solutions (sign-changing solutions) for problem (1.1) and state the complete multiplicity theorem.

First we consider the case where hypotheses H_1' hold.

Proposition 5.1. *If hypotheses H_0, H_1' hold, then problem* (1.1) *admits at least two nodal solutions*

$$y_0 \in \operatorname{int}_{C_0^1(\overline{\Omega})}[v_0, u_0],\ \hat{y} \in [v_0, u_0] \cap C_0^1(\overline{\Omega}),\ \hat{y} \neq y_0.$$

Proof. Let $u_* \in \operatorname{int} C_+$, $v_* \in -\operatorname{int} C_+$ be the two extremal constant sign solutions produced in Proposition 4.6. We introduce the Carathéodory function defined by

$$\gamma(z,x) = \begin{cases} f(z, v^*(z)) & \text{if } x < v^*(z), \\ f(z,x) & \text{if } v^*(z) \leq x \leq u^*(z), \\ f(z, u^*(z)) & \text{if } u^*(z) < x. \end{cases} \tag{5.1}$$

In addition, we introduce the positive and negative truncations of $\gamma_*(z,\cdot)$, namely the Carathéodory functions

$$\gamma_\pm(z,x) = \gamma(z, \pm x^\pm) \quad \text{for all } z \in \Omega,\ \text{all } x \in \mathbb{R}. \tag{5.2}$$

We set $\Gamma(z,x) = \int_0^x \gamma(z,s)\,ds$, $\Gamma_\pm(z,x) = \int_0^x \gamma_\pm(z,s)\,ds$ and consider the C^1-functionals $\mu, \mu_\pm : W_0^{1,p}(\Omega) \to \mathbb{R}$ defined by

$$\mu(u) = \frac{1}{p}\rho_{a_1,p}(Du) + \frac{1}{q}\rho_{a_2,q}(Du) - \int_\Omega \Gamma(z,u)\,dz,$$
$$\mu_\pm(u) = \frac{1}{p}\rho_{a_1,p}(Du) + \frac{1}{q}\rho_{a_2,q}(Du) - \int_\Omega \Gamma_\pm(z,u)\,dz \quad \text{for all } u \in W_0^{1,p}(\Omega).$$

Using (5.1) and (5.2), we can check easily that

$$K_\mu \subseteq [v_*, u_*] \cap C_0^1(\overline{\Omega}), \qquad K_{\mu_+} \subseteq [0, u_*] \cap C_+, \qquad K_{\mu_-} \subseteq [v_*, 0] \cap (-C_+).$$

The extremality of u^*, v^* implies that

$$K_\mu \subseteq [v_*, u_*] \cap C_0^1(\overline{\Omega}), \qquad K_{\mu_+} = \{0, u^*\}, \qquad K_{\mu_-} = \{0, v_*\}. \tag{5.3}$$

From (5.1) and (5.2) it is clear that $\mu_+(\cdot)$ is coercive. Also, it is sequentially weakly lower semicontinuous. So, we can find $\hat{u}^* \in W_0^{1,p}(\Omega)$ such that

$$\mu_+(\hat{u}^*) = \inf\left\{\mu_+(u) : u \in W_0^{1,p}(\Omega)\right\} < 0 = \mu_+(0),$$
$$\text{(since } \tau < q \text{ and } u^* \in \operatorname{int} C_+)$$
$$\Rightarrow \hat{u}^* \neq 0 \quad \text{and so } \hat{u}^* = u^* \quad \text{(see (5.3))}.$$

From (5.1) and (5.2) it is clear that

$$\mu_+\big|_{C_+} = \mu\big|_{C_+}.$$

Therefore, we have that

$$u^* \in \operatorname{int} C_+ \text{ is a local } C_0^1(\overline{\Omega})\text{-minimizer of } \varphi(\cdot),$$
$$\Rightarrow u^* \in \operatorname{int} C_+ \text{ is a local } W_0^{1,p}(\Omega)\text{-minimizer of } \mu(\cdot).$$

In a similar fashion, using this time the functional $\mu_-(\cdot)$, we show that

$$v^* \in -\operatorname{int} C_+ \text{ is a local } W_0^{1,p}(\Omega)\text{-minimizer of } \mu(\cdot).$$

We may assume that

$$\mu(v^*) \leq \mu(u^*) \quad \text{and} \quad K_\mu \text{ is finite} \tag{5.4}$$

(see (5.3) and note that any solution $y \notin \{0, u^*, v^*\}$ of (1.1) is necessarily nodal). The functional $\mu(\cdot)$ being coercive, satisfies the C-condition (see Hu–Papageorgiou [5], Proposition 3.19, p. 123). So, using Proposition 3.132, p. 179, of Hu–Papageorgiou [5], we can find $\rho \in (0,1)$ small such that

$$\rho < \|u^* - v^*\|, \ \mu(u^*) < \inf\{\mu(u) : \|u - u^*\| = \rho\} = m^*. \tag{5.5}$$

Then (5.4),(5.5) and the C-condition of $\mu(\cdot)$, permit the use of the mountain pass theorem. So, we can find $y_0 \in W_0^{1,p}(\Omega)$ such that

$$y_0 \in K_\mu \in [v^*, u^*] \cap C_0^1(\overline{\Omega}) \text{ (see(5.3))}, \quad m^* \leq \mu(y_0) \text{ (see (5.5))}. \tag{5.6}$$

We know that

$$C_1(\mu, y_0) \neq 0 \quad \text{(see [5, Corollary 3.123, p. 176])}. \tag{5.7}$$

Using the homotopy invariance property of critical groups (see [5, p. 179]) and the nonlinear regularity theory, we obtain

$$C_k(\mu, 0) = C_k(\varphi, 0) \quad \text{for all } k \in \mathbb{N}_0. \tag{5.8}$$

From hypothesis H_1'(iii) and Hu–Papageorgiou [5, Lemma 5.126, p. 462], we have

$$\begin{aligned} &C_k(\varphi, 0) = 0 \quad \text{for all } k \in \mathbb{N}_0, \\ \Rightarrow\ &C_k(\mu, 0) = 0 \quad \text{for all } k \in \mathbb{N}_0 \quad \text{(see (5.8))}. \end{aligned} \tag{5.9}$$

From (5.7) and (5.9), we infer that

$$y_0 \neq 0.$$

Also, $y_0 \notin \{u^*, v^*\}$ (see (5.5), (5.6)). Therefore

$$y_0 \in [v^*, u^*] \cap C_0^1(\overline{\Omega}) \text{ is a nodal solution of (1.1)}.$$

By Theorem 1.3 of Lucia–Prashanth [10] (see also Pucci–Serrin [15, p. 6]), we have

$$y_0(z) < u^*(z) \quad \text{for all } z \in \Omega. \tag{5.10}$$

Let $\rho = \max\{\|v^*\|_\infty, \|u^*\|_\infty\}$ and let $\hat{\xi}_\rho > 0$ be as postulated by hypothesis H_1(v). We take $\hat{\xi}^* > \hat{\xi}_\rho$ and have

$$\begin{aligned} -\Delta_p^{a_1} y_0 - \Delta_q^{a_2} y_0 + \hat{\xi}^* |y_0|^{p-2} y_0 &= f(z, y_0) + \hat{\xi}^* |y_0|^{p-2} y_0 \\ &\leq f(z, u^*) + \hat{\xi}^* (u^*)^{p-1} \\ &= -\Delta_p^{a_1} u^* - \Delta_q^{a_2} u^* + \hat{\xi}^* (u^*)^{p-1} \quad \text{in } \Omega. \end{aligned}$$

Note that

$$\begin{aligned} f(z, u^*) + \hat{\xi}^* (u^*)^{p-1} &= f(z, u^*) + \hat{\xi}_\rho (u^*) + (\hat{\xi}^* - \hat{\xi}_\rho)(u^*)^{p-1} \\ &\geq f(z, y_0) + \hat{\xi}_\rho |y_0|^{p-2} y_0 + (\hat{\xi}^* - \hat{\xi}_\rho)|y_0|^{p-2} y_0 \\ &= f(z, y_0) + \hat{\xi}^* |y_0|^{p-2} y_0 \quad \text{in } \Omega. \end{aligned} \tag{5.11}$$

From hypothesis H_1(v) and (5.10), we see that

$$f(\cdot, y_0(\cdot)) + \hat{\xi}^* |y_0(\cdot)|^{p-2} y_0(\cdot) \prec f(\cdot, u^*(\cdot)) + \hat{\xi}^* (u^*(\cdot))^{p-1}.$$

So, using Preccosition 3.1, from (5.11) we infer that

$$u^* - y_0 \in \operatorname{int} C_+.$$

Similarly we show that

$$y_0 - v^* \in \operatorname{int} C_+.$$

It follows that

$$y_0 \in \operatorname{int}_{C_0^1(\overline{\Omega})} [v^*, u^*]. \tag{5.12}$$

Moreover, using the flow invariance argument from the proof of Proposition 5.1 of Liu–Papageorgiou [8], we generate another solution

$$\hat{y} \in [v^*, u^*] \cap C_0^1(\overline{\Omega}) \setminus \operatorname{int}_{C_0^1(\overline{\Omega})} [v^*, u^*]. \tag{5.13}$$

From (5.12) and (5.13) it follows that

$$\hat{y} \in C_0^1(\overline{\Omega}) \text{ is a nodal solution of (1.1)}, \quad \hat{y} \neq y_0. \qquad \square$$

When hypotheses $H_2'(q=2)$ hold, since the behavior of $f(z,\cdot)$ near zero is different, we need to strengthen the hypotheses on the reaction $f(z,x)$ in order to have nodal solutions.

H_2'': $f:\Omega\times\mathbb{R}\to\mathbb{R}$ is a measurable function such that $f(z,\cdot)\in C^1(\mathbb{R})$ for a.a. $z\in\Omega$, hypotheses H_2''(i), (ii), (iv), (v) are the same as the corresponding hypotheses H_2(i), (ii), (iv), (v) and

(iii) there exists $k\geq 2$ such that

$$f_x'(z,0)\in\left[\hat{\lambda}_k^{a_2}(2),\hat{\lambda}_{k+1}^{a_2}(2)\right]\quad\text{for a.a. } z\in\Omega,$$
$$f_x'(\cdot,0)\not\equiv\hat{\lambda}_k^{a_2}(2),\ f_x'(\cdot,0)\not\equiv\hat{\lambda}_{k+1}^{a_2}(2).$$

Proposition 5.2. *If hypotheses H_0 and $H_2''(q\neq 2)$ hold, then problem (1.1) $(q=2)$ admits at least two nodal solutions*

$$y_0\in\operatorname{int}_{C_0^1(\overline{\Omega})}[v^*,u^*]$$
$$\hat{y}\in[v^*,u^*]\cap C_0^1(\overline{\Omega}),\quad \hat{y}\neq y_0.$$

Proof. The proof is similar to that of Proposition 5.1.

In this case (5.10) follows from the tangency principle (see Pucci–Serrin [16, p. 35]). Since $f(z,\cdot)$ exhibits a linear behavior near zero (see hypothesis H_2''(iii)) and so there is no resonance, we can not claim that $C_k(\mu,0)=0$ for all $k\in\mathbb{N}_0$. So, to prove the nontriviality of y_0, we have to use a different argument.

Let $\hat{l}:H_0^1(\Omega)\to\mathbb{R}$ be the C^2-functional defined by

$$\hat{l}(u)=\frac{1}{2}\rho_{a_2,2}(Du)-\int_\Omega F(z,u)\,dz\quad\text{for all } u\in H_0^1(\Omega).$$

We consider the following orthogonal direct sum decomposition

$$H_0^1(\Omega)=Y\oplus V$$

with $Y=\bigoplus_{i=1}^k E(\hat{\lambda}_i^{a_2}(2))$, $V=\overline{\bigoplus_{i\geq k} E(\hat{\lambda}_i^{a_2}(2))}$.

Hypotheses H_2''(i), (iii), imply that given $\varepsilon>0$, we can find $c_{10}=c_{10}(\varepsilon)>0$ such that

$$F(z,x)\leq\frac{1}{2}[f_x'(z,0)+\varepsilon]x^2+c_{10}|x|^p \tag{5.14}$$

If $u\in V$, then using (5.14), we have

$$\begin{aligned}\hat{l}(u)&\geq\frac{1}{2}\rho_{a_2,2}(Du)-\frac{1}{2}\int_\Omega f_x'(z,0)u^2\,dz-\frac{\varepsilon}{2\hat{\lambda}_1^{a_2}(2)}\rho_{a_2,2}(Du)-c_{11}\|u\|^p\quad\text{for some } c_{11}>0\\&\geq\frac{1}{2}\left[c_{12}-\varepsilon\right]\|u\|^2-c_{11}\|u\|^p\quad\text{for some } c_{12}>0 \text{ (see Proposition 2.2)}\end{aligned}$$

Choosing $\varepsilon\in(0,c_{12})$, we obtain

$$\hat{l}(u)\geq c_{13}\|u\|^2-c_{11}\|u\|^p\quad\text{for some } c_{13}>0.$$

Since $2<p$, for $\rho\in(0,1)$ small we can say that

$$\hat{l}(u)\geq 0\quad\text{for all } u\in V,\text{ with } \|u\|\leq\rho. \tag{5.15}$$

On the other hand, again from hypothesis H_2''(iii), we see that given $\varepsilon > 0$, we can find $\delta = \delta(\varepsilon) > 0$ such that

$$F(z,x) \geq \frac{1}{2}[f_x'(z,0) - \varepsilon]x^2 \quad \text{for a.a. } z \in \Omega, \text{ all } |x| \leq \delta. \tag{5.16}$$

The space Y is finite dimensional and so all norms are equivalent (see Papageorgiou–Winkert [14], p. 183). Therefore we can find $\hat{\rho} > 0$ such that

$$u \in Y, \ \|u\| \leq \hat{\rho} \Rightarrow |u(z)| \leq \delta \quad \text{for all } z \in \Omega. \tag{5.17}$$

Therefore if $u \in Y$ with $\|u\| \leq \hat{\rho}$, then

$$\begin{aligned} \hat{l}(u) &\leq \frac{1}{2}\rho_{a_2,2}(Du) - \frac{1}{2}\int_\Omega [f_x'(z,0) - \varepsilon]u^2\,dz \quad \text{(see (5.16), (5.17))} \\ &\leq \frac{1}{2}\left[\varepsilon - c_{14}\right]\|u\|^2 \quad \text{for some } c_{14} > 0 \quad \text{(see Proposition 2.2)} \end{aligned}$$

Choosing $\varepsilon \in (0, c_{14})$, we conclude that

$$\hat{l}(u) \leq 0 \quad \text{for all } u \in Y, \ \|u\| \leq \hat{\rho}. \tag{5.18}$$

From (5.15) and (5.18), we infer that $\hat{l}(\cdot)$ has a local linking at the origin with respect to the orthogonal decomposition (Y, V) (see [5, p. 145]). If $d_k = \dim Y \geq 2$ (recall $k \geq 2$), then

$$C_m(\hat{l}, 0) = \delta_{m,d_k}\mathbb{R} \quad \text{for all } m \in \mathbb{N}_0 \quad \text{(see [12], p. 539)}. \tag{5.19}$$

Let $l = \hat{l}|_{W_0^{1,p}(\Omega)}$ (recall $W_0^{1,p}(\Omega) \hookrightarrow H_0^1(\Omega)$ densely since $p > 2$). So by Proposition 3.128, p. 178, of Hu–Papageorgiou [5], we have

$$\begin{aligned} & C_m(l,0) = C_m(\hat{l},0) \quad \text{for all } m \in \mathbb{N}_0, \\ \Rightarrow\ & C_m(l,0) = \delta_{m,d_k}\mathbb{R} \quad \text{for all } m \in \mathbb{N}_0 \quad \text{(see (5.17))}. \end{aligned} \tag{5.20}$$

From the C^1-continuity property of critical groups (see Theorem 3.129, p. 179, of Hu–Papageorgiou [5]), we have

$$\begin{aligned} & C_m(\psi,0) = C_m(l,0) \quad \text{for all } m \in \mathbb{N}_0. \\ \Rightarrow\ & C_m(\psi,0) = \delta_{m,d_k}\mathbb{R} \quad \text{for all } m \in \mathbb{N}_0 \quad \text{(see (5.20))}. \end{aligned}$$

Recall via the homotopy invariance property of critical groups (see [5, p. 179]), we have

$$\begin{aligned} & C_m(\mu,0) = C_m(\psi,0)\mathbb{R} \quad \text{for all } m \in \mathbb{N}_0, \\ \Rightarrow\ & C_m(\mu,0) = \delta_{m,d_k}\mathbb{R} \quad \text{for all } m \in \mathbb{N}_0 \ (d_k \geq 2). \end{aligned} \tag{5.21}$$

We know that $C_1(\mu, y_0) \neq 0$. Comparing with (5.21) we deduce that

$$\begin{aligned} & y_0 \neq 0, \\ \Rightarrow\ & y_0 \in C_0^1(\overline{\Omega}) \text{ is a nodal solution of (1.1) } (q = 2). \end{aligned}$$

Since $y_0(z) < u^*(z)$ for all $z \in \Omega$, as in the proof of Proposition 5.1, we show that

$$y_0 \in \operatorname{int}_{C_0^1(\overline{\Omega})}[v^*, u^*].$$

Finally, the flow invariance argument of Liu–Papageorgiou [8], generates $\hat{y} \in [v^*, u^*] \cap C_0^1(\overline{\Omega})$ another nodal solution of problem (1.1) $(q = 2)$, $\hat{y} \neq y_0$. □

We conclude with the following two multiplicity theorems for

Theorem 5.3. *If hypotheses* H_0 *and* H_1' *hold, then problem* (1.1) *admits at least six nontrivial smooth solutions:*

$$\begin{aligned} &u_0, \hat{u} \in \operatorname{int} C_+, \quad u_0 \leq \hat{u}, \quad u_0(z) < \theta_+ \quad \textit{for all } z \in \overline{\Omega}, \\ &v_0, \hat{v} \in -\operatorname{int} C_+, \quad \hat{v} \leq v_0, \quad \theta_- < v_0(z) \quad \textit{for all } z \in \overline{\Omega}, \\ &y_0 \in \operatorname{int}_{C_0^1(\overline{\Omega})}[v_0, u_0], \quad \hat{y} \in [v_0, u_0] \cap C_0^1(\overline{\Omega}), \quad y_0 \neq \hat{y} \textit{ both nodal.} \end{aligned}$$

Theorem 5.4. *If hypotheses* H_0 *and* H_2' $(q = 2)$ *hold, then problem* (1.1) $(q = 2)$ *admits at least six nontrivial smooth solutions*

$$\begin{aligned} &u_0, \hat{u} \in \operatorname{int} C_+, \quad u_0 \leq \hat{u}, \quad u_0(z) < \theta_+ \quad \textit{for all } z \in \overline{\Omega}, \\ &v_0, \hat{v} \in -\operatorname{int} C_+, \quad \hat{v} \leq v_0, \quad \theta_- < v_0(z) \quad \textit{for all } z \in \overline{\Omega}, \\ &y_0 \in \operatorname{int}_{C_0^1(\overline{\Omega})}[v_0, u_0], \quad \hat{y} \in [v_0, u_0] \cap C_0^1(\overline{\Omega}), \quad y_0 \neq \hat{y} \textit{ both nodal.} \end{aligned}$$

Remark 5.5. We point out that in the above multiplicity theorems, we provide sign information for all the solutions produced. Moreover, the solutions $\{u_0, \hat{u}, v_0, \hat{v}, y_0\}$ and $\{u_0, \hat{u}, v_0, \hat{v}, y_0\}$ are ordered.

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