



Kolmogorov-type mathematical models with variable delay in population biology

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Abstract. We investigate several qualitative properties of scalar Kolmogorov-type models with time and state dependent delay, arising in population biology. Specifically, we shall focus on the analysis of boundedness and positivity of solutions, and the stability of equilibria. These results contribute to a deeper understanding of the long-term behavior of nonlinear delayed population models under biologically relevant assumptions.

Keywords: delay differential equations, state-dependent delay, global existence, stability, boundedness.

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1 Introduction

We study the scalar Kolmogorov-type delay differential equation

$$x'(t) = x(t) f(t, x(t), x(t - \tau(t, x(t)))), \quad (1.1)$$

where the delay function $\tau : [0, +\infty) \times \mathbb{R} \rightarrow [0, +\infty)$ depends on both time and the state variable. The function $f : [0, +\infty) \times \mathbb{R}^2 \rightarrow \mathbb{R}$ is assumed to be continuous and continuously differentiable with respect to its second and third arguments. Moreover, the delay function $\tau(t, x)$ is assumed to be continuous and bounded. We also assume that the equation satisfies the positivity condition

$$f(t, 0, 0) > 0 \quad \text{for all } t \geq 0.$$

From a biological perspective, only non-negative population densities are meaningful. Therefore, our analysis shall be restricted to non-negative equilibria and focus on the dynamics in the positive cone of the phase space.

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Delay differential equations (DDEs) arise naturally in a wide range of applied fields where the evolution of a quantity depends not only on its present state but also on its history. In particular, scalar Kolmogorov-type equations have been extensively used to model phenomena in population dynamics, epidemiology, physiology, and neuroscience. Based on the pioneering ideas of Malthus [6] and Verhulst [10], delays were typically introduced to represent gestation periods, maturation times, or processing lags, see e.g. [3, 5, 8, 9].

A well-studied example is the delayed logistic model analyzed in [7], which takes the form

$$x'(t) = r x(t) [1 - a_1 x(t) - a_2 x(t - \tau)],$$

where $a_1, a_2, r > 0$ are constants and $\tau \geq 0$ is a fixed delay. This equation admits a unique positive equilibrium $x^* = \frac{1}{a_1 + a_2}$, which is asymptotically stable in the absence of delay. In the general case, the stability of this equilibrium depends critically on the balance between the instantaneous feedback term a_1 and the delayed term a_2 . Specifically, when $a_1 \geq a_2$, the equilibrium remains locally asymptotically stable for all values of $\tau \geq 0$. Furthermore, when $a_1 > a_2$, the equilibrium is globally asymptotically stable. Conversely, if $a_1 < a_2$, then there exists a critical delay $\tau^* > 0$ such that x^* is asymptotically stable for $\tau \in [0, \tau^*)$, but becomes unstable when $\tau > \tau^*$, undergoing a Hopf bifurcation at $\tau = \tau^*$. These results underscore the stabilizing influence of instantaneous feedback in contrast with destabilizing delayed effects in population models.

In this work, we introduce a general class of delayed logistic-type equations inspired by models in cell biology. Our formulation includes both delayed and non-delayed interactions and incorporates a variable delay that depends on time and the current state. The main objective of this paper is to analyze the qualitative behavior of solutions to equation (1.1), with particular emphasis on the existence, boundedness and positivity of solutions, as well as the asymptotic stability of equilibria under appropriate assumptions.

Equation (1.1) extends the classical framework by allowing delays to depend on both time and the current state, thereby capturing more complex and realistic temporal dynamics. However, this added generality introduces significant analytical difficulties. While the theory of DDEs with constant or time-dependent delays is well developed [4, 7], the state-dependent case remains a challenging and active research area due to its strong nonlinear and frequently non-Lipschitz nature [11].

The paper is organized as follows. In Section 2, we introduce the notation, terminology, and several preliminary results that will be used throughout the work. In Section 3, we analyze the linearized equation to establish the local stability of the nontrivial equilibrium. Section 4 is devoted to proving the global stability of the nontrivial equilibrium. Finally, in Section 5, we present an example involving a delayed logistic equation with variable delay to illustrate the applicability of the theoretical results.

2 Notation and preliminaries

Let $\bar{\tau} := \|\tau\|_\infty$ be fixed. We denote by

$$C = C([-\bar{\tau}, 0], \mathbb{R})$$

the Banach space of continuous functions $\gamma : [-\bar{\tau}, 0] \rightarrow \mathbb{R}$, endowed with the uniform norm

$$\|\gamma\|_\infty := \max_{s \in [-\bar{\tau}, 0]} |\gamma(s)|.$$

For a given function $x : [-\bar{\tau}, \alpha) \rightarrow \mathbb{R}$, we define its history segment at time $t \geq 0$ as the element $x_t \in C$ given by

$$x_t(s) := x(t+s), \quad \text{for all } s \in [-\bar{\tau}, 0].$$

This settles the state space C , and each $x_t \in C$ represents the state of the system over the delay interval at time t .

In abbreviated form, problem (1.1) can be written as follows

$$x'(t) = F(t, x_t), \quad t \in [0, +\infty), \quad (2.1)$$

where

$$F : [0, \infty) \times C \rightarrow \mathbb{R}$$

is defined by

$$F(t, \phi) := \phi(0)f(t, \phi(0), \phi(-\tau(t, \phi(0)))).$$

For the sake of simplicity, as long as there are no ambiguities we shall write the delay as τ , thus omitting the dependence on the time and state variables.

An initial condition at an arbitrary $t_0 \geq 0$ for equation (1.1) is given by

$$x_{t_0}(t) = \phi(t), \quad t \in [-\bar{\tau}, 0]$$

where $\phi \in C$.

Due to the biological motivation for the models under consideration, only nonnegative solutions are of interest, so we shall define the positive cone

$$C^+ := \{\phi \in C([-\bar{\tau}, 0], \mathbb{R}), \phi(t) \geq 0, t \in [-\bar{\tau}, 0]\}$$

as the set of admissible initial conditions.

From the standard theory of delay differential equations, for each $t_0 \geq 0$ and each initial value $\phi \in C^+$ there exists at least one solution x of (2.1) defined over a maximal interval $[t_0 - \bar{\tau}, t_0 + \alpha)$, with $\alpha \in (0, +\infty]$. Furthermore, if there exists $\zeta : [t_0, +\infty) \rightarrow (0, +\infty)$ continuous such that $|x(t)| \leq \zeta(t)$ for all $t \in [t_0, \alpha)$ then $\alpha = +\infty$.

Definition 2.1. A constant solution x^* of (2.1) shall be called an equilibrium. We shall say that x^* is:

1. **Stable**, if for each $\varepsilon > 0$ and $t_0 \geq 0$ there exists $\delta(t_0, \varepsilon) > 0$ such that if a solution x satisfies $\|x_{t_0} - x^*\|_\infty < \delta$ then $|x(t) - x^*| < \varepsilon$ for all $t \geq t_0$.
2. **Asymptotically stable**, if it is stable and for each $t_0 \geq 0$ there exists $\delta(t_0) > 0$ such that, for any solution x with $\|x_{t_0} - x^*\|_\infty < \delta$ it is verified that $x(t) \rightarrow x^*$ as $t \rightarrow +\infty$.

3 Linearized equation

In this section, we shall assume that the delay is state independent, namely $\tau = \tau(t)$. According to the definition, an equilibrium x^* is defined as a solution of the equation

$$x^* f(t, x^*, x^*) = 0 \quad \text{for all } t \geq 0.$$

Next, for arbitrary fixed $t \geq 0$ we may consider the Taylor expansion of the map $(x, y) \mapsto xf(t, x, y)$ centered at (x^*, x^*) , that is

$$xf(t, x, y) = [f(t, x^*, x^*) + x^* \psi^x(t)](x - x^*) + x^* \psi^y(t)(y - x^*),$$

where

$$\psi^x(t) := f_x(t, x^*, x^*), \quad \psi^y(t) := f_y(t, x^*, x^*).$$

Thus, the linearized equation for $u(t) := x(t) - x^*$ is given by

$$u'(t) = f(t, 0, 0)u(t)$$

when $x^* = 0$ and

$$u'(t) = x^* \psi^x(t)u(t) + x^* \psi^y(t)u(t - \tau) \quad (3.1)$$

when $x^* > 0$. In the first case, the equilibrium cannot be asymptotically stable, due to the assumption $f(t, 0, 0) > 0$ and, furthermore, if $f(t, 0, 0)$ is not integrable on $[0, +\infty)$, then the equilibrium is unstable. Regarding the case $x^* > 0$, the following result can be directly deduced from [1].

Theorem 3.1. *Assume there exists $t_0 \geq 0$ such that one of the following conditions holds:*

1. *There exists $t_1 \geq t_0$ such that, for all $t \geq t_1$,*

$$\int_{t_0}^t (\psi^x(s) + |\psi^y(s)|) e^{x^* \int_s^t \psi^x(\theta) d\theta} ds \leq 0.$$

2. *For all $t \geq t_0$ it is verified that $\psi^x(t) \leq 0$, $\psi^y(t) < 0$ and*

$$x^* \int_{t-\rho}^t |\psi^y(s)| e^{x^* \int_s^t \psi^x(\theta) d\theta} ds \leq 1 \quad \text{for all } \rho \in [0, \tau(t)].$$

Then the zero solution of (3.1) is stable.

Theorem 3.2. *Assume that one of the following conditions holds:*

1.
$$\liminf_{t_0 \rightarrow +\infty} \left(\limsup_{t \rightarrow +\infty} \int_{t_0}^t (\psi^x(s) + |\psi^y(s)|) e^{x^* \int_s^t \psi^x(\theta) d\theta} ds \right) < 0.$$

2. *For all $t \gg 0$ it is verified that $\psi^x(t) \leq 0$, $\psi^y(t) < 0$, ψ^x or ψ^y is not integrable and*

$$x^* \int_{t-\rho}^t |\psi^y(s)| e^{x^* \int_s^t \psi^x(\theta) d\theta} ds \leq C \quad \text{for all } \rho \in [0, \tau(t)]$$

for some constant $C < 1$.

Then the zero solution of (3.1) is asymptotically stable.

In the preceding results, it is worth noticing that the first condition corresponds to the *absolute* stability while the second one corresponds to *conditional* stability. In other words, the second condition depends on τ , while the first one does not. For example, if $\psi^x \leq 0$, $\psi^y < 0$, and ψ^y is bounded, then it suffices to assume that its norm is smaller than $\frac{1}{\tau}$. On the other hand, absolute stability holds under the pointwise conditions

$$|f_y(t, x^*, x^*)| \leq -C f_x(t, x^*, x^*)$$

for $t \gg 0$ and some $C < 1$, or also, when

$$\limsup_{t \rightarrow +\infty} (f_x(t, x^*, x^*) + |f_y(t, x^*, x^*)|) < 0.$$

4 Global asymptotic stability

Throughout this section, we shall assume the following conditions:

(A1) There exists a constant $x^* > 0$ such that

$$f(t, x^*, x^*) = 0 \quad \text{for all } t \geq 0.$$

(A2) f satisfies the monotonicity conditions:

$$f_x(t, x, y) \leq 0, \quad f_y(t, x, y) < 0 \quad \text{for all } t \geq 0 \text{ and } x, y \geq 0. \quad (4.1)$$

A remarkable feature of equation (1.1) is that, for arbitrary $t_0 \geq 0$, its solutions can be written in the form

$$x(t) = x(t_0) e^{\int_{t_0}^t f(s, x(s), x(s-\tau)) ds}$$

and, in particular, $x(t) > 0$ for all $t > t_0$ whenever $x(t_0) > 0$. It is also verified that nonnegative solutions are globally defined on $[0, +\infty)$, due to the fact that

$$f(s, x(s), x(s-\tau)) \leq f(s, 0, 0)$$

and, consequently

$$x(t) \leq x(0) e^{\int_0^t f(s, 0, 0) ds}.$$

An extra assumption will ensure the permanence of the solutions, namely:

Proposition 4.1. *Assume that conditions (A₁) and (A₂) hold, and suppose there exists a constant $M > 0$ such that*

$$\int_t^{t+\bar{\tau}} f(s, x^*, 0) ds \leq M \quad \text{for all } t \geq 0. \quad (4.2)$$

Then all nonnegative solutions of equation (1.1) are bounded.

Proof. Assume firstly that x is eventually larger than x^* , that is, $x(t) > x^*$ for all t sufficiently large. Due to the monotonicity assumption (4.1), it follows that x is eventually nonincreasing and consequently bounded. Secondly, assume that $x(a) = x(b) = x^*$ and $x(t) > x^*$ for all $t \in (a, b)$. Fix a value $t_{\max} \in (a, b)$ in which the maximum x_{\max} of x over $[a, b]$ is achieved, so

$$0 = x'(t_{\max}) = x(t_{\max}) f(t_{\max}, x_{\max}, x(t_{\max} - \tau)).$$

Because $x_{\max} > x^*$, it follows that $x(t_{\max} - \tau) \leq x^*$ since otherwise

$$f(t_{\max}, x_{\max}, x(t_{\max} - \tau)) < f(t_{\max}, x^*, x^*) = 0,$$

a contradiction. As a consequence, it is deduced that $t_{\max} - \tau \leq a$, whence $t_{\max} \leq a + \bar{\tau}$. Moreover, for $t \in (a, b)$ it is verified that

$$f(t, x(t), x(t - \tau)) \leq f(t, x^*, 0),$$

so it is deduced that

$$x_{\max} = x^* e^{\int_a^{t_{\max}} f(s, x(s), x(s-\tau)) ds} \leq x^* e^{\int_a^{t_{\max}} f(s, x^*, 0) ds} \leq x^* e^M$$

and so completes the proof. \square

Remark 4.2. The preceding proof also shows that the trajectories eventually larger than x^* converge to some positive value and, in an analogous way, it is seen that the same happens with those nonnegative trajectories that are eventually smaller than x^* . However, when the equation is non-autonomous, an extra assumption is required to ensure that the limit value in both cases is equal to x^* .

Proposition 4.3. *In the previous context, assume also that*

$$\liminf_{t \rightarrow +\infty} |f(t, x, x)| > 0 \quad (4.3)$$

for all $x \neq x^*$. If $x(t)$ is a nonnegative trajectory that converges to some $x \geq 0$ as $t \rightarrow +\infty$, then $x = x^*$.

Proof. By the mean value theorem, for each $n \in \mathbb{N}$ there exists $t_n \in (n, n+1)$ such that $x'(t_n) = x(n+1) - x(n) \rightarrow 0$ as $n \rightarrow \infty$. From equation (1.1), it follows that $f(t_n, x(t_n), x(t_n - \tau)) \rightarrow 0$. Now assume for example that $x > x^*$, then $x(t) > \hat{x}$ for some $\hat{x} > x^*$ and any t sufficiently large. Thus,

$$\limsup_{n \rightarrow \infty} f(t_n, x(t_n), x(t_n - \tau)) \leq \limsup_{n \rightarrow \infty} f(t_n, \hat{x}, \hat{x}) < 0,$$

a contradiction. The proof is analogous if $x < x^*$. \square

With the previous result in mind, the stability analysis of the equilibrium x^* can be reduced to obtaining accurate bounds for the amplitude of the oscillations of $x(t)$ around x^* . To this end, the following lemma shall be crucial.

Lemma 4.4. *In the previous situation, let x be a nonnegative solution such that $x(t) - x^*$ oscillates and assume that $\alpha \leq x(t) \leq \beta$ for all $t \geq t_0$. Then there exists $t_1 \geq t_0$ such that*

$$g(\beta) \leq x(t) \leq g(\alpha)$$

for all $t \geq t_1$, where $g := g_{t_0}$ is defined by

$$g(\gamma) := \begin{cases} x^* \sup_{t \geq t_0} e^{\int_t^{t+\bar{\tau}} f(s, x^*, \gamma) ds} & \text{if } \gamma < x^* \\ x^* \inf_{t \geq t_0} e^{\int_t^{t+\bar{\tau}} f(s, x^*, \gamma) ds} & \text{if } \gamma > x^*. \end{cases}$$

Proof. Fix $t_1 \geq t_0 + \bar{\tau}$ such that $x(t_1) = x^*$, let $t > t_1$ be such that $x(t) \neq x^*$. Because $x(t)$ oscillates around x^* , there exists an interval $(a, b) \subset (t_1, +\infty)$ such that $x(a) = x(b) = x^*$ and $x(s) \neq x^*$ for all $s \in (a, b)$. If $x(t) > x^*$, then, we may proceed as in the proof of Proposition 4.1 to conclude that the maximum value x_{\max} of x over the interval $[a, b]$ satisfies

$$x_{\max} = x^* e^{\int_a^{t_{\max}} f(s, x(s), x(s-\tau)) ds} \leq x^* e^{\int_a^{a+\bar{\tau}} f(s, x^*, \alpha) ds} \leq g(\alpha).$$

In the same way, if $x(t) < x^*$ and x achieves over $[a, b]$ its minimum value x_{\min} at some $t_{\min} \in (a, b)$, then we obtain

$$x_{\min} = x^* e^{\int_a^{t_{\min}} f(s, x(s), x(s-\tau)) ds} \geq x^* e^{\int_a^{a+\bar{\tau}} f(s, x^*, \beta) ds} \geq g(\beta),$$

so the conclusion follows. \square

It is clear that the previous result might be useless in case that the new bounds $g(\beta) < x^*$ and $g(\alpha) > x^*$ do not improve the original ones. However, it is important to observe that if $\alpha \leq g(\beta)$ and $\beta \geq g(\alpha)$ then the procedure can be successfully iterated. In more precise terms,

Lemma 4.5. *In the context of the previous lemma, set $\alpha_1 := g_{t_0}(\beta)$ and $\beta_1 := g_{t_0}(\alpha)$. If $\alpha_1 \geq \alpha$ and $\beta_1 \leq \beta$, then $\alpha_2 := g_{t_1}(\beta_1) \geq \alpha_1$ and $\beta_2 := g_{t_1}(\alpha_1) \leq \beta_1$.*

Proof. It is clear that, for any fixed value $\gamma < x^*$, the map g is nonincreasing with respect to t_0 and, in particular, $g_{t_1}(\alpha_1) \leq g_{t_0}(\alpha_1)$. Furthermore, the fact that $\alpha_1 \geq \alpha$ implies that $f(s, x^*, \alpha_1) \leq f(s, x^*, \alpha)$ which, in turn, implies $g_{t_0}(\alpha_1) \leq g_{t_0}(\alpha)$ and, consequently, $\beta_2 = g_{t_1}(\alpha_1) \leq g_{t_0}(\alpha) = \beta_1$. The proof of the inequality $\alpha_2 \geq \alpha_1$ is analogous. \square

As a consequence, taking $\alpha = 0$ and $\beta = g_0(0)$ we obtain sequences

$$\alpha_1 \leq \alpha_2 \leq \dots < x^* < \dots \leq \beta_2 \leq \beta_1$$

and $t_j \nearrow +\infty$ such that any trajectory that oscillates around x^* satisfies $\alpha_j \leq x(t) \leq \beta_j$ for all $t \geq t_j$. In particular, any condition ensuring $\beta_j - \alpha_j \rightarrow 0$ will imply that x^* is asymptotically stable.

A specific instance of the preceding situation can be explicitly established when f is independent of t . Although this case cannot be considered autonomous due to the delay $\tau = \tau(t, x(t))$, condition (4.3) is trivially satisfied and, moreover, condition (4.2) is satisfied as well, due to the fact that the delay function is bounded. Another advantage is that the map g for arbitrary t_0 is easily computed and extended continuously to x^* by

$$g(\gamma) = x^* e^{\bar{\tau} f(x^*, \gamma)}.$$

Here, it is directly seen that $g'(\gamma) < 0$, so g is strictly nonincreasing and $g^2 := g \circ g$ is strictly nondecreasing. Furthermore, given a trajectory that oscillates around x^* , the previous computations imply, for each $n \in \mathbb{N}$, the existence of $t_n \nearrow +\infty$ such that

$$g^{2n}(0) \leq x(t) \leq g^{2n-1}(0), \quad t \geq t_n.$$

Let $L := \lim_{n \rightarrow \infty} g^{2n-1}(0)$, then the left-hand side converges to $g(L) \leq L$. Observe, furthermore, that L is a fixed point of g^2 , since

$$L = \lim_{n \rightarrow \infty} g^{2n+1}(0) = \lim_{n \rightarrow \infty} g^2(g^{2n-1}(0)) = g^2(L).$$

Thus, the asymptotic stability will be guaranteed if we are able to show that g^2 has no other fixed points besides x^* . The following result proves that this is always the case, provided that the delay is sufficiently small.

Theorem 4.6. *Let $f = f(x, y)$ and assume that conditions (A_1) and (A_2) hold. Then there exists $\tau^* > 0$ such that the equilibrium x^* is globally asymptotically stable when $\bar{\tau} < \tau^*$.*

Proof. In virtue of the previous computations, it suffices to show that the mapping $\varphi(s) := g^2(s) - s$ vanishes exactly once on $[0, +\infty)$. Observe that g is bounded, because $f(x^*, s) < 0$ for $s > x^*$. As a consequence, g^2 is also bounded, which yields

$$\varphi(s) < 0 \quad \text{for } s \gg 0.$$

Moreover, $\varphi(0) > 0$, so it suffices to verify that $\varphi'(s) < 0$ whenever $\varphi(s) = 0$. To this end, compute

$$\varphi'(s) = (\bar{\tau}x^*)^2 e^{\bar{\tau}[f(x^*,g(s))+f(x^*,s)]} f_y(x^*,g(s)) f_y(x^*,s) - 1.$$

It is noticed that the factor $f_y(x^*,s)$ might be unbounded but, since we assume $\varphi(s) = 0$, it can be replaced by the bounded quantity $f_y(x^*,g^2(s))$. Thus, the following upper bound for τ^* is obtained:

$$\tau^* < \frac{1}{\sqrt{S}}$$

where

$$S := (x^*)^2 \sup_{s \geq 0} e^{\bar{\tau}[f(x^*,g(s))+f(x^*,s)]} f_y(x^*,g(s)) f_y(x^*,g^2(s)). \quad \square$$

5 Example

Consider the delayed logistic equation with time-dependent delay

$$x'(t) = r_0(t) x(t) [1 - a_1 x(t) - a_2 x(t - \tau(t))], \quad (5.1)$$

where $a_1 \geq 0$, $a_2 > 0$, $r_0 : [0, \infty) \rightarrow (0, \infty)$ is a continuous growth rate, and $\tau : [0, \infty) \rightarrow [0, \infty)$ is a continuous and bounded time-dependent delay function.

It is straightforward to verify that, as in the autonomous case, the equation admits a unique positive equilibrium given by

$$x^* = \frac{1}{a_1 + a_2}.$$

We now analyze the stability of this equilibrium by linearizing the equation around x^* . Letting $u(t) := x(t) - x^*$, the linearized equation takes the form

$$u'(t) = x^* \psi^x(t) u(t) + x^* \psi^y(t) u(t - \tau(t)), \quad (5.2)$$

where

$$\psi^x(t) = f_x(t, x^*, x^*) = -a_1 r_0(t), \quad \psi^y(t) = f_y(t, x^*, x^*) = -a_2 r_0(t).$$

Note that both $\psi^x(t)$ and $\psi^y(t)$ are strictly negative.

We may now apply the stability result stated in Section 3. With this in mind observe, in the first place, that

$$\begin{aligned} \int_{t_0}^t (\psi^x(s) + |\psi^y(s)|) e^{x^* \int_s^t \psi^x(\theta) d\theta} ds &= \int_{t_0}^t (a_2 - a_1) r_0(s) e^{-a_1 x^* \int_s^t r_0(\theta) d\theta} ds \\ &= \frac{a_2 - a_1}{a_1 x^*} \left(1 - e^{-a_1 x^* \int_{t_0}^t r_0(\theta) d\theta} \right). \end{aligned}$$

Notice that, if r_0 is not integrable, then the last term tends to $\frac{a_2 - a_1}{a_1 x^*}$ as $t \rightarrow +\infty$; thus, the first condition in Theorem 3.2 is satisfied only when $a_2 < a_1$. In particular, when r_0 and τ are constant, this condition with non-strict inequality was already obtained in [7]. However, the second assumption in Theorem 3.2 deals with conditional stability and allows to improve the condition as follows. Let $\rho \in [0, \tau(t)]$ and compute

$$\begin{aligned} x^* \int_{t-\rho}^t |\psi^y(s)| e^{x^* \int_s^t \psi^x(\theta) d\theta} ds &= a_2 x^* \int_{t-\rho}^t r_0(s) e^{-a_1 x^* \int_s^t r_0(\theta) d\theta} ds \\ &= \frac{a_2}{a_1} \left(1 - e^{-a_1 x^* \int_{t-\rho}^t r_0(\theta) d\theta} \right) \leq \frac{a_2}{a_1} \left(1 - e^{-a_1 x^* \int_{t-\tau(t)}^t r_0(\theta) d\theta} \right), \end{aligned}$$

Thus, a sufficient condition for asymptotic stability is that

$$a_2 \left(1 - e^{-a_1 x^* R_{\text{sup}}}\right) < a_1$$

where

$$R_{\text{sup}} := \limsup_{t \rightarrow +\infty} \int_{t-\tau(t)}^t r_0(\theta) d\theta.$$

Clearly, when r_0 is constant it is verified that

$$R_{\text{sup}} = r_0 \limsup_{t \rightarrow +\infty} \tau(t).$$

Observe, furthermore, that if r_0 is integrable then the condition is fulfilled for arbitrary a_1 and a_2 , because $R_{\text{sup}} = 0$.

Next, we shall analyze the global asymptotic stability of the previous delayed logistic equation, now with constant growth and bounded variable delay,

$$x'(t) = r_0 x(t) [1 - a_1 x(t) - a_2 x(t - \tau(t, x(t)))]. \quad (5.3)$$

Define as before $\bar{\tau} := \|\tau\|_\infty$ and, in the context of Theorem 4.6, compute

$$g(s) = x^* e^{\bar{\tau} r_0 (1 - a_1 x^* - a_2 s)} = \frac{1}{a_1 + a_2} e^{\bar{\tau} r_0 \left(\frac{a_2}{a_1 + a_2} - a_2 s\right)},$$

so it is easy to prove that φ is nonincreasing if the following sufficient condition holds:

$$\frac{\bar{\tau} a_2 r_0}{a_1 + a_2} e^{\frac{\bar{\tau} a_2 r_0}{a_1 + a_2}} \sup_{s \geq 0} e^{-\bar{\tau} a_2 r_0 \frac{s + g(s)}{2}} \leq 1.$$

When $a_1 > a_2$, it was proven in [7] that global asymptotic stability holds for arbitrary (constant) delay $\tau \geq 0$, although the problem of finding an optimal estimate for τ^* when $a_1 \leq a_2$ is, to our knowledge, open. In this sense, Theorem 4.6 can be regarded as a first (non-sharp) approximation. For example, taking $a_1 = 0$ and $a_2 = r_0 = 1$, the previous condition reads

$$\tau e^\tau \sup_{s \geq 0} e^{-\tau \frac{s + g(s)}{2}} \leq 1,$$

where $g(s) = e^{\tau(1-s)}$. After a simple computation, it is readily verified that the condition is equivalent to $\tau \leq 1$, a value which is clearly worse than the optimal bound $\tau = \frac{\pi}{2}$ proposed by the well-known Wright conjecture. We recall that this problem remain unsolved for some decades, until the conjecture was finally proven in [2]. Such optimal value of τ coincides with the one obtained from the linearization; from this point of view, it would be interesting to investigate whether or not the result can be extended to the non-autonomous framework. In this direction, we remark that the bound given in Theorem 3.2 is not of any help when $a_1 = 0$. A deeper study of this problem shall be the subject of a forthcoming work.

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