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On nonlinear perturbations of a periodic integrodifferential Stein–Weiss equation with critical exponential growth

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Abstract. In this paper, we study the existence of solutions for integrodifferential Stein–Weiss equations of the form

$$-\mathcal{L}_K u + V(x)u = \frac{1}{|x|^\beta}\left(\int_{\mathbb{R}} \frac{F(y,u(y))}{|y-x|^\mu |y|^\beta}\,\mathrm{d}y\right) f(x,u(x)), \quad \text{in } \mathbb{R},$$

where $\mu > 0, \beta \geq 0$ such that $0 < \mu + 2\beta < 1$, $-\mathcal{L}_K$ is a nonlocal operator with a measurable kernel which satisfies "structural properties", more general than the standard kernel of the fractional Laplacian operator, V is a bounded potential which can change sign, the nonlinear term $f(x,u)$ has the critical exponential growth with respect to the Trudinger–Moser inequality and $F(x,t) = \int_0^t f(x,s)ds$.

Keywords: integrodifferential operators, Stein–Weiss equations, variational methods, critical points, Trudinger–Moser inequality.

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1 Introduction

In this paper, we are concerned with the existence of solutions for a class of integrodifferential Stein–Weiss equations

$$-\mathcal{L}_K u + V(x)u = \frac{1}{|x|^\beta}\left(\int_{\mathbb{R}} \frac{F(y,u(y))}{|y-x|^\mu |y|^\beta}\,\mathrm{d}y\right) f(x,u(x)), \quad \text{in } \mathbb{R}, \tag{1.1}$$

where V and f are functions that satisfy mild conditions, with $\mu > 0, \beta \geq 0$ such that $0 < \mu + 2\beta < 1$, $F(x,t) = \int_0^t f(x,s)ds$, and $\mathcal{L}_K u$ stands for the integrodifferential operator defined by

$$-\mathcal{L}_K u(x) = 2P.V.\int_{\mathbb{R}} (u(x) - u(y))K(x,y)\,\mathrm{d}y. \tag{1.2}$$

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Here $K(x,y) = K(x-y)$ and belongs to a class of singular symmetric kernels, and $P.V.$ means "the principal value sense". Nonlocal problems and operators have been extensively studied recently due to their applications in various branches of mathematical physics. These include phase transition models [3], image reconstruction problems [34], optimization, finance, stratified materials, anomalous diffusion, crystal dislocation, and more. For additional details, see [26] and the references therein.

Motivated by this, some authors have recently studied problems involving nonlocal integrodifferential operators where $K : \mathbb{R} \setminus \{0\} \to \mathbb{R}^+$ is a measurable kernel which satisfies

(K_1') $\gamma K \in L^1(\mathbb{R}^N)$, where $\gamma(x) = \min\{1, |x|^p\}$;

(K_2') there exists $\lambda > 0$ such that $K(x) \geq \lambda |x|^{-(N+\alpha p)}$, for all $x \in \mathbb{R}^N \setminus \{0\}$;

(K_3') $K(x) = K(-x)$, $\forall x \in \mathbb{R}^N \setminus \{0\}$.

See, for instance, [10–12, 28–32, 36, 42, 51–54, 57]. A standard example of this kind of operator is $K(x) = C_{N,\alpha}|x|^{-(N+2\alpha)}$, where

$$C_{N,\alpha} = \left(\int_{\mathbb{R}^N} \frac{1-\cos(\zeta_1)}{|\zeta|^{N+2\alpha}} \, \mathrm{d}\zeta \right)^{-1},$$

that is, when $-\mathcal{L}_K$ is the fractional Laplacian operator $(-\Delta)^\alpha$, $0 < \alpha < 1$, (see [26]). In this context, seeking solutions for an equation like (1.1) already presents both nonlocal and nonlinear challenges. However, there exists a large class of kernels satisfying (K_1')–(K_3') (see more general examples in [10, 32]). When working with fractional integrodifferential operators that are more general than the standard case, classical tools such as the Caffarelli–Silvestre extension (see [15]) or the commutator properties (see [50]) are no longer applicable. Thus, alternative strategies are required. Nonetheless, these operators are of particular interest because, within suitable functional spaces, variational methods can still be applied in a unified way.

For this reason, the literature contains numerous papers dedicated to the study of such problems, particularly emphasizing cases $N > \alpha p$ (see, for example [11, 28–31, 51–54, 57]). In these works, it is considered nonlinearities involving polynomial growth in terms of the Sobolev embedding. The pioneering work on treating the borderline case, when $N = \alpha p$, $\alpha \in (0,1)$ and $p \geq 2$, is [11], in which A. Bahrouni proved the existence of a solution of the following problem

$$\begin{cases} -\mathcal{L}_K u + |u|^{p-2}u + h(u) = f(u) & \text{in } \Omega, \\ u = 0 \quad \text{in } \mathbb{R}^N \setminus \Omega, \end{cases} \tag{1.3}$$

where Ω is a bounded domain of $\mathbb{R}^N$, (K_1')–(K_3') are assumed, $f \in L^\infty(\mathbb{R}^N)$ and h has a subcritical exponential growth. Subsequently, the papers [10, 12, 44] address problems establishing connections between a fractional integrodifferential operator and nonlinearities with critical exponential growth for $N = 1$, $p = 2$ and $s = 1/2$ in $\mathbb{R}$. In this context, the Sobolev embedding theorem asserts that $H^{1/2}(\mathbb{R}) \hookrightarrow L^q(\mathbb{R})$ for any $q \in [2, +\infty)$; however, $H^{1/2}(\mathbb{R})$ is not continuously embedded in $L^\infty(\mathbb{R})$. For further details, see [26, 46]. Then, in this case, the maximal growth enabling the variational treatment of this problem in $H^{1/2}(\mathbb{R})$ is inspired by the Trudinger–Moser inequality, initially established by T. Ozawa [46] and later refined by S. Iula [35], H. Kozono, T. Sato, and H. Wadade [37], and also by F. Takahashi [56]. By combining results from previous studies, it is established

$$\sup_{\substack{u \in H^{1/2}(\mathbb{R}) \\ \|u\|_{1/2} \leq 1}} \int_{\mathbb{R}} (e^{\alpha u^2} - 1)\,\mathrm{d}x < \infty, \quad \alpha \in [0, \pi], \tag{1.4}$$

where

$$\|u\|_{1/2} := \left(\frac{1}{2\pi} \int_{\mathbb{R}^2} \frac{|u(x) - u(y)|^2}{|x - y|^2}\,\mathrm{d}x\,\mathrm{d}y + \|u\|_2^2 \right)^{1/2}.$$

Moreover, it holds

$$\int_{\mathbb{R}} (e^{\alpha u^2} - 1)\,\mathrm{d}x \leq C(\alpha)\|u\|_2^2, \quad \text{for all } 0 < \alpha \leq \pi, \tag{1.5}$$

for details see [46, Theorem 1] and [56, Proposition 1.1]. Therefore, to tackle this class of problem using a variational approach in $H^{1/2}(\mathbb{R})$, the maximal growth of the nonlinearity $f(x, u)$ is characterized by $e^{\alpha_0 u^2}$ as $|u| \to +\infty$ for some $\alpha_0 > 0$ (refer also to the foundational works [45,58]).

On the other hand, recent studies have treated problems involving nonlinearities inspired by the weighted Hardy–Littlewood–Sobolev inequality, also known as the Stein–Weiss type inequality (see for example [8,14,17,27,55,59–62]). We recall this inequality, as it is frequently used throughout this paper.

Lemma 1.1 (Weighted Hardy–Littlewood–Sobolev inequality). *Let $1 < r, s < +\infty$, $0 < \mu < N$, $\gamma + \beta \geq 0$, $0 < \gamma + \beta + \mu \leq N$, $g \in L^r(\mathbb{R}^N)$ and $h \in L^s(\mathbb{R}^N)$. Then, there exists a sharp constant $C(r, s, N, \alpha, \beta, \mu)$ such that*

$$\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{g(x)h(y)}{|x|^\gamma |y - x|^\mu |y|^\beta}\,\mathrm{d}x\mathrm{d}y \leq C(r, s, N, \gamma, \beta, \mu)\|g\|_r \|h\|_s, \tag{1.6}$$

where

$$\frac{1}{r} + \frac{1}{s} + \frac{\gamma + \beta + \mu}{N} = 2$$

and

$$1 - \frac{1}{r} - \frac{\mu}{N} < \frac{\gamma}{N} < 1 - \frac{1}{r}.$$

In addition, for all $h \in L^s(\mathbb{R}^N)$, we have

$$\left\| \int_{\mathbb{R}^N} \frac{h(y)}{|x|^\gamma |y - x|^\mu |y|^\beta}\,\mathrm{d}y \right\|_t \leq C(t, s, N, \gamma, \beta, \mu)\|h\|_s, \tag{1.7}$$

where t verifies

$$1 + \frac{1}{t} = \frac{1}{s} + \frac{\gamma + \beta + \mu}{N} \quad \textit{and} \quad \frac{\gamma}{N} < \frac{1}{t} < \frac{\gamma\mu}{N}. \tag{1.8}$$

If we consider $g = h = F$, $\gamma = \beta$, $s = r$, we obtain

$$\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{F(u(x))F(u(y))}{|x|^\beta |y - x|^\mu |y|^\beta}\,\mathrm{d}x\mathrm{d}y \leq C(N, \beta, \mu)\|F(u)\|_s^2, \tag{1.9}$$

where $s > 1$ is defined by

$$\frac{2}{s} + \frac{2\beta + \mu}{N} = 2,$$

i.e.,

$$s = \frac{2N}{2N - (\mu + 2\beta)}.$$

Consequently, we have

$$F(u) \in L^{\frac{2N}{2N-(\mu+2\beta)}}(\mathbb{R}^N).$$

When $\beta = \gamma = 0$ in (1.6), we recover the well-known Hardy–Littlewood–Sobolev inequality (see [39]), which plays an important role in the study of equations with nonlinearities of the form

$$\left(\int_{\mathbb{R}^N} \frac{F(y, u(y))}{|y - x|^{\mu}} \, \mathrm{d}y \right) f(x, u(x)), \tag{1.10}$$

referred to as Choquard equations. This type of problem arises from the search for standing waves of the nonlinear nonlocal Schrödinger equation, which is known to affect the propagation of electromagnetic waves in plasmas [13]. It also plays a role in the theory of Bose–Einstein condensation [21]. Additionally, it was employed in the description of the quantum theory of a polaron at rest by S. Pekar in 1954 [48], and in the modeling of an electron trapped in its own hole in 1976, as explored by P. Choquard in a specific approximation to Hartree–Fock theory of one-component plasma [38]. For additional insights into the physical background, see, for instance, [18, 19]. Nowadays, some attention is given to problems with nonlinearities in form of (1.10) which present exponential growth. See for instance [5, 7, 20].

Recalling the Weighted Hardy–Littlewood–Sobolev inequality, we emphasize that (1.6) enables us to apply variational methods to a wide class of problems for a large range of nonlinearities f and dimension $N \geq 1$. So, it is mathematically natural to inquire if the existence of solutions persists for nonlinearities f exhibiting critical exponential growth in the sense of Trudinger–Moser Inequality. Recently, attention has been given to problems with nonlocal exponential nonlinearities, with kernels associated Stein–Weiss [8, 9, 62].

In parallel, problems with periodic or asymptotically periodic potentials, as well as nonlinearities exhibiting critical exponential growth, have been studied over the last decades. More precisely, let us consider the following class of problems

$$-\mathcal{L}_K u + V(x)u = f(x, u(x)), \quad \text{in } \mathbb{R}^{\mathbb{N}}. \tag{1.11}$$

In 2004, [6], Alves, do Ó and Miyagaki proved the existence of nonnegative solution for problem (1.11), considering the Laplacian operator and $N = 2$ with V non negative and periodic or asymptotically at infinity and f involving critical exponential growth. In 2016, [24], de Souza and Araújo proved similar results considering the $1/2$-Laplacian for $N = 1$ under analogous hypotheses on V and f. In 2018, [22], Albuquerque, Araújo and Clemente showed the existence of ground state solutions for a class of Kirchhoff with the $1/2$-Laplacian for $N = 1$, allowing that V can change sign. More recently, Barboza, Araújo and de Carvalho generalized some results of [22, 24] in [10, 12], respectively, in the sense that the authors concern with problems with a more general integrodifferential operator of order $1/2$. Still within this line of research, but involving the Laplacian operator, Alves and Yang [7] (2017) established the existence of a solution to a problem analogous to (1.11) when $N = 2$, V is positive, with a nonlocal nonlinearity as in (1.10) exhibiting exponential growth.

Here, motivated by (1.4) and the previous discussion, we focus on the case $N = 1$ considering a class of equations with integrodifferential fractional operators and nonlocal critical exponential nonlinearities. Thus, due to the potential and the general nature of the kernel, we need a version for the Trudinger–Moser inequality as in [10–12] for suitable solution spaces.

For details, see Lemma 2.3. Moreover, as we are assuming that the nonlinearity f exhibits critical exponential growth, it is necessary to impose conditions on its growth at infinity to control the minimax level of the associated functional. For this purpose, we may consider two kinds of assumptions. The first condition, established by D. M. Cao (see [16]), is widely utilized in the literature (see, for instance, [22, 24] and references therein). Specifically, it is assumed that there exist constants $p > 2$ and $C_p > 0$ such that

$$f(t) \geq C_p t^{p-1}, \quad \text{for all } t \text{ in domain of } f,$$

where C_p is chosen suitably. In this case, it is crucial to demonstrate that the embedding constant of the solution space into an appropriate Lebesgue space is attained, and a suitable version of Lions' Lemma plays a important role in this proof.

In the second case, a version of Moser functions sequence is used to estimate the minimax level of functional associated with problem. In studies of problems with approach, since [2, 23] it is extensively considered a heavy dependence of the nonlinearity on the asymptotic behavior of $h(t) = f(t)t/e^{\alpha_0 t^2}$ at infinity, which can appear in different ways. For example, in [23], the authors showed the existence of solution for an equation with Laplacian operator in a bounded domain of $\mathbb{R}^2$ for $\alpha_0 = 4\pi$, provided that f satisfies (among other conditions) that $\lim_{t\to\infty} h(t) = C(r)$, where r is the radius of the largest open ball in the domain. In [33], the authors looked for a solution of a equation with the $1/2$-Laplacian operator and $\alpha_0 = 1$ in $\mathbb{R}$. For this, they assumed that $\lim_{t\to\infty} h(t) = \infty$.

Talking specifically about works that treat with equations with nonlocal nonlinearity, based on (1.6), which has critical Trudinger–Moser, this kind of hypotheses needs to be adapted, due to the nature of the nonlinearity once we must control integrals where both the two nonlinearities $F(t)$ and $tf(t)$ appear simultaneously. More precisely, we need to control the behavior of $F(t)f(t)/e^{2\alpha_0 t^2}$ at infinity, where F the primitive of f. In the first two papers dealing with Choquard equations with critical exponential nonlinearity, there is a significant difference between the assumptions on this asymptotic behavior. Namely, in [5] it is assumed that there exists a positive constant γ_0 large enough such that

$$\lim_{t\to\infty} \frac{tf(t)F(t)}{e^{8\pi t^2}} \geq \gamma_0. \tag{1.12}$$

While, in [4] it is considered

$$\lim_{t\to+\infty} \frac{F(t)}{e^{4\pi t^2}} = \gamma > 0, \tag{1.13}$$

without requiring a constraint on constant $\gamma > 0$. See also [8, 20].

In this paper, we aim to establish a link between a nonlocal problem involving a general integrodifferential operator, a bounded potential, and a critical exponential nonlinearity for a Stein–Weiss equation in $\mathbb{R}$.

Motivated by [7, 12, 20], we first investigate (1.1) under the assumption that $V(x)$ and $f(x,u)$ are periodic functions with respect to x, and $f(x,u)$ behaves like $e^{\alpha_0 u^2}$ as $|u| \to +\infty$ for some $\alpha_0 > 0$. For this, we consider two types of hypotheses to control the minimax level of the functional associated with the problem.

Second, based on the results from the periodic case, we study a more general problem where $V(x)$ and $f(x,u)$ are merely asymptotically periodic at infinity. The novelty of this work lies in addressing Problem (1.11) with a general integrodifferential operator and a nonlocal nonlinearity in the context of the Weighted Hardy–Littlewood–Sobolev inequality. Consequently, our work complements some results in [7, 10, 12, 20, 24, 25]. For details, see Remark 1.6.

As usual, our main difficulties are related to unbounded domains and nonlinearities with critical growth. These challenges are compounded by the presence of the nonlocal nonlinearity, which introduces additional complications in estimating the minimax level of the functional associated with our problems. To address the lack of compactness, we must perform some suitable calculations.

For easy reference, we record problems, assumptions, and the main results.

1.1 A periodic problem

We now present the periodic problem for a bounded potential and a nonlocal nonlinearity with critical exponential growth. To this end, we first study the following problem

$$\begin{cases} \mathcal{L}_K u + V_0(x)u = \dfrac{1}{|x|^\beta}\left(\displaystyle\int_{\mathbb{R}} \frac{F_0(y,u(y))}{|y-x|^\mu |y|^\beta}\,\mathrm{d}y\right) f_0(x,u(x)) \quad \text{in } \mathbb{R}, \\ u \in X_0 \quad \text{and} \quad u \geq 0, \end{cases} \tag{1.14}$$

where $\mu > 0, \beta \geq 0$ such that $\mu + 2\beta < 1$, $F_0(t) = \int_0^t f_0(s)ds$, $-\mathcal{L}_K u$ is given in (1.2) and we assume that $K : \mathbb{R} \setminus \{0\} \to \mathbb{R}_+$ is a measurable function with the properties

(K_1) $\gamma K \in L^1(\mathbb{R})$, where $\gamma(x) = \min\{1, |x|^2\}$;

(K_2) there exists $\lambda > 0$ such that $K(x) \geq \lambda |x|^{-2}$, for all $x \in \mathbb{R} \setminus \{0\}$;

(K_3) $K(x) = K(-x)$, $\forall x \in \mathbb{R} \setminus \{0\}$.

These hypotheses enable us to work with various integrodifferential operators of orders different from $1/2$. As in the example provided in [10], where K is given by

$$K(x) = \begin{cases} C_1|x|^{-r} \text{ with } 1 < r \leq 2 \text{ if } |x| \geq 1; \\ C_2|x|^{-q} \text{ with } 2 \leq q < 3 \text{ if } |x| \leq 1. \end{cases} \tag{1.15}$$

As r and q can be chosen in the intervals $(1,2]$ and $[2,3)$, respectively, we obtain a wide range of nonlocal integrodifferential operators. Figure 1.1 illustrates that any function whose graphic falls within the hatched region satisfies conditions (K_1)–(K_2), demonstrating a greater variety of kernels than those presented in (1.15).

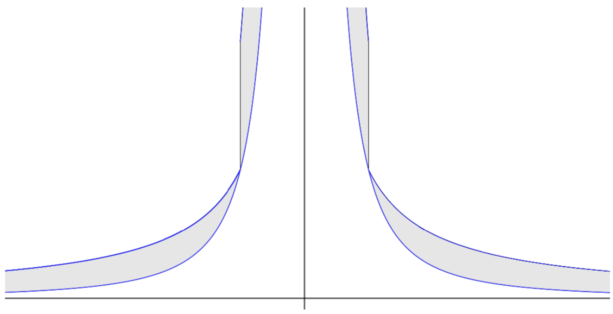

Figure 1.1: Range of Integrodifferential Operator of order different from 1/2

We assume that the function $V_0 : \mathbb{R} \to \mathbb{R}$ is a continuous 1-periodic function satisfying:

$(V_{0,1})$ there exists a positive constant v_0 such that $V_0(x) \geq -v_0$ for all $x \in \mathbb{R}$;

$(V_{0,2})$ the infimum

$$\xi_0 := \inf_{\substack{u\in X_0\\ \|u\|_2=1}} \left(\int_{\mathbb{R}^2} [u(x)-u(y)]^2 K(x,y)\,\mathrm{d}x\mathrm{d}y + \int_{\mathbb{R}} V_0(x)u^2(x)\,\mathrm{d}x \right)$$

is positive.

Moreover, we consider $f_0 : \mathbb{R}\times\mathbb{R} \to \mathbb{R}$ is a continuous 1-periodic function in x, which has critical exponential growth in s, that is,

$$\lim_{|s|\to+\infty} f_0(x,s)e^{-\alpha s^2} = \begin{cases} 0, & \text{for all } \alpha > \pi, \\ +\infty, & \text{for all } \alpha < \pi, \end{cases}$$

uniformly in $x \in \mathbb{R}$.

We emphasize that the notion of criticality considered here is determined by (1.4) and has already appeared in the literature on problems with exponential growth; see, e.g., [24, 49]. Since we are interested in the existence of nonnegative solutions, we set $f_0(x,s) = 0$ for all $(x,s) \in \mathbb{R}\times(-\infty, 0]$.

We also assume that the nonlinearity $f_0(x,u)$ satisfies the conditions

$(f_{0,1})$ $f_0(x,t) = o(s)$ as $s \to 0^+$ uniformly in $x \in \mathbb{R}$;

$(f_{0,2})$ there exists a constant $\theta > 1$ such that

$$0 < \theta F_0(x,s) := \theta \int_0^s f_0(x,t)\,\mathrm{d}t \le s f_0(x,s) \quad \text{for all } (x,s) \in \mathbb{R}\times(0,+\infty);$$

$(f_{0,3})$ for each fixed $x \in \mathbb{R}$, the function $f_0(x,s)/s$ is increasing with respect to $s \in \mathbb{R}$;

$(f_{0,4})$ there are constants $p > 2$ and $C_p > 0$ such that

$$f_0(x,s) \ge C_p s^{p-1}, \quad \text{for all } (x,s) \in \mathbb{R}\times[0,+\infty)$$

with

$$C_p > \frac{S_p^p}{p^{\frac{p}{2}}} \left[\frac{(p-1)2\theta\pi}{(\theta-1)(2-(\mu+2\beta))\omega} \right]^{\frac{p-2}{2}},$$

where ω is given precisely in Lemma 2.3.

Here, we define

$$X_0 := \left\{ u \in L^2(\mathbb{R});\ (u(x)-u(y))K(x,y)^{\frac{1}{2}} \in L^2(\mathbb{R}^2) \right\}$$

which is endowed with norm

$$\|u\|_{X_0} = \left([u]^2_{1/2,K} + \int_{\mathbb{R}} V_0(x)u(x)^2\,\mathrm{d}x \right)^{1/2}$$

where

$$[u]^2_{1/2,K} = \int_{\mathbb{R}^2} (u(x)-u(y))^2 K(x,y)\,\mathrm{d}x\,\mathrm{d}y.$$

We would like to point out that space X_0 has suitable properties which give to problem (1.14) a variational framework. More specifically, in light of results proved in [10], even with more general hypotheses under the potential, X_0 is uniformly convex Banach space and therefore is a reflexive space. Moreover, $C_0^\infty(\mathbb{R})$ is dense in X_0.

Throughout this paper, we say that $u \in X_0$ is a weak solution of (1.14) if the following equality holds:

$$\langle u,v\rangle_0 = \int_{\mathbb{R}} \frac{1}{|x|^\beta}\left(\int_{\mathbb{R}} \frac{F_0(y,u(y))}{|y-x|^\mu |y|^\beta}\,\mathrm{d}y\right) f_0(x,u(x))v(x)\,\mathrm{d}x,$$

for all $v \in X_0$ with

$$\langle u,v\rangle_0 := \int_{\mathbb{R}^2} (u(x)-u(y))(v(x)-v(y))K(x,y)\,\mathrm{d}x\,\mathrm{d}y + \int_{\mathbb{R}} V_0(x)uv\,\mathrm{d}x.$$

The main results of this subsection are presented bellow. The first result of the paper involves a classical assumption on the nonlinearity (see assumption $(f_{0,4})$ above) which was first introduced by D. M. Cao in [16].

Theorem 1.2. *Assume that (K_1)–(K_3), $(V_{0,1})$,$(V_{0,2})$ and $(f_{0,1})$–$(f_{0,4})$ hold. Then (1.14) has a nonnegative and nontrivial solution.*

The second theorem of the paper also deals with the critical growth nonlinearity but involves a little weaker assumption addressed by Adimurthi [1] and de Figueiredo, Miyagaki and Ruf [23] instead of D. M. Cao assumption $(f_{0,4})$. We assume that

$(\tilde{f}_{0,4})$ There exist t_0 and $M_0 > 0$ such that $0 < tF(t) \leq M_0 f(t)$, for all $t \geq t_0$ and

$$\liminf_{t\to+\infty} \frac{F(x,t)}{e^{\pi t^2}} = \sqrt{\beta_0} \quad \text{with} \quad \beta_0 > 0 \text{ uniformly in } x.$$

Remark 1.3. Assumption $(\tilde{f}_{0,4})$ implies the asymptotic behavior of $tf(t)F(t)/e^{2\pi t^2}$ at infinity. Precisely, for given $\varepsilon > 0$, there exists $t_0 > 0$ such that

$$tf(t)F(t) \geq (\beta_0 - \varepsilon)M_0 e^{2\pi t^2} t^2, \quad \text{for all } t > t_0. \tag{1.16}$$

We emphasize that this fact plays a crucial role in estimating the minimax level associated with Problem (1.14), utilizing a version of Moser functions.

In order to establish an existence result of a solution to problem (1.14), without the hypothesis $(f_{0,4})$, in addition to the hypothesis $(\tilde{f}_{0,4})$ we need the following hypothesis additional on the kernel

(K_4) There are $x_0, r, \lambda_0 \in \mathbb{R}$ such that if $x \in (x_0 - r, x_0 + r)$ then $K(x) \leq \dfrac{1}{\lambda_0}|x|^{-2}$.

The second result is the following.

Theorem 1.4. *Assume that (K_1)–(K_4), $(V_{0,1})$,$(V_{0,2})$, $(f_{0,1})$–$(f_{0,3})$ and $(\tilde{f}_{0,4})$ hold. Then (1.14) has a nonnegative and nontrivial solution.*

1.2 A nonperiodic problem

The second problem that we will study in this paper is the following,

$$\begin{cases} \mathcal{L}_{\mathcal{K}}u + V(x)u = \dfrac{1}{|x|^{\beta}}\left(\displaystyle\int_{\mathbb{R}} \frac{F(y,u(y))}{|y-x|^{\mu}|y|^{\beta}}\,\mathrm{d}y\right) f(x,u(x)) \quad \text{in } \mathbb{R}, \\ u \in X_1 \quad \text{and} \quad u \geq 0, \end{cases} \tag{1.17}$$

where $F(x,t) = \int_0^t f(x,s)ds$. Note that the terms $V(x)$ and $f(x,u)$ are not necessarily periodic anymore.

Here, we address the class of asymptotically periodic functions introduced by H. F. Lins and E. A. B. Silva [40]. Specifically, we define the set

$$\mathcal{F} := \{g \in C(\mathbb{R}) \cap L^{\infty}(\mathbb{R}) : |\{x \in \mathbb{R} : |g(x)| \geq \varepsilon\}| < \infty, \text{ for all } \varepsilon > 0\},$$

where $|A|$ denotes the Lebesgue measure of a set A. In order to deal with the challenges posed by the lack of periodicity, we impose conditions that compare the periodic terms with the asymptotically periodic terms. Regarding the potential $V(x)$, we assume the following:

(V_1) $V_0 - V \in \mathcal{F}$ and $V_0(x) \geq V(x) \geq -v_0$, for all $x \in \mathbb{R}$;

(V_2) The infimum

$$\xi_1 := \inf_{\substack{u \in X_1 \\ \|u\|_2 = 1}} \left(\int_{\mathbb{R}^2} |u(x)-u(y)|^2 K(x,y)\,\mathrm{d}x\mathrm{d}y + \int_{\mathbb{R}} V(x)u^2\,\mathrm{d}x\right)$$

is positive.

We assume that the nonlinearity $f : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is a continuous function that have critical exponential growth, $f(x,s) = 0$ for all $(x,s) \in \mathbb{R} \times (-\infty, 0]$ and satisfies the following conditions:

(f_1) $f(x,s) \geq f_0(x,s)$ for all $(x,s) \in \mathbb{R} \times [0,+\infty)$, and for all $\varepsilon > 0$, there exists $\nu > 0$ such that for $s \geq 0$ and $|x| \geq \nu$,

$$|f(x,s) - f_0(x,s)| \leq \varepsilon e^{\alpha_0 s^2};$$

(f_2) $\lim_{t\to 0} f(x,s) = o(s)$ as $s \to 0^+$ uniformly in $x \in \mathbb{R}$;

(f_3) there exists a constant $\tilde{\theta} \geq \theta > 1$ such that

$$0 < \tilde{\theta}F(x,s) := \tilde{\theta}\int_0^s f(x,t)\,\mathrm{d}t \leq sf(x,s), \quad \text{for all } (x,s) \in \mathbb{R} \times (0,+\infty);$$

(f_4) for each fixed $x \in \mathbb{R}$, the function $f(x,s)/s$ is increasing with respect to $s \in \mathbb{R}$;

(f_5) at least one of the nonnegative continuous functions $V_0(x) - V(x)$ and $f(x,s) - f_0(x,s)$ is positive on a set of positive measure.

In order to define the weak solution to problem (1.17), as in problem (1.14), we consider

$$X_1 := \left\{u \in L^2(\mathbb{R});\ (u(x)-u(y))K(x,y)^{\frac{1}{2}} \in L^2(\mathbb{R}^2)\right\}$$

which is a Hilbert space endowed with the inner product

$$\langle u,v\rangle_1 := \int_{\mathbb{R}^2}(u(x)-u(y))(v(x)-v(y))K(x,y)\,\mathrm{d}x\,\mathrm{d}y + \int_{\mathbb{R}} V(x)uv\,\mathrm{d}x$$

and the correspondent induced norm $\|u\|^2_{X_1} = \langle u,u\rangle$.

We would like to point out that space X_1 also has suitable properties which give to problem (1.17) a variational framework.

Throughout this paper, we say that $u \in X_1$ is a weak solution for (1.17) if the following equality holds:

$$\langle u,v\rangle_1 = \int_{\mathbb{R}} \frac{1}{|x|^\beta}\left(\int_{\mathbb{R}} \frac{F(u(y))}{|y-x|^\mu |y|^\beta}\,\mathrm{d}y\right) f(u(x))v(x)\,\mathrm{d}x,$$

for all $v \in X_1$.

Considering the functions V_0 and f_0 as in Theorems 1.2 and 1.4, the main result of this subsection is the following.

Theorem 1.5. *Assume that (K_1)–(K_3),(V_1),(V_2) and (f_1)–(f_5) hold. Then (1.17) has a nonnegative and nontrivial solution.*

Remark 1.6. As previously mentioned, the results of this paper are inspired by the works [7, 10, 12, 24, 25]. Specifically, our Theorems 1.2–1.5 can be seen as versions of Theorems 1.1, 1.2, and 1.3 from [10], without the Kirchhoff term, in the sense that we address a nonlocal nonlinearity of the Stein–Weiss type. Consequently, we obtain versions for the results in [10, 24, 25]. Moreover, Theorems 1.2 and 1.5 are integrodifferential and fractional versions of some results in [7]. We also improve some results presented in [20], as we consider a more general fractional operator, a potential that can change sign, and a nonlinearity kernel that encompasses the Choquard kernel.

The outline of this paper is as follows: Section 2 presents preliminary results necessary for establishing suitable properties for the solution spaces. Sections 3 through 5 address results related to the periodic problem. Specifically, in Section 3, we discuss its variational formulation. Section 4 involves estimating the minimax level of the associated functional, and Section 5 is dedicated to the proof of the main results for the periodic case. Lastly, Section 6 focuses on the results related to the nonperiodic problem.

2 Some preliminary results

We recall the definition of the fractional Sobolev space

$$H^{1/2}(\mathbb{R}) = \left\{u \in L^2(\mathbb{R}) : \int_{\mathbb{R}^2} \frac{|u(x)-u(y)|^2}{|x-y|^2}\,\mathrm{d}x\mathrm{d}y < \infty\right\},$$

which is endowed with the natural norm

$$\|u\|_{1/2} = \left([u]^2_{1/2} + \int_{\mathbb{R}} u^2\,\mathrm{d}x\right)^{1/2}.$$

We present some results proved in [10, 12].

Lemma 2.1. *Assume the conditions $(V_{0,1})$–$(V_{0,2})$ or (V_1)–(V_2) and (K_1)–(K_3). The space X_i is embedded in $H^{1/2}(\mathbb{R})$ and there exists $C(\lambda,\xi_i) > 0$ such that*

$$\|u\|_{1/2} \leq \sqrt{C(\lambda,\xi_i)}\|u\|_{X_i}, \quad \forall\, u \in X_i \text{ with } i = 0,1.$$

Corollary 2.2. *Let $q \in [2, +\infty)$, then the embedding $X_i \hookrightarrow L^q(\mathbb{R})$ is continuous with $i = 0, 1$. Moreover, if $q \in [1, 2]$ the embedding $X_i \hookrightarrow L^q_{\text{loc}}(\mathbb{R})$ is compact with $i = 0, 1$.*

Now we show a suitable version of Trudinger–Moser inequality for X_i with $i = 0, 1$.

Lemma 2.3. *Assume (K_1)–(K_3) and $(V_{0,1})$–$(V_{0,2})$ or (V_1)–(V_2), then there exists $\omega > 0$ such that if $0 < \alpha \leq \omega$, then one has a constant $C = C(\omega) > 0$, such that*

$$\sup_{\substack{u \in X_i \\ \|u\|_{X_i} \leq 1}} \int_{\mathbb{R}} (e^{\alpha u^2} - 1)\, \mathrm{d}x \leq C(\omega) \quad \text{for } i = 0, 1. \tag{2.1}$$

Moreover, for any $\alpha > 0$ and $u \in X_i$, for $i = 0, 1$, we have

$$\int_{\mathbb{R}} (e^{\alpha u^2} - 1)\, \mathrm{d}x < \infty.$$

Lemma 2.4. *If $\alpha > 0$, $q > 2$, $v \in X_i$ and $\|v\|_{X_i} \leq D$ with $\alpha D^2 < \omega$, then there exists $C = C(\alpha, D, q) > 0$, such that*

$$\int_{\mathbb{R}} (e^{\alpha v^2} - 1)|v|^q \mathrm{d}x \leq C\|v\|^q_{X_i} \quad \text{for } i = 0, 1.$$

3 A functional setting for the periodic problem

In order to use a variational framework considering the space X_0, we assume suitable conditions such that weak solutions of (1.14) become critical points of the Euler functional $I_0 : X_0 \to \mathbb{R}$ defined by

$$I_0(u) = \frac{1}{2}\|u\|^2_{X_0} - \frac{1}{2}\int_{\mathbb{R}}\int_{\mathbb{R}} \frac{F_0(x, u(y))F_0(y, u(x))}{|x|^\beta |y - x|^\mu |y|^\beta}\, \mathrm{d}y\, \mathrm{d}x \tag{3.1}$$

where $F_0(x, t) = \int_0^t f_0(x, \tau)\, \mathrm{d}\tau$. Notice that by the condition $(f_{0,1})$ and the fact that $f_0(x, s)$ has critical exponential growth, for each $\alpha > \pi$, $q > 2$ and $\varepsilon > 0$ there exists $C_\varepsilon > 0$ such that

$$f_0(x, t) \leq \varepsilon t + C_\varepsilon(e^{\alpha t^2} - 1)|t|^{q-1}, \quad \text{for all } (x, t) \in \mathbb{R}^2, \tag{3.2}$$

which implies that

$$F_0(x, t) \leq \frac{\varepsilon}{2}|t|^2 + C_\varepsilon(e^{\alpha t^2} - 1)|t|^q, \quad \text{for all } (x, t) \in \mathbb{R}^2. \tag{3.3}$$

From (3.3), we have

$$\|F_0(x, u)\|_{\frac{2}{2-(\mu+2\beta)}} \leq \frac{\varepsilon}{2}\|u\|^2_{\frac{2}{2-(\mu+2\beta)}} + C\left\{\int_{\mathbb{R}^2}\left[\left(e^{\alpha u^2} - 1\right)|u|^q\right]^{\frac{2}{2-(\mu+2\beta)}} \mathrm{d}x\right\}^{\frac{2-(\mu+2\beta)}{2}}. \tag{3.4}$$

By Hölder inequality we obtain

$$\int_{\mathbb{R}}\left[\left(e^{\alpha u^2} - 1\right)|u|^q\right]^{\frac{2}{2-(\mu+2\beta)}} \mathrm{d}x \leq \left[\int_{\mathbb{R}}\left(e^{\frac{4\alpha}{2-(\mu+2\beta)}u^2} - 1\right) \mathrm{d}x\right]^{\frac{1}{2}} \|u\|^{\frac{2q}{2-(\mu+2\beta)}}_{\frac{4q}{2-(\mu+2\beta)}}. \tag{3.5}$$

Then we are able to apply Lemma 2.3. Thus, (3.4), (3.5) jointly with Corollary 2.2 imply that

$$\|F_0(x, u)\|_{\frac{2}{2-(\mu-2\beta)}} \leq \frac{\varepsilon}{2}\|u\|^2_{X_0} + \tilde{C}\|u\|^q_{X_0}.$$

Hence, it follows from Weighted Hardy–Littlewood–Sobolev inequality that

$$\int_{\mathbb{R}}\int_{\mathbb{R}} \frac{F_0(y,u)F_0(x,u)}{|x|^\beta|x-y|^\mu|y|^\beta}\,\mathrm{d}x\mathrm{d}y \le \frac{\varepsilon^2}{4}\|u\|_{X_0}^4 + \tilde{C}\|u\|_{X_0}^{2q}. \tag{3.6}$$

By using the estimate above jointly with the continuous embedding $X_0 \hookrightarrow L^q(\mathbb{R})$, we can conclude that I_0 is well defined. Moreover, using standard arguments we can check that $I_0 \in C^1(X_0,\mathbb{R})$ with the derivative given by

$$I_0'(u)v = \langle u, v\rangle_0 - \int_{\mathbb{R}}\int_{\mathbb{R}} \frac{F_0(y,u(y))}{|x|^\beta|y-x|^\mu|y|^\beta} f_0(x,u(x))v(x)\,\mathrm{d}y\,\mathrm{d}x, \quad \forall v \in X_0.$$

Thus, critical points of I_0 are weak solutions of problem (1.14) and conversely.

3.1 The geometric condition

Next, using the hypotheses $(f_{0,1})$ and $(f_{0,2})$ we prove some facts about the geometric structure of I_0 required by the minimax procedure.

Lemma 3.1. *There exist $\gamma > 0$ and $\varrho > 0$ such that $I_0(u) \ge \gamma$, provided that $\|u\|_{X_0} = \varrho$.*

Proof. Initially, notice that for $\|u\|_{X_0} < \varrho_0 \le \sqrt{\frac{(2-(\mu+2\beta))\omega}{4\alpha}}$, where ω is given in Lemma 2.3. Then, by (2.1), we obtain

$$\int_{\mathbb{R}} \left(e^{\frac{4\alpha}{2-(\mu+2\beta)}u^2} - 1\right)\,\mathrm{d}x \le \int_{\mathbb{R}} \left(e^{\frac{4\alpha\|u\|_{X_0}^2}{2-(\mu+2\beta)}\left(\frac{u}{\|u\|_{X_0}}\right)^2} - 1\right)\,\mathrm{d}x < C.$$

Thus, (3.6) is available for each $u \in X_0$ such that $\|u\|_{X_0} < \varrho_0$. So, we have

$$I_0(u) \ge \frac{1}{2}\|u\|_{X_0}^2 - \frac{\varepsilon^2}{4}\|u\|_{X_0}^4 - C_\varepsilon\|u\|_{X_0}^{2q}.$$

Taking $q > 2$ and $\varepsilon > 0$ small enough, we may choose $0 < \varrho < \rho_0$ such that

$$\frac{\varepsilon}{2}\varrho^2 - \frac{\varepsilon^2}{4}\varrho^4 - C_2\varrho^q = \gamma > 0.$$

□

Lemma 3.2. *There exists $u_0 \in X_0$ with $\|u_0\|_{X_0} > \varrho$ such that $I_0(u_0) < 0$.*

Proof. Take $u_0 \in X_0 \setminus \{0\}$, and set

$$w(t) = \frac{1}{2}\int_{\mathbb{R}}\int_{\mathbb{R}} \frac{F_0\left(y, t\frac{u_0}{\|u_0\|_{X_0}}\right) F_0\left(x, t\frac{u_0}{\|u_0\|_{X_0}}\right)}{|x|^\beta|x-y|^\mu|y|^\beta}\,\mathrm{d}y\,\mathrm{d}x, \quad \text{for } t > 0.$$

It follows from $(f_{0,2})$ that

$$\frac{w'(t)}{w(t)} > \frac{2\theta}{t}, \quad \text{for } t > 0.$$

Thus, integrating this over $[1, s\|u_0\|_{1/2}]$ with $s > 1/\|u_0\|_{X_0}$, we can conclude that

$$\begin{aligned}&\frac{1}{2}\int_{\mathbb{R}}\int_{\mathbb{R}} \frac{F_0(y, su_0)\, F_0(x, su_0)}{|x|^\beta|x-y|^\mu|y|^\beta}\,\mathrm{d}y\,\mathrm{d}x \\ &\quad\ge \left(\frac{1}{2}\int_{\mathbb{R}}\int_{\mathbb{R}} \frac{F_0\left(y, t\frac{u_0}{\|u_0\|_{1/2}}\right) F_0\left(x, t\frac{u_0}{\|u_0\|_{1/2}}\right)}{|x|^\beta|x-y|^\mu|y|^\beta}\,\mathrm{d}y\,\mathrm{d}x\right)\|u_0\|_{X_0}^{2\theta}s^{2\theta}\end{aligned}$$

Therefore, from (3.1), we get

$$I(su_0) < Cs^2 - Cs^{2\theta}, \quad \text{for } s > \frac{1}{\|u_0\|_{X_0}}.$$

Since $\theta > 1$, taking $v = su_0$ with s large enough, the result follows. □

3.2 Palais–Smale sequence

By using the Mountain-Pass Theorem without the (PS) condition (see [43]), from Lemmas 3.1 and 3.2, there exists a sequence (u_k) in X_0 satisfying

$$I_0(u_k) \to c_0 \quad \text{and} \quad I_0'(u_k) \to 0, \tag{3.7}$$

where

$$c_0 = \inf_{g\in\Gamma} \max_{t\in[0,1]} I_0(g(t))$$

and $\Gamma = \{g \in C([0,1], X_0) : g(0) = 0 \text{ and } g(1) = e\}$.

Lemma 3.3. *Suppose that $(f_{0,1})$ and $(f_{0,2})$ hold. Then, the sequence (u_k) is bounded in X_0.*

Proof. Using well-known arguments it is not difficult to check that (u_k) is a bounded sequence in X_0. Indeed, by $(f_{0,2})$ we have

$$I_0(u_k) - \frac{1}{\theta} I_0'(u_k)u_k \geq \left(\frac{1}{2} - \frac{1}{2\theta}\right) \|u_k\|^2_{X_0}. \tag{3.8}$$

By (3.7), there exists $k_0 \in \mathbb{N}$ such that for all $k \geq k_0$, it holds

$$\left(\frac{1}{2} - \frac{1}{2\theta}\right) \|u_k\|^2_{X_0} \leq C + \|u_k\|_{X_0}.$$

Since $\theta > 1$, this implies that $\|u_k\|_{X_0} \leq C_1$. Thus, we have that this sequence is bounded. □

4 Minimax level for the periodic problem

As already mentioned, we highlight that the main difficulty in our work is the lack of compactness typical for elliptic problems in unbounded domains with nonlinearities with critical growth. To recover this, we will make use of assumptions $(f_{0,4})$ or $(\tilde{f}_{0,4})$ together with (K_4) to control the minimax level in a suitable range where we are able to recover some compactness. For this purpose, in the first case, we need a version of Lions's Lemma. In the second case, let us consider a version of Moser's functions sequence supported in a ball with an appropriate radius, which depends on (K_4).

Now, observe that as a consequence of Lemmas 3.1 and 3.2, the minimax level

$$c_0 = \inf_{g\in\Gamma} \max_{t\in[0,1]} I_0(g(t))$$

is positive, where $\Gamma = \{g \in C([0,1], X_0) : g(0) = 0 \text{ and } g(1) = e\}$.

4.1 Minimax estimate of Theorem 1.2

In order to provide an estimate of the minimax level of the functional associated with (1.14), in [10], the authors proved a version of a Lions's Lemma (see P. L. Lions [41]) for critical growth in $\mathbb{R}$, more specifically in [10, Lemma 3.1], for a bounded sequence in a suitable space they guarantee that if

$$\lim_{n\to\infty} \sup_{y\in\mathbb{R}} \int_{B_R(y)} |u_n(x)|^2 \, \mathrm{d}x = 0, \tag{4.1}$$

for some $R > 0$, then $u_n \to 0$ strongly in $L^q(\mathbb{R})$ for $2 < q < \infty$. This result is also available for bounded sequence in X_0.

Now, we consider the embedding constant, given by

$$S_p := \inf_{u\in X_0} \left(\frac{\left(\int_{\mathbb{R}^2} |u(x)-u(y)|^2 K(x,y)\, \mathrm{d}x\mathrm{d}y + \int_{\mathbb{R}} V_0(x) u^2(x)\, \mathrm{d}x\right)^{1/2}}{\left(\int_{\mathbb{R}}\int_{\mathbb{R}} \frac{|u|^p|u|^p}{|x|^\beta|x-y|^\mu|y|^\beta} \mathrm{d}y\, \mathrm{d}x\right)^{\frac{1}{2p}}} \right),$$

which is achieved by a nonnegative function u_p in X_0. From this, we may estimate the level.

Proposition 4.1. *Suppose that $(f_{0,4})$ holds. Then*

$$\gamma \le c_0 < \frac{(\theta-1)(2-(\mu+2\beta))\omega}{4\pi\theta},$$

where ω is given in Lemma 2.3

Proof. Let $u_p \in X_0$ such that $\|u_p\|^2_{X_0} = S_p$ and $\|u_p\|_p = 1$. Then, by $(f_{0,4})$, we obtain

$$\begin{aligned} c_0 &\le \max_{t\ge 0} I_0(tu_p) = \max_{t\ge 0} \left\{ \frac{1}{2} t^2 \|u_p\|^2_{X_0} - \frac{1}{2} \int_{\mathbb{R}}\int_{\mathbb{R}} \frac{F_0(y,tu_p)F_0(x,tu_p)}{|x|^\beta|x-y|^\mu|y|^\beta} \mathrm{d}x\mathrm{d}y \right\} \\ &\le \max_{t\ge 0} \left\{ \frac{1}{2} t^2 \|u_p\|^2_{X_0} - \frac{t^{2p}C_p^2}{2} \int_{\mathbb{R}}\int_{\mathbb{R}} \frac{|u|^p|u|^p}{|x|^\beta|x-y|^\mu|y|^\beta} \mathrm{d}y\, \mathrm{d}x \right\} \\ &= \frac{(p-1) S_p^{\frac{2p}{p-1}}}{2p^{\frac{p}{p-1}} C_p^{\frac{2}{p-1}}}. \end{aligned}$$

Then, taking

$$C_p > \frac{S_p^p}{p^{\frac{p}{2}}} \left[\frac{(p-1)2\theta\pi}{(\theta-1)(2-(\mu+2\beta))\omega} \right]^{\frac{p-2}{2}},$$

we have

$$c_0 < \frac{(\theta-1)(2-(\mu+2\beta))\omega}{4\pi\theta}.$$

□

4.2 Minimax estimate of Theorem 1.4

In this section, to estimate the minimax level by replacing $(f_{0,4})$ with $(\tilde{f}_{0,4})$, we require the additional hypothesis (K_4) for the kernel. To control the minimax level when assuming a hypothesis of the type $(\tilde{f}_{0,4})$, it is customary to consider the following sequence of nonnegative functions:

$$v_n(y) = \begin{cases} (\ln n)^{1/2}, & 0 \le |y| < \dfrac{1}{n}, \\ \dfrac{\ln \frac{1}{|y|}}{(\ln n)^{1/2}}, & \dfrac{1}{n} \le |y| \le 1, \\ 0, & |y| \ge 1, \end{cases}$$

well-known as Moser's sequence. By changing of variable, we obtain the following sequence of functions supported in $(x_0 - r_0, x_0 + r_0)$ given by

$$u_n(x) = \begin{cases} (\ln n)^{1/2}, & 0 \le |x - x_0| < \dfrac{r_0}{n}, \\ \dfrac{\ln \frac{r_0}{|x-x_0|}}{(\ln n)^{1/2}}, & \dfrac{r_0}{n} \le |x - x_0| \le r_0, \\ 0, & |x - x_0| \ge r_0, \end{cases}$$

with $r_0 = \min\{r, \frac{\lambda_0 \omega}{2\pi^2}\}$, where r and λ_0 are given in (K_4). We may notice that the restriction of u_n to $(x_0 - r_0, x_0 + r_0)$ belongs to $H^{1/2}((x_0 - r_0, x_0 + r_0))$ (see [47]). The following lemma deals with the asymptotic estimate on this version of Moser's sequence.

Lemma 4.2. *Suppose that (K_4) holds, then there exist convergent sequences $\tilde{C}_n$ and δ_n such that*

$$\|u_n\|_{X_0}^2 \le 2\pi \tilde{C}_n + \delta_n.$$

Proof. Note that $\|(-\Delta)^{1/4} u_n\|_2^2 = r_0 \|(-\Delta)^{1/4} v_n\|_2^2$, so by F. Takahashi [56], we have

$$\|(-\Delta)^{1/4} u_n\|_2^2 \le \pi r_0 \left(1 + \frac{1}{C \ln(n)}\right).$$

From (K_4),

$$\frac{[u_n]_{1/2,K}}{2\pi} \le \frac{1}{\lambda_0} \|(-\Delta)^{1/4} u_n\|_2^2 \le \frac{\pi r_0}{\lambda_0} \left(1 + \frac{1}{C \ln(n)}\right) \le \frac{\omega}{2\pi} \left(1 + \frac{1}{C \ln(n)}\right) =: \widetilde{C}_n. \tag{4.2}$$

Thus, for n large enough, we have

$$\begin{aligned} \|u_n\|_{X_0}^2 \le 2\pi \widetilde{C}_n &+ 2V r_0 \left[\int_{-\frac{1}{n}}^{\frac{1}{n}} \ln(n)\, dx + \frac{1}{\ln(n)} \int_{-1}^{-\frac{1}{n}} (\ln|x|)^2\, dx\right] \\ &+ 2V r_0 \left[\frac{1}{\ln(n)} \int_{\frac{1}{n}}^{1} (\ln|x|)^2\, dx\right], \end{aligned}$$

where $V := \max_{x \in \mathbb{R}} V_0(x)$, which implies that $\|u_n\|_{X_0}^2 \le 2\pi \widetilde{C}_n + \delta_n$, with

$$\delta_n := 4V r_0 \left(\frac{n - 1 - \ln(n)}{n \ln(n)}\right).$$

Notice that

$$\delta_n \to 0 \quad \text{and} \quad \tilde{C}_n \to \frac{\omega}{2\pi} \quad \text{as } n \to +\infty.$$ □

Now we consider $v_n = \frac{u_n}{\sqrt{2\pi \widetilde{C}_n + \delta_n}}$, so by Lemma 4.2 $\|v_n\|_{X_0} \le 1$, which will help us to estimate the minimax level.

Proposition 4.3. *Suppose that* (K_4) *and* $(\tilde{f}_{0,4})$ *are satisfied, then*

$$\gamma \le c_0 < \frac{(2-(\mu+2\beta))\omega}{4\pi},$$

Proof. By applying Lemma 3.1 we have that $c_0 \ge \gamma$. In order to get an upper estimate, it is enough to prove that there exists a function $v \in X_0$, $\|v\|_{X_0} \le 1$, such that

$$\max_{t\in[0,1]} I_0(tv) < \frac{(2-(\mu+2\beta))\omega}{4\pi}.$$

Let us argue by contradiction and suppose that for all $n \in \mathbb{N}$ there exists $t_n > 0$ such that

$$I_0(t_n v_n) = \max_{t\in[0,+\infty)} I_0(tv_n) \ge \frac{(2-(\mu+2\beta))\omega}{4\pi}. \tag{4.3}$$

As $F_0 \ge 0$, we obtain

$$t_n^2 \ge \frac{(2-(\mu+2\beta))\omega}{2\pi}. \tag{4.4}$$

Since t_n satisfies

$$\frac{d}{dt} I_0(tv_n)\Big|_{t=t_n} = 0,$$

it follows that

$$t_n^2 \ge \int_{\mathbb{R}} \left(\int_{\mathbb{R}} \frac{F_0(y, t_n v_n(y))}{|y-x|^\mu |y|^\beta}\, dy \right) \frac{f_0(x, t_n v_n(x)) t_n v_n(x)}{|x|^\beta}\, dx. \tag{4.5}$$

On the other hand, from Remark 1.3, we obtain

$$t f(t) F(t) \ge (\beta_0 - \varepsilon) M_0^{-1} e^{2\pi t^2} t^2, \quad \text{for all } t \ge t_0. \tag{4.6}$$

Thus, for $n \in \mathbb{N}$ large enough, by using (4.5) and (4.6), we have

$$\begin{aligned}
t_n^2 &\ge \int_{x_0-\frac{r_0}{n}}^{x_0+\frac{r_0}{n}} \frac{t_n\sqrt{\ln n}}{\sqrt{2\pi\tilde{C}_n+\delta_n}} f_0\left(x, \frac{t_n\sqrt{\ln n}}{\sqrt{2\pi\tilde{C}_n+\delta_n}}\right) \frac{1}{|x|^\beta}\, dx \int_{x_0-\frac{r_0}{n}}^{x_0+\frac{r_0}{n}} \frac{F_0\left(y, \frac{t_n\sqrt{\ln n}}{2\pi\tilde{C}_n+\delta_n}\right)}{|x-y|^\mu |y|^\beta}\, dy \\
&\ge (\beta_0-\varepsilon) M_0^{-1} \exp\left(\frac{2\pi t_n^2 \ln(n)}{2\pi\tilde{C}_n+\delta_n}\right) \left(\frac{t_n^2\ln(n)}{2\pi\tilde{C}_n+\delta_n}\right) n^{2\beta} \int_{x_0-\frac{r_0}{n}}^{x_0+\frac{r_0}{n}} \int_{x_0-\frac{r_0}{n}}^{x_0+\frac{r_0}{n}} \frac{dx\, dy}{|x-y|^\mu} \\
&= (\beta_0-\varepsilon) \frac{2^{2-\mu}}{(1-\mu)(2-\mu)M_0} \exp\left(\frac{2\pi t_n^2 \ln(n)}{2\pi\tilde{C}_n+\delta_n}\right) \left(\frac{t_n^2\ln(n)}{2\pi\tilde{C}_n+\delta_n}\right) \frac{1}{n^{2-(\mu+2\beta)}} \\
&= (\beta_0-\varepsilon) \frac{2^{2-\mu}}{(1-\mu)(2-\mu)M_0} \exp\left(\left[\frac{2\pi t_n^2}{2\pi\tilde{C}_n+\delta_n} - (2-(\mu+2\beta))\right]\ln(n) + \ln(\ln(n))\right)\left(\frac{t_n^2}{2\pi\tilde{C}_n+\delta_n}\right).
\end{aligned} \tag{4.7}$$

Consequently, as $\ln(\ln(n)) > 0$

$$1 \ge C(\beta_0-\varepsilon) \exp\left(\left[\frac{2\pi t_n^2}{2\pi\tilde{C}_n+\delta_n} - (2-(\mu+2\beta))\right]\ln(n)\right)\left(\frac{1}{2\pi\tilde{C}_n+\delta_n}\right). \tag{4.8}$$

Thus, we conclude that t_n^2 is bounded. Moreover, as $\delta_n \to 0$ as $n \to \infty$ and from (4.4), it follow that

$$t_n^2 \to \frac{(2-(\mu+2\beta))\omega}{2\pi}.$$

By (4.7), we have

$$1 \ge C(\beta_0-\varepsilon) \exp\left(\left[\frac{2\pi t_n^2}{2\pi\tilde{C}_n+\delta_n} - (2-(\mu+2\beta))\right]\ln(n) + \ln(\ln(n))\right)\left(\frac{1}{2\pi\tilde{C}_n+\delta_n}\right).$$

Passing to the limit as $n \to \infty$, there holds

$$1 \ge C(\beta_0-\varepsilon) \exp\left(\lim_{n\to\infty} \ln(\ln(n))\right).$$

Therefore, as $\ln(\ln(n)) \to +\infty$, we have a contradiction. □

5 Existence of a solution for the periodic problem

In the subsection 3.2, we guarantee that a Palais–Smale sequence is bounded in X_0. Since X_0 is a Hilbert space, up to a subsequence, we can assume that there exists $u_0 \in X_0$ such that

$$\begin{cases} u_k \rightharpoonup u_0 \ \text{ weakly in } X_0, \\ u_k \to u_0 \ \text{ in } L^q_{\mathrm{loc}}(\mathbb{R}) \ \text{ for all } q \geq 1, \\ u_k(x) \to u_0(x) \ \text{ almost everywhere in } \ \mathbb{R}. \end{cases}$$

In order to ensure that the weak limit of the Palais–Smale sequence is a solution of (1.14), we need the following auxiliary results, which holds under all hypotheses already cited.

Lemma 5.1. *Assume $(f_{0,1})$–$(f_{0,3})$ and $(f_{0,4})$ or $(\tilde{f}_{0,4})$. Let (u_k) a Palais–Smale sequence in X_0, such that $u_k \rightharpoonup u_0$ weakly in X_0, then*

$$\int_{\mathbb{R}}\int_{\mathbb{R}} \frac{F_0(y,u_k(y))f_0(x,u_k(x))\phi(x)}{|x|^\beta|y-x|^\mu|y|^\beta}\,\mathrm{d}y\,\mathrm{d}x \to \int_{\mathbb{R}}\int_{\mathbb{R}} \frac{F_0(y,u_0(y))f_0(x,u_0(x))\phi(x)}{|x|^\beta|y-x|^\mu|y|^\beta}\,\mathrm{d}y\,\mathrm{d}x, \tag{5.1}$$

for all $\phi \in C_0^\infty(\mathbb{R})$.

Proof. Similarly to [5, Lemma 2.4], we conclude that

$$\int_{\mathbb{R}} \frac{F_0(y,u_k(y))F_0(x,u_k(x))}{|x|^\beta|y-x|^\mu|y|^\beta}\,\mathrm{d}y \to \int_{\mathbb{R}} \frac{F_0(y,u_0(y))F_0(x,u_0(x))}{|x|^\beta|y-x|^\mu|y|^\beta}\,\mathrm{d}y, \quad \text{in } L^1_{\mathrm{loc}}(\mathbb{R}).$$

Now, let $u_k = u_k^+ - u_k^-$, where $u_k^+(x) = \max\{u_k(x), 0\}$ and $u_k^- = -\min\{u_n(x), 0\}$. Since $f(t) = 0$ for all $t \leq 0$, by taking $v_k = -u_k^-$ and using the fact that (u_k) is a Palais–Smale sequence, we obtain

$$\begin{aligned} o_n(1) =& I_0'(u_k)(-u_k^-) \\ =& -\int_{\mathbb{R}}\int_{\mathbb{R}} [u_k(x)-u_k(y)][u_k^-(x)-u_k^-(y)]K(x,y)\,\mathrm{d}x\mathrm{d}y - \int_{\mathbb{R}} V_0(x)u_k u_k^-\,\mathrm{d}x \\ \geq& \|u_k^-\|_{X_0}, \end{aligned}$$

where we have used that $u_n^+, u_n^- \geq 0$. Thus, $\|u_k^-\|_{X_0} \to 0$, as $k \to \infty$. Hence, we have that

$$\int_{\mathbb{R}}\int_{\mathbb{R}} [u_k^+(x)-u_k^+(y)][u_k^-(x)-u_k^-(y)]K(x,y)\,\mathrm{d}x\mathrm{d}y \to 0, \quad \text{as } k \to \infty,$$

which implies that $\|u_k\|_{X_0} = \|u_k^+\|_{X_0} + o_n(1)$. Therefore, (u_k^+) is also a Palais–Smale sequence for functional I_0. For this, we may suppose, without loss of generality, that (u_k) is a nonnegative Palais–Smale sequence.

Now, in the same spirit as the proof of [5, Lemma 2.4], let $\phi \in C_0^\infty(\mathbb{R})$ be such that $\operatorname{supp}\phi \subset \Omega'$ satisfying $0 \leq \phi \leq 1$ and $\phi \equiv 1$ in $\Omega \subset \Omega'$ and define $v_k = \phi/(1+u_k)$. In view of Young's inequality one has

$$\begin{aligned} \int_{\mathbb{R}}\int_{\mathbb{R}} [u_k(x)-u_k(y)][v_k(x)-v_k(y)]K(x,y)\,\mathrm{d}x\mathrm{d}y &\leq \frac{1}{2}[u_k]^2_{1/2,K} + \frac{1}{2}[v_k]^2_{1/2,K} \\ &= \frac{1}{2}[u_k]^2_{1/2,K} + \frac{1}{2}\int_{\mathbb{R}}\int_{\mathbb{R}} \frac{[(1+u_k(y))\phi(x)-(1+u_k(x))\phi(y)]^2}{(1+u_k(x))^2(1+u_k(y))^2}K(x,y)\,\mathrm{d}x\mathrm{d}y \\ &\leq \frac{1}{2}[u_k]^2_{1/2,K} + \frac{1}{2}\int_{\mathbb{R}}\int_{\mathbb{R}} [(1+u_k(y))\phi(x)-(1+u_k(x))\phi(y)]^2K(x,y)\,\mathrm{d}x\mathrm{d}y \\ &\leq C\left([u_k]^2_{1/2,K} + [\phi]^2_{1/2,K} + [u_n\phi]^2_{1/2,K}\right). \end{aligned} \tag{5.2}$$

Observe that

$$\begin{aligned}
&\int_{\mathbb{R}}\int_{\mathbb{R}} |u_k(x)\phi(x) - u_k(y)\phi(y)|^2 K(x,y)\,\mathrm{d}x\mathrm{d}y \\
&\quad \le C_1 \int_{\mathbb{R}}\int_{\mathbb{R}} \left[|u_k(x)\phi(x) - u_k(y)\phi(x)|^2 K(x,y) + V_0(x)|u_k(x)|^2|\phi(x)-\phi(y)|^2K(x,y)\right]\,\mathrm{d}x\mathrm{d}y \\
&\quad \le \tilde{C}_1 [u_k]^2_{1/2,K} + C_2(\phi)\int_{\mathbb{R}} V_0(x)u_k^2\,\mathrm{d}x \\
&\quad \le C(\phi)\|u_k\|^2_{X_0},
\end{aligned} \tag{5.3}$$

where $C_2(\phi)$ is a constant which depends on ϕ. By using (3.7), (5.2) and (5.3) we obtain

$$\begin{aligned}
\int_{\Omega}\int_{\mathbb{R}} \frac{F_0(y,u_k(y))}{|x|^\beta|y-x|^\mu|y|^\beta}\frac{f_0(u_k)}{1+u_k}\,\mathrm{d}y\,\mathrm{d}x &\le \int_{\mathbb{R}}\int_{\mathbb{R}} \frac{F_0(y,u_k(y))}{|x|^\beta|y-x|^\mu|y|^\beta}\frac{f_0(u_k)\phi}{1+u_k}\,\mathrm{d}y\,\mathrm{d}x \\
&= \int_{\mathbb{R}}\int_{\mathbb{R}} [u_k(x)-u_k(y)][v_k(x)-v_k(y)]K(x,y)\,\mathrm{d}x\mathrm{d}y + \int_{\mathbb{R}} V_0(x)u_kv_k\,\mathrm{d}x + \epsilon_k\|v_k\|_{1/2,K} \\
&\le C\left([u_k]^2_{1/2,K} + [\phi]^2_{1/2,K} + [u_k\phi]^2_{1/2,K}\right) + C\int_{\Omega'} u_k\,\mathrm{d}x + \epsilon_k\|v_k\|_{1/2,K} \\
&\le \tilde{C}(\phi)\|u_k\|^2_{1/2,K} + C_3(\phi) + C\int_{\Omega'} u_k\,\mathrm{d}x + \epsilon_k\|v_k\|_{1/2,K}.
\end{aligned}$$

Since (u_k) is bounded in X_0 and $u_k \to u_0$ in $L^1(\Omega')$ we conclude that

$$\int_{\Omega}\int_{\mathbb{R}} \frac{F_0(y,u_k(y))}{|x|^\beta|y-x|^\mu|y|^\beta}\frac{f_0(u_k)}{1+u_k}\,\mathrm{d}y\,\mathrm{d}x \le \overline{C}(\phi).$$

Thus, by a Radon–Nikodym argument, we can conclude that

$$\lim_{k\to\infty}\int_{\mathbb{R}}\int_{\mathbb{R}} \frac{F_0(y,u_k(y))f_0(x,u_k(x))\phi(x)}{|x|^\beta|y-x|^\mu|y|^\beta}\,\mathrm{d}y\,\mathrm{d}x = \int_{\mathbb{R}}\int_{\mathbb{R}} \frac{F_0(y,u_0(y))f_0(x,u_0(x))\phi(x)}{|x|^\beta|y-x|^\mu|y|^\beta}\,\mathrm{d}y\,\mathrm{d}x,$$

for all $\phi \in C_c^\infty(\mathbb{R})$. □

5.1 Proof of Theorems 1.2 and 1.4

From Lemma 5.1, we conclude that u is a weak solution of Problem (1.14). If $u \neq 0$, then the proof is done. Suppose that $u = 0$. We claim that there exists $R, a > 0$ and a sequence $(y_n) \subset \mathbb{Z}$ such that

$$\lim_{n\to+\infty}\int_{y_n-R}^{y_n+R} |u_k|^2\,\mathrm{d}x \ge a. \tag{5.4}$$

Suppose by contradiction that (5.4) does not hold. Thus, for any $R > 0$, there holds

$$\lim_{k\to+\infty}\sup_{y\in\mathbb{R}}\int_{y-R}^{y+R} |u_k|^2\,\mathrm{d}x = 0.$$

In view of (4.1), we obtain that $u_k \to 0$ strongly in $L^p(\mathbb{R})$, for $2 < p < \infty$. Similarly to [5], we may conclude that

$$\int_{\mathbb{R}} \frac{F_0(y,u_k(y))F_0(x,u_k(x))}{|x|^\beta|y-x|^\mu|y|^\beta}\,\mathrm{d}y \to 0, \quad \text{in } L^1(\mathbb{R}). \tag{5.5}$$

If we assume $(f_{0,4})$, by Proposition 4.1, (3.8) and $(f_{0,2})$, we obtain

$$\lim_{k\to+\infty}\|u_k\|^2_{X_0} < \frac{(2-(\mu+2\beta))\omega}{2\pi}.$$

On the other hand, if we assume $(\tilde{f}_{0,4})$ and (K_4), in view of Proposition 4.3, (3.7) and (5.5) one also has

$$\lim_{n\to+\infty}\|u_k\|^2_{X_0} = 2c_0 < \frac{(2-(\mu+2\beta))\omega}{2\pi}.$$

Thus, in both cases, there exists $\delta > 0$ small and $k_0 \in \mathbb{N}$ large such that

$$\|u_k\|^2_{X_0} \le \frac{(2-(\mu+2\beta))\omega}{2\pi}(1-\delta), \quad \text{for all } k \ge k_0. \tag{5.6}$$

In light of Weighted-Hardy–Littlewood–Sobolev inequality we have

$$\int_{\mathbb{R}}\int_{\mathbb{R}} \frac{F_0(y,u_k(y))f_0(x,u_k(x))u_k}{|x|^\beta|y-x|^\mu|y|^\beta}\,dy\,dx \le C\|F_0(x,u_k)\|_{\frac{2}{2-(\mu+2\beta)}}\|f_0(x,u_k)u_k\|_{\frac{2}{2-(\mu+2\beta)}}.$$

By using (3.2), for any $\varepsilon > 0$ and $q > 2$ such that

$$\|f_0(x,u_k)u_k\|_{\frac{2}{2-(\mu+2\beta)}} \le \varepsilon\|u_k\|^2_{\frac{2}{2-(\mu+2\beta)}} + C_\varepsilon\left[\int_{\mathbb{R}}(e^{\pi u_k^2}-1)^{\frac{2}{2-(\mu+2\beta)}}|u_k|^{\frac{2q}{2-(\mu+2\beta)}}\,dx\right]^{\frac{2-(\mu+2\beta)}{2}}.$$

Let us consider $\sigma > 1$ close to 1 and $r, r' > 1$ such that $1/r + 1/r' = 1$. Thus, one has

$$\left[\int_{\mathbb{R}}(e^{\pi u_k^2}-1)^{\frac{2}{2-(\mu+2\beta)}}|u_k|^{\frac{2q}{2-(\mu+2\beta)}}\,dx\right]^{\frac{2-(\mu+2\beta)}{2}} \le \|u_k\|^q_{\frac{2qr'}{2-(\mu+2\beta)}}\left[\int_{\mathbb{R}}(e^{\frac{2\sigma r}{2-(\mu+2\beta)}\pi u_k^2}-1)\,dx\right]^{\frac{2-(\mu+2\beta)}{2r}}.$$

By choosing $\sigma, r > 1$ sufficiently close to 1, $\delta > 0$ small enough and q such that

$$1 < \sigma r < \frac{1}{1-\delta} \quad \text{and} \quad \frac{2qr'}{2-(\mu+2\beta)} > 2,$$

it follows from (5.6) that

$$\frac{2\sigma r}{(2-(\mu+2\beta))}\|u_k\|^2_{X_0} < \frac{\omega}{\pi}, \quad \text{for all } k \ge k_0.$$

Thus, in view of Lemma 2.3, we obtain

$$\int_{\mathbb{R}}(e^{\frac{2\sigma r}{2-(\mu+2\beta)}\pi u_k^2}-1)\,dx = \int_{\mathbb{R}}(e^{\frac{2\sigma r}{2-(\mu+2\beta)}\|u_k\|^2_{X_0}\pi\frac{u_k^2}{\|u_k\|^2_{X_0}}}-1)\,dx < \int_{\mathbb{R}}(e^{\omega\frac{u_k^2}{\|u_k\|^2_{X_0}}}-1)\,dx \le C, \tag{5.7}$$

for all $k \ge k_0$. Therefore, by using the Lions Lemma for X_0 and combining (5.6)–(5.7) we conclude that

$$\int_{\mathbb{R}}\int_{\mathbb{R}} \frac{F_0(y,u_k(y))F_0(x,u_k(x))}{|x|^\beta|y-x|^\mu|y|^\beta}\,dydx \to 0, \quad \text{as } k \to \infty.$$

Since (u_k) is a Palais–Smale sequence we have that

$$0 < c = \frac{1}{2}\|u_k\|^2_{X_0} + o(1) \quad \text{and} \quad o(1) = \|u_k\|^2_{X_0},$$

which is not possible. Therefore, (5.4) is satisfied. We may assume, without loss of generality, that $(y_k) \subset \mathbb{Z}$. Letting $w_k(x) = u_k(x-y_k)$, since $V_0(\cdot)$, $f_0(\cdot,s)$ and $F_0(\cdot,s)$ are 1-periodic functions, by a careful calculation we obtain

$$\|u_k\|_{X_0} = \|w_k\|_{X_0},$$

$$I_0(u_k) = I_0(w_k) \to c_0 \quad \text{and} \quad I_0'(w_k) \to 0.$$

Consequently, by similar arguments made in the previous sections, we obtain that (w_k) is bounded in X_0 and there exists $w_0 \in X_0$ such that $w_k \rightharpoonup w_0$ weakly in X_0 and w_0 is a weak solution of the problem (1.14). Moreover, by (5.4), taking a subsequence and R sufficiently large, we get

$$a^{1/2} \leq \|w_k\|_{L^2(B_R(0))} \leq \|w_k - w_0\|_{L^2(B_R(0))} + \|w_0\|_{L^2(B_R(0))}. \tag{5.8}$$

Thus, by using Corollary 2.2 we conclude that w_0 is nontrivial. To finalize, notice that if u is a weak solution of (1.14), since $f_0(x,s) = 0$ for all $s \leq 0$ and $I_0'(u)v = 0$ for all $v \in X_0$, choosing the test function $v = -u^-$, by using the following inequality $|u^-(x) - u^-(y)|^2 \leq (u(x) - u(y))(u^-(y) - u^-(x))$ and the fact that m is nondecreasing we get that $\|u^-\|_{X_0} \leq 0$. Thus, u is a nonnegative function. This completes the proof of Theorems 1.2 and 1.4.

6 Existence of a solution for the nonperiodic problem

In this section, we are concerned with finding a nonnegative and nontrivial solution for (1.17). For this, we consider the functional $I : X_1 \to \mathbb{R}$ given by

$$I(u) = \frac{1}{2}\|u\|_{X_1}^2 - \frac{1}{2}\int_{\mathbb{R}}\int_{\mathbb{R}} \frac{F(y,u(y))F(x,u(x))}{|x|^\beta |y-x|^\mu |y|^\beta}\,\mathrm{d}y\,\mathrm{d}x.$$

From (f_1)–(f_2), Lemmas 2.3 and 2.4, similarly to Section 3, we can see that I is well defined and by using standard arguments $I \in C^1(X_1, \mathbb{R})$ with

$$I'(u)v = \langle u, v\rangle_1 - \int_{\mathbb{R}} \frac{1}{|x|^\beta}\left(\int_{\mathbb{R}} \frac{F(y,u(y))}{|y-x|^\mu |y|^\beta}\,\mathrm{d}y\right) f(x,u(x))v(x)\,\mathrm{d}x,.$$

for all $v \in X_1$. Thus, a critical point of I is a weak solution of (1.17) and reciprocally. Moreover, I has the geometry of the mountain-pass theorem, by analogous steps to Propositions 3.1 and 3.2, we obtain

Proposition 6.1. *If (f_2)–(f_3) and (V_1) hold, then*

i) there exist $\gamma_1, \rho_1 > 0$ such that $I(u) \geq \sigma_1$ if $\|u\|_{X_1} = \rho_1$;

ii) there exists $e_1 \in E$, with $\|e_1\|_{X_1} > \rho_1$, such that $I(e_1) < 0$.

As a consequence of Proposition 6.1, the minimax level

$$c_1 := \inf_{\gamma \in \Gamma} \max_{t \in [0,1]} I(\gamma(t))$$

is positive, where $\Gamma = \{\gamma \in C([0,1], X_1) : \gamma(0) = 0 \text{ and } \gamma(1) = e_1\}$.

Moreover, by applying the mountain-pass theorem without the (PS) condition (see [43]), there exists a sequence $(v_k) \subset X_1$ such that

$$I(v_k) \to c_1 \quad \text{and} \quad I'(v_k) \to 0.$$

Thus, similarly to Lemma 3.3, we obtain that (v_k) is a bounded sequence in X_1. Moreover, by using the arguments made in Proposition 5.1, we get the following result:

Proposition 6.2. *If* $(f_{0,4})$ *(or* $(\tilde{f}_{0,4})$ *and* (K_4)*),* (f_1)*–*(f_3) *and* (V_1) *hold, then* $v_k \rightharpoonup v_0$ *weakly in* X_1 *and* v_0 *is a critical point of functional* I*.*

Now, in order to prove that there exists a nontrivial critical point of I, we need some auxiliary results, among then a lemma of convergence. More specifically, assuming for the sake of contradiction that v_0 is trivial, following the same steps of [7, Section 3.2] and [24, Lemma 2.4], we obtain the following result.

Lemma 6.3. *If* (V_1)*,* $(f_{0,1})$*–*$(f_{0,2})$ *and* (f_1)*–*(f_3) *hold, then*

i) $\left[\int_{\mathbb{R}}\int_{\mathbb{R}} \frac{F_0(y,v_k(y))}{|x|^\beta|y-x|^\mu|y|^\beta} f_0(x,v_k(x)) - \frac{F(y,v_k(y))}{|x|^\beta|y-x|^\mu|y|^\beta} f(x,v_k(x))\right] v_k(x)\,\mathrm{d}y\,\mathrm{d}x \to 0;$

ii) $\int_{\mathbb{R}}\int_{\mathbb{R}} \left[\frac{F_0(y,v_k(y))F_0(x,v_k(x))}{|x|^\beta|y-x|^\mu|y|^\beta} - \frac{F(y,v_k(y))F(x,v_k(x))}{|x|^\beta|y-x|^\mu|y|^\beta}\right] \mathrm{d}y\,\mathrm{d}x \to 0;$

iii) $\int_{\mathbb{R}} [V_0(x) - V(x)] v_k^2\,\mathrm{d}x \to 0.$

6.1 Proof of Theorem 1.5

Assuming for the sake of contradiction that v_0 is trivial, as a consequence of Lemma 6.3, it follows that

$$|I_0(v_k) - I(v_k)| \to 0 \quad \text{and} \quad \|I_0'(v_k) - I'(v_k)\|_* \to 0. \tag{6.1}$$

Hence,

$$I_0(v_k) \to c_1 \quad \text{and} \quad I_0'(v_k) \to 0.$$

In addition, we obtained a version of Lions's result for a sequence in X_1 as in (4.1). From this, we conclude that there exist $(y_k) \subset \mathbb{Z}$ and $R, a > 0$ such that

$$\liminf \sup_{y_k \in \mathbb{R}} \int_{B_R(y_k)} |v_k|^2\,\mathrm{d}x > a.$$

Now consider $w_k(x) = v(x - y_k)$, since $V_0(x)$, $f_0(x,s)$ and $F_0(x,s)$ are 1-periodic functions in x, we get

$$\|v_k\|_{X_0} = \|w_k\|_{X_0},$$
$$I_0(v_k) = I_0(w_k) \to c_1 \quad \text{and} \quad I_0'(w_k) \to 0.$$

Then, there exists $w_0 \in X_0$ such that $w_k \rightharpoonup w_0$ weakly in X_0 and $I_0'(w_0) = 0$. Moreover, using (6.1) and Fatou's lemma we have

$$\begin{aligned}
I_0(w_0) &= I_0(w_0) - \frac{1}{2} I_0'(w_0) w_0 \\
&= \frac{1}{2}\int_{\mathbb{R}}\int_{\mathbb{R}} \frac{F_0(y,w_0(y))}{|x|^\beta|y-x|^\mu|y|^\beta}[f_0(x,w_0)w_0 - F_0(x,w_0)]\,\mathrm{d}y\,\mathrm{d}x \\
&\le \liminf \frac{1}{2}\int_{\mathbb{R}}\int_{\mathbb{R}} \frac{F_0(y,w_k(y))}{|x|^\beta|y-x|^\mu|y|^\beta}[f_0(x,w_k)w_k - F_0(x,w_k)]\,\mathrm{d}y\,\mathrm{d}x \\
&= \lim[I_0(w_k) - \frac{1}{2} I_0'(w_k) w_k] = c_1.
\end{aligned}$$

Arguing as in (5.8) we conclude that w_0 is nontrivial. On the one hand, by $(f_{0,3})$, we have that $\max\{I_0(tw_0) : t \ge 0\}$ is unique and then

$$c_0 \le \max_{t \ge 0} I_0(tw_0) = I_0(w_0) \le c_1. \tag{6.2}$$

On the other hand, considering u_0 the solution obtained in Theorem 1.2 (or Theorem 1.4), so from (V_1), (f_1), (f_5), (f_4) and $(f_{0,3})$, we have

$$c_1 \leq \max_{t\geq 0} I(tu_0) = I(t_1u_0) < I_0(t_1u_0) \leq \max_{t\geq 0} I_0(tu_0) = I_0(u_0) = c_0,$$

that is, $c_1 < c_0$, which is a contradiction with (6.2). Therefore, v_0 is nontrivial.

To finalize, notice that similarly to the proof of Theorems 1.2 and 1.4 if we have a weak solution of (1.17), then it is a nonnegative function. This completes the proof of Theorem 1.5.

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