Electronic Journal of Qualitative Theory of Differential Equations
2026, No. **42**, 1–15; https://doi.org/10.14232/ejqtde.2026.1.42 www.math.u-szeged.hu/ejqtde/

Oscillation of non-canonical odd-order equations with retarded arguments

Said R. Grace[1] and **Jehad Alzabut**[2,3]

[1]Department of Engineering Mathematics, Faculty of Engineering, Cairo University, Orman, Giza 12221, Egypt
[2]Department of Industrial Engineering, OSTİM Technical University, Ankara 06374, Türkiye
[3]Department of Mathematics and Sciences, Prince Sultan University, 11586 Riyadh, Saudi Arabia

Received 13 August 2025, appeared 17 August 2026
Communicated by Zuzana Došlá

Abstract. This paper establishes new oscillation criteria for odd-order half-linear differential equations with retarded arguments in the non-canonical case. By employing a linearization technique, the problem is reduced to studying an associated linear equation, allowing us to analyze oscillatory behavior via comparison with first-order delay differential equations whose properties are well known. The obtained results generalize, improve, and consolidate several existing criteria in the literature. Examples are included to illustrate the practical utility and effectiveness of the proposed method.

Keywords: oscillation, asymptotic behavior, non-canonical case, odd-order differential equations.

2020 Mathematics Subject Classification: 34N05, 39A10.

1 Introduction

The study focuses on analyzing the oscillatory nature of solutions associated with odd-order half-linear differential equations of the type

$$\left(p(\mathfrak{z})\left(\varphi^{(n-1)}(\mathfrak{z})\right)^{\varepsilon}\right)' + q(\mathfrak{z})\varphi^{\varepsilon}(w(\mathfrak{z})) = 0, \tag{1.1}$$

where $n \geq 3$ is an odd positive integer. We assume the following conditions:

(i) ε is the ratio of positive odd integers with $\varepsilon \geq 1$,

(ii) p and $q : [\mathfrak{z}_0, \infty) \to \mathbb{R}^+$ are continuous functions with $p'(\mathfrak{z}) \geq 0$,

(iii) $w : [\mathfrak{z}_0, \infty) \to \mathbb{R}$ is a continuous function with $w(\mathfrak{z}) \leq \mathfrak{z}$ and $\lim_{\mathfrak{z}\to\infty} w(\mathfrak{z}) = \infty$.

✉Corresponding author. Email: jalzabut@psu.edu.sa

This is accomplished by applying a linearization technique and utilizing comparison principles with first-order delay differential equations whose oscillatory behavior is well known.

We define $A(\mathfrak{z}, \mathfrak{z}_0) = \int_{\mathfrak{z}_0}^{\mathfrak{z}} p^{-1/\varepsilon}(s)\, ds$ and assume

$$A(\mathfrak{z}, \mathfrak{z}_0) < \infty \quad \text{as } \mathfrak{z} \to \infty. \tag{1.2}$$

A function φ is regarded as a solution of equation (1.1) if it is continuous on an interval $[\xi_\varphi, \infty)$ with $\xi_\varphi \geq \mathfrak{z}_0$ and satisfies equation (1.1) throughout that interval. In our analysis, we only consider solutions φ for which

$$\sup_{\mathfrak{z} \geq \xi} |\varphi(\mathfrak{z})| > 0, \quad \text{for every } \xi \geq \xi_\varphi,$$

which means that the function does not eventually vanish.

A solution is classified as *oscillatory* if it possesses zeros extending arbitrarily far along the $\mathfrak{z}$-axis; otherwise, it is referred to as *nonoscillatory*. The equation (1.1) itself is called oscillatory when all of its admissible solutions exhibit oscillations.

Understanding the oscillation characteristics of functional differential equations plays a key role in predicting the asymptotic dynamics of systems that incorporate retarded arguments, a common feature in many natural and engineered processes. These equations model processes in engineering, biology, and physics where the current state depends on past history, such as in control systems, population dynamics, and neural networks [22,24]. Studying their oscillatory behavior helps predict system stability, prevent undesirable fluctuations, and design effective control mechanisms.

The study of oscillatory behavior in odd-order, non-canonical, half-linear differential equations with retarded arguments remains a relatively unexplored direction within the qualitative analysis of functional differential equations. Considerable advancements have been reported for second-order cases [2,9,12] and for certain third-order formulations [4,7,8,14]; however, analogous results for higher odd-order half-linear problems subject to non-canonical conditions are still scarcely addressed in existing research. This paper addresses this gap by introducing a new analytical framework based on linearization and comparison with first-order delay equations, a method that, while classical for lower-order problems, has not been systematically applied to this generalized setting. While equation (1.1) is formulated for any odd positive integer n, the primary focus and novel contributions of this study are directed towards the more challenging and less explored cases where $n \geq 3$.

A key feature of this work is its focus on the non-canonical case, defined by the convergence of the integral $\int_{t_0}^{\infty} p^{-1/\varepsilon}(s)\, ds < \infty$. This condition introduces significant analytical challenges, as it invalidates many standard techniques such as Riccati transformations and asymptotic integration methods that rely on divergent integrals. In contrast, most existing results such as those in [2,7,14] assumed the canonical case, where $\int^{\infty} p^{-1/\varepsilon}(s)\, ds = \infty$. Furthermore, while studies like [2,4,7–9,12,14] and [20,21,25] provide foundational oscillation criteria for first, second and third-order equations, their extension to odd-order half-linear equations has not been rigorously established. Our results generalize these classical comparison principles to a more complex, nonlinear, and higher-order context, thereby offering a novel contribution to the field.

The assumptions employed—such as the positivity and monotonicity of p, the delayed argument $w(\mathfrak{z}) \leq \mathfrak{z}$, and the structure of the half-linear nonlinearity—are consistent with the literature, but their combination in the non-canonical, odd-order setting leads to non-trivial oscillation criteria. The derivation of theorems and corollaries relies on a careful classification of non-oscillatory solutions and the use of iterated integrals to estimate lower bounds on the solution and its derivatives. This approach extends the ideas of Koplatadze and Chanturiya [20] and Philos [25] beyond their original scope, demonstrating how oscillation can be deduced even in the presence of strong nonlinearity and delayed feedback.

2 Main results

In this section, we establish oscillation criteria for equation (1.1). Prior to the main results, we state the following two lemmas.

Lemma 2.1. *Let $q : [\mathfrak{z}_0, \infty) \to \mathbb{R}^+$ and $w : [\mathfrak{z}_0, \infty) \to \mathbb{R}$ be continuous functions, with $w(\mathfrak{z}) \le \mathfrak{z}$, $w(\mathfrak{z})$ is nondecreasing, and $\lim_{\mathfrak{z}\to\infty} w(\mathfrak{z}) = \infty$. If the first-order delay differential inequality*

$$\varphi'(\mathfrak{z}) + q(\mathfrak{z})\varphi(w(\mathfrak{z})) \le 0$$

has an eventually positive solution, then the delay equation

$$\varphi'(\mathfrak{z}) + q(\mathfrak{z})\varphi(w(\mathfrak{z})) = 0$$

also has an eventually positive solution.

This lemma extends the results introduced in [20, 21, 25, Corollary 1] and its proof follows directly as a consequence.

Lemma 2.2 ([19])**.** *Let conditions (i)–(iii) and (1.2) hold. Equation (1.1) has eventually positive solutions satisfying one of the following cases:*

(I) $\varphi(\mathfrak{z}) > 0$, $\varphi'(\mathfrak{z}) > 0$, $\varphi''(\mathfrak{z}) > 0, \dots, \varphi^{(n-1)}(\mathfrak{z}) > 0$, $\varphi^{(n)}(\mathfrak{z}) \le 0$, $\left(p(\mathfrak{z})\left(\varphi^{(n-1)}(\mathfrak{z})\right)^{\varepsilon}\right)' \le 0$,

(II) $\varphi(\mathfrak{z}) > 0$, $\varphi^{(j)}(\mathfrak{z}) > 0$, $\varphi^{(j+1)}(\mathfrak{z}) < 0$ *for every odd number* $j \in \{1, 2, \dots, n-3\}$, $\varphi^{(n-1)}(\mathfrak{z}) > 0$, $\varphi^{(n)}(\mathfrak{z}) \le 0$, $\left(p(\mathfrak{z})\left(\varphi^{(n-1)}(\mathfrak{z})\right)^{\varepsilon}\right)' \le 0$,

(III) $(-1)^i\varphi^{(i)}(\mathfrak{z}) > 0$, $i = 1, 2, \dots, n$,

(IV) $\varphi(\mathfrak{z}) > 0$, $\varphi^{(n-2)}(\mathfrak{z}) > 0$, $\varphi^{(n-1)}(\mathfrak{z}) < 0$, $\left(p(\mathfrak{z})\left(\varphi^{(n-1)}(\mathfrak{z})\right)^{\varepsilon}\right)' \le 0$.

Theorem 2.3. *Let conditions (i)–(iii) and (1.2) hold. Suppose that for any constant $\theta \in (0, 1)$, the linear delay differential equations*

$$\left(p^{1/\varepsilon}(\mathfrak{z})\varphi^{(n-1)}(\mathfrak{z})\right)' + \frac{1}{\varepsilon}\left(\frac{\theta}{(n-1)!}\right)^{\varepsilon-1} w^{(1-n)(1-\varepsilon)}(\mathfrak{z})p^{1/\varepsilon-1}(\mathfrak{z})q(\mathfrak{z})\varphi(w(\mathfrak{z})) = 0 \tag{2.1}$$

and

$$\left(p^{1/\varepsilon}(\mathfrak{z})\varphi^{(n-1)}(\mathfrak{z})\right)' + \frac{\theta^{\varepsilon-1}}{\varepsilon}w^{\varepsilon-1}(\mathfrak{z})\left(I_{n-2}(w(\mathfrak{z}))\right)^{\varepsilon-1} q(\mathfrak{z})\varphi(w(\mathfrak{z})) = 0 \tag{2.2}$$

are oscillatory, and the inequality

$$\left(p^{1/\varepsilon}(\mathfrak{z})\varphi^{(n-1)}(\mathfrak{z})\right)' + \frac{1}{\varepsilon}p^{(1-\varepsilon)/\varepsilon}(\mathfrak{z})I_{n-1}^{\varepsilon-1}(w(\mathfrak{z}))q(\mathfrak{z})\varphi(w(\mathfrak{z})) \le 0 \tag{2.3}$$

has no eventually positive decreasing solution, where

$$I_k(\mathfrak{z}) = \int_{\mathfrak{z}}^{\eta(\mathfrak{z})} \int_{s_1}^{\eta(s_1)} \cdots \int_{s_{k-1}}^{\eta(s_{k-1})} p^{-1/\varepsilon}(s_k)\, ds_k \cdots ds_1,$$

and $\eta(\mathfrak{z}) > \mathfrak{z}$ *is a continuous function with* $\lim_{\mathfrak{z}\to\infty}\eta(\mathfrak{z}) = \infty$. *Additionally, assume*

$$\limsup_{\mathfrak{z}\to\infty}\int_{\mathfrak{z}_1}^{\mathfrak{z}} q(s)\left(\frac{\theta}{(n-2)!}w^{n-2}(s)\right)^{\varepsilon} A^{\varepsilon}(w(s),\mathfrak{z}_0)\,ds > \varepsilon. \tag{2.4}$$

Then equation (1.1) *is oscillatory.*

Proof. Consider a solution φ of (1.1) that does not oscillate. Suppose there exists $\mathfrak{z}_1 \geq \mathfrak{z}_0$ such that $\varphi(\mathfrak{z}) > 0$ and $\varphi(w(\mathfrak{z})) > 0$ for every $\mathfrak{z} \geq \mathfrak{z}_1$. Under these circumstances, the term $p(\mathfrak{z})\big(\varphi^{(n-1)}(\mathfrak{z})\big)^{\varepsilon}$ cannot change sign. In fact, one can verify that $\varphi^{(n-1)}(\mathfrak{z})$ remains strictly positive. With this observation, we turn our attention to recasting (1.1) into an equivalent linear form.

From (1.1) we deduce that

$$\left(p(\mathfrak{z})\left(\varphi^{(n-1)}(\mathfrak{z})\right)^{\varepsilon}\right)' = -q(\mathfrak{z})\varphi^{\varepsilon}(w(\mathfrak{z})).$$

Differentiating, we obtain

$$\left(p(\mathfrak{z})\left(\varphi^{(n-1)}(\mathfrak{z})\right)^{\varepsilon}\right)' = \varepsilon\big(p^{\frac{1}{\varepsilon}}(\mathfrak{z})\varphi^{(n-1)}(\mathfrak{z})\big)^{\varepsilon-1}\left(p^{\frac{1}{\varepsilon}}(\mathfrak{z})\varphi^{(n-1)}(\mathfrak{z})\right)' = -q(\mathfrak{z})\varphi^{\varepsilon}(w(\mathfrak{z})), \tag{2.5}$$

where $\big(p(\mathfrak{z})\big(\varphi^{(n-1)}(\mathfrak{z})\big)^{\varepsilon}\big)'$ takes the form $\big(\big(p^{\frac{1}{\varepsilon}}(\mathfrak{z})\varphi^{(n-1)}(\mathfrak{z})\big)^{\varepsilon}\big)'$. Thus, we get

$$\left(p^{\frac{1}{\varepsilon}}(\mathfrak{z})\varphi^{(n-1)}(\mathfrak{z})\right)' = -\frac{1}{\varepsilon}\big(p^{\frac{1}{\varepsilon}}(\mathfrak{z})\varphi^{(n-1)}(\mathfrak{z})\big)^{1-\varepsilon}q(\mathfrak{z})\varphi^{\varepsilon}(w(\mathfrak{z})). \tag{2.6}$$

Assume that Case (I) of Lemma 2.2 holds. By [1, Lemma 2.2.3], there exists a constant $\theta \in (0,1)$ such that

$$\varphi(w(\mathfrak{z})) \geq \frac{\theta}{(n-1)!}w^{n-1}(\mathfrak{z})\varphi^{(n-1)}(w(\mathfrak{z})), \quad \text{for } \mathfrak{z} \geq \mathfrak{z}_1. \tag{2.7}$$

Rearranging, we get

$$\left(\frac{\theta}{(n-1)!}\right)^{-1} w^{1-n}(\mathfrak{z})\varphi(w(\mathfrak{z})) \geq \varphi^{(n-1)}(w(\mathfrak{z})) \quad \text{for } \mathfrak{z} \geq \mathfrak{z}_1. \tag{2.8}$$

Using (2.8) in (2.6), we have

$$\left(p^{\frac{1}{\varepsilon}}(\mathfrak{z})\varphi^{(n-1)}(\mathfrak{z})\right)' + \frac{1}{\varepsilon}\left(\frac{\theta}{(n-1)!}\right)^{\varepsilon-1} w^{(1-n)(1-\varepsilon)}(\mathfrak{z})p^{\frac{1}{\varepsilon}-1}(\mathfrak{z})q(\mathfrak{z})\varphi(w(\mathfrak{z})) \leq 0. \tag{2.9}$$

This outcome conflicts with the initial premise that equation (2.1) exhibits oscillatory behavior.

Next, suppose that Case (II) described in Lemma 2.2 is valid. Since $p'(\mathfrak{z}) \geq 0$, we see that φ satisfies $(-1)^{j+1}\varphi^{(j)}(\mathfrak{z}) > 0$ for $j = 2,\ldots,n-1$. Thus, we get

$$\begin{aligned}-\varphi^{(n-2)}(\mathfrak{z}) \geq \varphi^{(n-2)}(\eta(\mathfrak{z})) - \varphi^{(n-2)}(\mathfrak{z}) &= \int_{\mathfrak{z}}^{\eta(\mathfrak{z})}\varphi^{(n-1)}(s)\,ds \\ &\geq (\eta(\mathfrak{z}) - \mathfrak{z})\varphi^{(n-1)}(\eta(\mathfrak{z})) =: I_1(\mathfrak{z})\varphi^{(n-1)}(\eta(\mathfrak{z})).\end{aligned}$$

Repeated integration from $\mathfrak{z}$ to $\eta(\mathfrak{z})$ yields

$$\varphi'(\mathfrak{z}) \geq I_{n-2}(\mathfrak{z})\varphi^{(n-1)}(\eta_{n-2}(\mathfrak{z})),$$

where $\eta_k(\mathfrak{z})$ denotes the k-th iterated application of $\eta(\mathfrak{z})$, and

$$I_{n-2}(\mathfrak{z}) = \int_{\mathfrak{z}}^{\eta(\mathfrak{z})} \int_{s_1}^{\eta(s_1)} \cdots \int_{s_{n-3}}^{\eta(s_{n-3})} p^{-1/\varepsilon}(s_{n-2})\, ds_{n-2} \cdots ds_1.$$

Thus,

$$\varphi'(w(\mathfrak{z})) \geq I_{n-2}(w(\mathfrak{z}))\varphi^{(n-1)}(\mu(\mathfrak{z})), \tag{2.10}$$

where $\mu(\mathfrak{z}) = \eta_{n-2}(w(\mathfrak{z}))$. By [1, Lemma 2.2.3], there exists $\theta \in (0,1)$ such that

$$\varphi(w(\mathfrak{z})) \geq \theta w(\mathfrak{z})\varphi'(w(\mathfrak{z})), \quad \text{for } \mathfrak{z} \geq \mathfrak{z}_1. \tag{2.11}$$

Using this in the inequality (2.10), we get

$$\varphi(w(\mathfrak{z})) \geq \theta w(\mathfrak{z}) I_{n-2}(w(\mathfrak{z}))\varphi^{(n-1)}(\mu(\mathfrak{z})).$$

Since $\varphi^{(n-1)}(\mathfrak{z}) \leq \varphi^{(n-1)}(w(\mathfrak{z})) \leq \varphi^{(n-1)}(\mu(\mathfrak{z}))$, we have

$$\varphi^{(n-1)}(\mathfrak{z}) \leq \frac{\varphi(w(\mathfrak{z}))}{\theta w(\mathfrak{z}) I_{n-2}(w(\mathfrak{z}))}.$$

Hence,

$$p^{1/\varepsilon}(\mathfrak{z})\varphi^{(n-1)}(\mathfrak{z}) \leq \frac{p^{1/\varepsilon}(\mathfrak{z})\varphi(w(\mathfrak{z}))}{\theta w(\mathfrak{z}) I_{n-2}(w(\mathfrak{z}))}.$$

Thus, we have

$$\left(p^{\frac{1}{\varepsilon}}(\mathfrak{z})\varphi^{(n-1)}(\mathfrak{z})\right)' \geq \frac{\theta^{\varepsilon-1}}{\varepsilon} w^{\varepsilon-1}(\mathfrak{z}) \left(I_{n-2}(w(\mathfrak{z}))\right)^{\varepsilon-1} q(\mathfrak{z})\varphi(w(\mathfrak{z})) \tag{2.12}$$

contradicting the assumption that equation (2.2) is oscillatory.

Assume Case (III) of Lemma 2.2 holds. Since $p'(\mathfrak{z}) \geq 0$, it follows that φ obeys the sign pattern $(-1)^{j+1}\varphi^{(j)}(\mathfrak{z}) > 0$ for each $j = 2, \ldots, n-1$. Therefore,

$$-\varphi^{(n-2)}(\mathfrak{z}) \geq I_1(\mathfrak{z})\varphi^{(n-1)}(\eta(\mathfrak{z})).$$

Repeated integration yields

$$\varphi(\mathfrak{z}) \geq I_{n-1}(\mathfrak{z})\varphi^{(n-1)}(\eta_{n-1}(\mathfrak{z})).$$

Thus,

$$\varphi(w(\mathfrak{z})) \geq I_{n-1}(w(\mathfrak{z}))\varphi^{(n-1)}(\mu(\mathfrak{z})), \tag{2.13}$$

where $\mu(\mathfrak{z}) = \eta_{n-1}(w(\mathfrak{z}))$. Since $\varphi^{(n-1)}(\mathfrak{z}) \leq \varphi^{(n-1)}(w(\mathfrak{z})) \leq \varphi^{(n-1)}(\mu(\mathfrak{z}))$, we get

$$\varphi^{(n-1)}(\mathfrak{z}) \leq \frac{\varphi(w(\mathfrak{z}))}{I_{n-1}(w(\mathfrak{z}))}.$$

It follows that

$$p^{1/\varepsilon}(\mathfrak{z})\varphi^{(n-1)}(\mathfrak{z}) \leq \frac{p^{1/\varepsilon}(\mathfrak{z})\varphi(w(\mathfrak{z}))}{I_{n-1}(w(\mathfrak{z}))}. \tag{2.14}$$

Using (2.14) in (2.6), we obtain

$$-\left(p^{\frac{1}{\varepsilon}}(\mathfrak{z})\varphi^{(n-1)}(\mathfrak{z})\right)' \geq \frac{1}{\varepsilon}\left(p^{1/\varepsilon}(\mathfrak{z})\varphi(w(\mathfrak{z}))\right)^{1-\varepsilon} q(\mathfrak{z})\varphi^{\varepsilon}(w(\mathfrak{z})).$$

This simplifies to

$$\left(p^{\frac{1}{\varepsilon}}(\mathfrak{z})\varphi^{(n-1)}(\mathfrak{z})\right)' + \frac{1}{\varepsilon}\left(p^{1/\varepsilon}(\mathfrak{z})\varphi(w(\mathfrak{z}))\right)^{1-\varepsilon} q(\mathfrak{z})\varphi^{\varepsilon}(w(\mathfrak{z})) \leq 0$$

contradicting the assumption that inequality (2.3) has no eventually positive decreasing solution.

Finally, assume Case (IV) of Lemma 2.2 holds. It is clear that

$$p^{1/\varepsilon}(s)\varphi^{(n-1)}(s) \leq p^{1/\varepsilon}(\mathfrak{z})\varphi^{(n-1)}(\mathfrak{z}).$$

Integrating from $\mathfrak{z}$ to u and letting $u \to \infty$, we get

$$0 \leq \varphi^{(n-2)}(u) \leq \varphi^{(n-2)}(\mathfrak{z}) + A(\mathfrak{z}, \mathfrak{z}_0)p^{1/\varepsilon}(\mathfrak{z})\varphi^{(n-1)}(\mathfrak{z}).$$

Thus,

$$\varphi^{(n-2)}(\mathfrak{z}) \geq -p^{1/\varepsilon}(\mathfrak{z})\varphi^{(n-1)}(\mathfrak{z})A(\mathfrak{z}, \mathfrak{z}_0). \tag{2.15}$$

Since $\varphi^{(n-1)}(\mathfrak{z}) < 0$, the function $\varphi^{(n-2)}(\mathfrak{z})$ is decreasing. On the other hand, $A(\mathfrak{z}, \mathfrak{z}_0)$ is increasing in $\mathfrak{z}$. Therefore, using $w(\mathfrak{z}) \leq \mathfrak{z}$, we obtain

$$\frac{\varphi^{(n-2)}(w(\mathfrak{z}))}{A(w(\mathfrak{z}), \mathfrak{z}_0)} \geq \frac{\varphi^{(n-2)}(\mathfrak{z})}{A(\mathfrak{z}, \mathfrak{z}_0)}. \tag{2.16}$$

To justify (2.16), we note that since $\varphi^{(n-1)}(\mathfrak{z}) < 0$, the function $\varphi^{(n-2)}(\mathfrak{z})$ is decreasing. Moreover,

$$A(\mathfrak{z}, \mathfrak{z}_0) = \int_{\mathfrak{z}_0}^{\mathfrak{z}} p^{-1/\varepsilon}(s)\, ds$$

is increasing in $\mathfrak{z}$. Therefore, from $w(\mathfrak{z}) \leq \mathfrak{z}$, we obtain

$$\varphi^{(n-2)}(w(\mathfrak{z})) \geq \varphi^{(n-2)}(\mathfrak{z}) \quad \text{and} \quad A(w(\mathfrak{z}), \mathfrak{z}_0) \leq A(\mathfrak{z}, \mathfrak{z}_0).$$

Since $\varphi^{(n-2)}(\mathfrak{z}) > 0$ and $A(\mathfrak{z}, \mathfrak{z}_0) > 0$, we have

$$\frac{\varphi^{(n-2)}(w(\mathfrak{z}))}{A(w(\mathfrak{z}), \mathfrak{z}_0)} \geq \frac{\varphi^{(n-2)}(\mathfrak{z})}{A(w(\mathfrak{z}), \mathfrak{z}_0)} \geq \frac{\varphi^{(n-2)}(\mathfrak{z})}{A(\mathfrak{z}, \mathfrak{z}_0)},$$

and hence (2.16) holds.

Now, from [2, Lemma 2.2.3], there exists a constant $\theta \in (0, 1)$ such that

$$\varphi(w(\mathfrak{z})) \geq \frac{\theta}{(n-2)!} w^{n-2}(\mathfrak{z})\varphi^{(n-2)}(w(\mathfrak{z})), \quad \text{for } \mathfrak{z} \geq \mathfrak{z}_1. \tag{2.17}$$

From (2.16), one can deduce that

$$\varphi^{(n-2)}(w(\mathfrak{z})) \geq \varphi^{(n-2)}(\mathfrak{z})\frac{A(w(\mathfrak{z}), \mathfrak{z}_0)}{A(\mathfrak{z}, \mathfrak{z}_0)}.$$

Substituting this yields

$$\varphi(w(\mathfrak{z})) \geq \frac{\theta}{(n-2)!} w^{n-2}(\mathfrak{z}) \cdot \varphi^{(n-2)}(\mathfrak{z})\frac{A(w(\mathfrak{z}), \mathfrak{z}_0)}{A(\mathfrak{z}, \mathfrak{z}_0)}. \tag{2.18}$$

Recalling the linearized equation from (2.6)

$$z'(\mathfrak{z}) = \left(p^{1/\varepsilon}(\mathfrak{z})\varphi^{(n-1)}(\mathfrak{z})\right)' = -\frac{1}{\varepsilon}\left(p^{1/\varepsilon}(\mathfrak{z})\varphi^{(n-1)}(\mathfrak{z})\right)^{1-\varepsilon} q(\mathfrak{z})\varphi^{\varepsilon}(w(\mathfrak{z})). \tag{2.19}$$

Since $z(\mathfrak{z}) = p^{1/\varepsilon}(\mathfrak{z})\varphi^{(n-1)}(\mathfrak{z}) < 0$, we have $|z(\mathfrak{z})| = -z(\mathfrak{z})$. Therefore, equation (2.19) can be written as

$$-z'(\mathfrak{z}) = \frac{1}{\varepsilon}|z(\mathfrak{z})|^{1-\varepsilon} q(\mathfrak{z})\varphi^{\varepsilon}(w(\mathfrak{z})).$$

Substituting the lower bound for $\varphi(w(\mathfrak{z}))$ from (2.18) into this inequality, we get

$$-z'(\mathfrak{z}) \geq \frac{1}{\varepsilon}|z(\mathfrak{z})|^{1-\varepsilon} q(\mathfrak{z})\left(\frac{\theta}{(n-2)!}w^{n-2}(\mathfrak{z})\varphi^{(n-2)}(\mathfrak{z})\frac{A(w(\mathfrak{z}),\mathfrak{z}_0)}{A(\mathfrak{z},\mathfrak{z}_0)}\right)^{\varepsilon}.$$

From inequality (2.15), $\varphi^{(n-2)}(\mathfrak{z}) \geq -z(\mathfrak{z})A(\mathfrak{z}) = |z(\mathfrak{z})|A(\mathfrak{z})$. Substituting this we obtain

$$-z'(\mathfrak{z}) \geq \frac{1}{\varepsilon}|z(\mathfrak{z})|^{1-\varepsilon} q(\mathfrak{z})\left(\frac{\theta}{(n-2)!}w^{n-2}(\mathfrak{z})|z(\mathfrak{z})|A(\mathfrak{z},\mathfrak{z}_0)\frac{A(w(\mathfrak{z}),\mathfrak{z}_0)}{A(\mathfrak{z},\mathfrak{z}_0)}\right)^{\varepsilon}.$$

Simplifying the expression

$$\begin{aligned} -z'(\mathfrak{z}) &\geq \frac{1}{\varepsilon}|z(\mathfrak{z})|^{1-\varepsilon} q(\mathfrak{z})\left(\frac{\theta}{(n-2)!}w^{n-2}(\mathfrak{z})|z(\mathfrak{z})|A(w(\mathfrak{z}),\mathfrak{z}_0)\right)^{\varepsilon} \\ &= \frac{1}{\varepsilon}q(\mathfrak{z})\left(\frac{\theta}{(n-2)!}w^{n-2}(\mathfrak{z})A(w(\mathfrak{z}),\mathfrak{z}_0)\right)^{\varepsilon}|z(\mathfrak{z})| \end{aligned}$$

or

$$z'(\mathfrak{z}) \leq -\frac{1}{\varepsilon}q(\mathfrak{z})\left(\frac{\theta}{(n-2)!}w^{n-2}(\mathfrak{z})A(w(\mathfrak{z}),\mathfrak{z}_0)\right)^{\varepsilon} z(\mathfrak{z}).$$

Let $u(\mathfrak{z}) = -z(\mathfrak{z}) > 0$. Then $u'(\mathfrak{z}) = -z'(\mathfrak{z})$, and the above inequality becomes

$$u'(\mathfrak{z}) \geq \frac{1}{\varepsilon}q(\mathfrak{z})\left(\frac{\theta}{(n-2)!}w^{n-2}(\mathfrak{z})A(w(\mathfrak{z}),\mathfrak{z}_0)\right)^{\varepsilon} u(\mathfrak{z}). \tag{2.20}$$

Performing integration of (2.20) over the interval $[\mathfrak{z}_1, \mathfrak{z}]$ yields

$$\ln\left(\frac{u(\mathfrak{z})}{u(\mathfrak{z}_1)}\right) \geq \frac{1}{\varepsilon}\int_{\mathfrak{z}_1}^{\mathfrak{z}} q(s)\left(\frac{\theta}{(n-2)!}w^{n-2}(s)A(w(s),\mathfrak{z}_0)\right)^{\varepsilon} ds$$

or

$$u(\mathfrak{z}) \geq u(\mathfrak{z}_1)\exp\left(\frac{1}{\varepsilon}\int_{\mathfrak{z}_1}^{\mathfrak{z}} q(s)\left(\frac{\theta}{(n-2)!}w^{n-2}(s)A(w(s),\mathfrak{z}_0)\right)^{\varepsilon} ds\right).$$

Taking the limit superior as $\mathfrak{z} \to \infty$, if the integral in condition (2.4) is infinite, then $u(\mathfrak{z}) \to \infty$. Since $u(\mathfrak{z}) = -z(\mathfrak{z})$ is known to be bounded, the preceding estimate yields a contradiction. Therefore the assumed non-oscillatory solution cannot exist, and equation (1.1) must be oscillatory. □

Alternatively, we can combine the results in Theorem 2.3 by defining

$$Q(\mathfrak{z}) = \min\left\{\frac{1}{\varepsilon}\left(\frac{\theta}{(n-1)!}\right)^{\varepsilon-1} w^{(1-n)(1-\varepsilon)}(\mathfrak{z})p^{1/\varepsilon-1}(\mathfrak{z})q(\mathfrak{z}), \frac{\theta^{\varepsilon-1}}{\varepsilon}w^{\varepsilon-1}(\mathfrak{z})\left(I_{n-2}(w(\mathfrak{z}))\right)^{\varepsilon-1}q(\mathfrak{z})\right\}.$$

Then, Theorem 2.3 takes the following form.

Corollary 2.4. *Suppose that assumptions (i)–(iii), together with* (1.2) *and* (2.4)*, are satisfied. If, for every fixed constant $\theta \in (0,1)$, the associated linear problem*

$$\left(p^{\frac{1}{\varepsilon}}(\mathfrak{z})\,\varphi^{(n-1)}(\mathfrak{z})\right)' + q(\mathfrak{z})\,Q(\mathfrak{z})\,\varphi(w(\mathfrak{z})) = 0 \tag{2.21}$$

exhibits oscillatory behavior, then it follows that the original equation (1.1) *must also be oscillatory.*

Theorem 2.5. *Let conditions (i)—(iii),* (1.2) *and* (2.4) *hold. If for any constant $\theta \in (0,1)$, the linear first-order delay differential equations*

$$X'(\mathfrak{z}) + \frac{1}{\varepsilon}q(\mathfrak{z})\left(\frac{\theta}{(n-1)!}p^{-1/\varepsilon}(w(\mathfrak{z}))w^{n-1}(\mathfrak{z})\right)^{\varepsilon} X(w(\mathfrak{z})) = 0 \tag{2.22}$$

and

$$Y'(\mathfrak{z}) + \frac{1}{\varepsilon}q(\mathfrak{z})\left(\theta w(\mathfrak{z})p^{-1/\varepsilon}(w(\mathfrak{z}))I_{n-2}(w(\mathfrak{z}))\right)^{\varepsilon} Y(w(\mathfrak{z})) = 0 \tag{2.23}$$

are oscillatory, then equation (1.1) *is oscillatory.*

Proof. Consider a solution φ of equation (1.1) that is eventually positive, meaning there exists some $\mathfrak{z}_1 \geq \mathfrak{z}_0$ for which $\varphi(\mathfrak{z}) > 0$ and $\varphi(w(\mathfrak{z})) > 0$ whenever $\mathfrak{z} \geq \mathfrak{z}_1$. By applying the same reasoning as in the proof of Theorem 2.3, Case (I), one obtains the relations in (2.19) and (2.6). Combining these expressions by inserting (2.6) into (2.19) gives

$$-\left(p^{\frac{1}{\varepsilon}}(\mathfrak{z})\varphi^{(n-1)}(\mathfrak{z})\right)' \geq \frac{1}{\varepsilon}\left(p^{1/\varepsilon}(\mathfrak{z})\varphi^{(n-1)}(\mathfrak{z})\right)^{1-\varepsilon} q(\mathfrak{z})\left(\frac{\theta}{(n-1)!}w^{n-1}(\mathfrak{z})\varphi^{(n-1)}(w(\mathfrak{z}))\right)^{\varepsilon}.$$

Since $\varphi^{(n-1)}(w(\mathfrak{z})) = \frac{(p^{\frac{1}{\varepsilon}}(w(\mathfrak{z}))\varphi^{(n-1)}(w(\mathfrak{z})))'}{p^{1/\varepsilon}(w(\mathfrak{z}))}$, we get

$$-\left(p^{\frac{1}{\varepsilon}}(\mathfrak{z})\varphi^{(n-1)}(\mathfrak{z})\right)' \geq \frac{1}{\varepsilon}q(\mathfrak{z})\left(\frac{\theta}{(n-1)!}p^{-1/\varepsilon}(w(\mathfrak{z}))w^{n-1}(\mathfrak{z})\right)^{\varepsilon} p^{\frac{1}{\varepsilon}}(w(\mathfrak{z}))\varphi^{(n-1)}(w(\mathfrak{z})).$$

Let $X(\mathfrak{z}) = p^{1/\varepsilon}(\mathfrak{z})\varphi^{(n-1)}(\mathfrak{z})$. Then,

$$X'(\mathfrak{z}) + \frac{1}{\varepsilon}q(\mathfrak{z})\left(\frac{\theta}{(n-1)!}p^{-1/\varepsilon}(w(\mathfrak{z}))w^{n-1}(\mathfrak{z})\right)^{\varepsilon} X(w(\mathfrak{z})) \leq 0.$$

By Lemma 2.1, this contradicts the assumption that equation (2.22) is oscillatory.

For Case (II) of Lemma 2.2, proceeding as in Theorem 2.3 Case (II), we obtain (2.10) and (2.11), so

$$\varphi(w(\mathfrak{z})) \geq \theta w(\mathfrak{z})I_{n-2}(w(\mathfrak{z}))\varphi^{(n-1)}(w(\mathfrak{z})).$$

It follows that

$$-\left(p^{\frac{1}{\varepsilon}}(\mathfrak{z})\varphi^{(n-1)}(\mathfrak{z})\right)' \geq \frac{1}{\varepsilon}\left(p^{1/\varepsilon}(\mathfrak{z})\varphi^{(n-1)}(\mathfrak{z})\right)^{1-\varepsilon} q(\mathfrak{z})\left(\theta w(\mathfrak{z})I_{n-2}(w(\mathfrak{z}))\varphi^{(n-1)}(w(\mathfrak{z}))\right)^{\varepsilon}.$$

Again since $\varphi^{(n-1)}(w(\mathfrak{z})) = \frac{(p^{\frac{1}{\varepsilon}}(w(\mathfrak{z}))\varphi^{(n-1)}(w(\mathfrak{z})))'}{p^{1/\varepsilon}(w(\mathfrak{z}))}$, we obtain

$$-\left(p^{\frac{1}{\varepsilon}}(\mathfrak{z})\varphi^{(n-1)}(\mathfrak{z})\right)' \geq \frac{1}{\varepsilon}q(\mathfrak{z})\left(\theta w(\mathfrak{z})p^{-1/\varepsilon}(w(\mathfrak{z}))I_{n-2}(w(\mathfrak{z}))\right)^{\varepsilon}\left(p^{1/\varepsilon}(w(\mathfrak{z}))\varphi^{(n-1)}(w(\mathfrak{z}))\right).$$

Let $Y(\mathfrak{z}) = p^{1/\varepsilon}(\mathfrak{z})\varphi^{(n-1)}(\mathfrak{z})$. Then,

$$Y'(\mathfrak{z}) + \frac{1}{\varepsilon}q(\mathfrak{z})\left(\theta w(\mathfrak{z})p^{-1/\varepsilon}(w(\mathfrak{z}))I_{n-2}(w(\mathfrak{z}))\right)^{\varepsilon} Y(w(\mathfrak{z})) \leq 0.$$

By Lemma 2.1, this contradicts the assumption that equation (2.23) is oscillatory. The rest of the proof is similar and hence omitted. $\square$

As an application of the above theorem, we may have the following integral conditions.

Corollary 2.6. *Let conditions (i)–(iii), (1.2) and (2.4) hold. If for any constant $\theta \in (0,1)$,*

$$\lim_{\mathfrak{z}\to\infty} \int_{w(\mathfrak{z})}^{\mathfrak{z}} q(s)\left(\frac{\theta}{(n-1)!}p^{-1/\varepsilon}(w(s))w^{n-1}(s)\right)^{\varepsilon} ds > \frac{1}{e}, \tag{2.24}$$

and

$$\lim_{\mathfrak{z}\to\infty} \int_{w(\mathfrak{z})}^{\mathfrak{z}} q(s)\left(\theta w(s)p^{-1/\varepsilon}(w(s))I_{n-2}(w(s))\right)^{\varepsilon} ds > \frac{1}{e}, \tag{2.25}$$

then equation (1.1) is oscillatory.

Proof. The conditions (2.24) and (2.25) imply that equations (2.22) and (2.23) are oscillatory, respectively, by standard results in oscillation theory for first-order delay differential equations (see [20]). Thus, by Theorem 2.5, equation (1.1) is oscillatory. □

In comparison to Theorem 2.3, which requires the oscillation of two first-order linear delay equations, Theorem 2.7 replaces this dual requirement with a single oscillation condition based on a different comparison function involving the iterated integral $A_{n-2}(\mathfrak{z}, w(\mathfrak{z}))$. This alternative criterion provides a new pathway for establishing oscillation and leads to a distinct proof, particularly for Case (II) of Lemma 2.2.

Theorem 2.7. *Let conditions (i)–(iii), (1.2) and (2.4) hold. If for any constant $\theta \in (0,1)$, the linear delay differential equation*

$$\left(p^{1/\varepsilon}(\mathfrak{z})\varphi^{(n-1)}(\mathfrak{z})\right)' + \frac{1}{\varepsilon}\left(\theta w(\mathfrak{z})A_{n-2}(\mathfrak{z}, w(\mathfrak{z}))\right)^{\varepsilon-1} q(\mathfrak{z})\varphi(w(\mathfrak{z})) = 0 \tag{2.26}$$

is oscillatory, where

$$A_1(v,u) = \int_u^v p^{-1/\varepsilon}(s)\, ds, \quad A_i(v,u) = \int_u^v A_{i-1}(v,s)\, ds, \quad i = 2,3,\ldots,$$

then equation (1.1) is oscillatory.

Proof. We give the proof for Case (II) of Lemma 2.2. For $t > v > u$, we have

$$-\varphi^{(n-2)}(u) \geq \int_u^v p^{-1/\varepsilon}(s)p^{1/\varepsilon}(s)\varphi^{(n-1)}(s)ds > A_1(v,u)p^{1/\varepsilon}(v)\varphi^{(n-1)}(v).$$

By integrating this inequality successively $(n-3)$ times over the interval from u to v, we obtain

$$\varphi'(u) \geq A_{n-2}(v,u)p^{1/\varepsilon}(v)\varphi^{(n-1)}(v).$$

Applying Lemma 2.2.3 from [14], which states $\varphi(x) \geq \theta\omega(x)\varphi'(x)$ for a constant $\theta \in (0,1)$, and substituting the obtained bound for $\varphi'(u)$, we get

$$\varphi(u) \geq \theta\omega(u)A_{n-2}(v,u)p^{1/\varepsilon}(v)\varphi^{(n-1)}(v).$$

Now, substituting $u = \omega(\mathfrak{z})$ and $v = \mathfrak{z}$ (where $\mathfrak{z}$ denotes the independent variable), it follows that

$$\varphi(\omega(\mathfrak{z})) \geq \theta\omega(\omega(\mathfrak{z}))A_{n-2}(\mathfrak{z}, \omega(\mathfrak{z}))p^{1/\varepsilon}(\mathfrak{z})\varphi^{(n-1)}(\mathfrak{z}).$$

Using the preceding estimate together with (2.26), we obtain

$$-\left(p^{1/\varepsilon}(\mathfrak{z})\varphi^{(n-1)}(\mathfrak{z})\right)' \geq \frac{1}{\varepsilon}q(\mathfrak{z})\left(p^{1/\varepsilon}(\mathfrak{z})\varphi^{(n-1)}(\mathfrak{z})\right)^{1-\varepsilon} \times \left(\theta\omega(\mathfrak{z})A_{n-2}(\mathfrak{z},\omega(\mathfrak{z}))p^{1/\varepsilon}(\mathfrak{z})\varphi^{(n-1)}(\mathfrak{z})\right)^{\varepsilon-1}\varphi(\omega(\mathfrak{z})).$$

Hence,

$$\left(p^{1/\varepsilon}(\mathfrak{z})\varphi^{(n-1)}(\mathfrak{z})\right)' + \frac{1}{\varepsilon}\left(\theta\omega(\mathfrak{z})A_{n-2}(\mathfrak{z},\omega(\mathfrak{z}))\right)^{\varepsilon-1}q(\mathfrak{z})\varphi(\omega(\mathfrak{z})) \leq 0,$$

which contradicts the assumption that equation (2.26) is oscillatory. □

Corollary 2.8. *Let the assumptions of Theorem 2.7 be fulfilled. If, for every constant $\theta \in (0,1)$, the following holds:*

$$\limsup_{\mathfrak{z}\to\infty}\int_{w(\mathfrak{z})}^{\mathfrak{z}}\frac{1}{\varepsilon}\left[\theta\, w(s)\, A_{n-2}(s,w(s))\right]^{\varepsilon-1}q(s)\,ds > \frac{1}{e},$$

then equation (1.1) possesses only oscillatory solutions.

Unlike the earlier theorems, which establish oscillation via an associated linear equation, the next result formulates a criterion based on the absence of eventually positive and monotonically decreasing solutions to a certain differential inequality.

Theorem 2.9. *Let conditions (i)–(iii), together with (1.2) and (2.4), be satisfied. If, for any constant $\theta \in (0,1)$, the differential inequality*

$$\left(p^{1/\varepsilon}(\mathfrak{z})\varphi^{(n-1)}(\mathfrak{z})\right)' + \frac{1}{\varepsilon}A_{n-1}^{\varepsilon-1}(\mathfrak{z},w(\mathfrak{z}))\,q(\mathfrak{z})\,\varphi(w(\mathfrak{z})) \leq 0 \tag{2.27}$$

admits no solution that is eventually positive and monotonically decreasing, then equation (1.1) is oscillatory.

Proof. We present the argument for Case (III) of Lemma 2.2. In this setting, it follows that $(-1)^i\varphi^{(i)}(\mathfrak{z}) > 0$ for $i = 1,2,\ldots,n$. Consequently,

$$-\varphi^{(n-2)}(u) \geq \int_u^v p^{-1/\varepsilon}(s)\,p^{1/\varepsilon}(s)\,\varphi^{(n-1)}(s)\,ds \geq A_1(v,u)\,p^{1/\varepsilon}(v)\,\varphi^{(n-1)}(v).$$

By integrating this inequality $(n-2)$ times from u to v, we arrive at

$$\varphi(u) \geq A_{n-1}(v,u)\,p^{1/\varepsilon}(v)\,\varphi^{(n-1)}(v).$$

Choosing $u = w(\mathfrak{z})$ and $v = \mathfrak{z}$ gives

$$\varphi(w(\mathfrak{z})) \geq A_{n-1}(\mathfrak{z},w(\mathfrak{z}))\,p^{1/\varepsilon}(\mathfrak{z})\,\varphi^{(n-1)}(\mathfrak{z}).$$

From this, we deduce

$$-\left(p^{1/\varepsilon}(\mathfrak{z})\varphi^{(n-1)}(\mathfrak{z})\right)' \geq \frac{1}{\varepsilon}\left(p^{1/\varepsilon}(\mathfrak{z})\varphi^{(n-1)}(\mathfrak{z})\right)^{1-\varepsilon}q(\mathfrak{z})\left(A_{n-1}(\mathfrak{z},w(\mathfrak{z}))\,p^{1/\varepsilon}(\mathfrak{z})\,\varphi^{(n-1)}(\mathfrak{z})\right)^{\varepsilon}.$$

Simplifying yields

$$\left(p^{1/\varepsilon}(\mathfrak{z})\varphi^{(n-1)}(\mathfrak{z})\right)' + \frac{1}{\varepsilon}A_{n-1}^{\varepsilon-1}(\mathfrak{z},w(\mathfrak{z}))\,q(\mathfrak{z})\,\varphi(w(\mathfrak{z})) \leq 0,$$

which contradicts the hypothesis that (2.27) has no eventually positive decreasing solution. The remaining steps parallel the reasoning used in Theorem 2.3 Case (III) and are omitted. □

Corollary 2.10. *Assume the conditions of Theorem 2.9 hold. If for any constant $\theta \in (0,1)$,*

$$\int_{\mathfrak{z}_0}^{\infty} A_{n-1}^{\varepsilon-1}(\mathfrak{z},w(\mathfrak{z}))q(\mathfrak{z})d\mathfrak{z} = \infty, \tag{2.28}$$

then equation (1.1) is oscillatory.

3 Examples

Example 3.1. Consider the problem

$$\left(\mathfrak{z}^2 \left|\varphi''(\mathfrak{z})\right|^{\varepsilon-1} \varphi''(\mathfrak{z})\right)' + \frac{q_0}{\mathfrak{z}^{3\varepsilon}} \left|\varphi\left(\frac{t}{2}\right)\right|^{\beta-1} \varphi\left(\frac{\mathfrak{z}}{2}\right) = 0, \quad \mathfrak{z} \geq 2, \tag{E1}$$

where $\varepsilon > 0$ is a ratio of odd positive integers, $\beta > 0$, and $q_0 > 0$ is a constant.

From equation (E1), we identify $p(\mathfrak{z}) = \mathfrak{z}^2$, $q(\mathfrak{z}) = \frac{q_0}{\mathfrak{z}^{3\varepsilon}}$, $w(\mathfrak{z}) = \frac{\mathfrak{z}}{2}$ and $n = 3$. The non-canonical condition requires $\int_2^\infty s^{-2/\varepsilon} ds < \infty$, which is valid if $\varepsilon > 2$. Indeed, we get

$$A(\mathfrak{z}, 2) = \int_2^{\mathfrak{z}} s^{-2/\varepsilon} ds = \frac{\varepsilon}{\varepsilon - 2} \left(\mathfrak{z}^{1-2/\varepsilon} - 2^{1-2/\varepsilon}\right).$$

Next, we have

$$I_1(\mathfrak{z}) = \int_{w(\mathfrak{z})}^{\mathfrak{z}} p^{-1/\varepsilon}(s) ds = \int_{\mathfrak{z}/2}^{\mathfrak{z}} s^{-2/\varepsilon} ds = \frac{\varepsilon}{\varepsilon - 2} \left(\mathfrak{z}^{1-2/\varepsilon} - (\mathfrak{z}/2)^{1-2/\varepsilon}\right).$$

For

$$Q(\mathfrak{z}) = \min\left\{\frac{1}{\varepsilon}\left(\frac{\theta}{(n-1)!}\right)^{\varepsilon-1} w^{(1-n)(1-\varepsilon)}(\mathfrak{z}) p^{1/\varepsilon-1}(\mathfrak{z}) q(\mathfrak{z}),\ \frac{\theta^{\varepsilon-1}}{\varepsilon} w^{\varepsilon-1}(\mathfrak{z}) \left(I_{n-2}(w(\mathfrak{z}))\right)^{\varepsilon-1} q(\mathfrak{z})\right\},$$

we compute the two terms

$$T_1(\mathfrak{z}) = C_1(\theta, \varepsilon, q_0) \mathfrak{z}^{2/\varepsilon - 4}$$

and

$$T_2(\mathfrak{z}) = C_2(\theta, \varepsilon, q_0) \mathfrak{z}^{-\varepsilon - 4 + 2/\varepsilon},$$

where

$$C_1(\theta, \varepsilon, q_0) = \frac{q_0 \theta^{\varepsilon-1}}{\varepsilon} 2^{-4\varepsilon+3}, \quad C_2(\theta, \varepsilon, q_0) = \frac{\theta^{\varepsilon-1} q_0}{\varepsilon} 2^{-(\varepsilon-1)} C_3^{\varepsilon-1}(\varepsilon)$$

and

$$C_3(\varepsilon) = \frac{\varepsilon}{\varepsilon - 2} 2^{-(1-2/\varepsilon)} \left(1 - 2^{-(1-2/\varepsilon)}\right).$$

Since $\varepsilon > 2$, both $T_1(\mathfrak{z})$ and $T_2(\mathfrak{z})$ tend to zero as $\mathfrak{z} \to \infty$. However, for the oscillation criterion, we need the linear equation

$$\left(p^{1/\varepsilon}(\mathfrak{z}) \varphi''(\mathfrak{z})\right)' + q(\mathfrak{z}) Q(\mathfrak{z}) \varphi(w(\mathfrak{z})) = 0$$

to be oscillatory.

Let $Q(\mathfrak{z}) = \min\{T_1(\mathfrak{z}), T_2(\mathfrak{z})\}$. Then $q(\mathfrak{z})Q(\mathfrak{z})$ behaves like $\mathfrak{z}^{-\gamma}$ for some $\gamma > 1$. By standard oscillation criteria for first-order linear delay equations [20], if

$$\limsup_{\mathfrak{z} \to \infty} \int_{w(\mathfrak{z})}^{\mathfrak{z}} q(s) Q(s) ds > \frac{1}{e},$$

then the equation is oscillatory. For sufficiently large $q_0 > 0$, this integral condition can be satisfied. Therefore, by Corollary 2.4, if the above integral condition holds, then equation (E1) is oscillatory. Thus, for $\varepsilon > 2$ and q_0 sufficiently large, all solutions of equation (E1) are oscillatory.

Example 3.2. Consider the following third-order half-linear delay differential equation:

$$\left(\mathfrak{z}^4\left(\varphi''(\mathfrak{z})\right)^3\right)' + \frac{1}{\mathfrak{z}^2}\varphi^3\left(\frac{\mathfrak{z}}{2}\right) = 0, \quad \mathfrak{z} \geq 2. \tag{E2}$$

In view of equation (E2), $n = 3$, $\varepsilon = 3$, $p(\mathfrak{z}) = \mathfrak{z}^4$, $q(\mathfrak{z}) = \frac{1}{\mathfrak{z}^2}$ and $w(\mathfrak{z}) = \frac{\mathfrak{z}}{2}$. Moreover, we have $\int_2^\infty s^{-4/3}ds < \infty$ and $A_2(\mathfrak{z}, \mathfrak{z}/2) = C_1\mathfrak{z}^{2/3}$, where C_1 is a positive constant. FFurther, we have

$$\int_2^\infty C_1^2\mathfrak{z}^{-2/3}\,d\mathfrak{z} = \infty.$$

All conditions of Corollary 2.10 are satisfied. Therefore, all solutions of equation (E2) are oscillatory.

4 Conclusion

In this paper, we have studied the oscillation of odd-order half-linear differential equations with retarded arguments under the non-canonical condition $\int_{\mathfrak{z}_0}^\infty p^{-1/\varepsilon}(s)ds < \infty$. The equation under consideration generalizes several classical models by incorporating both nonlinearity (via the half-linear structure) and delay effects, making it applicable to a wide range of physical and biological systems with memory. Our main contribution lies in extending oscillation criteria to this broader class of equations, particularly for odd orders $n \geq 3$, where few results exist. By employing a linearization technique and comparison principles with first-order delay differential equations, we establish sufficient conditions for all solutions to be oscillatory, building upon foundational work in the field.

Compared to the existing literature, our results represent a significant generalization. While oscillation theory for second-order half-linear equations is well-developed (e.g., Agarwal et al. [2], Grace et al. [13]), and some results exist for third-order linear or canonical cases (e.g., Dzurina and Kotorova [7], Graef and Saker [15]), the non-canonical case for odd-order half-linear equations has been largely unexplored. Our work fills this gap by initiating a systematic study in this direction. Unlike many prior studies that assume the canonical condition, we focus precisely on the more challenging non-canonical scenario, which requires refined asymptotic analysis and new comparison functions involving iterated integrals of the coefficient $p(t)$. Furthermore, while earlier works often restrict to $n = 3$, our theorems apply to arbitrary odd orders, enhancing their applicability.

The methodology used—based on classifying possible non-oscillatory solution behaviors (Lemma 2.2) and reducing the half-linear problem to a linear comparison equation—is inspired by classical techniques from Koplatadze and Chanturiya [20] and Philos [25]. However, the extension to higher-order half-linear equations introduces technical challenges, particularly in justifying the sign patterns of derivatives and in the correct application of iterative integration. We note that some proofs in the current manuscript contain gaps or inaccuracies (e.g., incorrect sign assumptions in Case (III), undefined functions like $A_k(t,u)$, and missing assumption lists), which must be corrected to ensure mathematical rigor. Once revised, the proposed criteria could improve upon known results by relaxing conditions on the coefficients and delay functions.

In conclusion, this work advances the oscillation theory of functional differential equations by unifying several complex features: odd order, half-linearity, delay, and the non-canonical condition. Its value lies not only in the new theorems but also in the framework it proposes for analyzing higher-order nonlinear delay equations. Future research should focus on correcting the technical flaws, providing illustrative examples, and exploring extensions to neutral and superlinear/sublinear cases.

Acknowledgements

J. Alzabut would like to thank Prince Sultan University and Ostim Technical University for their help and support.

References

[1] R. P. Agarwal, M. Bohner, T. Li, C. Zhang, A new approach in the study of oscillatory behavior of even-order neutral delay differential equations, *Appl. Math. Comput.* **225**(2013), 787–794. https://doi.org/10.1016/j.amc.2013.09.037

[2] R. P. Agarwal, S. R. Grace, D. O'Regan, *Oscillation theory for second order linear, half-linear, superlinear and sublinear dynamic equations*, Kluwer, Dordrecht, 2002. https://doi.org/10.1007/978-94-017-2515-6

[3] R. P. Agarwal, S. R. Grace, D. O'Regan, The oscillation of certain higher-order functional differential equations, *Math. Comput. Model.* **37**(2003), 705–728. https://doi.org/10.1016/S0895-7177(03)00079-7

[4] J. Alzabut, S. R. Grace, S. S. Santra, G. N. Chhatria, Asymptotic and oscillatory behaviour of third-order nonlinear differential equations with canonical operator and mixed neutral terms, *Qual. Theory Dyn. Syst.* **22**(2023), Art. 15. https://doi.org/10.1007/s12346-022-00715-6

[5] D. D. Bainov, D. P. Mishev, *Oscillation theory of neutral differential equations with delay*, Adam Hilger Ltd., Bristol, 1991.

[6] J.-G. Dong, Oscillation behavior of second-order nonlinear neutral differential equations with deviating arguments, *Comput. Math. Appl.* **59**(2010), 3710–3717. https://doi.org/10.1016/j.camwa.2010.04.004

[7] J. Dzurina, R. Kotorova, Properties of the third-order trinomial differential equations with argument, *Nonlinear Anal.* **71**(2009), 1995–2002. https://doi.org/10.1016/j.na.2009.01.070

[8] S. R. Grace, New criteria on oscillatory behavior of third-order half-linear functional differential equations, *Mediterr. J. Math.* **20**(2023), Art. 180. https://doi.org/10.1007/s00009-023-02342-0

[9] S. R. Grace, R. P. Agarwal, M. Bohner, D. O'Regan, Oscillation of second-order strongly superlinear and strongly sublinear dynamic equations, *Commun. Nonlinear Sci. Numer. Simul.* **14**(2009), 3463–3471. https://doi.org/10.1016/j.cnsns.2009.01.003

[10] S. R. Grace, R. P. Agarwal, B. Kaymakçalan, W. Sae-jie, Oscillation theorems for second-order nonlinear dynamic equations, *J. Appl. Math. Comput.* **32**(2010), 205–218. https://doi.org/10.1007/s12190-009-0244-7

[11] S. R. Grace, E. Akin, M. Dikmen, On the oscillation of second-order nonlinear neutral dynamic equations with distributed deviating arguments on time scales, *Dynam. Systems Appl.* **23**(2014), 735–748.

[12] S. R. Grace, J. Alzabut, Oscillation results for nonlinear second-order difference equations with mixed neutral terms, *Adv. Differ. Equ.* **2020**, Paper No. 8, 12 pp. https://doi.org/10.1186/s13662-019-2472-y

[13] S. R. Grace, M. Bohner, R. P. Agarwal, On the oscillation of second-order half-linear dynamic equations, *J. Difference Equ. Appl.* **15**(2009), 451–460. https://doi.org/10.1080/10236190802125371

[14] S. R. Grace, J. R. Graef, M. A. El-Beltagy, On the oscillation of third-order neutral delay dynamic equations on time scales, *Comput. Math. Appl.* **63**(2012), 775–782. https://doi.org/10.1016/j.camwa.2011.11.042

[15] J. R. Graef, S. H. Saker, Oscillation theory of third-order nonlinear functional differential equations, *Hiroshima Math. J.* **43**(2013), 49–72. https://doi.org/10.32917/hmj/1368217950

[16] J. K. Hale, S. M. V. Lunel, *Introduction to functional differential equations*, Springer, New York, 1993. https://doi.org/10.1007/978-1-4612-4342-7

[17] G. H. Hardy, J. E. Littlewood, G. Pólya, *Inequalities*, Reprint of the 1952 edition, Cambridge University Press, Cambridge, UK, 1988.

[18] I. T. Kiguradze, On the oscillation of solutions of equation $\frac{d^m u}{dt^m} + a(t)u^m \operatorname{sgn} u = 0$, *Mat. Sb.* **65**(1964), 172–187.

[19] I. T. Kiguradze, T. A. Chanturia, *Asymptotic properties of solutions of nonautonomous ordinary differential equations*, Kluwer Academic Publishers, Dordrecht, 1993. https://doi.org/10.1007/978-94-011-1808-8

[20] R. G. Koplatadze, T. A. Chanturiya, Oscillating and monotone solutions of first-order differential equations with deviating argument, *Differ. Uravn.* **18**(1982), 1463–1465.

[21] G. Ladas, I. P. Stavroulakis, Oscillation caused by several retarded and advanced arguments, *J. Differential Equations* **44**(1982), 134–152. https://doi.org/10.1016/0022-0396(82)90029-8

[22] F. Lescher, C. Roos, Robust stability of time-delay systems with structured uncertainties: A μ-analysis based algorithm, in: *2011 50th IEEE Conference on Decision and Control and European Control Conference*, Orlando, FL, USA, 2011, pp. 4955–4960. https://doi.org/10.1109/CDC.2011.6161129.

[23] T. Li, Y. Rogovchenko, C. Zhang, Oscillation results for second-order nonlinear neutral differential equations, *Adv. Differ. Equ.* **2013**, Art. 336, 13 pp. https://doi.org/10.1186/1687-1847-2013-336

[24] S.-I. Niculescu, *Delay effects on stability: A robust control approach*, Springer, London, 2001. https://doi.org/10.1007/1-84628-553-4

[25] Ch. G. Philos, On the existence of nonoscillatory solutions tending to zero at infinity for differential equations with positive delays, *Arch. Math. (Basel)* **36**(1981), 168–178. https://doi.org/10.1007/BF01223686

[26] H. Qin, N. Shang, Y. Lu, A note on oscillation criteria of second-order nonlinear neutral delay differential equations, *Comput. Math. Appl.* **56**(2008), 2987–2992. https://doi.org/10.1016/j.camwa.2008.09.004

[27] J. S. W. Wong, Necessary and sufficient conditions for oscillation of second-order neutral differential equations, *J. Math. Anal. Appl.* **252**(2000), 342–352. https://doi.org/10.1006/jmaa.2000.7063