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Fake saddles and their transition maps

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Abstract. We study degenerate singular points of planar vector fields inside a (degenerate) flow box. These kind of singularities are called fake saddles and their linear parts are always zero. We characterize fake saddles with non-zero second-order jet and we give the first term of a uniform asymptotic expansion of the Poincaré map between two transverse sections to their corresponding singular fiber, determining its stability.

Keywords: singularities, Poincaré and Dulac transition maps, asymptotic expansion, uniform flatness, stability.

2020 Mathematics Subject Classification: 34D10, 34D20; Secondary: 34C05.

1 Introduction and statements of main results

Following [3,8] we define a *fake saddle* as a singular point having exactly two separatrices which are contained in a smooth invariant curve separating two hyperbolic sectors, see Figure 1.1. This kind of singularities are also known as impassable grains. Another way to think about a fake saddle is that the corresponding vector field can be put in a form that is like a degenerate flow box, i.e. near the singularity the phase portrait consists of parallel fibers, all but one of which have no singular points, and the singular fiber (the union of the singular point with its two separatrices) has a semi-stable equilibrium point, see [2]. Taking two transverse sections Σ_α and Σ_ω to the singular fiber outside the singular point there is a transition map $\Pi_\alpha^\omega : \Sigma_\alpha \to \Sigma_\omega$ which is well-defined on both sides of the singular fiber.

The local phase portrait of a singularity with non-zero linear part (for the nilpotent case see for instance [1, Theorem 3.5]) is well known and prevents the singular point from being a fake saddle. Notice that the smoothness of the singular fiber is strictly necessary because the Hamiltonian vector field $y\partial_x + x^2\partial_y$ has a nilpotent singularity at the origin with exactly two separatrices that are contained in the cuspidal curve $y^2 - \frac{2}{3}x^3 = 0$ separating two hyperbolic sectors.

The first non-trivial and generic case of a fake saddle arises when the second-order jet is non-zero. Our objective is to characterize these generic fake saddles and to analyze their transition maps in order to determine whether the behavior is attractive or repulsive on each side of the singular fiber. The techniques we employ allow us to derive the leading term of an asymptotic expansion that is uniform with respect to parameters.

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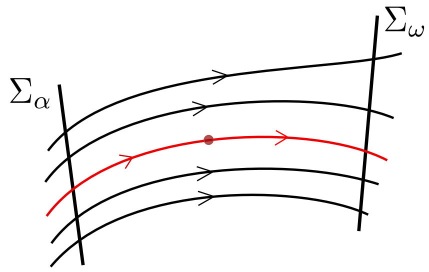

Figure 1.1: Fake saddle and the associated transition maps.

This work is mainly motivated by the paper [2], which provides a normal form for generic fake saddles that we recall in the sequel. First we can locate the fake saddle at the origin $(0,0)$. Secondly, if the vector field is at least of class $\mathscr{C}^3$ there exists a local diffeomorphism that rectifies the germ of singular fiber, so that it is contained in the line $y=0$ in a certain coordinate system (x,y), see [2, Lemma 2.1]. Thanks to the hypothesis that the second-order jet of the fake saddle is non-zero, after a suitable rescaling of the coordinates we can assume that the fake saddle is defined by a system of differential equations

$$\begin{cases} \dot{x} = x^2 + axy + y^2 + \mathrm{O}_3(x,y), \\ \dot{y} = y\big(cx + by + \mathrm{O}_2(x,y)\big), \end{cases}$$

where $a,b,c \in \mathbb{R}$. This normal form motivates us to consider a smooth family of planar vector fields $\{X_\mu\}_{\mu\in\Omega\subset\mathbb{R}^n}$ having the form

$$X_\mu(x,y) = (x^2 f_1(x,y;\mu) + a(\mu)xy + y^2 f_2(x,y;\mu))\partial_x + \big(xg_1(x,y;\mu) + yg_2(y;\mu)\big)y\partial_y, \quad (1.1)$$

where $f_1(x,y;\mu)$, $f_2(x,y;\mu)$, $g_1(x,y;\mu)$ and $g_2(y;\mu)$ are $\mathscr{C}^K$-functions fulfilling $f_1(0,0;\mu) = f_2(0,0;\mu) = 1$. We consider the following invariants

$$a(\mu), \quad b(\mu) := g_2(0;\mu), \quad c(\mu) := g_1(0,0;\mu) \quad (1.2)$$

of the family (1.1) and define the associated value $d(\mu) := 4(1-c(\mu)) - (a(\mu)-b(\mu))^2$, which will play a key role in the sequel.

In this paper we extend the main theorems of [2] and we give simpler proofs of them using the results stated in [5,7]. Our first result gives a characterization of generic fake saddles inside the family (1.1), generalizing [2, Theorem A] which only treats the case $a=0$.

Theorem 1.1. *If the invariants (a,b,c) given in (1.2) of a vector field X in the family (1.1) do not belong to $\{d=0\}\cap\{a^2-b^2=4\}$ then the origin is a fake saddle if and only if, either $d>0$, or $c=1$ and $a=b$. In both situations after blowing up the origin we have a single singular point on the exceptional divisor, which is a hyperbolic saddle of hyperbolicity ratio $1-c>0$ in the first one and a semi-hyperbolic saddle in the second one.*

The hypothesis $(a,b,c)\notin\{d=0\}\cap\{a^2-b^2=4\}$ in Theorem 1.1 (which is always verified when $a=0$) can not be removed as Example 3.1 will show.

Let Ω be an open subset of $\mathbb{R}^n$ and let W any subset of Ω. We recall (cf. [5, Definition 1.2]) that a $\mathscr{C}^K$-function $f(s;\mu)$ defined in the intersection of $(0,+\infty)\times\Omega$ with an open neighborhood of $\{0\} \times \Omega \subset \mathbb{R}^{n+1}$ belongs to the flat class $\mathcal{F}_L^K(W)$ if for each $\mu_0 \in W$ and $\nu = (\nu_0, \ldots, \nu_n) \in \mathbf{N}^{n+1}$ with $|\nu| = \nu_0 + \cdots + \nu_n \leq K$ there exists $\varepsilon > 0$ such that $|\partial_s^{\nu_0} \partial_{\mu_1}^{\nu_1} \cdots \partial_{\mu_n}^{\nu_n} f(s;\mu)| \leq C s^{L-\nu_0}$ for every $s \in (0,\varepsilon)$ and $\mu \in \mathbb{B}_\varepsilon(\mu_0) \cap \Omega$.

We consider the transverse sections $\Sigma_\star = \{x = \star\}$ to $y = 0$, for $\star \in \{\alpha, \omega\}$, where $\alpha < 0 < \omega$. We assume that $f_1(x,0) > 0$ for all $x \in [\alpha, \omega]$ and we consider the transition map $\Pi_\alpha^\omega : \Sigma_\alpha \to \Sigma_\omega$. Our second result extends [2, Theorem B], which only treats the case $a = b = 0$, $\partial_y f_1 = \partial_y g_1 = 0$, $f_2 = 1$ and $g_2 = 0$, when the transition map is restricted to $y > 0$. In that theorem the authors characterize the stability of the transition map from the sign of the Cauchy principal value

$$\mathrm{PV} \int_\alpha^\omega \frac{g_1(x,0;\mu)}{x f_1(x,0;\mu)} dx := \lim_{\varepsilon \to 0^+} \left(\int_\alpha^{-\varepsilon} \frac{g_1(x,0;\mu)}{x f_1(x,0;\mu)} dx + \int_\varepsilon^\omega \frac{g_1(x,0;\mu)}{x f_1(x,0;\mu)} dx \right).$$

Theorem 1.2. *Assume that $\mu_0 \in \Omega$ and that the invariants (1.2) of the vector field X_{μ_0} in (1.1) belong to $\{d = 4(1-c) - (a-b)^2 > 0\}$. If $\mu \approx \mu_0$ then the origin is a fake saddle of X_μ and its transition map $\Pi_\alpha^\omega : \Sigma_\alpha \to \Sigma_\omega$ satisfies $\Pi_\alpha^\omega(y;\mu) = e^{\gamma_\pm(\mu)} y + \mathcal{F}_{1+\epsilon}^K(\{d > 0\})$ on $\pm y \geq 0$, where $\epsilon > 0$ and*

$$\gamma_\pm(\mu) = \mathrm{PV} \int_\alpha^\omega \frac{g_1(x,0;\mu)}{x f_1(x,0;\mu)} dx \pm \frac{\pi(2b(\mu) - c(\mu)(a(\mu) + b(\mu)))}{\sqrt{d(\mu)}}.$$

Remark 1.3. Using that $f_1(0,0) = 1$ and $g_1(0,0) = c$ we can explicitly compute the previous Cauchy principal value using a convergent integral:

$$\mathrm{PV} \int_\alpha^\omega \frac{g_1(x,0)}{x f_1(x,0)} dx = c \log \left| \frac{\omega}{\alpha} \right| + \int_\alpha^\omega \left(\frac{g_1(x,0)}{f_1(x,0)} - c \right) \frac{dx}{x}.$$

Remark 1.4. Clearly the sign of $\gamma_\pm(\mu)$ determines the stability of the singular fiber $y = 0$ on the side $\pm y > 0$. In the case $a = b = 0$ the two values $\gamma_\pm(\mu)$ coincide with the principal value of the integral. Otherwise the stability of the two sides $\pm y > 0$ can be different, see Example 3.2.

It is worth to be noticed that the writing of $\Pi_\alpha^\omega(y;\mu)$ in the statement implies that it is of the form $e^{\gamma_\pm(\mu)} y + \mathrm{o}(y)$ but also that the remainder term $\mathrm{o}(y)$ is uniform with respect to the parameter μ. This uniformity allows to address cyclicity problems and not just study stability. For clarity in the exposition we will omit the dependence on μ when it is not essential.

2 Proofs of the main results

Proof of Theorem 1.1. A straightforward computation shows that

$$Y = \frac{(u, uv)^* X}{u} = P(u,v) u \partial_u + Q(u,v) v \partial_v$$

with $P(0,0) = 1$ and $Q(0,v) = -v^2 + (b-a)v + c - 1$. The point $(u,v) = (0,0)$ is always a singular point of Y. If $d < 0$ then Y has two other singular points on the exceptional divisor $u = 0$ with at least one non-zero eigenvalue. If $(a,b,c) \in \{d = 0\} \setminus (\{c = 1, a = b\} \cup \{a^2 - b^2 = 4\})$ then Y has another double singular point at $(u,v) = (0, (b-a)/2)$ with a non-zero eigenvalue. In these situations the transition map $\Pi_\alpha^\omega : \{x = \alpha\} \to \{x = \omega\}$ is not well defined so that the origin is not a fake saddle of X. The remaining assertions are easy to check. ☐

The proof of Theorem 1.2 is based on some results of [5,7]. For reader's convenience we recall here the notations that we will use in the sequel.

Consider a smooth family of planar vector fields $\{X_\mu\}_{\mu\approx\mu_0}$

$$X_\mu = P_1(x,y;\mu)x\partial_x + P_2(x,y;\mu)y\partial_y \tag{2.1}$$

having a saddle point at the origin with hyperbolicity ratio $\lambda(\mu) = -\frac{P_2(0,0;\mu)}{P_1(0,0;\mu)} > 0$ and separatrices contained in the coordinate axes $xy = 0$. Let $\sigma_1(s;\mu) = (\sigma_{11}(s;\mu), \sigma_{12}(s;\mu))$ and $\sigma_2(s;\mu) = (\sigma_{21}(s;\mu), \sigma_{22}(s;\mu))$ be parametrized smooth transverse sections to $x = 0$ and $y = 0$ respectively. Denote $\sigma_{ijk}(\mu) = \partial_s^k \sigma_{ij}(0;\mu)$ and assume that $\sigma_{110} = \sigma_{220} = 0$. We also assume that for all $\mu \approx \mu_0$ we have $P_2(0,y;\mu) \neq 0$ for all $y \in [0, \sigma_{120}(\mu)]$ and $P_1(x,0;\mu) \neq 0$ for all $x \in [0, \sigma_{210}(\mu)]$. Let us introduce the following auxiliary functions (see [7, p. 47])

$$\begin{aligned} L_1(z;\mu) &= \exp \int_0^z \left(\frac{P_1}{P_2}(0,y;\mu) + \frac{1}{\lambda(\mu)} \right) \frac{dy}{y}, \\ L_2(z;\mu) &= \exp \int_0^z \left(\frac{P_2}{P_1}(x,0;\mu) + \lambda(\mu) \right) \frac{dx}{x}. \end{aligned} \tag{2.2}$$

We will use the next result on the Dulac map of (2.1) which follows by applying [5, Theorem A] and [7, Theorem A].

Theorem 2.1. *The Dulac map $D(s;\mu)$ of the saddle at the origin of (2.1) associated to the parametrized transverse sections σ_1 and σ_2 admits the following asymptotic expansion*

$$D(s;\mu) = s^{\lambda(\mu)}(\Delta_{00}(\mu) + \mathcal{F}_\epsilon(\mu_0)),$$

where $\epsilon \in (0, \min(\lambda(\mu_0), 1))$ and

$$\Delta_{00} = \frac{\sigma_{111}\sigma_{120}^{\lambda}(L_2(\sigma_{210}))^{\lambda}}{\sigma_{221}^{\lambda}\sigma_{210}L_1(\sigma_{120})}. \tag{2.3}$$

We will also use the following auxiliary result that gathers some properties of the flat class $\mathcal{F}_L^K(W)$ (see [5, Lemma A.2]).

Lemma 2.2. For every $K \in \mathbf{Z}_{\geq 0} \cup \{\infty\}$ the following assertions hold:

(a) $\mathscr{C}^K(\Omega) \subset \mathcal{F}_0^K(\Omega)$.

(b) If $L \geq L'$ then $\mathcal{F}_L^K(W) \subset \mathcal{F}_{L'}^K(W)$.

(c) $\mathcal{F}_L^K(W)$ is closed under addition.

(d) $\mathcal{F}_L^K(W) \cdot \mathcal{F}_{L'}^K(W) \subset \mathcal{F}_{L+L'}^K(W)$.

(e) $\mathcal{F}_L^K(W) \circ \mathcal{F}_{L'}^K(W) \subset \mathcal{F}_{LL'}^K(W)$.

Proof of Theorem 1.2. Let us compute first the value γ_+, i.e. we assume that $y \geq 0$. For simplicity we omit the dependence on μ. Consider the charts $(x,y) = \pi_\pm(u,v) = (\pm u(1-v), uv)$ of the blow-up of the origin, see Figure 2.1. Notice that $v = \frac{y}{y \pm x}$ is a coordinate on the exceptional divisor $u = 0$. We assume that $u, v \geq 0$, we write

$$\frac{\pi_\pm^* X}{u} = P_\pm(u,v)u\partial_u + Q_\pm(u,v)v\partial_v$$

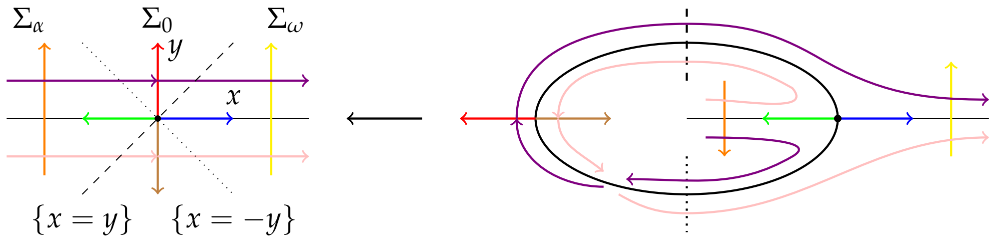

Figure 2.1: Blowup of the fake saddle at the origin.

and define

$$X_\pm(x,y) = P_1^\pm(x,y)x\partial_x + P_2^\pm(x,y)y\partial_y$$

with

$$P_1^-(x,y) = Q_-(y,x), \quad P_2^-(x,y) = P_-(y,x), \quad P_1^+(x,y) = P_+(x,y), \quad P_2^+(x,y) = Q_+(x,y).$$

It can be checked that $\lambda_\pm = -\frac{P_2^\pm(0,0)}{P_1^\pm(0,0)} > 0$, so that $(0,0)$ is a hyperbolic saddle of $X_\pm$. Moreover $\lambda_+\lambda_- = 1$. Consider the Dulac map $D_\pm$ of the origin of $X_\pm$ between the parametrized transverse sections

$$\sigma_1^-(s) = \left(\frac{s}{-\alpha+s}, -\alpha+s\right), \quad \sigma_2^-(s) = (1,s), \quad \sigma_1^+(s) = (s,1), \quad \sigma_2^+(s) = \left(\omega+s, \frac{s}{\omega+s}\right), \tag{2.4}$$

which come from the parametrizations by the y coordinate of the original transverse sections Σ_α, Σ_0 and Σ_ω to the invariant line $\{y=0\}$ of X.

By Theorem 2.1 it follows that $D_\pm(s;\mu) = s^{\lambda_\pm(\mu)}(\Delta_{00}^\pm(\mu) + \mathcal{F}_\epsilon)$ for some $\epsilon_\pm > 0$, where we write $\mathcal{F}_{\epsilon_\pm}$ instead of $\mathcal{F}_{\epsilon_\pm}^K(\{\lambda_\pm > 0\})$ for brevity. By applying Lemma 2.2 we get that $(1+s)^{\lambda_-} - 1 \in \mathcal{F}_1$ and $D_- \in \mathcal{F}_{\epsilon'_-}$ for some $\epsilon'_- > 0$. Then, using again Lemma 2.2 and the fact $\lambda_+\lambda_- = 1$, we can write

$$\Pi_\alpha^\omega(y) = (D_+\circ D_-)(y) = \left(y^{\lambda_-}(\Delta_{00}^- + \mathcal{F}_{\epsilon_-})\right)^{\lambda_+}(\Delta_{00}^+ + \mathcal{F}_{\epsilon_+}\circ D_-) = \Delta_{00}y + \mathcal{F}_{1+\epsilon},$$

with $\Delta_{00} = (\Delta_{00}^-)^{\lambda_+}\Delta_{00}^+$ and some $\epsilon > 0$. According to (2.3) and (2.2), in order to compute the coefficients $\Delta_{00}^\pm$ we must consider the following functions:

$$\begin{aligned}
R_{12}^-(y) &= \frac{P_1^-}{P_2^-}(0,y) = \frac{g(-y,0)}{f(-y,0)} - 1,\\
R_{21}^-(x) &= \frac{P_2^-}{P_1^-}(x,0) = \frac{1}{1-c}\left[\frac{(-a+b+c-2)\,x^2 + (a-c+2)\,x - 1}{(a-b-c+2)\,x^2 + (-a+b+2c-2)\,x + 1 - c}\right],\\
R_{12}^+(y) &= \frac{P_1^+}{P_2^+}(0,y) = \frac{1}{1-c}\left[\frac{(-a+b-c+2)\,y^2 + (a+c-2)\,y + 1}{(a-b+c-2)\,y^2 + (-a+b-2c+2)\,y + c - 1}\right],\\
R_{21}^+(x) &= \frac{P_2^+}{P_1^+}(x,0) = \frac{g(x,0)}{f(x,0)} - 1.
\end{aligned}$$

Notice that $\lambda := \lambda_+ = 1-c > 0$ and recall that $\lambda_- = \frac{1}{\lambda_+}$. According to (2.2) we consider the functions

$$\log L_1^-(u) := \int_0^u \left(R_{12}^-(y) + \frac{1}{\lambda_-} \right) \frac{dy}{y} = \int_0^{-u} \left(\frac{g(x,0)}{f(x,0)} - c \right) \frac{dx}{x},$$

$$\log L_2^-(u) := \int_0^u (R_{21}^-(x) + \lambda_-) \frac{dx}{x} = \frac{1}{1-c} \int_0^u \frac{\Big(c\,(a-b-c+2)\,x - \big(ac - c^2 - b + c\big) \Big) dx}{(a-b-c+2)\,x^2 - (a-b-2c+2)\,x - (c-1)},$$

$$\log L_1^+(u) := \int_0^u \left(R_{12}^+(y) + \frac{1}{\lambda_+} \right) \frac{dy}{y} = \frac{1}{1-c} \int_0^u \frac{\Big(c\,(a-b+c-2)\,y - \big(ac + c^2 - b - c\big) \Big) dy}{(a-b+c-2)\,y^2 - (a-b+2c-2)\,y + (c-1)},$$

$$\log L_2^+(u) := \int_0^u (R_{21}^+(x) + \lambda_+) \frac{dx}{x} = \int_0^u \left(\frac{g(x,0)}{f(x,0)} - c \right) \frac{dx}{x}.$$

From (2.4) we compute the partial derivatives $\sigma_{ijk}^{\pm} = \partial_s^k \sigma_{ij}^{\pm}(0)$ of the parametrizations $\sigma_i^{\pm}(s) = (\sigma_{i1}^{\pm}(s), \sigma_{i2}^{\pm}(s))$:

$$\sigma_{111}^- = -1/\alpha, \quad \sigma_{120}^- = -\alpha, \quad \sigma_{210}^- = 1, \quad \sigma_{221}^- = 1, \quad \sigma_{111}^+ = 1, \quad \sigma_{120}^+ = 1, \quad \sigma_{210}^+ = \omega, \quad \sigma_{221}^+ = 1/\omega.$$

Then, according to (2.3) and Remark 1.3, we obtain the following expression:

$$\Delta_{00} = (\Delta_{00}^-)^{\lambda_+} \Delta_{00}^+ = \frac{\sigma_{111}^- (\sigma_{120}^-)^{\lambda} (L_2^-(\sigma_{210}^-))^{\lambda} (\sigma_{111}^+)^{\lambda} \sigma_{120}^+ L_2^+(\sigma_{210}^+)}{L_1^-(\sigma_{120}^-)(\sigma_{221}^-)^{\lambda} \sigma_{210}^- (L_1^+(\sigma_{120}^+))^{\lambda} \sigma_{221}^+ (\sigma_{210}^+)^{\lambda}} = \frac{(-\alpha)^{-1+\lambda} L_2^+(\omega)}{\omega^{-1+\lambda} L_1^-(-\alpha)} \left(\frac{L_2^-(1)}{L_1^+(1)} \right)^{\lambda}$$
$$= \exp\left(\mathrm{PV} \int_{\alpha}^{\omega} \frac{g_1(x,0)}{x f_1(x,0)} dx + \gamma_0 \right),$$

where $\gamma_0 := \lambda \log\left(\frac{L_2^-(1)}{L_1^+(1)}\right)$. Let us prove now that

$$\gamma_0 = -\frac{\pi(c(a+b) - 2b)}{\sqrt{d}}.$$

With this aim notice that

$$\log L_1^+(u; a, b, c) = \log L_2^-(u; -a, -b, c) = \alpha(u; a, b, c) + \beta(u; a, b, c)$$

where

$$\alpha(u; a, b, c) = \frac{c}{2(1-c)} \log\left(\frac{1 - c + (a - b + 2c - 2)u + (-a + b - c + 2)u^2}{1-c} \right)$$

and

$$\beta(u; a, b, c) = -\tfrac{((a+b)c-2b)}{(1-c)\sqrt{d}} \left[\arctan\left(\tfrac{2(1-c)+b-a+2(a-b+c-2)u}{\sqrt{d}} \right) - \arctan\left(\tfrac{2(1-c)+b-a}{\sqrt{d}} \right) \right].$$

Since $\alpha(1; a, b, c) = -\frac{c}{2(1-c)} \log(1-c)$ we obtain that

$$\log\left(\frac{L_2^-(1)}{L_1^+(1)} \right) = \beta(1; -a, -b, c) - \beta(1; a, b, c) = \frac{((a+b)c - 2b)}{(1-c)\sqrt{d}} F(a, b, c),$$

where $F(a, b, c)$ is the following function

$$\arctan\left(\tfrac{b-a-2}{\sqrt{d}} \right) - \arctan\left(\tfrac{b-a+2-2c}{\sqrt{d}} \right) + \arctan\left(\tfrac{-b+a-2}{\sqrt{d}} \right) - \arctan\left(\tfrac{-b+a+2-2c}{\sqrt{d}} \right).$$

Notice that $F(a, b, c) = G(c, e)$ only depends on c and $e = b - a$ because $d = 4(1-c) - e^2$. It can be checked that $\partial_c G = \partial_e G = 0$ so that G, and consequently F, is constant. Evaluating at $e = 0$ we obtain that $F \equiv -\pi$.

To compute γ_- it suffices to apply to X the symmetry $(x, y) \mapsto (x, -y)$ that transforms the invariants (a, b, c) into $(-a, -b, c)$ and the value γ_0 into $-\gamma_0$. □

3 Examples and applications

Example 3.1. For each integer $n \geq 3$ we consider the vector field

$$X_n = (x+y)^2 \partial_x + y^n \partial_y$$

having a degenerate singularity at the origin with invariants $(a,b,c) = (2,0,0) \in \{d = 0\} \cap \{a^2 - b^2 = 4\}$.

It can be checked that in the resolution of singularities of X_3 appears a saddle-node whose weak separatrix is not the strict transform of $y = 0$ and meets transversely the exceptional divisor. Consequently the origin is not a fake saddle for the vector field X_3.

On the other hand, the blowup of the vector field X_4

$$Y_0 = \frac{(v, uv)^* X_4}{v} = (-(u+1)^2 + u^3 v^2) u \partial_u + (u+1)^2 v \partial_v$$

has two singular points on the exceptional divisor $v = 0$: $(u, v) = (0, 0)$ which is a hyperbolic saddle and $(u, v) = (-1, 0)$ which is degenerated. Moreover,

$$Y_1 = (u-1, v)^* Y_0 = (u^2 + v^2 - u^3 - 4uv^2 + 6u^2v^2 - 4u^3v^2 + u^4v^2) \partial_u + u^2 v \partial_v$$

has invariants $(a,b,c) = (0,0,0)$, so that $d = 4$ and the transition map along $v = 0$ is well-defined for Y_1. Thus, the origin is a fake saddle of X_4, see Figure 3.1. This example (for which

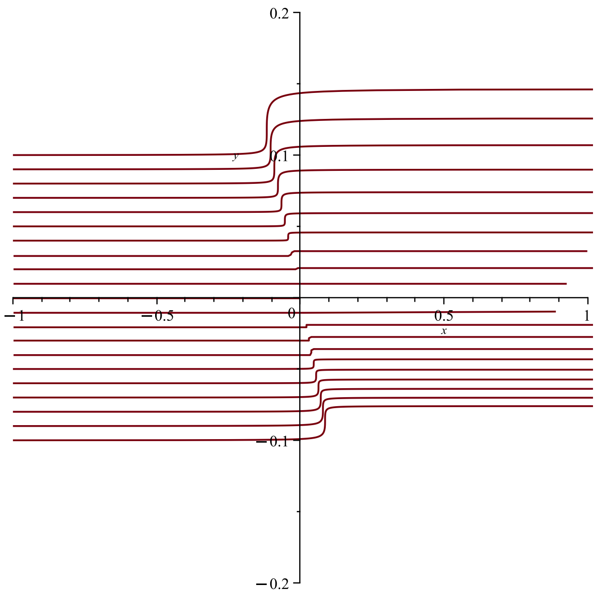

Figure 3.1: Numerical solutions of $X_4 = (x+y)^2 \partial_x + y^4 \partial_y$.

$a_{1,1} := a = 2$, $h_{0,1} := b = 0$ and $h_{1,0} := c = 0$) contradicts the following claim of [2]: "it is readily seen that a necessary condition for the origin to be a fake singularity on the singular fiber $y = 0$ is that [...] either $(h_{0,1} - a_{1,1})^2 + 4(h_{1,0} - 1) < 0$, or that $h_{1,0} = 1$ and $h_{0,1} = a_{1,1}$." Moreover, since $a = b = 0$, according to Theorem 1.2, the derivative at $v = 0$ of the transition map $\Pi_\alpha^\omega(v)$ of Y_1 is

$$\exp \mathrm{PV} \int_\alpha^\omega \frac{u^2}{u^2 - u^3} du = \left| \frac{1-\alpha}{1-\omega} \right|,$$

so that it is contractive or repulsive on both sides $\pm v > 0$. Notice that the transition map $\Pi_{-1}^{+1}(y)$ of X_4 is contractive on one side $y > 0$ and repulsive on the other side $y < 0$, see Figure 3.1.

Example 3.2. Consider the quadratic homogeneous vector field $X = (x^2 + y^2 + axy)\partial_x + (cx + by)y\partial_y$ with $d = 4(1 - c) - (a - b)^2 > 0$. Taking $\alpha = -1$ and $\omega = 1$ we have that $\mathrm{PV} \int_\alpha^\omega \frac{g_1(x,0)}{f_1(x,0)} \frac{dx}{x} = \mathrm{PV} \int_{-1}^{1} \frac{cx}{x^2} dx = 0$. Taking also $a = 1$, $b = -1$, $c = -1$ (so that $d = 4$) the transition map Π_{-1}^{+1} is contractive for $y > 0$ and expansive for $y < 0$ in accordance with $\Pi_{-1}^{+1}(y) = e^{\gamma_\pm} y + \mathcal{F}_{1+\epsilon}$, where $\gamma_\pm = 0 \pm \pi \frac{(-2(-1)-(1-1)(-1))}{2} = \pm\pi$ depending on the sign $\pm$ of y, see Figure 3.2.

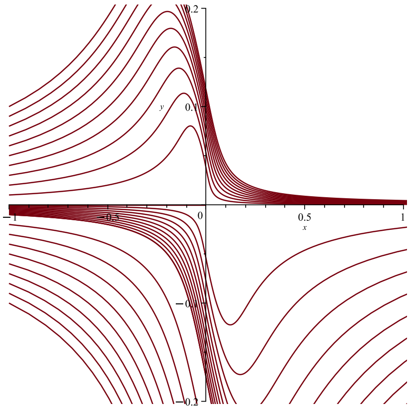

Figure 3.2: Local phase portrait of the vector field $(x^2 + y^2 + xy)\partial_x - (x + y)y\partial_y$ having first integral $\ln\big(y^2(2x^2 + 2xy + y^2)\big) - 2\arctan\big(\frac{x+y}{x}\big)$.

To finish this work we apply Theorems 1.1 and 1.2 to the study of the following family of degenerated singularities considered in [4]:

$$Z_\mu : \begin{cases} \dot{x} = \beta x^2 y + \alpha x y^2 - \beta y^3 - x^4, \\ \dot{y} = 4\beta x y^2 + \alpha y^3 + 2x^5, \end{cases} \qquad \mu = (\alpha, \beta) \in \mathbb{R}^2. \tag{3.1}$$

According to [4, p. 189], the origin is monodromic for Z_μ if and only if $\beta > 1/4$.

Blowing up the origin, see Figure 3.3, we have that

$$Y_\mu = \frac{(x, ux)^* Z_\mu}{x^2} = (3\beta u^2 + \beta u^4 + ux + 2x^2)\partial_u + (\beta u + \alpha u^2 - \beta u^3 - x)x\partial_x$$

and, assuming that $\beta > 0$

$$X_\mu = \left(\frac{x}{3\beta}, \frac{y}{\sqrt{6\beta}}\right)^* Y_\mu = \left(x^2 + y^2 + \frac{xy}{\sqrt{6\beta}} + \frac{x^4}{27\beta^2}\right)\partial_x + \left(\frac{x}{3} - \frac{y}{\sqrt{6\beta}} + \frac{\alpha x^2}{9\beta^2} - \frac{x^3}{27\beta^2}\right) y\partial_y$$

is of the form (1.1) with $f_1(x, y; \mu) = 1 + \frac{x^2}{27\beta^2}$, $f_2(x, y; \mu) \equiv 1$, $a(\mu) = \frac{1}{\sqrt{6\beta}}$, $g_1(x, y; \mu) = \frac{1}{3} + \frac{\alpha x}{9\beta^2} - \frac{x^2}{27\beta^2}$, $g_2(y; \mu) \equiv b(\mu) = -\frac{1}{\sqrt{6\beta}}$ and $c(\mu) = \frac{1}{3}$.

Notice that $f_1(x, 0; \mu) \neq 0$ for all $x \in \mathbb{R}$ and $d(\mu) = 4(1 - c(\mu)) - (a(\mu) - b(\mu))^2 = \frac{2}{3}(4 - \frac{1}{\beta})$ is positive if and only if $\beta > \frac{1}{4}$. By applying Theorem 1.1 we also deduce that the origin is

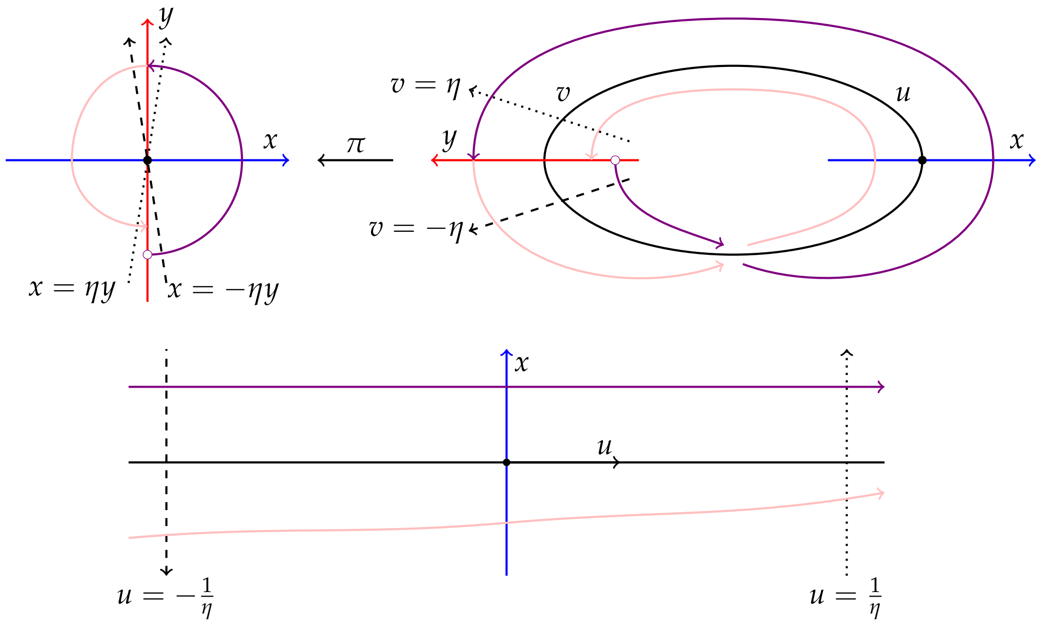

Figure 3.3: Blowup of the origin for the family Z_μ in the charts $\pi(x,u) = (x, ux)$ and $\pi(v,y) = (vy, y)$.

monodromic if and only if $\beta > \frac{1}{4}$. In that case we can apply Theorem 1.2 and consider the limit (as $\eta \to 0^+$) transition map $\Pi_{-\infty}^{\pm\infty}(y;\mu) = e^{\gamma_\pm(\mu)}y + \mathcal{F}_{1+\epsilon}^\infty(\{\beta > \frac{1}{4}\})$ of X_μ, with

$$\gamma_\pm(\mu) = \mathrm{PV}\int_{-\infty}^{+\infty} \frac{\frac{1}{3} + \frac{\alpha x}{9\beta^2} - \frac{x^2}{27\beta^2}}{1 + \frac{x^2}{27\beta^2}} \frac{dx}{x} \mp \frac{\pi}{\sqrt{4\beta - 1}} = \pi\left(\frac{\alpha}{\beta\sqrt{3}} \mp \frac{1}{\sqrt{4\beta-1}}\right).$$

Assuming that $\beta > \frac{1}{4}$, the Poincaré return map $R(x;\mu)$ of Z_μ around $(0,0)$ is the composition of two transition maps $\Pi_{-\infty}^{\pm\infty}(x;\mu) = e^{\gamma_\pm(\mu)}x + \mathcal{F}_{1+\epsilon}^\infty(\{\beta > \frac{1}{4}\})$ of X_μ, one for $x > 0$ and the other for $x < 0$, see Figure 3.3, so that $R(x;\mu) = e^{\gamma(\mu)}x + \mathcal{F}_{1+\epsilon}^\infty(\{\beta > \frac{1}{4}\})$ with $\gamma(\mu) = \gamma_+(\mu) + \gamma_-(\mu) = \frac{2\pi\alpha}{\beta\sqrt{3}}$, in agreement with [4, p. 189].

Notice that for any $\beta > \frac{1}{4}$ the origin is a center of $Z_{(0,\beta)}$ because it is reversible via $(x,y,t) \mapsto (-x,y,-t)$. Moreover, thanks to the uniform properties of the flat remainder class $\mathcal{F}_{1+\epsilon}^\infty$ we can assert that the ciclicity of the origin in the family (3.1) is zero at any $\mu_0 = (\alpha_0, \beta_0) \in \mathbb{R}^2$ with $\beta_0 > \frac{1}{4}$. This last assertion can not be deduced from the results in [4] because they are not uniform with respect to parameters. Indeed, this property is clear for $\alpha_0 \neq 0$. Assume now that $\alpha_0 = 0$. We can express the displacement function

$$R(x;\mu) - x = x(e^{\frac{2\pi\alpha}{\beta\sqrt{3}}} - 1 + f(x;\mu)),$$

with a remainder term $f(x;\mu) \in \mathcal{F}_\epsilon^\infty$ which vanishes identically along $\alpha = 0$. By using the division result [6, Lemma 4.1] in the flat class $\mathcal{F}_L^K$, we can write $f(x;\mu) = \alpha g(x;\mu)$ for some $g \in \mathcal{F}_\epsilon^\infty$. If $\alpha x \neq 0$ then

$$\frac{R(x;\mu) - x}{\alpha x} = \frac{e^{\frac{2\pi\alpha}{\beta\sqrt{3}}} - 1}{\alpha} + g(x;\mu)$$

tends to $\frac{2\pi}{\beta_0\sqrt{3}} \neq 0$ as $(x,\mu) \to (0,\mu_0)$. Hence there is no limit cycle for Z_μ near the origin for any $\mu \approx \mu_0$.

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