Electronic Journal of Qualitative Theory of Differential Equations
2026, No. **61**, 1–16; https://doi.org/10.14232/ejqtde.2026.1.61 www.math.u-szeged.hu/ejqtde/

Multiple solutions for an anisotropic $\overrightarrow{p}$-Laplacian problem involving two parameters

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Received 26 August 2025, appeared 15 November 2026
Communicated by Gabriele Bonanno

Abstract. The goal of this paper is to establish multiplicity results for an anisotropic $\overrightarrow{p}$-Laplacian problem involving two parameters. More specifically, two critical point theorems are employed to prove the existence of at least three distinct weak solutions for the problem subject to suitable conditions on the nonlinear term. We also provide some corollaries and examples to illustrate the obtained results.

Keywords: anisotropic operator, critical point theory, $\overrightarrow{p}$-Laplacian operator.

2010 Mathematics Subject Classification: 35A15, 35J66, 35J05.

1 Introduction

In this work, we study the following anisotropic problem

$$\begin{cases} -\Delta_{\overrightarrow{p}} u = \lambda f_1(x,u) + \mu f_2(x,u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where $\Omega \subseteq \mathbb{R}^N$ $(N \geq 2)$ represents a bounded open set with C^1-boundary, $f_1, f_2 : \Omega \times \mathbb{R} \to \mathbb{R}$ denote two L^1-Carathéodory functions, λ, μ are constrained within the intervals $(0,\infty)$ and $[0,\infty)$ and $\overrightarrow{p} = (p_1, p_2, \ldots, p_N) \in \mathbb{R}^N$ is a constant vector, together with

$$p^- = \min_{1\leq j\leq N} p_j > N, \qquad p^+ = \max_{1\leq j\leq N} p_j.$$

The anisotropic $\overrightarrow{p}$-Laplacian operator is given by

$$\Delta_{\overrightarrow{p}} u = \sum_{j=1}^{N} \frac{\partial}{\partial x_j} \left(\left| \frac{\partial u}{\partial x_j} \right|^{p_j - 2} \frac{\partial u}{\partial x_j} \right), \tag{1.2}$$

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of course, whenever $p_j = 2$ for all $j = 1, 2, \ldots, N$, the operator in (1.2) reduces to the usual Laplacian. Besides, operator (1.2) is extended to the anisotropic variable exponent $\overrightarrow{p}(.)$-Laplacian case, that is,

$$\Delta_{\overrightarrow{p}(x)} u = \sum_{j=1}^{N} \frac{\partial}{\partial x_j} \left(\left| \frac{\partial u}{\partial x_j} \right|^{p_j(x)-2} \frac{\partial u}{\partial x_j} \right), \tag{1.3}$$

where $\overrightarrow{p}(\cdot) = (p_1(\cdot), \ldots, p_N(\cdot))$ is a vector with variable components. When choosing $p_j(\cdot) = p(\cdot)$ for all $j = 1, 2, \ldots, N$, operator (1.3) transforms into the pseudo-$p(x)$-Laplacian operator with similar properties to the variable exponent $p(.)$-Laplace operator. We refer to the literature [14, 15, 20, 22] and references therein for the study of problems including operator of type (1.3).

In recent years, researchers have studied boundary value problems involving anisotropic operator, because of their applications (see [3, 9, 13, 17, 21, 26, 27, 30]). Anisotropic problems can be strongly justified from two fundamental perspectives in mathematical research. The first is that this class of problems is strongly motivated by physical phenomena and models phenomena wherein partial derivatives change by direction. For example, such models appear in biology (see [5] and [6]) as a model for the spread of an epidemic in heterogeneous environments. They also occur, as shown in [2] and [4], in mathematical models of fluid dynamics where conductivities differ across directions, and they find applications in image processing (see [31]).

The second reason for the significance of anisotropic problems lies in the mathematical methods required to study them. In certain cases, more delicate estimates are necessary because the operator involves derivatives of different orders in various directions.

The study of anisotropic problems has extended to the case involving $\overrightarrow{p}$-Laplacian operator. In the setting of anisotropic $\overrightarrow{p}$-Laplacian problem, researchers in [16, 18, 29] and their references have focused on $\overrightarrow{p}$-Laplacian problems. In [16], the authors studied the following anisotropic equation

$$\begin{cases} -\Delta_{\overrightarrow{p}} u = \lambda g_\lambda(u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

where $1 < p_1 \leq p_2 \leq \cdots \leq p_N$ and λ denotes a positive parameter. Using genus theory, they established some multiplicity results to the problem in two cases when the function g_λ has subcritical or critical growth.

It is worth noting that the authors in [10] investigated problem (1.1) on condition $\mu = 0$, that is,

$$\begin{cases} -\Delta_{\overrightarrow{p}} u = \lambda f(x, u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.4}$$

where $\Omega \subseteq \mathbb{R}^N$ $(N \geq 2)$ represents a nonempty open set with C^1-boundary, $f : \Omega \times \mathbb{R} \to \mathbb{R}$ denotes a L^1-Carathéodory function, λ is a positive real parameter and $\overrightarrow{p} = (p_1, p_2, \ldots, p_N) \in \mathbb{R}^N$, together with

$$p^- = \min_{1 \leq j \leq N} p_j > N, \qquad p^+ = \max_{1 \leq j \leq N} p_j.$$

The authors investigated the existence and multiplicity of solutions for (1.4), finding three different solutions. Their strategy was fundamentally built upon variational approaches and three critical points theory [7].

In [19], the authors established the existence of infinitely many solutions for a class of anisotropic $\overrightarrow{p}$-Laplacian system of the form

$$\begin{cases} -\Delta_{\overrightarrow{p}} u = \lambda f(x,u) + \mu g(x,u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.5}$$

where $\Omega \subseteq \mathbb{R}^N$ $(N \geq 2)$ represents a nonempty open set possessing a C^1-boundary, $f, g : \Omega \times \mathbb{R} \to \mathbb{R}$ denote two L^1-Carathéodory functions, λ, μ are positive real parameters. They used a smooth version of Theorem 2.1 from [12] to obtain infinitely many solutions for the problem (1.5).

Within the framework of anisotropic $\overrightarrow{p}(x)$-Laplacian problem, in [18], the authors studied the following anisotropic variable exponent problem

$$\begin{cases} -\Delta_{\overrightarrow{p}(x)} u = \rho\alpha(x)u^q + \beta(x)u^\theta & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.6}$$

where $\Omega \subseteq \mathbb{R}^N$ represents a bounded domain with smooth boundary, ρ denotes a positive parameter, $\alpha, \beta \in L^\infty(\Omega)$ with $\alpha(x), \beta(x) > 0$ a.e. $x \in \Omega$ and $p_j \in C^1(\overline{\Omega})$, $1 \leq j \leq N$ with $2 \leq p_j \leq p_M(x) := \max\{p_j(x);\ 1 \leq j \leq N\} < \overline{p^*}(x)$ where

$$\overline{p^*}(x) = \begin{cases} \frac{N\overline{p}(x)}{N-\overline{p}(x)} & \text{if } \overline{p}(x) < N, \\ +\infty & \text{if } \overline{p}(x) \geq N, \end{cases} \quad \text{and} \quad \frac{1}{\overline{p}(x)} = \frac{1}{N}\sum_{j=1}^{N} \frac{1}{p_j(x)}.$$

They studied the existence and nonexistence of (positive solution), also positive minimal solution of equation (1.6), by using the sub-supersolution method combined with the iterative method. Via variational methods, they also established the existence and nonexistence of positive solution when the right-hand side of equation (1.6) is given by $\rho\alpha(x)u^q - \beta(x)u^\theta$. Tavares [29] considered an anisotropic variable exponent system of the form

$$\begin{cases} -\Delta_{\overrightarrow{p}(x)} u = a(x)u^{\alpha(x)-1} + F_u(x,u,v) & \text{in } \Omega, \\ -\Delta_{\overrightarrow{q}(x)} v = b(x)v^{\beta(x)-1} + F_v(x,u,v) & \text{in } \Omega, \\ u = v = 0 & \text{on } \partial\Omega. \end{cases} \tag{1.7}$$

where $p_j, q_j \in C(\overline{\Omega}), 2 \leq p_j(x) \leq p_M(x) \leq \overline{p^*}(x)$ and $2 \leq q_j(x) \leq q_M(x) \leq \overline{q^*}(x)$, for $j = 1, 2, \ldots, N$. Under specified conditions on F_u and F_v, by using a suitable L^∞ estimates combined with the Mountain Pass Theorem and sub-supersolution argument, he established the existence and multiplicity results for (1.7). Tavares in [28], had considered before a scalar version of the system (1.7).

Using Neumann boundary conditions, existence of three solutions for an anisotropic $\overrightarrow{p}(x)$-Laplacian problem involving two parameters was studied in the work of Ahmed et al. [1]. Based on a recent three critical points theorem due to Ricceri [25], they obtained at least three solutions for the following Neumann $\overrightarrow{p}(x)$-elliptic problem

$$\begin{cases} -\Delta_{\overrightarrow{p}(x)} u + \lambda(x)|u|^{p_0(x)-2}u = \alpha f(x,u) + \beta g(x,u) & \text{in } \Omega, \\ \sum_{j=1}^{N} \left|\frac{\partial u}{\partial x_j}\right|^{p_j-2} \frac{\partial u}{\partial x_j} \overrightarrow{\gamma} = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.8}$$

where $\overrightarrow{\gamma} = (\gamma_1, \gamma_2, \ldots, \gamma_N)$ denotes the outward unit normal vector on the smooth boundary of a bounded domain $\Omega \subseteq \mathbb{R}^N$ $(N \geq 3)$ and $\alpha, \beta > 0$ denote real parameters and $p_j \in C(\overline{\Omega})$ with $p_j(x) \geq 1$ for all $x \in \overline{\Omega}$.

Inspired by the aforementioned papers, especially by [10], in this work we attain multiplicity of weak solutions for problem (1.1) based on two critical point theorems 2.4 and 2.5 introduced in [11]. Indeed, we achieve at least three weak solutions for problem (1.1) under independent conditions, see Theorems 3.1 and 3.2. In Theorem 3.2, we obtain at least three nonnegative weak solutions by supposing a suitable sign and growth conditions on the potential f_1 not having the asymptotic condition at infinity on f_1 for any continuous nonnegative function f_2. Corollary 3.3 follows from Theorems 3.1. Finally we provide two Examples to illustrate the obtained results. We point out that our results extend the obtained result in [10]. The construction of this paper contains four sections in such a way: Section 2 presents essential notations to defining our working space and some crucial properties of the functionals corresponding to the problem (1.1). The multiplicity results and their proofs are stated in Sections 3. Finally, Section 4 presents the main conclusions.

2 Functional context

In the present section, first, we define our working space and recall some crucial attributes concerning anisotropic variable exponent Sobolev space. In that context, we refer to the books of Rákosník [23, 24], the papers of Bonanno et al. [9, 10]. If $\overrightarrow{p} = (p_0, p_1, p_2, \ldots, p_N)$ denotes a vector with $p_j \geq 1$ for $j = 0, 1, 2, \ldots, N$. The anisotropic variable exponent Sobolev space [23, Definition 7] is described as

$$W^{1,\overrightarrow{p}}(\Omega) = \left\{ u \in L^{p_0}(\Omega); \frac{\partial u}{\partial x_j} \in L^{p_j}(\Omega),\ j = 1, 2, \ldots, N \right\},$$

including the norm

$$\|u\|_{\overrightarrow{p}} = \|u\|_{W^{1,\overrightarrow{p}}(\Omega)} = \|u\|_{L^{p_0}(\Omega)} + \sum_{j=1}^{N} \left\| \frac{\partial u}{\partial x_j} \right\|_{L^{p_j}(\Omega)}.$$

Our working space $Z = W_0^{1,\overrightarrow{p}}(\Omega)$ is described as the complement of $C_0^\infty(\Omega)$ with respect to the norm

$$\|u\|_Z = \sum_{j=1}^{N} \left\| \frac{\partial u}{\partial x_j} \right\|_{L^{p_j}(\Omega)},$$

which is equivalent to the usual one $\|u\|_{\overrightarrow{p}}$ (see [10]). The space $(Z, \|\cdot\|_Z)$ denotes a separable and reflexive Banach space [23, 24]. Hereafter, suppose that $p^- = \min_{1 \leq j \leq N} p_j > N$. So, according to Propositions 2.1 in [9, 10], we have

$$\|u\|_\infty \leq C_0 \|u\|_Z, \tag{2.1}$$

in which $\|u\|_\infty = \sup_{x \in \overline{\Omega}} |u(x)|$ and

$$C_0 = 2^{\frac{(N-1)(p^- - 1)}{p^-}} m_{p^-} \max_{1 \leq j \leq N} \left(|\Omega|^{\frac{p_j - p^-}{p_j p^-}} \right)$$

with

$$m_{p^-} = \frac{N^{-\frac{1}{p^-}}}{\sqrt{\pi}} \left[\Gamma(1+\frac{N}{2})\right]^{\frac{1}{N}} \left(\frac{p^- - 1}{p^- - N}\right)^{1-\frac{1}{p^-}} |\Omega|^{\frac{1}{N}-\frac{1}{p^-}},$$

where Γ denotes the Gamma function described by $\Gamma(s) = \int_0^\infty x^{s-1}e^{-x}dx$ for all $s > 0$ and $|\Omega|$ is the Lebesgue measure of Ω. Besides, based on the same Propositions 2.1 of [9, 10], the assumption $p^- > N$ ensures that the embedding of Z into $C^0(\overline{\Omega})$ is compact.

Proposition 2.1 (see [10, Proposition 2.2]). *Fix $\sigma > 0$. If $u \in Z$ with $\sum_{j=1}^N \frac{1}{p_j} \left\| \frac{\partial u}{\partial x_j} \right\|^{p_j}_{L^{p_j}(\Omega)} < \sigma$, then*

$$\|u\|_\infty < C_{\overrightarrow{p}} \max\left\{\sigma^{\frac{1}{p^+}}, \sigma^{\frac{1}{p^-}}\right\}, \tag{2.2}$$

in which $C_{\overrightarrow{p}} = C_0 \sum_{j=1}^N p_j^{\frac{1}{p_j}}$.

Next, we define two functionals $E_1, E_2 : Z \to \mathbb{R}$ by

$$E_1(u) = \sum_{j=1}^N \frac{1}{p_j} \int_\Omega \left|\frac{\partial u}{\partial x_j}\right|^{p_j} dx = \sum_{j=1}^N \frac{1}{p_j} \left\|\frac{\partial u}{\partial x_j}\right\|^{p_j}_{L^{p_j}(\Omega)},$$

$$E_2(u) = \int_\Omega \left[F_1(x,u) + \frac{\mu}{\lambda} F_2(x,u)\right] dx,$$

where $F_j(x,t) = \int_0^t f_j(x,\tau)d\tau$ for every $t \in \mathbb{R}$ and $j = 1, 2$.

The Gâteaux derivatives of E_1 and E_2 are given by

$$\langle E_1'(u), v\rangle = \sum_{j=1}^N \int_\Omega \left|\frac{\partial u}{\partial x_j}\right|^{p_j-2} \frac{\partial u}{\partial x_j} \frac{\partial v}{\partial x_j} dx,$$

$$\langle E_2'(u), v\rangle = \int_\Omega \left[f_1(x,u) + \frac{\mu}{\lambda} f_2(x,u)\right] v dx.$$

We recall that any function $u \in Z$ obeying the following relationship is named a weak solution of problem (1.1):

$$\langle E_1'(u), v\rangle = \lambda \langle E_2'(u), v\rangle$$

for any $v \in Z$. As in Proposition 2.3 and 2.4 in [10], we can obtain the following propositions.

Proposition 2.2. *E_1 is a sequentially weakly lower semicontinuous and coercive functional. Besides, $E_1' : Z \to Z^*$ admits a continuous inverse over Z^* in which Z^* is the dual space of Z.*

Proposition 2.3. *E_2 denotes a continuously Gâteaux differentiable functional whose derivative is compact.*

Finally, we recall the following three critical point theorems which are our basic tools to prove the main results. In the first item, the coercivity condition of $E_1 - \lambda E_2$ is necessary, in the second item an appropriate sign condition is supposed.

Theorem 2.4 (see [11, Theorem 3.6]). *Assume that $E_1 : Z \to \mathbb{R}$ denotes a continuously Gâteaux differentiable, sequentially weakly lower semicontinuous and coercive functional on the real Banach space Z whose derivative possesses a continuous inverse over Z^*, $E_2 : Z \to \mathbb{R}$ denotes a continuously Gâteaux differentiable functional with compact derivative together with $E_1(0) = E_2(0) = 0$.*

Assume that there exist $\sigma > 0$ and $z_0 \in Z$, with $E_1(z_0) > \sigma$ such that

($\mathbf{c_1}$) $\frac{\sup_{u\in E_1^{-1}(-\infty,\sigma]} E_2(u)}{\sigma} < \frac{E_2(z_0)}{E_1(z_0)}$;

($\mathbf{c_2}$) *for each* $\lambda \in \Lambda_\sigma = \left(\frac{E_1(z_0)}{E_2(z_0)}, \frac{\sigma}{\sup_{u\in E_1^{-1}(-\infty,\sigma]} E_2(u)}\right)$, $E_1 - \lambda E_2$ *denotes a coercive functional.*

Then, for all $\lambda \in \Lambda_\sigma$, *the functional* $E_1 - \lambda E_2$ *possesses at least three distinct critical points over* Z.

Theorem 2.5 (see [8, Corollary 3.1]). *If* $E_1 : Z \to \mathbb{R}$ *denotes a coercive, convex and continuously Gâteaux differentiable functional over real Banach space* Z *whose derivative possesses a continuous inverse over* Z^*. *Besides, if* $E_2 : Z \to \mathbb{R}$ *denotes a continuously Gâteaux differentiable functional with compact derivative together with*

(**i**) $\inf_Z E_1 = E_1(0) = E_2(0) = 0$;

(**ii**) *for all* $r > 0$ *and for all* $z_1, z_2 \in Z$ *which denote local minima for* $E_1 - \lambda E_2$ *and so that* $E_1(z_1) \geq 0$ *and* $E_2(z_2) \geq 0$, *one has*

$$\inf_{r\in[0,1]} E_2(rz_1 + (1-r)z_2) \geq 0.$$

Suppose that there are two positive constants σ_1, σ_2 *and* $z_0 \in Z$, *with* $2\sigma_1 < E_1(z_0) < \frac{\sigma_2}{2}$, *such that*

($\mathbf{d_1}$) $\frac{\sup_{u\in E_1^{-1}(-\infty,\sigma_1)} E_2(u)}{\sigma_1} < \frac{2}{3}\frac{E_2(z_0)}{E_1(z_0)}$;

($\mathbf{d_2}$) $\frac{\sup_{u\in E_1^{-1}(-\infty,\sigma_2)} E_2(u)}{\sigma_2} < \frac{1}{3}\frac{E_2(z_0)}{E_1(z_0)}$;

Then, for all

$$\lambda \in \left(\frac{2}{3}\frac{E_1(z_0)}{E_2(z_0)}, \min\left\{\frac{\sigma_1}{\sup_{u\in E_1^{-1}(-\infty,\sigma_1)} E_2(u)}, \frac{\frac{\sigma_2}{2}}{\sup_{u\in E_1^{-1}(-\infty,\sigma_2)} E_2(u)}\right\}\right),$$

the functional $E_1 - \lambda E_2$ *possesses at least three distinct critical points on* $E_1^{-1}(-\infty, \sigma_2)$.

3 Main results

For the sake of describing our main results, we require to introduce some notations involving our assumptions. Put

$$D = \sup_{x\in\Omega} \operatorname{dist}(x, \partial\Omega);$$

so, there is $\overline{x} \in \Omega$ with $B(\overline{x}, D) \subseteq \Omega$ in which $B(\overline{x}, D)$ denotes the open ball in $\mathbb{R}^N$ with radius D centered at $\overline{x}$ and we denote by

$$V_D = |B(\overline{x}, D)| = \frac{\pi^{\frac{N}{2}}}{\Gamma(1+\frac{N}{2})} D^N,$$

the measure of N-dimensional ball $B(\overline{x}, D)$. Let $x = (x_1, x_2, \ldots, x_N), \overline{x} = (\overline{x}_1, \overline{x}_2, \ldots, \overline{x}_N) \in \mathbb{R}^N$ and put

$$\alpha_j = \int_{B(\overline{x},D)\setminus B(\overline{x},\frac{D}{2})} \frac{|x_j - \overline{x}_j|^{p_j}}{|x - \overline{x}|_N^{p_j}} dx, \tag{3.1}$$

and

$$K = \sum_{j=1}^{N} \left[\frac{1}{p_j} \left(\frac{2}{D} \right)^{p_j} \alpha_j \right], \tag{3.2}$$

in which $|\cdot|_N$ denotes the Euclidean norm in $\mathbb{R}^N$. For any $y > 0$, set

$$(y)_{\min} = \min\{y^{p^+}, y^{p^-}\}, \quad (y)_{\max} = \max\{y^{p^+}, y^{p^-}\},$$

$$(F_j)^y = \int \sup_{|t| \le y} F_j(x,t)dx \quad \forall j = 1,2;$$

$$(F_2)_y = \inf_{\Omega \times [0,y]} F_2,$$

$$[F_1]_{y,D} = \int_{B(\overline{x}, \frac{D}{2})} F_1(x,y)dx.$$

Note that if f_2 is sign-changing, then $(F_2)^y \ge 0$ and $(F_2)_y \le 0$. Let us fix two positive constants a and b and choosing

$$\lambda \in \Lambda := \left(\frac{K(b)_{\max}}{[F_1]_{b,D}}, \frac{(a)_{\min}}{(F_1)^a (C_{\overrightarrow{p}})_{\max}} \right).$$

Now, put

$$\theta_{\lambda,f_2} := \min \left\{ \frac{(a)_{\min} - \lambda (C_{\overrightarrow{p}})_{\max} (F_1)^a}{(C_{\overrightarrow{p}})_{\max} (F_2)^a}, \frac{K(b)_{\max} - \lambda [F_1]_{b,D}}{|\Omega| (F_2)_b} \right\}$$

and

$$\theta^*_{\lambda,f_2} := \min \left\{ \theta_{\lambda,f_2}, \frac{1}{\max \left\{ 0, p^+ N^{p_j} C_0^{p^-} |\Omega| \overline{\lim}_{|t| \to \infty} \frac{F_2(x,t)}{t^{p^-}} \right\}} \right\} \tag{3.3}$$

in which $\theta^*_{\lambda,f_2} = +\infty$ if

$$\overline{\lim}_{|t| \to \infty} \frac{F_2(x,t)}{t^{p^-}} \le 0 \quad \text{uniformly a.e. in } \Omega,$$

and $(F_2)^a = (F_2)_b = 0$.

Our first main result is described by the subsequent theorem.

Theorem 3.1. *Suppose that $f_1 : \Omega \times \mathbb{R} \to \mathbb{R}$ denotes an L^1-Carathéodory function and there are two positive constants a and b verifying*

$$(a)_{\min} < K(C_{\overrightarrow{p}})_{\min} (b)_{\min} \tag{3.4}$$

so that

$$F_1(x,t) \ge 0, \qquad \forall (x,t) \in \Omega \times [0,b] \tag{3.5}$$

and

$$\frac{(F_1)^a}{(a)_{\min}} < \frac{1}{K(C_{\overrightarrow{p}})_{\max}} \frac{[F_1]_{b,D}}{(b)_{\max}}. \tag{3.6}$$

Besides, assume that

$$\overline{\lim}_{|t| \to \infty} \frac{F_1(x,t)}{|t|^{p^-}} = 0, \quad \textit{uniformly a.e. in } \Omega. \tag{3.7}$$

Then, for all $\lambda \in \Lambda$ and for each L^1-Carathéodory function $f_2 : \Omega \times \mathbb{R} \to \mathbb{R}$ verifying the relationship

$$\overline{\lim}_{|t|\to\infty} \frac{F_2(x,t)}{t^{p^-}} < +\infty \quad \text{uniformly a.e. in } \Omega,$$

there is $\theta^*_{\lambda,f_2} > 0$ as defined by (3.3) such that, problem (1.1) possesses at least three distinct weak solutions over Z for all $\mu \in [0, \theta^*_{\lambda,f_2})$.

Proof. We are going to apply Theorem 2.4. Thanks to the Propositions 2.2 and 2.3, E_1 and E_2 verify all regularity conditions demanded in Theorem 2.4 and besides we have $E_1(0) = E_2(0) = 0$. Put $\sigma = \left(\frac{a}{C_{\overrightarrow{p}}}\right)_{\min}$ and consider the function u^* by setting

$$u^*(x) = \begin{cases} 0 & \text{if } x \in \Omega \backslash B(\overline{x}, D), \\ \frac{2b}{D}(D - |x - \overline{x}|) & \text{if } x \in B(\overline{x}, D) \backslash B(\overline{x}, \frac{D}{2}) \\ b & \text{if } x \in B(\overline{x}, \frac{D}{2}). \end{cases} \tag{3.8}$$

Obviously, $u^* \in Z$. Using (3.1), (3.2) and (3.4), and following the same arguments as in [9, 10], it follows that

$$\begin{aligned} E_1(u^*) &= \sum_{j=1}^{N} \frac{1}{p_j} \left(\frac{2b}{D}\right)^{p_j} \int_{B(\overline{x},D)\backslash B(\overline{x},\frac{D}{2})} \frac{|x_j - \overline{x}_j|^{p_j}}{|x - \overline{x}|_N^{p_j}} dx \\ &\geq \sum_{j=1}^{N} \left[\frac{1}{p_j} (\frac{2}{D})^{p_j} \alpha_j\right] (b)_{\min} \\ &= K(b)_{\min} > \left(\frac{a}{C_{\overrightarrow{p}}}\right)_{\min} = \sigma. \end{aligned} \tag{3.9}$$

So $E_1(u^*) > \sigma$. Besides, for any $u \in Z$, by Proposition 2.1, we obtain

$$\begin{aligned} E_1^{-1}(-\infty, \sigma] &= \{u \in Z : E_1(u) \leq \sigma\} \\ &\subseteq \left\{u \in Z : \|u\|_\infty < C_{\overrightarrow{p}} \max\left\{\sigma^{\frac{1}{p^+}}, \sigma^{\frac{1}{p^-}}\right\} = a\right\} \\ &\subseteq \{u \in Z : |u(x)| \leq a, \ \ \forall x \in \Omega\}. \end{aligned}$$

Hence

$$\sup_{u \in E_1^{-1}(-\infty,\sigma]} E_2(u) \leq (F_1)^a + \frac{\mu}{\lambda}(F_2)^a.$$

Thus, we infer that

$$\frac{\sup_{u \in E_1^{-1}(-\infty,\sigma]} E_2(u)}{\sigma} \leq (C_{\overrightarrow{p}})_{\max} \frac{(F_1)^a + \frac{\mu}{\lambda}(F_2)^a}{(a)_{\min}}. \tag{3.10}$$

Additionally, by (3.1) and (3.2), we obtain

$$E_1(u^*) \leq \sum_{j=1}^{N} \left[\frac{1}{p_j} \left(\frac{2}{D}\right)^{p_j} \alpha_j\right] (b)_{\max} = K(b)_{\max}, \tag{3.11}$$

and thanks to (3.5), we obtain

$$\begin{aligned} E_2(u^*) &\geq \int_{B(x^0,\frac{D}{2})} F_1(x,b)dx + \frac{\mu}{\lambda} \int_\Omega F_2(x,u^*)dx \\ &\geq [F_1]_{b,D} + |\Omega| \frac{\mu}{\lambda} (F_2)_b. \end{aligned}$$

So, we get

$$\frac{E_2(u^*)}{E_1(u^*)} \geq \frac{1}{K}\frac{[F_1]_{b,D} + |\Omega|\frac{\mu}{\lambda}(F_2)_b}{(b)_{\max}}. \tag{3.12}$$

Because $\mu < \theta_{\lambda,f_2}$, we obtain

$$\frac{1}{\lambda} > (C_{\overrightarrow{p}})_{\max}\frac{(F_1)^a + \frac{\mu}{\lambda}(F_2)^a}{(a)_{\min}}. \tag{3.13}$$

and

$$\frac{1}{\lambda} < \frac{1}{K}\frac{[F_1]_{b,D} + |\Omega|\frac{\mu}{\lambda}(F_2)_b}{(b)_{\max}}. \tag{3.14}$$

By (3.10)–(3.14), we obtain

$$\begin{aligned}\frac{\sup_{u\in E_1^{-1}(-\infty,\sigma]} E_2(u)}{\sigma} &\leq (C_{\overrightarrow{p}})_{\max}\frac{(F_1)^a + \frac{\mu}{\lambda}(F_2)^a}{(a)_{\min}} \\ &< \frac{1}{\lambda} < \frac{1}{K}\frac{[F_1]_{b,D} + |\Omega|\frac{\mu}{\lambda}(F_2)_b}{(b)_{\max}} \leq \frac{E_2(u^*)}{E_1(u^*)}.\end{aligned} \tag{3.15}$$

So, the condition $(\mathbf{c_1})$ of Theorem 2.4 holds with $z_0 = u^*$.

Finally, since $\mu < \theta^*_{\lambda,f_2}$, we can fix $\beta > 0$ obeying the following relationship

$$\overline{\lim}_{|t|\to\infty}\frac{F_2(x,t)}{t^{p^-}} < \beta \quad \text{uniformly a.e. in } \Omega \text{ and } \quad \mu\beta < \frac{1}{p^+N^{p_j}C_0^{p^-}|\Omega|}.$$

Thus, by a similar reasoning as in [10], there is a function $w \in L^1(\Omega)$ so that

$$F_2(x,t) < \beta t^{p^-} + w(x) \qquad \forall (x,t) \in \Omega \times \mathbb{R}.$$

Now, let us fix $0 < \epsilon < \frac{1}{p^+\lambda N^{p_j}C_0^{p^-}|\Omega|} - \frac{\mu}{\lambda}\beta$. By (3.7) and since f denotes an L^1-Carathéodory function, we deduce that there is a function $\nu_\epsilon \in L^1(\Omega)$ obeying the relationship

$$F_1(x,t) < \epsilon|t|^{p^-} + \nu_\epsilon(x) \ \ \forall (x,t) \in \Omega \times \mathbb{R}.$$

Thus, thanks to (2.1) and the fact that $E_1(u) \geq \frac{1}{p^+N^{p_j}}\|u\|_Z^{p_j}$, we obtain

$$\begin{aligned}E_1(u) - \lambda E_2(u) \geq \frac{1}{p^+N^{p_j}}\|u\|_Z^{p_j} - \lambda\epsilon C_0^{p^-}|\Omega|\|u\|_Z^{p^-} - \lambda\|\nu_\epsilon\|_{L^1(\Omega)} \\ - \mu\beta C_0^{p^-}|\Omega|\|u\|_Z^{p^-} - \mu\|w\|_{L^1(\Omega)}.\end{aligned}$$

Now, let $u \in Z$ be so that $\|u\|_Z \to \infty$. If $p_j \neq p^-$, the coercivity of $E_1 - \lambda E_2$ is obvious. If not, it follows because $0 < \epsilon < \frac{1}{p^+\lambda N^{p_j}C_0^{p^-}|\Omega|} - \frac{\mu}{\lambda}\beta$. So, the condition $(\mathbf{c_2})$ of Theorem 2.4 holds.

Now, by (3.15), we obtain

$$\lambda \in \left(\frac{E_1(u^*)}{E_2(u^*)}, \frac{\sigma}{\sup_{u\in E_1^{-1}(-\infty,\sigma]} E_2(u)}\right).$$

Therefore, Theorem 2.4 yields the desired result. □

The second result is a modification of Theorem 3.1 in the case that no asymptotic condition on f_2 is demanded. Instead, the functions $f_1, f_2 : \Omega \times \mathbb{R} \to \mathbb{R}$ are considered nonnegative.

Theorem 3.2. *Suppose that $f_1 : \Omega \times \mathbb{R} \to \mathbb{R}$ denotes an L^1-Carathéodory function obeying the relationship $f_1(x,t) \geq 0$. If there are $a_1, a_2, b > 0$ verifying*

$$(a_1)_{\min} < \frac{K}{2}(C_{\overrightarrow{p}})_{\min}(b)_{\min} \tag{3.16}$$

and

$$2K(b)_{\max}(C_{\overrightarrow{p}})_{\max} < (a_2)_{\max} \tag{3.17}$$

such that

$$\max\left\{\frac{(F_1)^{a_1}(C_{\overrightarrow{p}})_{\max}}{(a_1)_{\min}}, \frac{2(F_1)^{a_2}(C_{\overrightarrow{p}})_{\min}}{(a_2)_{\max}}\right\} < \frac{2}{3}\frac{[F_1]_{b,D}}{K(b)_{\max}}. \tag{3.18}$$

Then, for each

$$\lambda \in \Lambda' := \left(\frac{3}{2}\frac{K(b)_{\max}}{[F_1]_{b,D}}, \min\left\{\frac{(a_1)_{\min}}{(F_1)^{a_1}(C_{\overrightarrow{p}})_{\max}}, \frac{(a_2)_{\max}}{2(F_1)^{a_2}(C_{\overrightarrow{p}})_{\min}}\right\}\right),$$

and for every nonnegative L^1-Carathéodory function $f_2 : \Omega \times \mathbb{R} \to \mathbb{R}$, there exists $\theta^{**}_{\lambda,f_2} > 0$ which is defined by

$$\theta^{**}_{\lambda,f_2} = \min\left\{\frac{(a_1)_{\min} - \lambda(C_{\overrightarrow{p}})_{\max}(F_1)^{a_1}}{(C_{\overrightarrow{p}})_{\max}(F_2)^{a_1}}, \frac{(a_2)_{\max} - 2\lambda(C_{\overrightarrow{p}})_{\min}(F_1)^{a_2}}{2(C_{\overrightarrow{p}})_{\min}(F_2)^{a_2}}\right\}$$

such that, for each $\mu \in [0, \theta^{**}_{\lambda,f_2})$, problem (1.1) possesses at least three distinct weak solutions $\tilde{u}_i$ for $i = 1,2,3$, obeying the following relationship

$$0 \leq \tilde{u}_i(x) \leq a_2 \qquad \forall x \in \Omega.$$

Proof. We aim to apply Theorem 2.5. Since for the given $q > 2$, the function $g : [0,\infty) \to \mathbb{R}$, $g(t) = t^q$ is convex, the functional E_1 is convex and by Propositions 2.2 and 2.3, all demanded regularity conditions in Theorem 2.5 are verified.
Next, we will check remainder conditions. To this end, let u^* be the same function defined by (3.8) and put

$$\sigma_1 = \left(\frac{a_1}{C_{\overrightarrow{p}}}\right)_{\min}, \qquad \sigma_2 = \left(\frac{a_2}{C_{\overrightarrow{p}}}\right)_{\max}.$$

By (3.16) and (3.17), we obtain

$$2\sigma_1 < E_1(u^*) < \frac{1}{2}\sigma_2.$$

Indeed, by (3.9) and (3.11), we have

$$E_1(u^*) > 2\left(\frac{a_1}{C_{\overrightarrow{p}}}\right)_{\min} = 2\sigma_1,$$

and

$$E_1(u^*) < \frac{1}{2}\left(\frac{a_2}{C_{\overrightarrow{p}}}\right)_{\max} = \frac{1}{2}\sigma_2.$$

Because $\mu < \theta^{**}_{\lambda,f_2}$ and $(F_2)_b = 0$, we deduce that

$$\begin{aligned}\frac{\sup_{u\in E_1^{-1}(-\infty,\sigma_1]} E_2(u)}{\sigma_1} &= \frac{(F_1)^{a_1} + \frac{\mu}{\lambda}(F_2)^{a_1}}{\left(\frac{a_1}{C_{\overrightarrow{p}}}\right)_{\min}} \\ &\leq (C_{\overrightarrow{p}})_{\max}\frac{(F_1)^{a_1} + \frac{\mu}{\lambda}(F_2)^{a_1}}{(a_1)_{\min}} \\ &< \frac{1}{\lambda} \\ &< \frac{2}{3}\frac{[F_1]_{b,D} + |\Omega|\frac{\mu}{\lambda}(F_2)_b}{K(b)_{\max}} \\ &\leq \frac{2}{3}\frac{E_2(u^*)}{E_1(u^*)},\end{aligned}$$

and

$$\begin{aligned}\frac{2\sup_{u\in E_1^{-1}(-\infty,\sigma_2]} E_2(u)}{\sigma_2} &\leq \frac{2(F_1)^{a_2} + 2\frac{\mu}{\lambda}(F_2)^{a_2}}{\left(\frac{a_2}{C_{\overrightarrow{p}}}\right)_{\max}} \\ &\leq (C_{\overrightarrow{p}})_{\min}\frac{2(F_1)^{a_2} + 2\frac{\mu}{\lambda}(F_2)^{a_2}}{(a_2)_{\max}} \\ &< \frac{1}{\lambda} \\ &< \frac{2}{3}\frac{[F_1]_{b,D} + |\Omega|\frac{\mu}{\lambda}(F_2)_b}{K(b)_{\max}} \\ &\leq \frac{2}{3}\frac{E_2(u^*)}{E_1(u^*)}.\end{aligned}$$

So, the conditions $(\mathbf{d_1})$ and $(\mathbf{d_2})$ of Theorem 2.5 with $z_0 = u^*$. are fulfilled. Besides, to prove the condition $(\mathbf{ii})$ of Theorem 2.5, let w_1 and w_2 be two local minima for $E_1 - \lambda E_2$. Therefore, they are two nonnegative weak solutions of problem (1.1). Indeed, arguing by contradiction, consider the case that $M = \{x \in \Omega;\ w_1(x) < 0\}$. Then, we get

$$\sum_{j=1}^{N}\int_\Omega \left|\frac{\partial w_1}{\partial x_j}\right|^{p_j-2}\frac{\partial w_1}{\partial x_j}\frac{\partial w_1^-}{\partial x_j}dx - \lambda\int_\Omega f_1(x,w_1)w_1^- dx - \mu\int_\Omega f_2(x,w_1)w_1^- dx = 0,$$

in which $w_1^-(x) = \min\{0, w_1(x)\}$. Thanks to our sign conditions on the data, we obtain

$$0 \leq \sum_{j=1}^{N}\int_M \left|\frac{\partial w_1}{\partial x_j}\right|^{p_j} dx = \lambda\int_M f_1(x,w_1)w_1 dx + \mu\int_M f_2(x,w_1)w_1 dx \leq 0.$$

Hence

$$\|w_1\|_{(M)} = \sum_{j=1}^{N}\int_M \left|\frac{\partial w_1}{\partial x_j}\right|^{p_j} dx = 0.$$

So $w_1 = 0$ in M which is impossible. Hence w_1 is nonnegative. Similarly, $w_2(x) \geq 0$ for all $x \in \Omega$. So, we obtain $rw_1 + (1-r)w_2 \geq 0$ for every $r \in [0,1]$, and that

$$(\lambda f_1 + \mu f_2)(x, rw_1 + (1-r)w_2) \geq 0,$$

and thus $\inf_{r\in[0,1]} E_2(rw_1 + (1-r)w_2) \geq 0$. So, the condition $(\mathbf{ii})$ of Theorem 2.5 holds with $z_1 = w_1$, $z_2 = w_2$ and we obtain the desired result. □

Now, we point out the subsequent multiplicity results, as the particular case of Theorem 3.1.

Corollary 3.3. *If $f_1 : \mathbb{R} \to \mathbb{R}$ denotes a continuous function, and denote $F_j(t) = \int_0^t f_j(\tau)d\tau$ for each $t \in \mathbb{R}$ and $j = 1, 2$. Suppose that $F_1(t) \geq 0$ in $[0, b]$ for some $b > 0$ and $F_1(b) > 0$ and*

$$\underline{\lim}_{t\to 0} \frac{F_1(t)}{t^{p^+}} = 0, \tag{3.19}$$

$$\overline{\lim}_{t\to \infty} \frac{F_1(t)}{t^{p^-}} = 0. \tag{3.20}$$

Then, there exists $\lambda_0 > 0$ such that, for every $\lambda > \lambda_0$ and for every L^1-Carathéodory function $f_2 : \Omega \times \mathbb{R} \to \mathbb{R}$ verifying the asymptotical condition

$$\overline{\lim}_{|t|\to \infty} \frac{F_2(x,t)}{t^{p^-}} < +\infty \quad \text{uniformly a.e. in } \Omega,$$

there is $\theta^*_{\lambda,f_2} > 0$ such that the problem

$$\begin{cases} -\Delta_{\overrightarrow{p}} u = \lambda f_1(u) + \mu f_2(x,u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{3.21}$$

possesses at least three distinct weak solutions over Z for all $\mu \in [0, \theta^*_{\lambda,f_2})$.

Proof. Fix $\lambda > \lambda_0 := \frac{K(b_{\max})}{F_1(b)V_{\frac{D}{2}}}$ for some $b > 0$. By (3.19), there exists a sequence $\{a_n\} \subset (0, \infty)$ obeying the relationship

$$a_n \to 0 \quad \text{and} \quad \lim_{n\to\infty} \frac{\sup_{|t|\leq a_n} F_1(t)}{a_n^{p^+}} = 0.$$

Indeed, we have

$$\lim_{n\to\infty} \frac{\sup_{|t|\leq a_n} F_1(t)}{a_n^{p^+}} = \lim_{n\to\infty} \frac{F_1(t_{a_n})}{(t_{a_n})^{p^+}} \frac{(t_{a_n}^{p^+})}{a_n^{p^+}} = 0,$$

in which $F_1(t_{a_n}) = \sup_{|t|\leq a_n} F_1(t)$. Thus, there exists $a_0 > 0$ small enough such that (3.4) be verified and

$$\frac{\sup_{|t|\leq a_0} F_1(t)}{(a_0)^{p^+}} < \min\left\{ \frac{V_{\frac{D}{2}}}{K|\Omega|(C_{\overrightarrow{p}})_{\max}} \frac{F_1(b)}{(b)_{\max}}, \frac{1}{\lambda|\Omega|(C_{\overrightarrow{p}})_{\max}} \right\}.$$

By using Theorem 3.1, the proof is achieved. □

Example 3.4. Fix $N = 2$ and $\Omega = B(0, 2)$. Suppose that $p = (p_1, p_2)$ with $p_1 = 4, p_2 = 8$ and define $f_1 : \mathbb{R} \to \mathbb{R}$ by

$$f_1(t) = \begin{cases} 9t^8 & \text{if } t < 1, \\ \frac{9}{t^4} & \text{if } t \geq 1, \end{cases}$$

and $f_2(t) = t^2$ for each $t \in \mathbb{R}$. We get

$$F_1(t) = \begin{cases} t^9 & \text{if } t < 1, \\ 4 - \frac{3}{t^3} & \text{if } t \geq 1. \end{cases}$$

It is obvious that

$$\underline{\lim}_{t\to 0}\frac{F_1(t)}{t^8} = \overline{\lim}_{t\to\infty}\frac{F_1(t)}{t^4} = 0 \quad \text{and} \quad \overline{\lim}_{t\to\infty}\frac{F_2(t)}{t^4} < +\infty.$$

Hence, Corollary 3.3 establishes the existence of $\lambda_0 > 0$ and $\theta^*_{\lambda,f_2} > 0$ such that the following problem

$$\begin{cases} -\sum_{j=1}^{N}\frac{\partial}{\partial x_j}(|\frac{\partial u}{\partial x_j}|^{p_j-2}\frac{\partial u}{\partial x_j}) = \lambda f_1(u) + \mu f_2(u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{3.22}$$

has at least three distinct weak solutions.

Besides, we present an example of the application of Theorem 3.2.

Example 3.5. If $f_1 : \mathbb{R} \to \mathbb{R}$ denotes a continuous non-negative function obeying the following relationship $\lim_{t\to 0^+}\frac{f_1(t)}{t^3} = 0$ and $\int_0^{10^2} f_1(\tau)d\tau < \frac{10^8\int_0^1 f_1(\tau)d\tau}{54\sqrt[4]{\pi}\sqrt{2}(\sqrt[3]{3}+\sqrt{2})^3}$. Then, for every

$$\lambda \in \left(\frac{27}{64\int_0^1 f_1(\tau)d\tau}, \frac{10^8}{128\sqrt[4]{\pi}\sqrt{2}(\sqrt[3]{3}+\sqrt{2})^3\int_0^{10^2} f_1(\tau)d\tau}\right)$$

and for all non-negative continuous function $f_2 : \mathbb{R} \to \mathbb{R}$, there is $\theta^*_{\lambda,f_2} > 0$ such that problem (3.21) possesses at least three distinct non-negative solutions.

Proof. Our goal is to apply Theorem 3.2 by selecting $\Omega = \{(x,y) \in \mathbb{R}^2; x^2+y^2 < 4\}, b = 1, a_2 = 10^2, x^0 = (0,0), p_1 = 3, p_2 = 4$. By simple calculations, we obtain

$$K = \frac{3^2}{2^5}\pi,$$

$$\frac{3}{2}\frac{K(b)_{\max}}{[F_1]_{b,D}} = \frac{27}{64\int_0^1 f_1(\tau)d\tau}$$

$$m_{p^-} = m_3 = \sqrt[3]{\frac{4}{\pi}}$$

$$C_0 = \frac{2\sqrt{2}}{\sqrt[4]{\pi}}$$

$$C_{\overrightarrow{p}} = \frac{2\sqrt{2}}{\sqrt[4]{\pi}}(\sqrt[3]{3}+\sqrt{2}) > 1$$

$$2K(b)_{\max}(C_{\overrightarrow{p}})_{\max} = 36(\sqrt[3]{3}+\sqrt{2})^4$$

$$\frac{K}{2}(C_{\overrightarrow{p}})_{\min}(b)_{\min} = \frac{3^2}{2\sqrt{2}}\sqrt[4]{\pi}(\sqrt[3]{3}+\sqrt{2})^3$$

and

$$\frac{(a_2)_{\max}}{2(F_1)^{a_2}(C_{\overrightarrow{p}})_{\min}} = \frac{10^8}{128\sqrt[4]{\pi}\sqrt{2}(\sqrt[3]{3}+\sqrt{2})^3\int_0^{10^2} f_1(\tau)d\tau}.$$

Besides, since $\lim_{t\to 0^+}\frac{f(t)}{t^3} = 0$, we obtain

$$\lim_{t\to 0^+}\frac{\int_0^1 f_1(\tau)d\tau}{t^4} = 0.$$

So, there exists $0 < a_1 < 1 < \left(\frac{K}{2}(C_{\overrightarrow{p}})_{\min}(b)_{\min}\right)^{\frac{1}{3}} = \frac{\sqrt[12]{\pi}\sqrt[3]{9}(\sqrt[3]{3}+\sqrt{2})}{\sqrt{2}}$ such that

$$\frac{\int_0^{a_1} f_1(\tau)d\tau}{a_1^4} < \frac{\int_0^1 f_1(\tau)d\tau}{108(\sqrt[3]{3}+\sqrt{2})^4}$$

and

$$\frac{a_1^4}{\int_0^{a_1} f_1(\tau)d\tau} > \frac{10^8\sqrt{2}(\sqrt[3]{3}+\sqrt{2})}{\sqrt[4]{\pi}\int_0^{10^2} f_1(\tau)d\tau}.$$

Hence, all hypothesis of Theorem 3.2 are fulfilled and we obtain the desired result. □

4 Conclusions

A new result for an anisotropic $\overrightarrow{p}$-Laplacian equation consisting of two parameters is proposed in this paper. Motivated by the paper [10], the basic contribution of this paper is that we attain multiplicity results for a class of anisotropic problems involving two parameters that have been rarely considered. We also provide some corollaries and examples to display the obtained results.

Acknowledgements

B. Zhang was supported by the Natural Science Foundation of Shandong Provincial, PR China (No. ZR2023MA090).

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