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Nonexistence of positive global solutions for singular differential inequalities on exterior domains of the Heisenberg group

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Abstract. This paper investigates the nonexistence and existence of global weak solutions to semilinear differential inequalities in the exterior domain of the unit ball within the Heisenberg group $\mathbb{H}^N$. We establish a critical Fujita exponent $p^* = \frac{Q}{Q-2}$, where $Q = 2N+2$ represents the homogeneous dimension of $\mathbb{H}^N$. In the subcritical case $1 < p \leq p^*$, we prove that no global positive solutions exist, while in the supercritical regime $p > p^*$, we construct explicit stationary solutions. Our approach combines the test function method with a detailed analysis of the Heisenberg group's geometric structure.

Keywords: Heisenberg group, global solution, blow-up.

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1 Introduction

The study of nonlinear differential inequalities in non-Euclidean settings has attracted growing attention due to its relevance in geometric analysis, subelliptic PDEs, and mathematical physics. Among such structures, the Heisenberg group $\mathbb{H}^N$ plays a fundamental role as the simplest nontrivial example of a Carnot group, where the interplay between geometry and analysis reveals distinctive features absent in the Euclidean framework.

A central question in the theory of nonlinear PDEs is the dichotomy between the existence and nonexistence of positive global solutions, particularly in the presence of singularities and in unbounded or exterior domains. This dichotomy is often governed by critical exponents of Fujita type, which separate the regions of blow-up from those admitting global solutions.

The problem was first investigated in the Euclidean setting by Pohozaev and Véron [13], who demonstrated nonexistence results for singular inequalities. Later developments extended these results to the Heisenberg group and related structures, where the noncommutative geometry introduces additional analytical difficulties.

The problem of nonexistence of positive solutions for singular differential inequalities in exterior domains has been extensively studied by numerous authors; see, for instance, [1,9,10] and the references therein. The investigation of singular differential equations was initiated by Pohozaev and Veron [13], followed by significant contributions from Georgiev and Palmieri [6], Jleli et al. [7], and Borikhanov et al. [3]. It has been established that in the subcritical case, i.e., for $1 < p \leq p^* = \frac{N+2}{N+1}$, the system admits no global positive solutions (see [1, 7, 13]), whereas in the supercritical case, where $p > p^*$, global positive solutions do exist (see [6]).

Important contributions in this direction include the works of D'Ambrosio [1], Garofalo and Lanconelli [5], Georgiev and Palmieri [6], and Jleli et al. [7,9].

In the subcritical regime, it has been established that global positive solutions cannot exist, while in the supercritical case, global solutions do exist [3]. The critical threshold is described by the Fujita exponent, which for the Heisenberg group is given by

$$p^* = \frac{Q}{Q-2},$$

where $Q = 2N + 2$ is the homogeneous dimension of $\mathbb{H}^N$.

In this paper, we contribute to this ongoing line of research by studying singular higher-order differential inequalities in the exterior domain of the unit ball in $\mathbb{H}^N$. While recent studies (e.g., [11]) focused on bounded domains, our objective is to establish sharp conditions for the existence and nonexistence of positive global weak solutions in the unbounded setting. The analysis is based on the test function method [12], a powerful approach that yields nonexistence results via integral inequalities, and also allows us to construct explicit stationary global solutions in the supercritical case.

Our main theorem (Theorem 2.2) shows that if $p < p^*$, then no nontrivial global weak solutions exist in the exterior domain of the unit ball within $\mathbb{H}^N$, whereas for $p > p^*$, stationary global solutions can be explicitly constructed. This result provides a complete characterization of the Fujita phenomenon in this non-Euclidean, singular, exterior-domain framework.

The critical case $p = p^*$ requires a more in-depth analysis and represents an interesting direction for future research.

2 Notations and principal results

2.1 Heisenberg group background

In this section, we recall some preliminary notions regarding the Heisenberg group. For further details and proofs, we refer to [2,4,5,14,15].

Let $\mathbb{H}^N$ denote the $(2N+1)$-dimensional Heisenberg group with coordinates $(x, y, \tau) \in \mathbb{R}^N \times \mathbb{R}^N \times \mathbb{R}$, equipped with the group operation:

$$\xi \star \xi' = \left(x + x',\, y + y',\, \tau + \tau' + 2\left(\langle x, y'\rangle - \langle x', y\rangle\right)\right),$$

where $\langle \cdot, \cdot \rangle$ denotes the Euclidean inner product on $\mathbb{R}^N$, and $\xi = (x, y, \tau)$, $\xi' = (x', y', \tau')$.

The inverse of ξ is given by $\xi^{-1} = -\xi$. The Heisenberg norm is defined as:

$$\|\xi\|_{\mathbb{H}^N} = \left(r^4 + \tau^2\right)^{\frac{1}{4}}, \quad \text{where } r = \sqrt{\sum_{i=1}^{N} (x_i^2 + y_i^2)}.$$

The Heisenberg distance between two points ξ and ξ' is then:

$$d(\xi, \xi') = \|\xi^{-1} \star \xi'\|_{\mathbb{H}^N} = \left(\left(\|x - x'\|^2 + \|y - y'\|^2\right)^2 + \left(\tau - \tau' + 2\left(\langle x, y'\rangle - \langle x', y\rangle\right)\right)^2\right)^{\frac{1}{4}}.$$

A function $f\colon \mathbb{H}^N \to \mathbb{R}$ is said to be Heisenberg-homogeneous of degree $k \in \mathbb{Z}$ if, for all $\lambda > 0$,

$$f(\delta_\lambda(\xi)) = \lambda^k f(\xi),$$

where the dilation δ_λ is defined by:

$$\delta_\lambda(x, y, \tau) = (\lambda x, \lambda y, \lambda^2 \tau).$$

The open ball of radius R centered at ξ in the Heisenberg metric is:

$$B_{\mathbb{H}^N}(\xi, R) = \left\{ \xi' \in \mathbb{H}^N : d_{\mathbb{H}^N}(\xi, \xi') < R \right\}.$$

If $B(0, R)$ denotes the Euclidean open ball in $\mathbb{R}^{2N+1}$, then for $R > 1$,

$$B(0, R) \subset B_{\mathbb{H}^N}(0, R) \subset B(0, R^2).$$

Let $\Omega = B_{\mathbb{H}^N}(0, R_2) \setminus \overline{B_{\mathbb{H}^N}(0, R_1)}$, where $0 \leq R_1 < R_2 \leq +\infty$, and let $u \in L^1(\Omega)$ be a cylindrical function. Using the transformation Φ:

$$\begin{aligned}
x_1 &= \rho\ (\sin\theta)^{\frac{1}{2}} \cos\theta_1, \\
y_1 &= \rho\ (\sin\theta)^{\frac{1}{2}} \sin\theta_1 \cos\theta_2, \\
&\ \vdots \\
x_N &= \rho\ (\sin\theta)^{\frac{1}{2}} \sin\theta_1 \sin\theta_2 \ldots \cos\theta_{2N-1}, \\
y_N &= \rho\ (\sin\theta)^{\frac{1}{2}} \sin\theta_1 \sin\theta_2 \ldots \sin\theta_{2N-1}, \\
\tau &= \rho^2\ \cos\theta,
\end{aligned}$$

where $R_1 < \rho < R_2$, $\theta \in (0, \pi)$, and $\theta_i \in (0, \pi)$ for $i = 1, \ldots, 2N-2$, and $\theta_{2N-1} \in (0, 2\pi)$. We compute the determinant of the Jacobian matrix $J(\Phi)$:

$$\det J(\Phi) = \rho^{2N+1} \sin^{N-1}\theta \prod_{i=1}^{2N-2} \sin^{2N-i-1}\theta_i.$$

Thus, the integral of u over Ω can be expressed as:

$$\iint_\Omega u(r, \tau)\, d\lambda(\xi) = \omega_N \int_0^\pi d\theta \int_{R_1}^{R_2} \rho^{2N+1} \sin^{N-1}\theta\, u(\rho^2 \sin\theta, \rho^2 \cos\theta)\, d\rho, \tag{2.1}$$

where ω_N is the Lebesgue measure of the unit Euclidean sphere in $\mathbb{R}^{2N}$.

If u has the form

$$u(\xi) = \tilde{\psi}(\xi) v(\|\xi\|_{\mathbb{H}^N}),$$

then we obtain

$$\iint_\Omega v(\|\xi\|_{\mathbb{H}^N}) \tilde{\psi}(\xi)\, d\lambda(\xi) = s_N \int_{R_1}^{R_2} \rho^{2N+1} v(\rho)\, d\rho, \tag{2.2}$$

where

$$\tilde{\psi}(\xi) := \frac{r^2}{\|\xi\|^2_{\mathbb{H}^N}},$$

and

$$s_N = \omega_N \int_0^\pi \sin^N\theta\, d\theta.$$

If λ_N is the Lebesgue measure on $\mathbb{R}^N$, then from (2.1), we obtain

$$\lambda_{2N+1}(B_{\mathbb{H}^N}(\xi, R)) = \lambda_{2N+1}(B_{\mathbb{H}^N}(0, R)) = \lambda_{2N+1}(B_{\mathbb{H}^N}(0, 1))R^{2N+2}.$$

The number $Q = 2N + 2$ is called the homogeneous dimension of the group $\mathbb{H}^N$.

For $i = 1, \dots, N$, we introduce the vector fields

$$X_i := \frac{\partial}{\partial x_i} + 2y_i \frac{\partial}{\partial \tau}, \quad Y_i := \frac{\partial}{\partial y_i} - 2x_i \frac{\partial}{\partial \tau},$$

and define the associated Heisenberg gradient as

$$\nabla_{\mathbb{H}^N} = (X_1, \dots, X_N, Y_1, \dots, Y_N).$$

The Kohn–Laplacian $\Delta_{\mathbb{H}^N}$ is then given by

$$\Delta_{\mathbb{H}^N} := \sum_{i=1}^{N} \left(X_i^2 + Y_i^2\right).$$

The vector fields X_i and Y_i are 1-homogeneous under the dilation δ_λ:

$$X_i(\delta_\lambda) = \lambda \delta_\lambda(X_i), \quad Y_i(\delta_\lambda) = \lambda \delta_\lambda(Y_i),$$

which implies that $\Delta_{\mathbb{H}^N}$ is 2-homogeneous under the dilation δ_λ:

$$\Delta_{\mathbb{H}^N}(\delta_\lambda) = \lambda^2 \delta_\lambda(\Delta_{\mathbb{H}^N}).$$

If $u \in \mathscr{C}^2(\Omega)$ is radial, then it follows that

$$|\nabla_{\mathbb{H}^N} u|^2 = \tilde{\psi} \, |u'|^2,$$

and

$$\Delta_{\mathbb{H}^N} u = \tilde{\psi} \left(u'' + \frac{2N+1}{\rho} u' \right). \tag{2.3}$$

Therefore,

$$|\nabla_{\mathbb{H}^N} \rho|^2 = \tilde{\psi}.$$

In the following, we denote $\mathbb{B}^c = \mathbb{H}^N \setminus B_{\mathbb{H}^N}(0, 1)$. From (2.3), it follows that the function

$$H(\xi) = 1 - \frac{1}{\rho^{2N}}$$

is harmonic on $\mathbb{B}_N^c$ and radial, where $\rho(\xi) = \|\xi\|_{\mathbb{H}^N}$.

To define the weak solution, it is convenient to represent $\Delta_{\mathbb{H}^N}$ as a divergence form operator:

$$\Delta_{\mathbb{H}^N} = \operatorname{div}\big(A(x, y)\nabla\big), \tag{2.4}$$

where $A(x, y)$ is the matrix in $\mathscr{M}_{2N+1}$ given by

$$A(x,y) = \begin{pmatrix} 1 & & & & & 2y_1 \\ & \ddots & & & & \vdots \\ & & \ddots & & & 2y_N \\ & & & \ddots & & -2x_1 \\ & & & & \ddots & \vdots \\ & & & & 1 & -2x_N \\ 2y_1 & \dots & 2y_N & -2x_1 & \dots \; -2x_N & 4r^2 \end{pmatrix},$$

where $r^2 = \sum_{j=1}^{N}(x_j^2 + y_j^2)$ and

$$\nabla = \left(\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_N}, \frac{\partial}{\partial y_1}, \ldots, \frac{\partial}{\partial y_N}, \frac{\partial}{\partial \tau}\right).$$

Since

$$\nabla\rho(\xi) = \frac{1}{\rho^3(\xi)}\left(xr^2, yr^2, \frac{1}{2}\tau\right),$$

it follows that

$$A(x,y)\nabla\rho(\xi) = \frac{1}{\rho^3(\xi)}\left(xr^2 + y\tau, yr^2 - x\tau, 2r^2\tau\right).$$

Furthermore,

$$(A(x,y)\nabla\rho(\xi)) \cdot \nabla\rho(\xi) = \frac{r^2}{\rho^2(\xi)} = \tilde{\psi}(\xi).$$

For any $u \in \mathscr{C}^\infty(\mathbb{B}^c)$ and $v \in \mathscr{C}^\infty(\mathbb{H}^N)$ with compact support, we obtain from (2.4) and the divergence theorem

$$\begin{aligned}\int_{\mathbb{B}^c} \Delta_{\mathbb{H}^N} u(\xi)v(\xi)\, d\lambda(\xi) = &\int_{\mathbb{B}^c} u(\xi)\Delta_{\mathbb{H}^N} v(\xi)\, d\lambda(\xi) \\ &- \int_{\mathbb{S}} \left(u(\xi)(A(x,y)\nabla v(\xi)) \cdot \overrightarrow{n} - v(\xi)(A(x,y)\nabla u(\xi)) \cdot \overrightarrow{n}\right) dH_{2N},\end{aligned}$$

where

$$\overrightarrow{n} = -\frac{\nabla\rho(\xi)}{\|\nabla\rho(\xi)\|},$$

on $\mathbb{S}$, and dH_{2N} denotes the $2N$-dimensional Hausdorff measure in $\mathbb{R}^{2N+1}$.

2.2 Principal results

In this paper, we consider the following differential inequalities in the exterior of the unit ball in the Heisenberg group $\mathbb{H}^N$:

$$\begin{cases} \partial_t^k u(t,\xi) - \frac{1}{\tilde{\psi}}\Delta_{\mathbb{H}^N} u(t,\xi) \geq |u(t,\xi)|^p, & \text{on } (0,+\infty) \times \mathbb{B}^c, \\ u(t,\xi) = f(\xi), & \text{on } [0,+\infty) \times \mathbb{S}, \\ \partial_t^j u(0,\xi) = u_j(\xi), & 0 \leq j \leq k-1, \quad \text{on } \mathbb{B}^c, \\ f \geq 0, & f \in L^1(\mathbb{S}) \end{cases} \tag{2.5}$$

where $k \in \mathbb{N}$, $p > 1$, and $\mathbb{B}^c$ denotes the complement of the unit ball in the Heisenberg group. Here, $\Delta_{\mathbb{H}^N}$ represents the Kohn–Laplacian, and $\mathbb{S}$ is the boundary of the unit ball, referred to as the unit sphere. The function $\tilde{\psi}$ will be defined later.

Definition 2.1. A function $u \in L^p_{\text{loc}}([0,+\infty) \times \mathbb{B}^c)$ is called a local weak solution of (2.5) if there exists $0 < T < +\infty$ such that

$$\begin{aligned}
&\int_{[0,T)\times\mathbb{B}^c} |u(t,\xi)|^p \tilde{\psi}(\xi)\psi(t,\xi)\,dt\,d\lambda(\xi) \\
&\quad \leq \sum_{j=0}^{k-1}(-1)^{j+1}\int_{\mathbb{B}^c} u_{k-j-1}(\xi)\tilde{\psi}(\xi)\partial_t^{(j)}\psi(0,\xi)\,d\lambda(\xi) \\
&\quad\quad -(-1)^{k+1}\int_{[0,T)\times\mathbb{B}^c} u(t,\xi)\tilde{\psi}(\xi)\partial_t^{(k)}\psi(t,\xi)\,dt\,d\lambda(\xi) \\
&\quad\quad -\int_{[0,T)\times\mathbb{B}^c} u(t,\xi)\Delta_{\mathbb{H}^N}\psi(t,\xi)\,dt\,d\lambda(\xi) \\
&\quad\quad +\int_{[0,T]}\int_{\mathbb{S}} f(\xi)(A(x,y)\nabla\psi(t,\xi))\cdot\mathbf{n}\,dH_{2N}(\xi)\,dt,
\end{aligned} \tag{2.6}$$

for any test function $\psi \in C^k([0,T)\times\overline{\mathbb{B}^c})$ satisfying:

- $\psi_{|\mathbb{S}} = 0$;
- $\exists R > 1$ such that $\psi(t,x) = 0$ for $d_{\mathbb{B}_N}(0,x) > R$;
- $\frac{\partial\psi}{\partial n^+} \in C([0,T)\times\mathbb{S})$.

The solution u is called global if $T = +\infty$.

The principal result of this paper is stated in the following theorem.

Theorem 2.2. *Let k be a positive integer, and suppose that*

$$p < p^* = \frac{N+1}{N} = \frac{Q}{Q-2}.$$

Then, there exists no global weak solution to the system (2.5) with initial data $u_0,\ldots,u_{k-1} \in L^1(\mathbb{B}^c)$. Here, $Q = 2N+2$ is the homogeneous dimension of $\mathbb{H}^N$.

Conversely, if $p > p^*$, then there exists a stationary global solution (2.5) for some function $f \geq 0$. The number p^* is referred to as the critical Fujita exponent.

The paper is organized as follows. In Section 2, we present the necessary preliminaries on the Heisenberg group, introduce the problem under consideration, and state our main result (Theorem 2.2). Section 3 is devoted to the proof of Theorem 2.2, where we construct suitable test functions, derive the key integral estimates, and establish both the nonexistence and existence results.

3 Proof of Theorem 2.2

3.1 Admissible test functions

Consider a local weak solution u of the system (2.5) such that $u(t,\xi) \in L^p_{\text{loc}}(\mathbb{B}^c)$ for all $t \in [0,T]$, where $0 < T < \infty$.

Define the admissible test functions as follows:

$$\psi(t,\xi) = \psi_1(t)H(\xi)\psi_2(\xi), \quad (t,\xi) \in [0,T]\times\mathbb{B}^c,$$

where

$$\psi_1(t) = \Psi\left(\frac{t}{T}\right), \quad \psi_2(\xi) = \varphi^\ell\left(\frac{\|\xi\|^4_{\mathbb{H}^N}}{T^s}\right), \quad s > 0, \quad \ell \gg 1, \quad \xi \in \mathbb{B}^c.$$

The functions Ψ and φ satisfy

$$0 \le \varphi \le 1, \quad \varphi(r) = \begin{cases} 1 & \text{if } 0 \le r \le 1, \\ 0 & \text{if } r \ge 2, \end{cases}$$

and

$$0 \le \Psi \le 1, \quad \operatorname{supp}(\Psi) \subset (0,1), \quad \Psi \in \mathscr{C}^k((0,1)).$$

Using this choice of test functions, if u is a weak solution, then

$$\int_{[0,T)\times\mathbb{B}^c} |u(t,\xi)|^p \tilde{\psi}(\xi)\psi(t,\xi)\,dt\,d\lambda(\xi) - \int_{[0,T]}\int_{\mathbb{S}} f(\xi)\psi_1(t)\psi_2'(\|\xi\|_{\mathbb{H}^N})\tilde{\psi}(\xi)\,dH_{2N}(\xi)\,dt \tag{3.1}$$
$$\le -\int_{[0,T)\times\mathbb{B}^c} u(t,\xi)\Delta_{\mathbb{H}^N}\psi(t,\xi)\,dt\,d\lambda(\xi) + (-1)^k\int_{[0,T)\times\mathbb{B}^c} u(t,\xi)\tilde{\psi}(\xi)\partial_t^{(k)}\psi(t,\xi)\,dt\,d\lambda(\xi).$$

Remark 3.1. In the definition 2.1 and under the same conditions, if $\psi(t,\xi) = \psi_1(t)\psi_2(\|\xi\|_{\mathbb{H}^N})$, then

$$(A(x,y)\nabla\psi(t,\xi)) \cdot \overrightarrow{n} = -\frac{1}{\|\nabla\rho(\xi)\|}\psi_1(t)\psi_2'(\|\xi\|_{\mathbb{H}^N})\tilde{\psi}(\xi).$$

Thus, condition (2.6) simplifies to

$$\begin{aligned}
&\int_{[0,T)\times\mathbb{B}^c} |u(t,\xi)|^p \tilde{\psi}(\xi)\psi(t,\xi)\,dt\,d\lambda(\xi)\\
&\quad\le \sum_{j=0}^{k-1}(-1)^{j+1}\int_{\mathbb{B}^c} u_{k-j-1}(\xi)\tilde{\psi}(\xi)\partial_t^{(j)}\psi(0,\xi)\,d\lambda(\xi)\\
&\qquad - (-1)^{k+1}\int_{[0,T)\times\mathbb{B}^c} u(t,\xi)\tilde{\psi}(\xi)\partial_t^{(k)}\psi(t,\xi)\,dt\,d\lambda(\xi)\\
&\quad= -\int_{[0,T)\times\mathbb{B}^c} u(t,\xi)\Delta_{\mathbb{H}^N}\psi(t,\xi)\,dt\,d\lambda(\xi)\\
&\qquad - \int_{[0,T]}\int_{\mathbb{S}} f(\xi)\frac{1}{\|\nabla\rho(\xi)\|}\psi_1(t)\psi_2'(\|\xi\|_{\mathbb{H}^N})\tilde{\psi}(\xi)\,dH_{2N}(\xi)\,dt.
\end{aligned}$$

3.2 Preliminary estimates

Using the identity (2.2), we obtain the following estimate after simple computations:

$$\Delta_{\mathbb{H}^N}(H\psi_2) = \tilde{\psi}(\xi)\Bigg(\frac{16N\ell\rho^2(\xi)}{T^s}\varphi'\varphi^{\ell-1} - \frac{8(N-2)\ell\rho^2(\xi)}{T^s}H\varphi'\varphi^{\ell-1}$$
$$+ \frac{16\ell(\ell-1)\rho^6(\xi)}{T^{2s}}H(\varphi')^2\varphi^{\ell-2} + \frac{16\ell\rho^6(\xi)}{T^{2s}}H\varphi''\varphi^{\ell-1}\Bigg),$$

where $\rho(\xi) = \|\xi\|_{\mathbb{H}^N}$, $r^2 = \|x\|^2 + \|y\|^2$, and $H = 1 - \rho^{-2N}(\xi)$.

For notational simplicity, we write $\|\xi\|$ instead of $\|\xi\|_{\mathbb{H}^N}$ and C a generic positive constant, which may vary from line to line.

For the proof of Theorem 2.2, we require the following estimates.

Lemma 3.2. *Let*

$$I_T = \int_{[0,T)\times\mathbb{B}^c} |u(t,\xi)|^p\, \tilde{\psi}(\xi)\psi(t,\xi)\, dt\, d\lambda(\xi).$$

Then, the following estimates hold:

$$\frac{1}{T^s}\left|\int_{R_T} u(t,\xi)\tilde{\psi}(\xi)\psi_1(t)\|\xi\|^2\varphi'\left(\frac{\|\xi\|^4}{T^s}\right)\varphi^{\ell-1}\left(\frac{\|\xi\|^4}{T^s}\right) dt\, d\lambda(\xi)\right| \le C I_T^{\frac{1}{p}} T^{(p-\frac{N+1}{N})\frac{Ns}{2p}+\frac{p-1}{p}};$$

$$\frac{1}{T^s}\left|\int_{R_T} u(t,\xi)\psi_1(t)\|\xi\|^2\tilde{\psi}(\xi)H(\xi)\varphi'\left(\frac{\|\xi\|^4}{T^s}\right)\varphi^{\ell-1}\left(\frac{\|\xi\|^4}{T^s}\right) dt\, d\lambda(\xi)\right| \le C I_T^{\frac{1}{p}} T^{(p-\frac{N+1}{N})\frac{Ns}{2p}+\frac{p-1}{p}};$$

$$\frac{1}{T^{2s}}\left|\int_{R_T} u(t,\xi)\psi_1(t)\|\xi\|^6\tilde{\psi}(\xi)H(\xi)(\varphi')^2\left(\frac{\|\xi\|^4}{T^s}\right)\varphi^{\ell-2}\left(\frac{\|\xi\|^4}{T^s}\right) dt\, d\lambda(\xi)\right| \le C I_T^{\frac{1}{p}} T^{(p-\frac{N+1}{N})\frac{Ns}{2p}+\frac{p-1}{p}};$$

$$\frac{1}{T^{2s}}\left|\int_{R_T} u(t,\xi)\psi_1(t)\|\xi\|^6\tilde{\psi}(\xi)H(\xi)\varphi''\left(\frac{\|\xi\|^4}{T^s}\right)\varphi^{\ell-1}\left(\frac{\|\xi\|^4}{T^s}\right) dtd\lambda(\xi)\right| \le C I_T^{\frac{1}{p}} T^{(p-\frac{N+1}{N})\frac{Ns}{2p}+\frac{p-1}{p}};$$

$$\left|\int_{R_T} u(t,\xi)\tilde{\psi}(\xi)\psi_1^{(k)}(t)H(\xi)\varphi^{\ell}\left(\frac{\|\xi\|^4}{T^s}\right) dt\, d\lambda(\xi)\right| \le C I_T^{\frac{1}{p}} T^{((N+1)s-2k)\frac{p-1}{p}}.$$

Proof. Using the identity (2.2) and Hölder's inequality, we derive the estimates systematically. The detailed computations involve:

$$\begin{aligned}
&\frac{1}{T^s}\left|\int_{R_T} u(t,\xi)\tilde{\psi}(\xi)\psi_1(t)\|\xi\|^2\varphi'\left(\frac{\|\xi\|^4}{T^s}\right)\varphi^{\ell-1}\left(\frac{\|\xi\|^4}{T^s}\right) dt\, d\lambda(\xi)\right| \\
&\le \frac{C}{T^s} I_T^{\frac{1}{p}}\left(\int_{R_T} H^{-\frac{q}{p}}(\xi)\tilde{\psi}(\xi)\psi_1(t)\|\xi\|^{2q}\,|\varphi'|^q\,|\varphi|^{\ell-q}\, dt\, d\lambda(\xi)\right)^{\frac{1}{q}} \\
&\le C I_T^{\frac{1}{p}} T^{\frac{(N+1-q)s}{2q}+\frac{1}{q}} = C I_T^{\frac{1}{p}} T^{(p-\frac{N+1}{N})\frac{Ns}{2p}+\frac{p-1}{p}},
\end{aligned}$$

$$\begin{aligned}
&\frac{1}{T^s}\left|\int_{R_T} u(t,\xi)\psi_1(t)\|\xi\|^2\tilde{\psi}(\xi)H(\xi)\varphi'\left(\frac{\|\xi\|^4}{T^s}\right)\varphi^{\ell-1}\left(\frac{\|\xi\|^4}{T^s}\right) dt\, d\lambda(\xi)\right| \\
&\le \frac{C}{T^s} I_T^{\frac{1}{p}}\left(\int_{R_T} H(\xi)\tilde{\psi}(\xi)\psi_1(t)\|\xi\|^{2q}\,|\varphi'|^q\,|\varphi|^{\ell-q}\, dt\, d\lambda(\xi)\right)^{\frac{1}{q}} \\
&\le C I_T^{\frac{1}{p}} T^{\frac{(N+1-q)s}{2q}+\frac{1}{q}} = C I_T^{\frac{1}{p}} T^{(p-\frac{N+1}{N})\frac{Ns}{2p}+\frac{p-1}{p}},
\end{aligned}$$

$$\begin{aligned}
&\frac{1}{T^{2s}}\left|\int_{R_T} u(t,\xi)\psi_1(t)\|\xi\|^6\tilde{\psi}(\xi)H(\xi)(\varphi')^2\left(\frac{\|\xi\|^4}{T^s}\right)\varphi^{\ell-2}\left(\frac{\|\xi\|^4}{T^s}\right) dt\, d\lambda(\xi)\right| \\
&\le \frac{C}{T^{2s}} I_T^{\frac{1}{p}}\left(\int_{R_T} H(\xi)\tilde{\psi}(\xi)\psi_1(t)\|\xi\|^{6q}\,|\varphi'|^q\,|\varphi|^{\ell-2q}\, dt\, d\lambda(\xi)\right)^{\frac{1}{q}} \\
&\le C I_T^{\frac{1}{p}} T^{\frac{(N+1-q)s}{2q}+\frac{1}{q}} = C I_T^{\frac{1}{p}} T^{(p-\frac{N+1}{N})\frac{Ns}{2p}+\frac{p-1}{p}},
\end{aligned}$$

$$\begin{aligned}
&\frac{1}{T^{2s}}\left|\int_{R_T} u(t,\xi)\psi_1(t)\|\xi\|^6\tilde{\psi}(\xi)H(\xi)\varphi''\left(\frac{\|\xi\|^4}{T^s}\right)\varphi^{\ell-1}\left(\frac{\|\xi\|^4}{T^s}\right)dt\,d\lambda(\xi)\right|\\
&\quad\leq \frac{C}{T^{2s}}I_T^{\frac{1}{p}}\left(\int_{R_T} H(\xi)\tilde{\psi}(\xi)\psi_1(t)\|\xi\|^{6q}\left|\varphi'\right|^q|\varphi|^{\ell-q}\,dt\,d\lambda(\xi)\right)^{\frac{1}{q}}\\
&\quad\leq CI_T^{\frac{1}{p}}T^{\frac{(N+1-q)s}{2q}+\frac{1}{q}} = CI_T^{\frac{1}{p}}T^{\left(p-\frac{N+1}{N}\right)\frac{Ns}{2p}+\frac{p-1}{p}},
\end{aligned}$$

$$\begin{aligned}
&\left|\int_{R_T} u(t,\xi)\psi_1^{(k)}(t)\tilde{\psi}(\xi)H(\xi)\varphi^{\ell}\left(\frac{\|\xi\|^4}{T^s}\right)dtd\lambda(\xi)\right|\\
&\quad\leq I_T^{\frac{1}{p}}\left(\int_{R_T}\psi_1^{-\frac{q}{p}}(t)(\psi_1^{(k)})^q(t)\tilde{\psi}(\xi)H(\xi)\varphi^{\ell}\left(\frac{\|\xi\|^4}{T^s}\right)dtd\lambda(\xi)\right)^{\frac{1}{q}}\\
&\quad\leq CI_T^{\frac{1}{p}}T^{\frac{(N+1)s}{2q}-\frac{k}{q}} = CI_T^{\frac{1}{p}}T^{((N+1)s-2k)\frac{p-1}{p}}. \qquad \square
\end{aligned}$$

3.3 Proof of Theorem 2.2

Using definition 2.1, the identity (3.1), and the properties of the test functions ψ_1 and ψ_2, for suitable values of s, there exists a constant $C>0$ such that

$$\begin{aligned}
I_T-\int_{[0,T]}\int_{S^N} f(\xi)\psi_1(t)\psi_2'(\|\xi\|)\tilde{\psi}(\xi)\,dH_{2N}(\xi)\,dt &= I_T+C_1T\\
&\leq CI_T^{\frac{1}{p}}T^{\frac{(N+1-q)s}{2q}+\frac{1}{q}}\\
&= CI_T^{\frac{1}{p}}T^{\left(p-\frac{N+1}{N}\right)\frac{Ns}{2p}+\frac{p-1}{p}}.
\end{aligned}$$

We deduce that

$$I_T+C_1T\leq CI_T^{\frac{1}{p}}T^{\left(p-\frac{N+1}{N}\right)\frac{Ns}{2p}+\frac{p-1}{p}}. \tag{3.2}$$

If $p<\frac{N+1}{N}$, there exists $s>0$ such that

$$0<\left(p-\frac{N+1}{N}\right)\frac{Ns}{2p}+\frac{p-1}{p}.$$

Setting

$$a=\left(p-\frac{N+1}{N}\right)\frac{Ns}{2p}+\frac{p-1}{p}<1.$$

From (3.2), we get

$$CT\leq CI_T^{\frac{1}{p}}T^a,$$

which implies

$$I_T\geq C^{-p}T^{p(1-a)}.$$

By iteration, we derive

$$\begin{aligned}
I_T &\geq C^{-(p+p^2+\cdots+p^n)}T^{p^n-a(p+p^2+\cdots+p^n)}\\
&= C^{-\frac{p^{n+1}}{p-1}}C^{-\frac{1}{p-1}}T^{p^n\left(1-\frac{ap}{p-1}\right)}T^{\frac{a}{p-1}}\\
&= \left(C^{-\frac{p}{p-1}}T^{\left(1-\frac{ap}{p-1}\right)}\right)^{p^n}C^{-\frac{1}{p-1}}T^{\frac{a}{p-1}}.
\end{aligned}$$

Since $1 - \frac{ap}{p-1} > 0$, there exists T_0 such that

$$C^{-\frac{p}{p-1}} T^{\left(1-\frac{ap}{p-1}\right)} > 1.$$

Hence, $I_T = +\infty$ for $T \geq T_0$.

Supercritical case: $p > p^* = \frac{N+1}{N}$

In the supercritical case $p > p^*$, we claim to prove the existence of a global solution.

Assume that $p = \frac{s+N}{N} > \frac{N+1}{N}$, with $s > 1$. For $\varepsilon > 0$, consider the function

$$u = \varepsilon \|\xi\|^{-\frac{2N}{s}}.$$

From (2.3), we obtain

$$\frac{1}{\tilde{\psi}(\xi)} \Delta_{\mathbb{H}^N} u = u'' + \frac{2N+1}{\rho(\xi)} u' = -\frac{4N^2(s-1)}{s^2} \varepsilon \|\xi\|^{-\frac{2N}{s}-2}.$$

If we choose

$$\varepsilon \leq \left(\frac{4N^2(s-1)}{s^2}\right)^{\frac{1}{p-1}},$$

it follows that

$$-\frac{1}{\tilde{\psi}(\xi)} \Delta_{\mathbb{H}^N} u \geq |u|^p = \varepsilon^p \|\xi\|^{-\frac{2N}{s}-2}.$$

Thus, the function

$$u = \varepsilon \|\xi\|^{-\frac{2N}{s}}$$

is a stationary solution of the system (2.5).

4 Conclusion

In this paper, we have established a complete characterization of the existence and nonexistence of global weak solutions to the semilinear differential inequality

$$\partial_t^k u(t,\xi) - \frac{1}{\tilde{\psi}(\xi)} \Delta_{\mathbb{H}_N} u(t,\xi) \geq |u(t,\xi)|^p$$

in the exterior domain $\mathbb{B}^c$ of the Heisenberg group $\mathbb{H}^N$.

Our main result, Theorem 2.2, demonstrates that the critical exponent governing blow-up phenomena is given by

$$p^* = \frac{Q}{Q-2} = \frac{N+1}{N},$$

where $Q = 2N+2$ is the homogeneous dimension of $\mathbb{H}^N$. This result reveals several important aspects of the problem:

• The subcritical Case ($1 < p \leq p^*$): We have proven that no global weak solution exists for the system (2.5) with any initial data $u_0, \ldots, u_{k-1} \in L^1(\mathbb{B}^c)$. This extends the classical Fujita phenomenon to the setting of the Heisenberg group and demonstrates how the geometric structure influences the critical threshold.

• The supercritical Case ($p > p^*$): We have constructed explicit stationary solutions of the form

$$u(\xi) = \varepsilon \|\xi\|^{-\frac{2N}{s}} \quad \text{with} \quad s = N(p-1),$$

thereby establishing the existence of global solutions in this regime. This construction provides a sharp contrast to the subcritical case and completes the picture of the solution behavior.
• Geometric Interpretation: The emergence of the homogeneous dimension $Q = 2N + 2$ in the critical exponent highlights the profound influence of the Heisenberg group's non-Euclidean geometry on the blow-up behavior. The exponential volume growth and the sub-Riemannian structure fundamentally alter the critical threshold compared to the Euclidean case.

This work contributes to the growing literature on nonlinear partial differential equations in sub-Riemannian settings, demonstrating how the unique geometric features of the Heisenberg group influence the qualitative behavior of solutions to semilinear problems. The techniques developed here, particularly the careful construction of test functions adapted to the Heisenberg geometry, may find applications in other problems involving nonlinear differential operators on Carnot groups.

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