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Existence of positive solutions of a semipositone problem with Minkowski-curvature operator

Zhongzi Zhao ✉

School of Mathematics and Statistics, Shanxi Datong University, Datong, 037009, P. R. China

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Abstract. We are concerned with the semipositone problem with Minkowski-curvature operator

$$\begin{cases} -\operatorname{div}\left(\dfrac{\nabla u}{\sqrt{1-|\nabla u|^2}}\right) = \lambda a(|x|)(u^\gamma - \epsilon), & x \in B(R), \\ u = 0, & x \in \partial B(R), \end{cases}$$

where $B(R) = \{x \in \mathbb{R}^N : |x| < R\}$, $N \geq 2$, $R > 0$, $\epsilon > 0$, $\gamma > 1$ and $a : [0,\infty) \to (0,\infty)$ is a continuous function. We show that there exist the constants $\Lambda > 2N/(\max_{r\in[0,R]} a(r)R^{\gamma+1})$ and $\epsilon_0(\Lambda) > 0$ such that, if $\epsilon < \epsilon_0(\Lambda)$, then the problem has least two positive radial solution for any $\lambda \in (\Lambda, \Lambda^*)$, where Λ^* is a constant that depends on ϵ. The proof of our main result is based upon degree theory.

Keywords: mean curvature operator, semipositone, positive solutions, degree theory.

2020 Mathematics Subject Classification: 35B09, 35A16, 35J93.

1 Introduction

In this paper, we are concerned with the problem with Minkowski-curvature operator

$$\begin{cases} -\operatorname{div}\left(\frac{\nabla u}{\sqrt{1-|\nabla u|^2}}\right) = \lambda a(|x|)(u^\gamma - \epsilon), & x \in B(R), \\ u = 0, & x \in \partial B(R), \end{cases} \tag{1.1}$$

where $B(R) = \{x \in \mathbb{R}^N : |x| < R\}$, $N \geq 2$, $R > 0$, $\epsilon > 0$, $\gamma > 1$ and $a : [0,\infty) \to (0,\infty)$ is a continuous function.

Boundary value problem of differential equation

$$\begin{cases} -\Delta u = f(x,u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega \end{cases} \tag{1.2}$$

✉ Email: zhaozz@sxdtdx.edu.cn

is called a semipositone problem if $f(\cdot,0) < 0$; and it is called a positone problem if $f(\cdot,0) \geq 0$, where Ω is a strictly convex bounded domain in $\mathbb{R}^N$, $N \geq 1$, with C^2 boundary $\partial\Omega$. Semipositone problem has been largely investigated for semilinear elliptic equations (1.2) since the late 1980s, with increasing interest after the paper Castro *et al.* [11] and Brown *et al.* [9], see [4,16,20] and the references therein.

It is well documented (see [8,21]) that, if compared with the positone problems, the study of positive solutions of the semipositone problem is mathematically challenging. For example, the absence of maximum principle, the difficulty in constructing sub and super solutions, and the trivial solution $u \equiv 0$ is not a solution of problem (1.2) with $f(\cdot,0) < 0$. These three factors make it unclear how to apply the methods based on the theory of positive operators for cones in Banach spaces, the method of sub and super solutions and bifurcation from the trivial solution.

For quasilinear semipositone problems, there have been some recent papers on the problems with the p-Laplacian operator, see Alves, de Holanda and Santos [2, 3], Castro, de Figueredo and Lopera [10], Chhetri, Drábek and Shivaji [12], Morris, Shivaji and Sim [23]. In particular, Chhetri, Drábek and Shivaji [12] considered a quasilinear elliptic problem of the form

$$\begin{cases} -\Delta_p u = \lambda g(u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.3}$$

where $\Delta_p u := \operatorname{div}(|\nabla u|^{p-2}\nabla u)$ is the p-Laplacian operator with $1 < p < 2$, and $\lambda > 0$ is a parameter. The nonlinearity $g : [0,\infty) \to \mathbb{R}$ is a continuous function that satisfies $g(0) < 0, g(s) > 0$ for $s \gg 1$. They said that λ_∞ is a bifurcation point from infinity if the solution set $S := \{(\lambda,u) \in \mathbb{R} \times W_0^{1,p}(\Omega) : \lambda \text{ and } u \text{ solves (1.3)}\}$ contains a sequence $\{(\lambda_n,u_n)\}$ such that

$$\lambda_n \to \lambda_\infty \quad \text{and} \quad \|u_n\|_\infty \to \infty. \tag{1.4}$$

And they proved the following result.

Theorem A. *Assume $1 < p < 2$ and $N > p$. Let $g : [0,\infty) \to \mathbb{R}$ be a continuous function satisfying the following conditions:*

(i) $g(0) < 0$;

(ii) there exist $b > 0$ and $q \in (p-1, N(p-1)/(N-p)]$ such that

$$\lim_{s\to\infty} \frac{g(s)}{s^q} = b.$$

Then (1.1) has a positive solution (λ,u) for $\lambda > 0$ small.

The key points of the proof of Theorem A are that rescale the parameter and the solution variable using $\gamma = \lambda^{\frac{1}{q-p+1}}$ and $w = \gamma u$, respectively, then w formally satisfies

$$-\Delta_p w = -\operatorname{div}(|\nabla \gamma u|^{p-2}\nabla \gamma u) = -\gamma^{p-1}\operatorname{div}(|\nabla u|^{p-2}\nabla u) = \gamma^q g(w/\gamma). \tag{1.5}$$

The theory of positive operators for (1.5) can be applied when $\gamma > 0$ is small (i.e., $\lambda > 0$ is small). This is the standard technique of dealing with semipositone elliptic problems.

However, the mean curvature operator in semipositone problem (1.1) prevents the application of the above standard technique, since $\varphi(ks) \neq k\varphi(s)$ (where $\varphi(s) = \frac{s}{\sqrt{1-s^2}}$) for any constant $k \in \mathbb{R}\backslash\{0,1\}$ and any solution u of (1.1) is bounded (see Remark 2.1 below). So although there have been significant paper on the problems with mean curvature operator (see Azzollini [5], Bereanu and Mawhin [7], Bereanu, Jebelean and Torres [6], Ma, Gao and Lu [22] and Dai [15] and the references therein) in the last decade, they have had to require that the nonlinearity f be non-negative at 0. In particular, Bereanu, Jebelean and Torres [6] studied the problem (1.1) with $\epsilon = 0$ and proved that there exists $\Lambda > 2N/(\max_{r\in[0,R]} a(r)R^{\gamma+1})$ such that the problem has at least two positive radial solutions according to $\lambda > \Lambda$.

Thus, this paper appears to be the first contribution where consider the problem with mean curvature operator is faced in a semipositone setting. And the aim of this brief paper is to provide a result about the existence of positive solution of the semipositone problem (1.1).

To state our results precisely we recall the following hypothesis,

(H0) $\epsilon > 0$, $\gamma > 1$ and $a : [0,\infty) \to (0,\infty)$ is a continuous function.

Theorem 1.1. *Let (H0) hold. There exist the constants $\Lambda > 2N/(\max_{r\in[0,R]} a(r)R^{\gamma+1})$ and $\epsilon_0(\Lambda) > 0$ such that, for any $\epsilon < \epsilon_0(\Lambda)$ there is $\Lambda^*(\epsilon)$ such that problem (1.1) has at least two positive radial solutions for any $\lambda \in (\Lambda, \Lambda^*)$.*

Remark 1.2. The nonlinearity in (1.1) is related to *pure-power* model $g(s) = s^\gamma$, see W. Allegretto, P. Nistri, P. Zecca [1], D. Costa, H. Ramos Quoirin, H. Tehrani [14], Q. Morris, R. Shivaji, I. Sim [23]. In recent years, for the existence of positive solutions to the semipositone elliptic equation, see, D. D. Hai, A. Muthunayake, R. Shivaji [17], A. Joseph, L. Sankar [18], R. Kajikiya, E. Ko [19] and their references.

2 Preliminary results

Setting, as usual, $r = |x|$ and $u(x) = u(r)$, we reduce the Dirichlet problem (1.1) to the mixed boundary value problem

$$\begin{cases} -(r^{N-1}\varphi(u'(r)))' = \lambda r^{N-1}a(r)(u^\gamma - \epsilon), & r \in (0,R), \\ u'(0) = u(R) = 0, \end{cases} \tag{2.1}$$

where $\varphi(s) = \frac{s}{\sqrt{1-s^2}}$. The space $C := C[0,R]$ will be endowed with the usual sup-norm $\|\cdot\|_\infty$ and $C^1 := C^1[0,R]$ will be considered with the norm $\|u\| = \|u\|_\infty + \|u'\|_\infty$. Also, we shall use the closed subspace of C^1 defined by

$$C^1_M = \{u \in C^1 : u'(0) = 0 = u(R)\}.$$

Let

$$W := \Big\{w \in C^1_M : \max_{[0,R]} |w'| < 1 \text{ and } \varphi(w') \in C^1[0,R]\Big\}.$$

A solution of (2.1) is a function $u \in W$ satisfying (2.1). Function $\Psi = \varphi^{-1}$ is smooth, $w' \in C^1$ can be obtained immediately.

Remark 2.1. Combined with the boundary condition $u(R) = 0$, it is easy to find that the solution $u \in W$ of (2.1) is bounded.

In the sequel, let us consider an auxiliary problem

$$\begin{cases} -(r^{N-1}\varphi(u'(r)))' = \lambda r^{N-1}a(r)(u(r))^{\gamma}, & r \in (0,R), \\ u'(0) = u(R) = 0. \end{cases} \tag{2.2}$$

We introduce the linear operators

$$S : C \to C, \qquad Su(r) = \frac{1}{r^{N-1}} \int_0^r t^{N-1} u(t)dt \qquad (r \in (0,R]), \quad Su(0) = 0;$$

$$K : C \to C^1, \qquad Ku(r) = \int_r^R u(t)dt \qquad (r \in [0,R]).$$

It is easy to see that K is bounded and standard arguments, invoking the Arzelà–Ascoli theorem, show that S is compact. This implies that the nonlinear operator $K \circ \varphi^{-1} \circ S : C \to C^1$ is compact. Let N_λ be the Nemytskii operator associated to $\lambda a(r)u^\gamma$, i.e,

$$N_\lambda : C \to C, \qquad N_\lambda(u) = \lambda a(\cdot)u^\gamma(\cdot).$$

From [6, Lemma 1], a function $u \in C^1_M$ is a solution of (2.2) if and only if it is a fixed point of the compact nonlinear operator

$$\mathcal{N}_\lambda : C^1_M \to C^1_M, \qquad \mathcal{N}_\lambda = K \circ \varphi^{-1} \circ S \circ N_\lambda. \tag{2.3}$$

A lower solution of (2.2) is a function $\alpha \in C^1$ such that $\|\alpha'\|_\infty < 1$, $r^{N-1}\varphi(\alpha') \in C^1$ and

$$(r^{N-1}\varphi(\alpha'(r)))' + \lambda r^{N-1}a(r)\alpha(r)^\gamma \geq 0, \qquad r \in [0,R], \quad \alpha(R) \leq 0.$$

Further, we say that a lower solution α of (2.2) is strict if every solution u of (2.2) with $u \geq \alpha$ in $[0,R]$ satisfies $u \gg \alpha$ in $[0,R)$ (i.e., $u(r) > \alpha(r)$ for any $r \in [0,R)$).

Similarly, a upper solution of (2.2) is a function $\beta \in C^1$ such that $\|\beta'\|_\infty < 1$, $r^{N-1}\varphi(\beta') \in C^1$ and

$$(r^{N-1}\varphi(\beta'(r)))' + \lambda r^{N-1}a(r)\beta(r)^\gamma \leq 0, \qquad r \in [0,R], \quad \beta(R) \geq 0.$$

Further, we say that a upper solution β of (2.2) is strict if every solution u of (2.2) with $u \leq \beta$ in $[0,R]$ satisfies $u \ll \beta$ in $[0,R)$ (i.e., $u(r) < \beta(r)$ for any $r \in [0,R)$).

Lemma 2.2 ([13]). *For any fixed $\lambda > 0$, suppose that there exist a lower solution α and an upper solution β of (2.2) with $\alpha \leq \beta$ in $[0,R]$. Then problem (2.2) has solutions v, w with $\alpha \leq v \leq w \leq \beta$ in $[0,R]$ such that every solution u of (2.2) with $\alpha \leq u \leq \beta$ in $[0,R]$ satisfies $v \leq u \leq w$ in $[0,R]$. Further, if α and β are strict, then*

$$\deg(I - \mathcal{N}_\lambda, U_\alpha^\beta, 0) = 1,$$

where $\mathcal{N}_\lambda$ is defined by (2.3) and

$$U_\alpha^\beta = \{z \in C^1_M : \alpha \ll z \ll \beta \text{ in } [0,R) \text{ and } \|z'\|_\infty < 1\}.$$

From [6, Theorem 1.1], there exists

$$\Lambda > 2N/(a_M R^{\gamma+1}),$$

where $a_M := \max_{r\in[0,R]} a(r)$, such that problem (2.2) has at least one positive solutions according to $\lambda = \Lambda$. Let $\lambda_0 > \Lambda$ be arbitrarily chosen. Let $\widetilde{\alpha}$ be a positive solution for (2.2) corresponding to $\lambda = \Lambda$. It is easy to see that $\widetilde{\alpha}$ is a lower solution for (2.2) with $\lambda = \lambda_0$. For fixed $\lambda_2 > \lambda_0$, let $\widetilde{\beta}$ be the unique (positive) solution of

$$\begin{cases}(r^{N-1}\varphi(u'(r)))' + \lambda_2 r^{N-1} a_M \widetilde{R}^\gamma = 0, & r \in (0, \widetilde{R}),\\ u'(0) = u(\widetilde{R}) = 0,\end{cases}$$

where $\widetilde{R} > R$. Using that $\widetilde{\beta}(R) > 0$ and

$$\lambda_0 a(r) u^\gamma(r) < \lambda_2 a_M \widetilde{R}^\gamma, \qquad r \in [0, R],$$

it follows that $\widetilde{\beta}$ is an upper solution for (2.2) with $\lambda = \lambda_0$. We can find $\widetilde{R}$ sufficiently large, such that $\widetilde{\alpha}(0) < \widetilde{\beta}(R)$. For any $r \in [0, R)$,

$$\widetilde{\alpha}'(r) = -\varphi^{-1}\Big(\frac{1}{r^{N-1}}\int_0^r t^{N-1}[\Lambda a(t)(\widetilde{\alpha}(t))^\gamma]dt\Big) < 0$$

and

$$\widetilde{\beta}'(r) = -\varphi^{-1}\Big(\frac{1}{r^{N-1}}\int_0^r t^{N-1}[\lambda_2 a_M \widetilde{R}^\gamma]dt\Big) < 0.$$

Then, taking into account that $\widetilde{\alpha}, \widetilde{\beta}$ are strictly decreasing, we infer that $\widetilde{\alpha} < \widetilde{\beta}$ on $[0, R]$.

From Lemma 2.2, problem (2.2) has a solution u_0 satisfies $\widetilde{\alpha} \le u_0 \le \widetilde{\beta}$ in $[0, R]$, i.e., $u_0 \in U_{\widetilde{\alpha}}^{\widetilde{\beta}}$. We will show that $\widetilde{\alpha}$ and $\widetilde{\beta}$ are strict. In fact, for all $r \in [0, R)$, one has

$$\begin{aligned}\widetilde{\beta}(r) &= \int_r^{\widetilde{R}} \varphi^{-1}\Big(\frac{1}{t^{N-1}}\int_0^t s^{N-1}[\lambda_2 a_M \widetilde{R}^\gamma]ds\Big)dt\\ &> \int_r^R \varphi^{-1}\Big(\frac{1}{t^{N-1}}\int_0^t s^{N-1}[\lambda_0 a(s)(u_0(s))^\gamma]ds\Big)dt = u_0(r).\end{aligned}$$

Similarly, for all $r \in [0, R)$,

$$\begin{aligned}\widetilde{\alpha}(r) &= \int_r^R \varphi^{-1}\Big(\frac{1}{t^{N-1}}\int_0^t s^{N-1}[\Lambda a(s)(\widetilde{\alpha}(s))^\gamma]ds\Big)dt\\ &< \int_r^R \varphi^{-1}\Big(\frac{1}{t^{N-1}}\int_0^t s^{N-1}[\lambda_0 a(s)(u_0(s))^\gamma]ds\Big)dt = u_0(r).\end{aligned}$$

This suggests that

$$\deg\Big(I - \mathcal{N}_{\lambda_0}, U_{\widetilde{\alpha}}^{\widetilde{\beta}}, 0\Big) = 1.$$

For any $\rho > 0$, set $B_\rho := \{u \in C_M^1 : \ \|u\| < \rho\}$. Consider $\rho_1 > 0$ sufficiently small and $\rho_2 \ge R + 1$ such that $U_{\widetilde{\alpha}}^{\widetilde{\beta}} \cap B_{\rho_1} = \emptyset$ and $(U_{\widetilde{\alpha}}^{\widetilde{\beta}} \cup B_{\rho_1}) \subset B_{\rho_2}$. By [6, Lemma 1 and Lemma 4], one has

$$\deg(I - \mathcal{N}_{\lambda_0}, B_{\rho_1}, 0) = 1$$

and

$$\deg(I - \mathcal{N}_{\lambda_0}, B_{\rho_2}, 0) = 1.$$

Then, from the additivity-excision property of the Leray–Schauder degree it follows that

$$\deg(I - \mathcal{N}_{\lambda_0}, B_{\rho_2} \setminus [\overline{U}_{u_1}^{u_2} \cup \overline{B}_{\rho_1}], 0) = -1.$$

3 The proof of main result

Lemma 3.1. *For any fixed $\lambda_0 > \Lambda$, there exists $\delta > 0$ such that if $F_1 : C^1 \to C^1_M$ is a compact map with $\|F_1(v)\| < \delta$ for v in $U^{\tilde{\beta}}_{\tilde{\alpha}}$ and $\|F_1(w)\| < \delta$ for w in $B_{\rho_2} \setminus [\overline{U}^{\tilde{\beta}}_{\tilde{\alpha}} \cup \overline{B}_{\rho_1}]$, $v, w > 0$, then there exist $u_1 \in U^{\tilde{\beta}}_{\tilde{\alpha}}$, $u_2 \in B_{\rho_2} \setminus [\overline{U}^{\tilde{\beta}}_{\tilde{\alpha}} \cup \overline{B}_{\rho_1}]$ and $u_1, u_2 > 0$ such that $u_i = \mathcal{N}_{\lambda_0}(u_i) + F_1(u_i)$, $i = 1, 2$.*

Proof. Denote

$$S_1 := \left\{ u \in U^{\tilde{\beta}}_{\tilde{\alpha}} : u \text{ is a solution of (2.2) with } \lambda = \lambda_0 \right\}.$$

Choose $\varepsilon_1 > 0$ small and set $\overline{N}_{1,\varepsilon_1} = \bigcup_{u\in S_1} B_{\varepsilon_1}(u)$, where $B_{\varepsilon_1}(u)$ denotes the open ball of radius ε_1 in C^1_M centered at u. Obviously, $\overline{N}_{1,\varepsilon_1}$ is open. Since $\mathcal{N}_{\lambda_0} : C^1_M \to C^1_M$ is compact, there exists $\delta > 0$ such that $\|(I - \mathcal{N}_{\lambda_0})(v)\|_{C^1} > \delta$ if $v \in \partial\overline{N}_{1,\varepsilon_1}$.

Claim. if $v \in \overline{N}_{1,\varepsilon_1}$ and ε_1 is small enough, then $v > 0$ in the interval $[0, R)$.

Suppose on the contrary that there exists $v_\varepsilon \in C^1_M$, $u_\varepsilon \in S_1$ such that $\|v_\varepsilon - u_\varepsilon\| \to 0$ and $v_\varepsilon(r_0) \le 0$ for some $r_0 \in [0, R)$. Since S_1 is compact, without loss of generality, we can assume that $u_\varepsilon \to u$ (as $\varepsilon \to 0$) for some $u \in S_1$, (i.e. $v_\varepsilon \to u$). From $u \in S_1$ and the monotonicity of u, we obtain that $u > 0$ in $[0, R)$ and $u'(R) < 0$. Since $v'_\varepsilon(0) = v_\varepsilon(R) = 0$, it follows that $v_\varepsilon > 0$ in $[0, R)$ for ε small enough. This contradiction ends the proof of Claim.

If $\|F_1(v)\| < \delta$ for v in $U^{\tilde{\beta}}_{\tilde{\alpha}}$, we conclude that

$$\deg\left((I - \mathcal{N}_{\lambda_0} - F_1), \overline{N}_{1,\varepsilon_1}, 0\right) = \deg(I - \mathcal{N}_{\lambda_0}, \overline{N}_{1,\varepsilon_1}, 0) = \deg(I - \mathcal{N}_{\lambda_0}, U^{\tilde{\beta}}_{\tilde{\alpha}}, 0) = 1.$$

Similarly, let

$$S_2 := \left\{ u \in B_{\rho_2} \setminus [\overline{U}^{\tilde{\beta}}_{\tilde{\alpha}} \cup \overline{B}_{\rho_1}] : u \text{ is a solution of (2.2) with } \lambda = \lambda_0 \right\}.$$

Choose $\varepsilon_2 > 0$ small and set $\overline{N}_{2,\varepsilon_2} = \bigcup_{u\in S_2} B_{\varepsilon_2}(u)$. If $\|F_1(v)\| < \delta$ for v in $B_{\rho_2} \setminus [\overline{U}^{\tilde{\beta}}_{\tilde{\alpha}} \cup \overline{B}_{\rho_1}]$, we conclude that

$$\deg\left((I - \mathcal{N}_{\lambda_0} - F_1), \overline{N}_{2,\varepsilon_2}, 0\right) = \deg(I - \mathcal{N}_{\lambda_0}, \overline{N}_{2,\varepsilon_2}, 0) = \deg(I - \mathcal{N}_{\lambda_0}, B_{\rho_2} \setminus [\overline{U}^{\tilde{\beta}}_{\tilde{\alpha}} \cup \overline{B}_{\rho_1}], 0) = -1.$$

□

The proof of Theorem 1.1. Select δ as in Lemma 3.1. Define

$$\Lambda^* := \frac{N\varphi(\delta/(R+1))}{a_M \epsilon R},$$

where $\epsilon > 0$ is defined in (1.1) and $a_M = \max_{r\in[0,R]} a(r)$. There exist a constant $\epsilon_0(\Lambda) > 0$ such that $\Lambda < \Lambda^*$ for any $\epsilon < \epsilon_0(\Lambda)$, where $\Lambda > 2N/(a_M R^{\gamma+1})$ is the constant that appears in Theorem 1.1 of [6]. Let $\lambda_0 \in [\Lambda, \Lambda^*]$ be arbitrarily chosen.

Let

$$F_1(u)(r) = -\int_r^R \varphi^{-1}\left(\frac{\lambda_0 \epsilon}{t^{N-1}} \int_0^t \tau^{N-1} a(\tau) d\tau\right) dt.$$

With these choices, we obtain that

$$\begin{aligned}\|F_1(u)\| &= \max_{r\in[0,R]}\left|\int_r^R \varphi^{-1}\left(\frac{\lambda_0\epsilon}{t^{N-1}}\int_0^t \tau^{N-1}a(\tau)d\tau\right)dt\right| + \max_{r\in[0,R]}\left|\varphi^{-1}\left(\frac{\lambda_0\epsilon}{r^{N-1}}\int_0^r \tau^{N-1}a(\tau)d\tau\right)\right| \\ &\le \max_{r\in[0,R]}\left|\int_r^R \varphi^{-1}\left(\frac{\lambda_0 a_M\epsilon R}{N}\right)dt\right| + \varphi^{-1}\left(\frac{\lambda_0 a_M\epsilon R}{N}\right) \\ &\le \varphi^{-1}\left(\frac{\lambda_0 a_M\epsilon R}{N}\right)(R+1) < \delta.\end{aligned}$$

By Lemma 3.1, we can conclude the existence of the $\widetilde{u}_1, \widetilde{u}_2 > 0$ such that

$$\widetilde{u}_i = \int_r^R \varphi^{-1}\left(\frac{\lambda_0}{t^{N-1}}\int_0^t \tau^{N-1}a(\tau)(\widetilde{u}_i(\tau)-\epsilon)d\tau\right)dt, \qquad i=1,2.$$

This suggests that $\widetilde{u}_1, \widetilde{u}_2 > 0$ are the solutions of (2.1). The proof of Theorem 1.1 is complete. □

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References

[1] W. Allegretto, P. Nistri, P. Zecca, Positive solutions of elliptic non-positone problems, *Differential Integral Equations* **5**(1992), No. 1, 95–101. Zbl 0758.35032

[2] C. O. Alves, A. R. F. de Holanda, J. A. Santos, Existence of positive solutions for a class of semipositone quasilinear problems through Orlicz–Sobolev space, *Proc. Amer. Math. Soc.* **147**(2019), No. 1, 285–299. https://doi.org/10.1090/proc/14212; MR3876749; Zbl 1405.35049

[3] C. O. Alves, A. R. F. de Holanda, J. A. Santos, Existence of positive solutions for a class of semipositone problem in whole $\mathbb{R}^N$, *Proc. Roy. Soc. Edinburgh Sect. A* **150**(2020), No. 5, 2349–2367. https://doi.org/10.1017/prm.2019.20; MR4153614; Zbl 1459.35206

[4] A. Ambrosetti, D. Arcoya, B. Buffoni, Positive solutions for some semi-positone problems via bifurcation theory, *Differential Integral Equations* **7**(1994), No. 3–4, 655–663. MR1270096 ; Zbl 0808.35030

[5] A. Azzollini, Ground state solution for a problem with mean curvature operator in Minkowski space, *J. Funct. Anal.* **266**(2014), No. 4, 2086–2095. https://doi.org/10.1016/j.jfa.2013.10.002; MR3150152; Zbl 1305.35082

[6] C. Bereanu, P. Jebelean, P. J. Torres, Multiple positive radial solutions for a Dirichlet problem involving the mean curvature operator in Minkowski space, *J. Funct. Anal.* **256**(2013), No. 4, 644–659. https://doi.org/10.1016/j.jfa.2013.04.006; MR3062540; Zbl 1285.35051

[7] C. Bereanu, J. Mawhin, Existence and multiplicity results for some nonlinear problems with singular ϕ-Laplacian, *J. Differential Equations* **243**(2007), No. 2, 536–557. https://doi.org/10.1016/j.jde.2007.05.014; MR2371799; Zbl 1148.34013

[8] H. Berestycki, L. A. Caffarelli, L. Nirenberg, Inequalities for second order elliptic equations with applications to unbounded domains, *Duke Math. J.* **81**(1996), No. 2, 467–494. https://doi.org/10.1215/S0012-7094-96-08117-X; MR1395408; Zbl 0860.35004

[9] K. J. Brown, A. Castro, R. Shivaji, Non-existence of radially symmetric non-negative solutions for a class of semi-positone problems, *Differential and Integral Equations* **2**(1989), No. 4, 541–545. Zbl 0736.35039

[10] A. Castro, D. G. de Figueredo, E. Lopera, Existence of positive solutions for a semipositone p-Laplacian problem, *Proc. Roy. Soc. Edinburgh Sect. A* **146**(2016), No. 3, 475–482. https://doi.org/10.1017/S0308210515000657; MR3507282; Zbl 1358.35045

[11] A. Castro, R. Shivaji, Nonnegative solutions for a class of nonpositone problems, *Proc. Roy. Soc. Edinburgh Sect. A* **108**(1988), No. 3–4, 291–302. https://doi.org/10.1017/S0308210500014670; MR0943804

[12] M. Chhetri, P. Drábek, R. Shivaji, Existence of positive solutions for a class of p-Laplacian superlinear semipositone problems, *Proc. Roy. Soc. Edinburgh Sect. A* **145**(2015), No. 5, 925–936. https://doi.org/10.1017/S0308210515000220; MR3406455; Zbl 1335.35080

[13] C. Corsato, F. Obersnel, P. Omari, The Dirichlet problem for gradient dependent prescribed mean curvature equations in the Lorentz–Minkowski space, *Georgian Math. J.* **24**(2017), No. 1, 113–134. https://doi.org/10.1515/gmj-2016-0078; MR3607245; Zbl 1360.35079

[14] D. Costa, H. Ramos Quoirin, H. Tehrani, A variational approach to superlinear semipositone elliptic problems, *Proc. Amer. Math. Soc.* **145**(2017), No. 6, 2662–2675. https://doi.org/10.1090/proc/13426; MR3626519; Zbl 1367.35070

[15] G. Dai, Bifurcation and nonnegative solutions for problems with mean curvature operator on general domain, *Indiana Univ. Math. J.* **67**(2018), No. 6, 2103–2121. https://doi.org/10.1512/iumj.2018.67.7546; MR3900363; Zbl 1420.35101

[16] E. N. Dancer, J. Shi, Uniqueness and nonexistence of positive solutions to semipositone problems, *Bull. London Math. Soc.* **38**(2006), No. 6, 1033–1044. https://doi.org/10.1112/S0024609306018984; MR2285257; Zbl 1194.35164

[17] D. D. Hai, A. Muthunayake, R. Shivaji, A uniqueness result for a class of infinite semipositone problems with nonlinear boundary conditions, *Positivity* **25** (2021), No. 4, 1357–1371. https://doi.org/10.1007/s11117-021-00820-x; MR4301140; Zbl 1492.34025

[18] A. Joseph, L. Sankar, Sublinear positone and semipositone problems on the exterior of a ball in $\mathbb{R}^2$, *J. Math. Anal. Appl.* **548** (2025), No. 2, 129423, 13 pp. https://doi.org/10.1016/j.jmaa.2025.129423; MR4873731; Zbl 1562.35188

[19] R. Kajikiya, E. Ko, Existence of positive radial solutions for a semipositone elliptic equation, *J. Math. Anal. Appl.* **484** (2020), No. 2, 123735, 19 pp. https://doi.org/10.1016/j.jmaa.2019.123735; MR4040129; Zbl 1439.35177

[20] U. Kaufmann, H. Ramos Quoirin, Positive solutions of indefinite semipositone problems via sub-super solutions, *Differential Integral Equations* **31**(2018), No. 7–8, 497–506. https://doi.org/10.57262/die/1526004027; MR3801821; Zbl 1449.35200

[21] P. L. Lions, On the existence of positive solutions of semilinear elliptic equations, *SIAM Rev.* **24**(1982), No. 4, 441–461. https://doi.org/10.1137/1024101; MR0678562; Zbl 0511.35033

[22] R. Ma, H. Gao, Y. Lu, Global structure of radial positive solutions for a prescribed mean curvature problem in a ball, *J. Funct. Anal.* **270**(2016), No. 7, 2430–2455. https://doi.org/10.1016/j.jfa.2016.01.020; MR3464046; Zbl 1342.34044

[23] Q. Morris, R. Shivaji, I. Sim, Existence of positive radial solutions for a superlinear semipositone p-Laplacian problem on the exterior of a ball, *Proc. Roy. Soc. Edinburgh Sect. A* **148**(2018), No. 2, 409–428. https://doi.org/10.1017/S0308210517000452; MR3782023; Zbl 1393.35063