Electronic Journal of Qualitative Theory of Differential Equations
2026, No. **46**, 1–20; https://doi.org/10.14232/ejqtde.2026.1.46 www.math.u-szeged.hu/ejqtde/

On the logarithmic Schrödinger–Bopp–Podolsky system with steep potential

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Received 23 August 2025, appeared 29 August 2026
Communicated by Dimitri Mugnai

Abstract. This paper deals with the following logarithmic Schrödinger–Bopp–Podolsky system

$$\begin{cases} -\Delta u + \lambda V(x)u + q(x)\phi u = u\log u^2, & \text{in } \mathbb{R}^3, \\ -\Delta\phi + \Delta^2\phi = 4\pi q(x)u^2, & \text{in } \mathbb{R}^3, \end{cases}$$

where $\lambda > 0$, $V(x) \in C(\mathbb{R}^3, \mathbb{R})$ and $q(x) \geq 0$. Under suitable conditions on potentials $V(x)$ and $q(x)$, we demonstrate the existence of a positive solution $u_\lambda \in H^1(\mathbb{R}^3)$ of the above problem for $\lambda > 0$ large enough via the variational method developed by Szulkin for the functional that is the sum of a smooth and a convex lower semicontinuous term, and we also study the concentration behavior of positive solutions $\{u_\lambda\}$ as $\lambda \to +\infty$.

Keywords: Schrödinger–Bopp–Podolsky system, logarithmic nonlinearity, variational methods, concentration behavior.

2020 Mathematics Subject Classification: 35A15, 35J10, 35Q60.

1 Introduction

In the past few years, the Schrödinger–Bopp–Podolsky system

$$\begin{cases} -\Delta u + V(x)u + K(x)\phi u = f(x,u), & \text{in } \mathbb{R}^3, \\ -\Delta\phi + a^2\Delta^2\phi = 4\pi K(x)u^2, & \text{in } \mathbb{R}^3, \end{cases} \tag{1.1}$$

where $a > 0$, $K(x) > 0$ and u, $\phi : \mathbb{R}^3 \to \mathbb{R}$, has been extensively studied by many authors. Such a system appears when coupling a Schrödinger field $\psi = \psi(t,x)$ with its electromagnetic field. For further physical explanation, see [4,12,24]. In 2019, d'Avenia–Siciliano [8] initially studied system (1.1) and obtained the existence and nonexistence of solutions, where $V(x)$ is a positive constant, $K(x) = q^2$ and $f(x,u) = |u|^{p-2}u$ for $p \in (2,6)$. Besides, they proved the asymptotical behavior of solutions as $a \to 0$. Subsequently, more and more scholars have become interested in this system, and they devoted themselves to exploring the existence

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of positive solutions, ground state solutions, multiple solutions and normalized solutions by setting different hypotheses on the potential $V(x)$ and the nonlinear term $f(x,u)$, respectively.

Li–Pucci–Tang [17] generalized the existing results for system (1.1) to the critical case, assuming $K(x) = q^2$ for $q \neq 0$ and $f(x,u) = \mu |u|^{p-1} u + |u|^4 u$ for $p \in (2,5)$, and used the Pohožaev–Nehari manifold method to prove the existence of a ground state solution. Chen–Tang [5] improved the nonlinear term in [17] to a general nonlinear term, i.e., $f(x,u) = \mu g(u) + u^5$, and adopted some new analytical techniques and new inequalities to prove the existence of ground state solutions for any $\mu > 0$ if $p \in (4,6)$, and for any $\mu > \mu_0$ if $p \in (2,4]$. Yang–Chen–Liu [27] assumed that $V(x)$ is coercive, $K(x) = 1$, $f(x,u) = \lambda g(u) + |u|^4 u$ and proved the existence of a nontrivial solution via using a cut-off function, the mountain pass theorem and Moser iteration.

Later, Jia–Li–Chen [16] established the existence of a positive solution for system (1.1), where $V(x) = 1$, $K(x) = \lambda q(x)$ and $f(x,u) = b(x) |u|^{p-2} u$ for $p \in (2,4]$. They also obtained ground state solutions for the case $3.18 \approx \frac{1+\sqrt{73}}{3} < p \leq 4$. Liu–Chen [19] investigated the existence, nonexistence and asymptotic behavior of ground state solutions for problem (1.1) with asymptotically constant potential, $K(x) = 1$ and $f(x,u) = \lambda q(x)g(u) + |u|^4 u$. Zhang [29] studied the existence of a sign-changing solution for system (1.1), where $V(x)$ is coercive, $K(x) = 1$ and $f(x,u) = g(u)$, in particular, a sequence of high energy sign-changing solutions was obtained if g is odd. Moreover, they also considered the asymptotic behavior of the solution as $a \to 0$. Yang–Yuan–Liu [28] focused on system (1.1) and assumed $V(x)$ is asymptotically periodic potential, $K(x) = 1$ and $f(x,u) = Q(x) |u|^{p-2} u$ for $p \in [4,6)$; as a consequence, they proved the existence of ground states. For more information on this system, see [6,13,20,21,30] and the references therein.

Recently, the Schrödinger problem with logarithmic terms expressed as

$$i\varepsilon \frac{\partial \Phi}{\partial t} = -\varepsilon^2 \Delta \Phi + W(x)\Phi - \Phi \log |\Phi|^2, \quad N \geq 3, \tag{NLS}$$

with $\Phi : [0,+\infty) \times \mathbb{R}^N \to \mathbb{C}$, has attracted much attention due to its special properties and has appeared in some physical phenomena, such as quantum mechanics, quantum optics, nuclear physics, effective quantum and Bose–Einstein condensation (see [31]). The standing wave solution for (*NLS*) possesses the form $\Phi(t,x) = u(x)e^{-i\omega t/\varepsilon}$, where $\omega \in \mathbb{R}$. This substitution transforms equation (*NLS*) into

$$-\varepsilon^2 \Delta u + V(x)u = u \log u^2, \quad \text{in } \mathbb{R}^N, \tag{1.2}$$

where $V(x) = W(x) - \omega$. We find this class of problems particularly remarkable because the associated energy functional

$$\tilde{J}_\varepsilon(u) = \frac{1}{2} \int_{\mathbb{R}^N} \left(\varepsilon^2 |\nabla u|^2 + (V(x)+1)u^2\right) \mathrm{d}x - \frac{1}{2} \int_{\mathbb{R}^N} u^2 \log u^2 \, \mathrm{d}x$$

lacks smoothness. As a result, we cannot discuss the existence of solutions using general critical point theory, which poses a considerable challenge for us. To overcome this difficulty, many scholars have conducted research on it and proposed some solutions. In [7], the authors employed the non-smooth critical point theory developed by Degiovanni–Zani [10] to establish the existence of multiple radial solutions for equation (1.2) when $\varepsilon \equiv 1$ and $V(x) \equiv 1$. d'Avenia–Squassina–Zenari [9] extended the same non-smooth critical point approach to investigate a fractional version of the logarithmic Schrödinger equation. By using the minimax principles for lower semicontinuous functionals developed by Szulkin [26], Squassina–Szulkin

[25] obtained the existence of multiple solutions of problem (1.2) when the right-hand side of (1.2) is $Q(x)u\log u^2$, and $V, Q : \mathbb{R}^N \to \mathbb{R}$ are 1-periodic continuous functions.

In the recent work of Ji–Szulkin [15], inspired by [25], they considered the existence of multiple solutions for problem (1.2) with $\varepsilon = 1$. When $V(x)$ is coercive, they obtained infinitely many solutions, and in the case of bounded potential they obtained a ground state solution. Alves–de Morais Filho [1] studied equation (1.2) and assumed $V(x)$ to be an asymptotically constant potential. Consequently, they proved the existence of a positive solution for small $\varepsilon > 0$ and concentration of the solution as $\varepsilon \to 0$. Moreover, they also discussed the cases where $V(x)$ is periodic or asymptotically periodic. Alves–Ji [3] continued to study problem (1.2) when $V(x)$ satisfies some local conditions. They proved the existence and concentration of positive solutions via combining the variational method with a penalization scheme. For the steep potential case, Alves–de Morais Filho–Figueiredo [2] established the existence of one positive solution of the problem as follows

$$-\Delta u + \lambda V(x)u = u\log u^2, \quad \text{in } \mathbb{R}^N, \tag{1.3}$$

where $\lambda > 0$ and $V(x) \geq 0$. Moreover, they investigated the concentration behavior of solutions as $\lambda \to +\infty$. Later, Ji [14] obtained at least $2^k - 1$ nodal solutions of problem (1.3) by using a penalization method, and they also verified that the solutions possess the concentration phenomena as $\lambda \to +\infty$.

More recently, inspired by [2], Peng–Jia [23] studied the existence and concentration behavior of solutions for the logarithmic Schrödinger–Poisson system

$$\begin{cases} -\Delta u + \lambda V(x)u + q(x)\phi u = u\log u^2, & \text{in } \mathbb{R}^3, \\ -\Delta\phi = q(x)u^2, & \text{in } \mathbb{R}^3, \end{cases} \tag{1.4}$$

where $\lambda > 0$, V and q satisfy the following conditions, respectively:

$(V1)$ $V(x) \in C\left(\mathbb{R}^3, \mathbb{R}\right)$ with $V(x) \geq 0$, and there exists $V_\infty > 0$ such that the set

$$\mathcal{V}_\infty = \left\{x \in \mathbb{R}^3 : V(x) < V_\infty\right\}$$

is non-empty and has finite measure;

$(V2)$ $\Lambda := \operatorname{int}\left\{x \in \mathbb{R}^3 : V(x) = 0\right\}$ is a non-empty bounded open set with smooth boundary

$$\bar{\Lambda} = \left\{x \in \mathbb{R}^3 : V(x) = 0\right\};$$

$(V3)$ $\operatorname{dist}\left(\Lambda, \mathbb{R}^3 \setminus \mathcal{V}_\infty\right) = \inf\left\{|x-y| : x \in \Lambda, y \in \mathbb{R}^3 \setminus \mathcal{V}_\infty\right\} > 0$;

(Q) $q(x) \geq 0$ for all $x \in \mathbb{R}^3$, $q(x) \not\equiv 0$ with $q(x) \in L^2\left(\mathbb{R}^3\right)$, and

$$\Lambda_q := \operatorname{int}\left\{x \in \mathbb{R}^3 : q(x) = 0\right\}$$

is a non-empty open set and $\operatorname{dist}\left(\Lambda, \mathbb{R}^3 \setminus \Lambda_q\right) > 0$.

Motivated by the above literature, in this article, we consider the existence and concentration behavior of a positive solution for the following logarithmic Schrödinger–Bopp–Podolsky system with steep potential:

$$\begin{cases} -\Delta u + \lambda V(x)u + q(x)\phi u = u\log u^2, & \text{in } \mathbb{R}^3, \\ -\Delta\phi + \Delta^2\phi = 4\pi q(x)u^2, & \text{in } \mathbb{R}^3, \end{cases} \tag{1.5}$$

where $\lambda > 0$ describes the steepness of the potential, V and q satisfy $(V1) - (V3)$ and (Q), respectively. Here we point out that there is a certain gap in the research on such problems addressed in reference [23]. To address this, our paper will adopt a completely different proof method from reference [23] to establish theoretical results. Specifically, when verifying the geometric structure of the mountain pass theorem, there is no need to truncate the Bopp–Podolsky term when handling the interaction between the Bopp–Podolsky term and the logarithmic term. Instead, we directly control the Bopp–Podolsky term using the steep potential condition and (Q), ensuring that the energy functional tends to negative infinity at infinity (see Lemma 3.3).

Now, we state the main results.

Theorem 1.1. *Assume that $V(x)$ and $q(x)$ satisfy the conditions $(V1)$–$(V3)$ and (Q), respectively. Then, there exists $\lambda^* > 0$ such that, for all $\lambda \geq \lambda^*$, the system (1.5) has a positive solution $(u_\lambda, \phi_{u_\lambda}) \in H^1(\mathbb{R}^3) \times \mathcal{D}^{1,2}(\mathbb{R}^3)$ (see Section 2).*

Theorem 1.2. *Assume that $V(x)$ and $q(x)$ satisfy the conditions $(V1)$–$(V3)$ and (Q), respectively. If $(u_\lambda, \phi_{u_\lambda}) \in H^1(\mathbb{R}^3) \times \mathcal{D}^{1,2}(\mathbb{R}^3)$ is the solution of problem (1.5) found in Theorem 1.1, then*

$$(u_\lambda, \phi_{u_\lambda}) \to (u_\Lambda, 0) \in H^1(\mathbb{R}^3) \times \mathcal{D}^{1,2}(\mathbb{R}^3)$$

as $\lambda \to +\infty$, where u_Λ is a nontrivial solution of the equation

$$\begin{cases} -\Delta u = u \log u^2, & \text{in } \Lambda, \\ u = 0, & \text{on } \partial\Lambda \end{cases} \tag{S}$$

and Λ is defined in $(V2)$.

Remark 1.3. Our approach is based on a new version of the minimax method developed in [26].

(1) Firstly, due to the existence of the logarithmic term, the energy functional $\mathcal{I}_\lambda(u)$ (see (2.3)) is not of class C^1. Therefore, we will make a technical decomposition of $\mathcal{I}_\lambda(u)$ into the sum of a smooth term and a convex lower semicontinuous term. In this case, we can apply the mountain pass theorem without (PS) condition, which was first mentioned in [1].

(2) Secondly, since the logarithmic nonlinear term is sign-changing, which neither satisfies the monotonicity condition nor the Ambrosetti–Rabinowitz condition, proving the boundedness of the (PS) sequence poses significant difficulties. To address this challenge, we will utilize the logarithmic inequality introduced in [11].

(3) Finally, owing to the lack of compactness in the whole space $\mathbb{R}^3$, we need to invoke the Vanishing Lemma (see Lemma 4.1) to recover the compactness properties of the energy functional $\mathcal{I}_\lambda(u)$ (see (2.3)).

The paper is organized as follows: In Section 2, we give some preliminary knowledge which will be frequently used later, and show some technical results. In Section 3, we introduce the mountain pass theorem without (PS) condition developed by Alves–de Morais Filho [1]. In Section 4, we focus on completing the proof of the main theorem.

Notation. In this paper, we adopt the following notations:

- $L^s(\mathbb{R}^3)$ $(1 \le s \le \infty)$ denotes the Lebesgue space with the norm

$$\|u\|_s = \left(\int_{\mathbb{R}^3} |u|^s \, \mathrm{d}x\right)^{1/s}, \quad \text{for } 1 \le s < \infty$$

  and

$$\|u\|_\infty := \inf\left\{C > 0 : |u(x)| \le C \text{ almost everywhere on } \mathbb{R}^3\right\}, \quad \text{for } s = \infty.$$

- $B_r(x)$ denotes the ball centered at x with radius r.
- C, C_i $(i = 1, 2, \dots)$ denote positive constants, which may vary in different places.
- $o_n(1)$ denotes a real sequence with $o_n(1) \to 0$ as $n \to \infty$.
- $H_c^1\left(\mathbb{R}^3\right) = \left\{u \in H^1\left(\mathbb{R}^3\right) \mid u \text{ has a compact support }\right\}$.

2 Preliminaries

We start with some preliminary basic results. Let $H^1(\mathbb{R}^3)$ denote the Sobolev space equipped with the norm

$$\|u\|_H = \left(\int_{\mathbb{R}^3} \left(|\nabla u|^2 + u^2\right) \mathrm{d}x\right)^{\frac{1}{2}},$$

and $\mathcal{D}^{1,2}\left(\mathbb{R}^3\right) = \left\{u \mid u \in L^6\left(\mathbb{R}^3\right), \nabla u \in L^2\left(\mathbb{R}^3\right)\right\}$ be endowed with the norm

$$\|u\|_{\mathcal{D}^{1,2}} = \left(\int_{\mathbb{R}^3} |\nabla u|^2 \, \mathrm{d}x\right)^{\frac{1}{2}}.$$

Let $\mathcal{D}$ be the completion of $C_0^\infty\left(\mathbb{R}^3\right)$ with respect to the norm $\|\cdot\|_{\mathcal{D}}$ induced by the scalar product

$$\langle \phi, \xi \rangle_{\mathcal{D}} := \int_{\mathbb{R}^3} \nabla\phi \nabla\xi \mathrm{d}x + \int_{\mathbb{R}^3} \Delta\phi \Delta\xi \mathrm{d}x.$$

It is easy to see that $\mathcal{D}$ is a Hilbert space and $\mathcal{D} \hookrightarrow \mathcal{D}^{1,2}\left(\mathbb{R}^3\right) \hookrightarrow L^6\left(\mathbb{R}^3\right)$. Moreover, $\mathcal{D} \hookrightarrow L^\infty\left(\mathbb{R}^3\right)$ (see Lemma 3.1 in [8]).

Next, we list the basic lemma.

Lemma 2.1 ([8, Lemma 3.2]). *The space $C_0^\infty\left(\mathbb{R}^3\right)$ is dense in*

$$\mathcal{A} := \left\{\phi \in \mathcal{D}^{1,2}\left(\mathbb{R}^3\right) : \Delta\phi \in L^2\left(\mathbb{R}^3\right)\right\}$$

normed by $\sqrt{\langle \phi, \phi \rangle_{\mathcal{D}}}$ and, therefore, $\mathcal{D} = \mathcal{A}$.

In view of the Riesz Theorem, for fixed $u \in H^1\left(\mathbb{R}^3\right)$, the equation

$$-\Delta\phi + \Delta^2\phi = 4\pi q(x) u^2$$

has a unique solution $\phi_u \in \mathcal{D}$ given by

$$\phi_u = \frac{1 - \mathrm{e}^{-|x|}}{|x|} * \left(q(x) u^2\right). \tag{2.1}$$

Moreover, ϕ_u has the following fundamental properties.

Lemma 2.2 ([8, Lemma 3.4] and [27, Lemma 2.2]). *For every $u \in H^1\left(\mathbb{R}^3\right) \setminus \{0\}$ we have:*

(i) $\phi_u \geq 0$ *in* $\mathbb{R}^3$;

(ii) $\phi_{u(\cdot+y)} = \phi_u(\cdot + y)$, $\forall y \in \mathbb{R}^3$;

(iii) $\phi_u \in L^s\left(\mathbb{R}^3\right) \cap C_0\left(\mathbb{R}^3\right)$, $\forall s \in (3, +\infty]$;

(iv) $\|\phi_u\|_{\mathcal{D}} \leq C\|u\|_H^2$, $\int_{\mathbb{R}^3} \phi_u u^2 \,\mathrm{d}x \leq C\|u\|_{12/5}^4$;

(v) *if* $u_n \rightharpoonup u$ *in* $H^1\left(\mathbb{R}^3\right)$, *then* $\phi_{u_n} \rightharpoonup \phi_u$ *in* $\mathcal{D}$;

(vi) *if* $u_n \to u$ *in* $L^{\frac{12}{5}}\left(\mathbb{R}^3\right)$, *then* $\phi_{u_n} \to \phi_u$ *in* $\mathcal{D}$ *and* $\int_{\mathbb{R}^3} \phi_{u_n} u_n^2 \,\mathrm{d}x \to \int_{\mathbb{R}^3} \phi_u u^2 \,\mathrm{d}x$;

(vii) ϕ_u *is the unique minimizer of the functional*

$$X(\phi) = \frac{1}{2}\|\nabla\phi\|_2^2 + \frac{1}{2}\|\Delta\phi\|_2^2 - 4\pi \int_{\mathbb{R}^3} q(x)\phi u^2 \,\mathrm{d}x, \quad \phi \in \mathcal{D}.$$

In order to find the positive solution of system (1.5), we use the form of a reduced functional introduced in [8] to write system (1.5) as the following equation

$$-\Delta u + \lambda V(x)u + q(x)\phi_u u = u^+ \log\left|u^+\right|^2, \quad \text{in } \mathbb{R}^3. \tag{2.2}$$

The associated functional is

$$\begin{aligned} \mathcal{I}_\lambda(u) := &\frac{1}{2}\int_{\mathbb{R}^3} \left(|\nabla u|^2 + (\lambda V(x) + 1)\, u^2\right) \mathrm{d}x \\ &+ \frac{1}{4}\int_{\mathbb{R}^3} q(x)\phi_u u^2 \,\mathrm{d}x - \frac{1}{2}\int_{\mathbb{R}^3} \left|u^+\right|^2 \log\left|u^+\right|^2 \,\mathrm{d}x \end{aligned} \tag{2.3}$$

for all $u \in E_\lambda$, where

$$E_\lambda := \left\{u \in H^1\left(\mathbb{R}^3\right) : \int_{\mathbb{R}^3} \lambda V(x) u^2 \mathrm{d}x < +\infty\right\}$$

is endowed with the norm

$$\|u\|_\lambda = \left(\int_{\mathbb{R}^3} \left(|\nabla u|^2 + (\lambda V(x) + 1)u^2\right) \mathrm{d}x\right)^{\frac{1}{2}}.$$

In view of $(V1)$, E_λ is a Hilbert space, and the embedding $E_\lambda \hookrightarrow H^1\left(\mathbb{R}^3\right)$ is continuous for all $\lambda > 0$. Then, for any $s \in [2, 6]$, the embedding $E_\lambda \hookrightarrow L^s\left(\mathbb{R}^3\right)$ is continuous and there exists $C > 0$ such that for all $u \in E_\lambda$,

$$\|u\|_s \leq C\|u\|_H \leq C\|u\|_\lambda.$$

Now, we give the definition of the weak solution of problem (2.2).

Definition 2.3. A weak solution for problem (2.2) means $u \in E_\lambda$ satisfying $u^2 \log u^2 \in L^1\left(\mathbb{R}^3\right)$ (i.e., $\mathcal{I}_\lambda(u) < +\infty$) and

$$\int_{\mathbb{R}^3} \left(\nabla u \nabla \zeta + \lambda V(x) u\zeta + q(x)\phi_u u\zeta\right) \mathrm{d}x = \int_{\mathbb{R}^3} u\zeta \log u^2 \,\mathrm{d}x, \quad \forall \zeta \in C_0^\infty\left(\mathbb{R}^3\right).$$

From the previously mentioned content, we know that the functional $\mathcal{I}_\lambda$ is not of class C^1. Therefore, according to the research techniques in [15], we need to decompose $\mathcal{I}_\lambda$ into a sum of a C^1 functional plus a convex lower semicontinuous functional. For $\delta > 0$ sufficiently small, we set:

$$F_1(\kappa) = \begin{cases} 0, & \kappa = 0, \\ -\frac{1}{2}\kappa^2 \log \kappa^2, & 0 < |\kappa| < \delta, \\ -\frac{1}{2}\kappa^2 \left(\log \delta^2 + 3\right) + 2\delta|\kappa| - \frac{1}{2}\delta^2, & |\kappa| \geq \delta \end{cases}$$

and

$$F_2(\kappa) = \begin{cases} 0, & |\kappa| < \delta, \\ \frac{1}{2}\kappa^2 \log\left(\kappa^2/\delta^2\right) + 2\delta|\kappa| - \frac{3}{2}\kappa^2 - \frac{1}{2}\delta^2, & |\kappa| \geq \delta. \end{cases}$$

Then,

$$F_2(\kappa) - F_1(\kappa) = \frac{1}{2}\kappa^2 \log \kappa^2, \quad \forall \kappa \in \mathbb{R} \tag{2.4}$$

and the functional $\mathcal{I}_\lambda : E_\lambda \to (-\infty, +\infty]$ takes the form

$$\mathcal{I}_\lambda(u) = \Phi_\lambda(u) + \Psi(u), \quad u \in E_\lambda, \tag{2.5}$$

where

$$\Phi_\lambda(u) := \frac{1}{2}\int_{\mathbb{R}^3} \left(|\nabla u|^2 + (\lambda V(x) + 1)u^2\right) \, \mathrm{d}x + \frac{1}{4}\int_{\mathbb{R}^3} q(x)\phi_u u^2 \, \mathrm{d}x - \int_{\mathbb{R}^3} F_2(u^+) \, \mathrm{d}x$$

and

$$\Psi(u) := \int_{\mathbb{R}^3} F_1(u^+) \, \mathrm{d}x.$$

According to [15], for fixed $\delta > 0$, we can observe that F_1 and F_2 have the properties as follows:

- $F_1, F_2 \in C^1(\mathbb{R}, \mathbb{R})$.
- If $\delta > 0$ small, then F_1 is convex, even,
$$F_1(\kappa) \geq 0 \quad \text{and} \quad F_1'(\kappa)\kappa \geq 0, \quad \forall \kappa \in \mathbb{R}.$$
- $F_2'(\kappa) \geq 0$ for $\kappa > 0$ and $F_2'(\kappa) > 0$ for $\kappa > \delta$, and for any $p \in (2, 6)$, there exists $C > 0$ such that
$$\left|F_2'(\kappa)\right| \leq C|\kappa|^{p-1}, \quad \forall \kappa \in \mathbb{R}. \tag{2.6}$$
- $\kappa \mapsto \frac{F_2'(\kappa)}{\kappa}$ is nondecreasing for $\kappa > 0$ and strictly increasing for $\kappa > \delta$. Moreover, $\lim_{\kappa \to +\infty} \frac{F_2'(\kappa)}{\kappa} = +\infty$.

Referring to [15], we can get that Ψ is nonnegative, convex and lower semicontinuous, and $\Phi_\lambda \in C^1(E_\lambda, \mathbb{R})$.

Now, we shall need the following definitions, essentially taken from [26]:

Definition 2.4. Let E be a Banach space, E^* be the dual space of E and $\langle \cdot, \cdot \rangle$ be the duality pairing between E and E^*. Let $\mathcal{I} : E \to \mathbb{R}$ be a functional and $\mathcal{I}(u) = \Phi(u) + \Psi(u)$, where $\Phi \in C^1(E, \mathbb{R})$ and Ψ is convex and lower semicontinuous. Then the following results are given:

(i) The set $D(\mathcal{I}) := \{u \in E \colon \mathcal{I}(u) < +\infty\}$ is called the effective domain of $\mathcal{I}$.

(ii) The sub-differential $\partial\mathcal{I}(u)$ of the functional $\mathcal{I}$ at a point $u \in E$ is the following set

$$\left\{w \in E^* : \left\langle \Phi'(u), v-u \right\rangle + \Psi(v) - \Psi(u) \geq \langle w, v-u \rangle,\ \forall v \in E \right\}.$$

(iii) $u \in E$ is a critical point of $\mathcal{I}$ such that $u \in D(\mathcal{I})$ and $0 \in \partial\mathcal{I}(u)$, namely,

$$\left\langle \Phi'(u), v-u \right\rangle + \Psi(v) - \Psi(u) \geq 0, \quad \forall v \in E.$$

(iv) A Palais–Smale sequence at level c for $\mathcal{I}$ is a sequence $\{u_n\} \subset E$ such that $\mathcal{I}(u_n) \to c$ and there exists a numerical sequence $\tau_n \to 0^+$ with

$$\left\langle \Phi'(u_n), v-u_n \right\rangle + \Psi(v) - \Psi(u_n) \geq -\tau_n \|v-u_n\|, \quad \forall v \in E. \tag{2.7}$$

(v) The functional $\mathcal{I}$ satisfies the Palais–Smale condition at level c ($(PS)_c$ condition, for short) if all Palais–Smale sequences of $\mathcal{I}$ have convergent subsequences in E.

For further discussion, we now show some useful information. For each $u \in D(\mathcal{I}_\lambda)$, we define $\mathcal{I}'_\lambda(u) : H^1_c(\mathbb{R}^3) \to \mathbb{R}$ with

$$\left\langle \mathcal{I}'_\lambda(u), z \right\rangle = \left\langle \Phi'_\lambda(u), z \right\rangle + \int_{\mathbb{R}^3} F'_1(u^+) z \,\mathrm{d}x, \quad \forall z \in H^1_c(\mathbb{R}^3)$$

and

$$\left\|\mathcal{I}'_\lambda(u)\right\| := \sup\left\{\left\langle \mathcal{I}'_\lambda(u), z \right\rangle : z \in H^1_c(\mathbb{R}^3) \text{ with } \|z\|_\lambda \leq 1\right\}.$$

If $\left\|\mathcal{I}'_\lambda(u)\right\| < +\infty$, then $\mathcal{I}'_\lambda(u)$ may be extended to a bounded operator in E_λ, and it can also be considered as an element of E^*_λ.

Similar to [22], we present some useful results that can better assist us in addressing the research problem.

Lemma 2.5. *Suppose that $\mathcal{I}_\lambda(u)$ satisfies (2.5). Then*

(i) if $u \in D(\mathcal{I}_\lambda)$ is a critical point of $\mathcal{I}_\lambda$, we have

$$\left\langle \Phi'_\lambda(u), v-u \right\rangle + \Psi(v) - \Psi(u) \geq 0, \quad \forall v \in E_\lambda$$

namely,

$$\int_{\mathbb{R}^3} \nabla u \nabla(v-u)\mathrm{d}x + \int_{\mathbb{R}^3} (\lambda V(x)+1)u(v-u)\mathrm{d}x + \int_{\mathbb{R}^3} q(x)\phi_u u(v-u)\,\mathrm{d}x$$
$$+ \int_{\mathbb{R}^3} F_1(v)\mathrm{d}x - \int_{\mathbb{R}^3} F_1(u)\mathrm{d}x \geq \int_{\mathbb{R}^3} F'_2(u)(v-u)\mathrm{d}x, \quad \forall v \in E_\lambda;$$

(ii) for every $u \in D(\mathcal{I}_\lambda)$ such that $\left\|\mathcal{I}'_\lambda(u)\right\| < +\infty$, we have $\partial\mathcal{I}_\lambda(u) \neq \emptyset$, i.e., there is $w \in E^*_\lambda$, which is denoted by $w = \mathcal{I}'_\lambda(u)$, such that

$$\left\langle \Phi'_\lambda(u), v-u \right\rangle + \int_{\mathbb{R}^3} F_1(v)\mathrm{d}x - \int_{\mathbb{R}^3} F_2(u)\mathrm{d}x \geq \langle w, v-u \rangle, \quad \forall v \in E_\lambda;$$

(iii) if $u \in D(\mathcal{I}_\lambda)$ is a critical point of $\mathcal{I}_\lambda$, then (u, ϕ_u) is a solution of equation (2.2);

(iv) if $\{u_n\} \subset E_\lambda$ *is a Palais–Smale sequence, then*

$$\left\langle \mathcal{I}_\lambda'\left(u_n\right), z\right\rangle = o_n(1)\|z\|_\lambda, \quad \forall z \in H_c^1\left(\mathbb{R}^3\right);$$

(v) if Ω *is a bounded domain with regular boundary, then* Ψ *(and hence* $\mathcal{I}_\lambda$*) is of class* C^1 *in* $H^1(\Omega)$*. Precisely, the functional*

$$\Psi(u) = \int_\Omega F_1(u)\mathrm{d}x, \quad \forall u \in H^1(\Omega)$$

belongs to $C^1\left(H^1(\Omega), \mathbb{R}\right)$*.*

Based on the above properties, we obtain the following result.

Lemma 2.6. *If* $u \in D(\mathcal{I}_\lambda)$ *and* $\left\|\mathcal{I}_\lambda'(u)\right\| < +\infty$*, then* $F_1'(u)u \in L^1(\mathbb{R}^3)$*.*

Proof. This proof can be modified from Lemma 2.2 in [1], which we omit here. □

Furthermore, for each $u \in D(\mathcal{I}_\lambda)$, we can directly set the functional $\mathcal{I}_\lambda'(u) : E_\lambda \to \mathbb{R}$ given by

$$\begin{aligned}
\mathcal{I}_\lambda'(u)u = & \int_{\mathbb{R}^3} \left(|\nabla u|^2 + (\lambda V(x)+1)u^2\right) \mathrm{d}x + \int_{\mathbb{R}^3} q(x)\phi_u u^2 \mathrm{d}x \\
& - \int_{\mathbb{R}^3} F_2'(u^+)u^+ \mathrm{d}x + \int_{\mathbb{R}^3} F_1'(u^+)u^+ \mathrm{d}x \\
= & \int_{\mathbb{R}^3} \left(|\nabla u|^2 + \lambda V(x)u^2\right) \mathrm{d}x + \int_{\mathbb{R}^3} q(x)\phi_u u^2 \mathrm{d}x - \int_{\mathbb{R}^3} \left|u^+\right|^2 \log\left|u^+\right|^2 \mathrm{d}x.
\end{aligned} \tag{2.8}$$

3 Mountain pass theorem without (*PS*) condition

In this section, we will introduce an abstract theorem for the functional of the type $\mathcal{I}(u) = \Phi(u) + \Psi(u)$, developed by Alves–de Morais Filho [1], where $\Phi \in C^1$ and Ψ is convex and lower semicontinuous.

Theorem 3.1 ([1])**.** *Let E be a Banach space and* $\mathcal{I}\colon E \to (-\infty, +\infty]$ *be a functional such that:*

(i) $\mathcal{I}(u) = \Phi(u) + \Psi(u)$*, where* $\Phi(u) \in C^1(E, \mathbb{R})$ *and* $\Psi\colon E \to (-\infty, +\infty]$ *is convex, lower semicontinuous and* $\Psi(u) \not\equiv +\infty$*;*

(ii) $\mathcal{I}(0) = 0$ *and* $\mathcal{I}|_{\partial B_r(0)} \geq b$ *for some* $b, r > 0$*;*

(iii) $\mathcal{I}(e) \leq 0$ *for some* $e \notin \overline{B}_r(0)$*.*

Then for fixed $\tau > 0$*, there is* $u_\tau \in E$ *satisfying*

$$\left\langle \Phi'\left(u_\tau\right), v - u_\tau\right\rangle + \Psi(v) - \Psi\left(u_\tau\right) \geq -3\tau\left\|v - u_\tau\right\|, \quad \forall v \in E$$

and

$$\mathcal{I}\left(u_\tau\right) \in \left[c - \tau, c + \tau\right],$$

where

$$c := \inf_{\gamma \in \Gamma} \max_{t \in [0,1]} \mathcal{I}(\gamma(t)) \tag{3.1}$$

and

$$\Gamma = \{\gamma \in C([0,1], E) : \gamma(0) = 0, \mathcal{I}(\gamma(1)) < 0\}.$$

Corollary 3.2. *From Theorem 3.1, it is clear that there exists a $(PS)_c$ sequence $\{u_n\} \subset E$ for $\mathcal{I}$, namely, $\mathcal{I}(u_n) \to c$ and*

$$\left\langle \Phi'(u_n), v - u_n \right\rangle + \Psi(v) - \Psi(u_n) \geq -\tau_n \|v - u_n\|, \quad \forall v \in E$$

with $\tau_n \to 0^+$.

In the following, we will apply Theorem 3.1 and Corollary 3.2 to obtain our results. The most crucial point is to prove that $\mathcal{I}_\lambda(u)$ satisfies the (PS) condition. Firstly, we verify that $\mathcal{I}_\lambda$ possesses the mountain pass geometry.

Lemma 3.3. *Suppose that $(V1)$–$(V3)$ and (Q) are valid. Then*

(i) $\mathcal{I}_\lambda(0) = 0$;

(ii) there are b_0, $r_0 > 0$ satisfying $\mathcal{I}_\lambda(u) \geq b_0$, where $\|u\|_\lambda = r_0$;

(iii) there is $\tilde{e} \in \mathbb{R}^3 \setminus B_{r_0}(0)$ satisfying $\mathcal{I}_\lambda(\tilde{e}) < 0$, where r_0 is given in (i).

Proof. (i) Indeed, it is clear that $\mathcal{I}_\lambda(0) = 0$.

(ii) According to (2.5) and (2.6) for $p \in (2, 6)$, together with $F_1 \geq 0$ and $\phi_u \geq 0$, we find

$$\begin{aligned} \mathcal{I}_\lambda(u) &= \frac{1}{2}\|u\|_\lambda^2 + \frac{1}{4}\int_{\mathbb{R}^3} q(x)\phi_u u^2 \mathrm{d}x - \int_{\mathbb{R}^3} F_2(u^+)\,\mathrm{d}x + \int_{\mathbb{R}^3} F_1(u^+)\,\mathrm{d}x \\ &\geq \frac{1}{2}\|u\|_H^2 - C\|u\|_H^p \geq b_0 > 0, \end{aligned}$$

where $b_0 > 0$ and $\|u\|_H \leq \|u\|_\lambda = r_0$ is small enough.

(iii) In view of $(V3)$ and (Q), there holds $\Lambda \subset \left(\mathcal{V}_\infty \cap \Lambda_q\right)$. Now, let $0 < \tilde{u} \in C_0^\infty(\Lambda)$ and $t > 0$. By (2.3), $(V2)$ and (Q), there holds

$$\begin{aligned} \mathcal{I}_\lambda(t\tilde{u}) &= \frac{t^2}{2}\|\tilde{u}\|_\lambda^2 + \frac{t^4}{4}\int_{\mathbb{R}^3} q(x)\phi_{\tilde{u}}\tilde{u}^2\,\mathrm{d}x - \frac{1}{2}\int_{\mathbb{R}^3} t^2\tilde{u}^2\log\left(|t\tilde{u}|^2\right)\,\mathrm{d}x \\ &= \frac{t^2}{2}\int_{\mathbb{R}^3}\left(|\nabla\tilde{u}|^2 + \tilde{u}^2\right)\,\mathrm{d}x - \frac{1}{2}\int_{\mathbb{R}^3} t^2\tilde{u}^2\log\left(|t\tilde{u}|^2\right)\,\mathrm{d}x \\ &= t^2\left(\frac{1}{2}\|\tilde{u}\|_H^2 - \frac{1}{2}\int_{\mathbb{R}^3}\tilde{u}^2\log\tilde{u}^2\,\mathrm{d}x - \log t\int_{\mathbb{R}^3}\tilde{u}^2\,\mathrm{d}x\right) \to -\infty, \end{aligned}$$

as $t \to +\infty$. Then, we can certainly find $\tilde{e} = t_*\tilde{u}$ such that $\|\tilde{e}\|_\lambda > r_0$ for $t_* > 0$ large enough and $\mathcal{I}_\lambda(\tilde{e}) < 0$. The proof is complete. □

Remark 3.4. According to Theorem 3.1 and Lemma 3.3, there exists a (PS) sequence $\{u_n\} \subset E_\lambda$ of $\mathcal{I}_\lambda(u)$ at the level $c_\lambda > 0$, where

$$c_\lambda = \inf_{\gamma\in\Gamma}\max_{t\in[0,1]} \mathcal{I}_\lambda(\gamma(t)) \tag{3.2}$$

and

$$\Gamma = \left\{\gamma \in C\left([0,1], E_\lambda\right) : \gamma(0) = 0, \gamma(1) = \tilde{e}\right\}.$$

The next lemma gives both lower and upper bounds for the value c_λ, which will be important in later analysis.

Lemma 3.5. *There exist positive constants b_1 and b_2, independent of λ, such that*

$$b_1 \leq c_\lambda \leq b_2, \quad \forall \lambda > 0.$$

Proof. Based on Lemma 3.3-(ii), it is easy to find $b_1 \in (0, b_0)$. Now, we need to verify that $\mathcal{I}_\lambda$ is bounded above. Fixing $\tilde{u} \in C_0^\infty(\Lambda)$ and $t > 0$, by Lemma 3.3-(iii), it holds that

$$\mathcal{I}_\lambda \left(t\tilde{u} \right) \to -\infty,$$

as $t \to +\infty$. Consequently, setting $b_2 := \max_{t>0} \mathcal{I}_\lambda \left(t\tilde{u} \right) > 0$, we obtain

$$c_\lambda \leq b_2, \quad \forall \lambda > 0,$$

thus completing the proof. □

Next, we are dedicated to obtaining the boundedness of (PS) sequence $\{u_n\}$. To this aim, we will invoke the logarithmic inequality from [11].

Lemma 3.6. *There exist constants A_1, $A_2 > 0$ such that for any $u \in H^1\left(\mathbb{R}^3\right) \setminus \{0\}$,*

$$\int_{\mathbb{R}^3} |u|^2 \log\left(|u|^2\right) \, \mathrm{d}x \leq A_1 + A_2 \log\left(\|u\|\right).$$

Lemma 3.7. *Suppose that $(V1)$–$(V3)$ and (Q) are valid. For $\lambda > 0$, all $(PS)_{c_\lambda}$ sequences of $\mathcal{I}_\lambda$ are bounded (independent of λ) in E_λ.*

Proof. Let $\{u_n\}$ be a $(PS)_{c_\lambda}$ sequence of $\mathcal{I}_\lambda$, where c_λ is given in Remark 3.4. According to Lemma 3.5, there exists a constant $b_2 > 0$ (independent of λ) such that for any $n \in \mathbb{N}$,

$$\mathcal{I}_\lambda \left(u_n \right) \leq b_2.$$

Since $\phi_u \geq 0$ and the condition (Q) holds, it follows that

$$\begin{aligned} b_2 \geq \mathcal{I}_\lambda \left(u_n \right) &= \frac{1}{2} \left\| u_n \right\|_\lambda^2 + \frac{1}{4} \int_{\mathbb{R}^3} q(x) \phi_{u_n} u_n^2 \, \mathrm{d}x - \frac{1}{2} \int_{\mathbb{R}^3} \left| u_n^+ \right|^2 \log \left| u_n^+ \right|^2 \, \mathrm{d}x \\ &\geq \frac{1}{2} \left\| u_n \right\|_\lambda^2 - \frac{1}{2} \int_{\mathbb{R}^3} \left| u_n^+ \right|^2 \log \left| u_n^+ \right|^2 \, \mathrm{d}x. \end{aligned}$$

Consequently, for all $n \in \mathbb{N}$,

$$\left\| u_n \right\|_\lambda^2 \leq 2b_2 + \int_{\mathbb{R}^3} \left| u_n^+ \right|^2 \log \left| u_n^+ \right|^2 \, \mathrm{d}x. \tag{3.3}$$

Without loss of generality, we assume that $u_n^+ \neq 0$, because otherwise, the inequality

$$\left\| u_n \right\|_\lambda^2 \leq 2b_2$$

holds directly. Next, combining Lemma 3.6 with (3.3) yields

$$\left\| u_n \right\|_\lambda^2 \leq 2b_2 + A_1 + A_2 \log\left(\left\| u_n \right\|_\lambda \right).$$

Based on the above analysis, we conclude that $\{u_n\}$ is bounded in E_λ. □

4 Proof of main results

In this section, we aim to prove Theorems 1.1 and 1.2. To this end, we will divide it into several parts to complete the proof.

4.1 Proof of Theorem 1.1

We first state the vanishing lemma for Sobolev spaces, which will be crucial to the proof of Theorems 1.1 and 1.2.

Lemma 4.1 (Vanishing lemma, [18]). *Suppose that $\{u_n\}$ is a bounded sequence in $H^1(\mathbb{R}^3)$ such that*

$$\lim_{n\to\infty} \sup_{y\in\mathbb{R}^3} \int_{B_r(y)} |u_n(x)|^2 \, dx = 0$$

for some $r > 0$. Then $u_n \to 0$ in $L^q(\mathbb{R}^3)$ for $q \in (2, 6)$.

Lemma 4.2. *Suppose that $(V1)$–$(V3)$ and (Q) are valid. Then, there exists $\lambda^* > 0$ such that for any $\lambda \geq \lambda^*$, $\mathcal{I}_\lambda$ satisfies the $(PS)_{c_\lambda}$ condition, where c_λ is given in (3.2).*

Proof. Let $\{u_n\} \subset E_\lambda$ be a (PS) sequence of $\mathcal{I}_\lambda$ at the level c_λ, then Lemma 3.7 implies that $\{u_n\}$ is bounded in E_λ. Up to a subsequence, we can suppose that there exists $u_\lambda \in E_\lambda$ such that, as $n \to \infty$,

$$\begin{aligned} u_n &\rightharpoonup u_\lambda && \text{in } E_\lambda, \\ u_n &\to u_\lambda && \text{in } L^2_{\text{loc}}\left(\mathbb{R}^3\right), \\ u_n(x) &\to u_\lambda(x) && \text{a.e. in } \mathbb{R}^3. \end{aligned}$$

Set $v_n = u_n - u_\lambda$. Next, we claim that $v_n \to 0$ in E_λ as $n \to \infty$. Indeed, for some $r > 0$, only one of two following cases can occur:

$$(a) \lim_{n\to\infty} \sup_{y\in\mathbb{R}^3} \int_{B_r(y)} |v_n(x)|^2 \, dx > 0; \qquad (b) \lim_{n\to\infty} \sup_{y\in\mathbb{R}^3} \int_{B_r(y)} |v_n(x)|^2 \, dx = 0.$$

If (a) holds, then there exists $\beta > 0$ such that

$$\lim_{n\to\infty} \sup_{y\in\mathbb{R}^3} \int_{B_r(y)} |v_n(x)|^2 \, dx = \beta > 0. \tag{4.1}$$

Following the weakly lower semicontinuous of norm, we have

$$\|v_n\|_\lambda = \|u_n - u_\lambda\|_\lambda \leq \|u_n\|_\lambda + \|u_\lambda\|_\lambda \leq \|u_n\|_\lambda + \liminf_{n\to\infty} \|u_n\|_\lambda .$$

It follows from Lemma 3.7 that for some $C_1 > 0$

$$\|v_n\|_\lambda \leq C_1,$$

where C_1 is independent of λ. Now, for any given $\mathcal{R} > 0$, we define

$$A(\mathcal{R}) := \left\{x \in \mathbb{R}^3 \setminus B_{\mathcal{R}}(0) : V(x) \geq V_\infty\right\}$$

and

$$B(\mathcal{R}) := \left\{x \in \mathbb{R}^3 \setminus B_{\mathcal{R}}(0) : V(x) < V_\infty\right\},$$

where V_∞ is given in $(V1)$. Note that $\{v_n\}$ is bounded, then for any $n \in \mathbb{N}$, we have

$$\int_{A(\mathcal{R})} |v_n|^2 \, dx \leq \frac{\|v_n\|_\lambda^2}{\lambda V_\infty + 1} \leq \frac{C_1}{\lambda V_\infty + 1} \leq \frac{\beta}{4}, \tag{4.2}$$

if $\lambda \geq \lambda^*$, where $\lambda^* \geq V_\infty^{-1}\left(\frac{4C_1}{\beta} - 1\right)$.

On the other hand, it follows from the Hölder and Sobolev inequalities that

$$\int_{B(\mathcal{R})} |v_n|^2 \, dx \leq \|v_n\|_6^2 \, |B(\mathcal{R})|^{\frac{2}{3}} \leq C_2 \, |B(\mathcal{R})|^{\frac{2}{3}} \to 0, \tag{4.3}$$

where we use the fact that $|B(\mathcal{R})| \to 0$ as $\mathcal{R} \to +\infty$. Then, by (4.2) and (4.3), we get that for any $\lambda \geq \lambda^*$ and $\mathcal{R} > 0$ large enough,

$$\begin{aligned}
\lim_{n\to\infty} \sup_{y\in\mathbb{R}^3} \int_{B_r(y)} |v_n|^2 \, dx &\leq \limsup_{n\to\infty} \int_{\mathbb{R}^3} |v_n|^2 \, dx \\
&= \limsup_{n\to\infty} \int_{\mathbb{R}^3 \setminus B_\mathcal{R}(0)} |v_n|^2 \, dx + \lim_{n\to\infty} \int_{B_\mathcal{R}(0)} |v_n|^2 \, dx \\
&= \limsup_{n\to\infty} \left(\int_{A(\mathcal{R})} |v_n|^2 \, dx + \int_{B(\mathcal{R})} |v_n|^2 \, dx \right) \leq \frac{\beta}{2},
\end{aligned}$$

which is contradict to (4.1). Hence, (b) is valid. Then, from Lemma 4.1, we have $v_n \to 0$ in $L^q(\mathbb{R}^3)$ with $q \in (2,6)$. That is,

$$u_n \to u_\lambda \quad \text{in } L^q\left(\mathbb{R}^3\right) \text{ for } q \in (2,6). \tag{4.4}$$

Next, combining the Hölder inequality with (4.4), one can infer

$$\begin{aligned}
\left| \int_{\mathbb{R}^3} q(x)\phi_{u_n} \left(u_n^2 - u_\lambda^2\right) dx \right| &\leq \int_{\mathbb{R}^3} \left| q(x)\phi_{u_n} \left(u_n^2 - u_\lambda^2\right) \right| dx \\
&\leq \|\phi_{u_n}\|_6 \, \|u_n + u_\lambda\|_6 \left(\int_{\mathbb{R}^3} |q(x) \left(u_n - u_\lambda\right)|^{\frac{3}{2}} dx \right)^{\frac{2}{3}} \\
&\leq C_3 \left(\int_{\mathbb{R}^3} |q(x) \left(u_n - u_\lambda\right)|^{\frac{3}{2}} dx \right)^{\frac{2}{3}} \\
&\leq C_3 \, \|q(x)\|_2^{\frac{2}{3}} \, \|u_n - u_\lambda\|_3 \\
&\to 0, \quad \text{as } n \to \infty.
\end{aligned} \tag{4.5}$$

Using the fact that $\mathcal{D} \hookrightarrow L^6\left(\mathbb{R}^3\right)$, together the Hölder inequality and Lemma 2.2-(vi) with (4.4) and (4.5), we have

$$\begin{aligned}
&\left| \int_{\mathbb{R}^3} q(x)\phi_{u_n} u_n^2 dx - \int_{\mathbb{R}^3} q(x)\phi_{u_\lambda} u_\lambda^2 dx \right| \\
&\leq \int_{\mathbb{R}^3} \left| q(x)\phi_{u_n} \left(u_n^2 - u_\lambda^2\right) \right| dx + \int_{\mathbb{R}^3} \left| q(x) \left(\phi_{u_n} - \phi_{u_\lambda}\right) u_\lambda^2 \right| dx \\
&\leq o_n(1) + \|q(x)\|_2 \, \|\phi_{u_n} - \phi_{u_\lambda}\|_6 \, \|u_\lambda\|_6^2 \\
&\leq o_n(1) + C_4 \, \|\phi_{u_n} - \phi_{u_\lambda}\|_\mathcal{D} \\
&\to 0, \quad \text{as } n \to \infty.
\end{aligned} \tag{4.6}$$

By (2.6) and (4.4), we have

$$\int_{\mathbb{R}^3} F_2'\left(u_n^+\right) u_n^+ dx \to \int_{\mathbb{R}^3} F_2'\left(u_\lambda^+\right) u_\lambda^+ dx \tag{4.7}$$

and

$$\int_{\mathbb{R}^3} F_2\left(u_n^+\right) \mathrm{d}x \to \int_{\mathbb{R}^3} F_2\left(u_\lambda^+\right) \mathrm{d}x. \tag{4.8}$$

Furthermore, using the standard argument, we can deduce that $\mathcal{I}'_\lambda(u_\lambda)\eta = 0$ for all $\eta \in C_0^\infty(\mathbb{R}^3)$, and so, $\mathcal{I}'_\lambda(u_\lambda)u_\lambda = 0$, namely,

$$\begin{aligned}&\int_{\mathbb{R}^3} \left(|\nabla u_\lambda|^2 + (\lambda V(x)+1)u_\lambda^2\right) \mathrm{d}x + \int_{\mathbb{R}^3} q(x)\phi_{u_\lambda} u_\lambda^2 \,\mathrm{d}x \\ &\quad + \int_{\mathbb{R}^3} F_1'\left(u_\lambda^+\right) u_\lambda^+ \mathrm{d}x = \int_{\mathbb{R}^3} F_2'\left(u_\lambda^+\right) u_\lambda^+ \mathrm{d}x.\end{aligned} \tag{4.9}$$

Since $\mathcal{I}'_\lambda\left(u_n\right) u_n = o_n(1)$, namely,

$$\begin{aligned}&\int_{\mathbb{R}^3} \left(|\nabla u_n|^2 + (\lambda V(x)+1)u_n^2\right) \mathrm{d}x + \int_{\mathbb{R}^3} q(x)\phi_{u_n} u_n^2 \,\mathrm{d}x \\ &\quad + \int_{\mathbb{R}^3} F_1'\left(u_n^+\right) u_n^+ \mathrm{d}x = \int_{\mathbb{R}^3} F_2'\left(u_n^+\right) u_n^+ \mathrm{d}x + o_n(1).\end{aligned} \tag{4.10}$$

Then, it follows from (4.6), (4.7), (4.9) and (4.10) that

$$\begin{aligned}&\int_{\mathbb{R}^3} \left(|\nabla u_n|^2 + (\lambda V(x)+1)u_n^2\right) \mathrm{d}x + \int_{\mathbb{R}^3} F_1'\left(u_n^+\right) u_n^+ \mathrm{d}x \\ &\quad = \int_{\mathbb{R}^3} \left(|\nabla u_\lambda|^2 + (\lambda V(x)+1)u_\lambda^2\right) \mathrm{d}x + \int_{\mathbb{R}^3} F_1'\left(u_\lambda^+\right) u_\lambda^+ \mathrm{d}x + o_n(1),\end{aligned} \tag{4.11}$$

which leads to

$$u_n \to u_\lambda \quad \text{in } E_\lambda \tag{4.12}$$

and

$$F_1'\left(u_n^+\right) u_n^+ \to F_1'\left(u_\lambda^+\right) u_\lambda^+ \quad \text{in } L^1\left(\mathbb{R}^3\right).$$

From the definition of $F_1(\kappa)$, we can directly calculate that $0 \le F_1(\kappa) \le F_1'(\kappa)\kappa$ for all $\kappa \in \mathbb{R}$. Then, by the dominated convergence theorem, one has

$$F_1\left(u_n^+\right) \to F_1\left(u_\lambda^+\right) \quad \text{in } L^1\left(\mathbb{R}^3\right). \tag{4.13}$$

Thus, (4.6), (4.8), (4.12) and (4.13) together guarantee that

$$0 \in \partial \mathcal{I}_\lambda\left(u_\lambda\right) \quad \text{and} \quad \mathcal{I}_\lambda\left(u_n\right) \to \mathcal{I}_\lambda\left(u_\lambda\right) = c_\lambda.$$

The proof is completed. □

Proof of Theorem 1.1. Assume that $(V1)$–$(V3)$ and (Q) hold, according to Lemma 4.2, the problem (2.2) has a positive solution u_λ for any $\lambda \ge \lambda^*$, this gives the proof of Theorem 1.1. □

4.2 Concentration behavior of u_λ as $\lambda \to +\infty$

Before the proof of Theorem 1.2, we need to highlight some facts related to the problem (S) discussed in Section 1. Let $\mathcal{I}_\infty : H_0^1(\Lambda) \to \mathbb{R}$ be the energy functional of problem (S). According to (2.4), $\mathcal{I}_\infty$ can be expressed as

$$\begin{aligned}\mathcal{I}_\infty(u) &:= \frac{1}{2}\int_\Lambda \left(|\nabla u|^2 + u^2\right) \mathrm{d}x - \frac{1}{2}\int_\Lambda u^2 \log u^2 \mathrm{d}x \\ &= \frac{1}{2}\int_\Lambda \left(|\nabla u|^2 + u^2\right) \mathrm{d}x + \int_\Lambda F_1(u)\mathrm{d}x - \int_\Lambda F_2(u)\mathrm{d}x.\end{aligned}$$

It is not difficult to verify that $\mathcal{I}_\infty$ satisfies the mountain pass geometry, and the corresponding mountain pass level can be defined by

$$c_\infty := \inf_{\gamma\in\Gamma_\Lambda} \max_{t\in[0,1]} \mathcal{I}_\infty(\gamma(t))$$

where

$$\Gamma_\Lambda = \left\{\gamma \in C\left([0,1], H_0^1(\Lambda)\right) : \gamma(0) = 0, \mathcal{I}_\infty(\gamma(1)) < 0\right\}.$$

Remark 4.3. For any sequence $\{\lambda_n\} \subset [\lambda^*, +\infty)$ (see Lemma 4.2) with $\lambda_n \to +\infty$ as $n \to \infty$, there exists a sequence of positive solutions u_{λ_n} to equation (2.2) with $\lambda = \lambda_n$, which satisfies $\mathcal{I}_{\lambda_n}(u_{\lambda_n}) = c_{\lambda_n}$ and $\mathcal{I}'_{\lambda_n}(u_{\lambda_n}) = 0$.

Lemma 4.4. *Suppose that $(V1)$–$(V3)$ and (Q) are valid. If $\lambda_n \to +\infty$ and u_{λ_n} is the solution for equation (2.2) with $\lambda = \lambda_n$, then there exists $u_\Lambda \in H^1(\mathbb{R}^3)$ such that*

$$u_{\lambda_n} \to u_\Lambda \quad \text{in } H^1\left(\mathbb{R}^3\right).$$

Moreover, $u_\Lambda = 0$ a.e. $x \in \mathbb{R}^3 \setminus \Lambda$ and u_Λ is a nontrivial solution of (S), where Λ is given in $(V2)$.

Proof. We can repeat the method of Lemma 3.7 to prove that $\{u_{\lambda_n}\}$ is bounded in $H^1(\mathbb{R}^3)$. If necessary, by passing to a subsequence, there exists $u_\Lambda \in H^1(\mathbb{R}^3)$ such that

$$\begin{aligned} u_{\lambda_n} &\rightharpoonup u_\Lambda && \text{in } H^1\left(\mathbb{R}^3\right), \\ u_{\lambda_n} &\to u_\Lambda && \text{in } L^r_{\mathrm{loc}}\left(\mathbb{R}^3\right) \text{ for } r \in [2,6), \\ u_{\lambda_n}(x) &\to u_\Lambda(x) && \text{a.e. in } \mathbb{R}^3. \end{aligned}$$

Using the same methods as (4.2) and (4.3), we can get

$$\int_{A(\mathcal{R})} |u_{\lambda_n}|^2 \, dx \le \frac{\|u_{\lambda_n}\|^2_{\lambda_n}}{\lambda_n V_\infty + 1} \le \frac{C_1}{\lambda_n V_\infty + 1} \to 0$$

and

$$\int_{B(\mathcal{R})} |u_{\lambda_n}|^2 \, dx \le \|u_{\lambda_n}\|_6^2 |B(\mathcal{R})|^{\frac{2}{3}} \le C_2 |B(\mathcal{R})|^{\frac{2}{3}} \to 0,$$

where $\lambda_n \to +\infty$ and $|B(\mathcal{R})| \to 0$ as $\mathcal{R} \to +\infty$. This leads to

$$\limsup_{n\to\infty} \int_{\mathbb{R}^3\setminus B_{\mathcal{R}}(0)} |u_{\lambda_n}|^2 \, dx = \limsup_{n\to\infty} \left(\int_{A(\mathcal{R})} |u_{\lambda_n}|^2 \, dx + \int_{B(\mathcal{R})} |u_{\lambda_n}|^2 \, dx\right) < \varepsilon$$

for any $\varepsilon > 0$. Together with interpolation inequality, we conclude that for any $r \in [2,6)$,

$$\limsup_{n\to\infty} \int_{\mathbb{R}^3\setminus B_{\mathcal{R}}(0)} |u_{\lambda_n}|^r \, dx < \varepsilon. \tag{4.14}$$

Next, according to (4.14), we deduce that for any $\varepsilon > 0$, when n and $\mathcal{R}$ are large enough,

$$\begin{aligned} \|u_{\lambda_n} - u_\Lambda\|_{L^r(\mathbb{R}^3)} &= \|u_{\lambda_n} - u_\Lambda\|_{L^r(B_{\mathcal{R}}(0))} + \|u_{\lambda_n} - u_\Lambda\|_{L^r(\mathbb{R}^3\setminus B_{\mathcal{R}}(0))} \\ &\le \varepsilon + \|u_{\lambda_n}\|_{L^r(\mathbb{R}^3\setminus B_{\mathcal{R}}(0))} + \|u_\Lambda\|_{L^r(\mathbb{R}^3\setminus B_{\mathcal{R}}(0))} \\ &\le 3\varepsilon. \end{aligned}$$

Hence,

$$u_{\lambda_n} \to u_\Lambda \quad \text{in } L^r\left(\mathbb{R}^3\right) \text{ for } r \in [2,6). \tag{4.15}$$

For every $m \in \mathbb{N} \setminus \{0\}$, let us define the set

$$\Lambda_m = \left\{ x \in \mathbb{R}^3 : V(x) \geq \frac{1}{m} \right\}.$$

Then, as $n \to \infty$,

$$\int_{\Lambda_m} |u_{\lambda_n}|^2 \, \mathrm{d}x \leq \frac{m}{\lambda_n} \int_{\Lambda_m} \lambda_n V(x) |u_{\lambda_n}|^2 \, \mathrm{d}x \leq \frac{m}{\lambda_n} \|u_{\lambda_n}\|_{\lambda_n}^2 \to 0. \tag{4.16}$$

It follows from Fatou's lemma, (4.15) and (4.16) that

$$\int_{\Lambda_m} |u_\Lambda|^2 \, \mathrm{d}x = 0. \tag{4.17}$$

Recalling that $\bigcup_{m=1}^{\infty} \Lambda_m = \mathbb{R}^3 \setminus \Lambda$, which together with (4.17) yields

$$\int_{\mathbb{R}^3 \setminus \Lambda} |u_\Lambda|^2 \, \mathrm{d}x = 0.$$

Additionally, the smoothness of the boundary of Λ implies that $u_\Lambda \in H_0^1(\Lambda)$.

On the other hand, $\mathcal{I}'_{\lambda_n}(u_{\lambda_n}) u_{\lambda_n} = 0$, namely,

$$\begin{aligned} &\int_{\mathbb{R}^3} \left(|\nabla u_{\lambda_n}|^2 + (\lambda_n V(x) + 1) u_{\lambda_n}^2 \right) \mathrm{d}x + \int_{\mathbb{R}^3} q(x) \phi_{u_{\lambda_n}} u_{\lambda_n}^2 \, \mathrm{d}x \\ &\quad + \int_{\mathbb{R}^3} F_1' \left(u_{\lambda_n}^+ \right) u_{\lambda_n}^+ \mathrm{d}x = \int_{\mathbb{R}^3} F_2' \left(u_{\lambda_n}^+ \right) u_{\lambda_n}^+ \mathrm{d}x. \end{aligned}$$

In view of (Q) and for any $\eta \in H_0^1(\Lambda)$, there holds

$$\int_{\mathbb{R}^3} q(x) \phi_{u_{\lambda_n}} u_{\lambda_n} \eta \, \mathrm{d}x = 0. \tag{4.18}$$

Then, by the condition $(V2)$ and (4.18), we can conclude

$$\begin{aligned} 0 &= \lim_{n \to \infty} \mathcal{I}'_{\lambda_n}(u_{\lambda_n}) \eta \\ &= \lim_{n \to \infty} \left(\int_{\mathbb{R}^3} (\nabla u_{\lambda_n} \nabla \eta + (\lambda_n V(x) + 1) u_{\lambda_n} \eta) \, \mathrm{d}x + \int_{\mathbb{R}^3} q(x) \phi_{u_{\lambda_n}} u_{\lambda_n} \eta \, \mathrm{d}x \right. \\ &\qquad \left. + \int_{\mathbb{R}^3} F_1' \left(u_{\lambda_n}^+ \right) \eta \mathrm{d}x - \int_{\mathbb{R}^3} F_2' \left(u_{\lambda_n}^+ \right) \eta \mathrm{d}x \right) \\ &= \lim_{n \to \infty} \left(\int_{\mathbb{R}^3} (\nabla u_{\lambda_n} \nabla \eta + u_{\lambda_n} \eta) \, \mathrm{d}x + \int_{\mathbb{R}^3} F_1' \left(u_{\lambda_n}^+ \right) \eta \mathrm{d}x - \int_{\mathbb{R}^3} F_2' \left(u_{\lambda_n}^+ \right) \eta \mathrm{d}x \right) \\ &= \int_{\mathbb{R}^3} (|\nabla u_\Lambda \nabla \eta| + u_\Lambda \eta) \, \mathrm{d}x + \int_{\mathbb{R}^3} F_1' (u_\Lambda^+) \eta \mathrm{d}x - \int_{\mathbb{R}^3} F_2' (u_\Lambda^+) \eta \mathrm{d}x \\ &= \mathcal{I}'_\infty (u_\Lambda) \eta, \end{aligned}$$

which implies that u_Λ is a weak solution of (S), and so, $\mathcal{I}'_\infty(u_\Lambda) u_\Lambda = 0$, namely,

$$\int_{\mathbb{R}^3} \left(|\nabla u_\Lambda|^2 + u_\Lambda^2 \right) \mathrm{d}x + \int_{\mathbb{R}^3} F_1' (u_\Lambda^+) u_\Lambda^+ \mathrm{d}x - \int_{\mathbb{R}^3} F_2' (u_\Lambda^+) u_\Lambda^+ \mathrm{d}x = 0.$$

Furthermore, using the same method as (4.11), one has

$$\begin{aligned} &\int_{\mathbb{R}^3} \left(|\nabla u_{\lambda_n}|^2 + (\lambda_n V(x) + 1) u_{\lambda_n}^2 \right) \mathrm{d}x + \int_{\mathbb{R}^3} F_1' \left(u_{\lambda_n}^+ \right) u_{\lambda_n}^+ \mathrm{d}x \\ &\quad = \int_{\mathbb{R}^3} \left(|\nabla u_\Lambda|^2 + u_\Lambda^2 \right) \mathrm{d}x + \int_{\mathbb{R}^3} F_1' (u_\Lambda^+) u_\Lambda^+ \mathrm{d}x. \end{aligned}$$

The above equation gives that

$$u_{\lambda_n} \to u_\Lambda \quad \text{in } H^1\left(\mathbb{R}^3\right) \quad \text{and} \quad \lambda_n \int_{\mathbb{R}^3} V(x) u_{\lambda_n}^2 \, \mathrm{d}x \to 0.$$

Finally, we prove $u_\Lambda \neq 0$. Arguing by contradiction, we assume that $u_\Lambda = 0$ which implies that $u_{\lambda_n} \to 0$ in $H^1\left(\mathbb{R}^3\right)$. Then, based on (2.3), (2.8) and the fact that $\mathcal{D} \hookrightarrow L^6\left(\mathbb{R}^3\right)$, using Hölder inequality and Lemma 2.2-(iv), we obtain

$$\begin{aligned}
0 \le \left|\mathcal{I}_{\lambda_n}\left(u_{\lambda_n}\right)\right| &= \left|\mathcal{I}_{\lambda_n}\left(u_{\lambda_n}\right) - \frac{1}{2}\mathcal{I}'_{\lambda_n}\left(u_{\lambda_n}\right) u_{\lambda_n}\right| \\
&\le \frac{1}{4}\int_{\mathbb{R}^3} \left|q(x)\phi_{u_{\lambda_n}} u_{\lambda_n}^2\right| \, \mathrm{d}x + \frac{1}{2}\int_{\mathbb{R}^3} \left|u_{\lambda_n}^2\right| \, \mathrm{d}x \\
&\le \frac{1}{4}\|q(x)\|_2 \left\|\phi_{u_{\lambda_n}}\right\|_6 \|u_{\lambda_n}\|_6^2 + \frac{1}{2}\|u_{\lambda_n}\|_2^2 \\
&\le C_1 \left\|\phi_{u_{\lambda_n}}\right\|_{\mathcal{D}} + C_2 \|u_{\lambda_n}\|_{\lambda_n}^2 \\
&\le C_3 \|u_{\lambda_n}\|_{\lambda_n}^2 + C_2 \|u_{\lambda_n}\|_{\lambda_n}^2 \\
&\to 0, \quad \text{as } n \to \infty.
\end{aligned}$$

Thus, we have

$$\mathcal{I}_{\lambda_n}\left(u_{\lambda_n}\right) \to 0, \quad \text{as } n \to \infty. \tag{4.19}$$

Moreover, we get $\mathcal{I}_{\lambda_n}\left(u_{\lambda_n}\right) = c_{\lambda_n} \ge b_1 > 0$ by virtue of u_{λ_n} be critical point of $\mathcal{I}_{\lambda_n}$ obtained by Lemma 3.5. It contradicts (4.19). Thus, u_Λ is a nontrivial weak solution of (S). The proof is completed. □

Proof of Theorem 1.2. Assume that $(V1)$–$(V3)$ and (Q) hold, from Lemma 4.4 we know that for all $\lambda \ge \lambda^*$, if u_λ is a positive solution of equation (2.2), then there exists $u_\Lambda \in H^1(\mathbb{R}^3)$ such that $u_\lambda \to u_\Lambda$ in $H^1(\mathbb{R}^3)$ as $\lambda \to +\infty$ and u_Λ is a nontrivial solution of (S). □

Acknowledgements

This work is supported by the Research Fund of the National Natural Science Foundation of China (No. 12361024), the Team Building Project for Graduate Tutors in Chongqing (No. yds223010), the Natural Science Foundation of Chongqing, China (No. 2024NSCQ-MSX4817) and Chongqing Technology and Business University Graduate Research Innovation Project (No. yjscxx2025-269-243). We also wish to thank the referee for their careful reading of the manuscript which greatly improved the exposition.

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