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Insensitizing controls for a nonlocal-in-time semilinear heat equation

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Abstract. In this paper, a nonlocal-in-time semilinear heat equation is studied. The goal of this paper is prove the existence of insensitizing controls for a function that depends on the L^2-norm of solutions of the nonlocal-in-time semilinear heat equation. This problem is equivalent to the null controllability result for a nonlinear cascade system of the heat equations with terms that are nonlocal in time. To prove the null controllability of this nonlinear cascade system, we first obtained a Carleman inequality for an adjoint system, and after, we use Liusternik's Theorem.

Keywords: nonlinear parabolic equation, null controllability, Carleman inequality, insensitizing controls.

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1 Introduction

One of the issues related to controllability includes: the problem of insensitizing control, which was originally addressed by J. L. Lions in [23,24], leading to numerous papers on this topic both in hyperbolic and parabolic equations.

Regarding the semi-linear heat equation, the initial findings were presented in [5], focusing on a distributed control, specifically

$$\begin{cases} y_t - \Delta y + f(y) = \xi + v\chi_\omega & \text{in } \Omega \times (0,T), \\ y(x,t) = 0 & \text{on } \partial\Omega \times (0,T), \\ y(x,0) = y_0(x) + \tau \hat{y}_0(x) & \text{in } \Omega, \end{cases} \tag{1.1}$$

where Ω is an open set in $\mathbb{R}^N$; χ_ω denotes the characteristic function of the open set ω, $\omega \subset \Omega$; ξ, $\hat{y}_0$ are given in Y_1, Y_0 (certain spaces). The data of the state equation (1.1) is incomplete in the following sense:

- $\hat{y}_0 \in Y_0$ is unknown and $\|\hat{y}_0\|_{Y_0} = 1$.
- $\tau \in \mathbb{R}$ is unknown and small enough.

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Additionally, taking into account the energy functional:

$$\Phi(y) := \frac{1}{2}\iint_{\mathcal{O}\times(0,T)} |y(x,t;\tau,v)|^2\,dx\,dt, \tag{1.2}$$

the problem of insensitizing controls can be stated, roughly, as follows:

We say that the control v insensitizes Φ if

$$\left|\frac{\partial\Phi}{\partial\tau}(y(\cdot,\cdot;\tau,v))\Big|_{\tau=0}\right| = 0, \tag{1.3}$$

when (1.2) holds, the functional Φ is locally insensitive to the perturbation $\tau\hat{y}_0$.

In [5], the authors introduced and studied the notion of approximate insensitizing controls (ϵ-insensitizing controls) of (1.1). Also, we can mentioned that in [6] the authors studied the existence of insensitizing controls for a semilinear heat equation when we consider nonlinearities that behave superlinearly at infinity. In order to get rid of the condition $y_0 = 0$, the authors prove that the problem of ϵ-insensitizing controls is equivalent to an approximate controllability result for a cascade system, which is established therein.

Furthermore, the insensitizing problem, as we have seen in this special case, is directly related to control problems for coupled systems. In particular, one could ask whether it is possible to control both states of a coupled system just by acting on one equation. In [17] and [16], as well as some insensitizing problems, the author studied this problem respectively for Stokes and heat systems in a more general framework.

In the study presented in [31], the authors studied the insensitizing control problem with constraints on the control for a nonlinear heat equation for more general cost functionals, by means of Kakutani's Fixed Point Theorem combined with an adapted Carleman inequality, with respect to the insensitizing controllability quasi-linear equations have been studied in [25] and [28]. Additionally, some results regarding the controllability of nonlocal-in-time semilinear heat equation types have been examined in [27], where the author focused on the following system:

$$\begin{cases} y_t - \Delta y + \varphi\left(\int_0^T \alpha(x,t,r)y(x,r)\,dr\right)y = v\chi_\omega & \text{in } \Omega\times(0,T), \\ y(x,t) = 0 & \text{on } \partial\Omega\times(0,T), \\ y(x,0) = y_0(x) & \text{in } \Omega. \end{cases} \tag{1.4}$$

Due to the limited number of controllability results for the system described in (1.4), the main contribution of this paper is our investigation into the insensitizing controllability of systems of the type (1.4), particularly focusing on the energy functional (1.2).

2 The problem and main result

2.1 The problem

Let $\Omega \subset \mathbb{R}^N (N = 1,2,3)$ be a bounded domain whose boundary Γ is regular enough. We fix $T > 0$ and denote by $Q := \Omega\times(0,T)$ and $\Sigma = \partial\Omega\times(0,T)$. The symbols C_i and $\hat{C}_i$ are used to design generic positive constants for $i \in \mathbb{N}$. Let us assume that ω and $\mathcal{O}$ are two given non-empty open subsets of Ω. In the sequel, we will denote by χ_ω the characteristic function of the subset ω; $(\cdot,\cdot)$ and $||\cdot||_{L^2(\Omega)}$ respectively the L^2 scalar product and the norm in Ω.

We consider the nonlocal-in-time semilinear parabolic system.

$$\begin{cases} y_t - \Delta y + \varphi\left(\int_0^T \alpha(x,t,r)y(x,r)\,dr\right) y = \xi + v\chi_\omega & \text{in } Q, \\ y(x,t) = 0 & \text{on } \Sigma, \\ y(x,0) = y_0(x) + \tau\hat{y}_0(x) & \text{in } \Omega, \end{cases} \tag{2.1}$$

where v is the control and y is the associated state; ξ and y_0 are known functions, τ is an unknown small real number, $\hat{y}_0$ is a known function, let us assume that

$$\begin{cases} \varphi \in C^2(\mathbb{R}),\, 0 \le \varphi(r) \le a(1+|r|)\ \forall\, r \in \mathbb{R} \quad \text{for some } a > 0, \\ |\varphi'(r)| \le M_0\ \forall\, r \in \mathbb{R}, \end{cases} \tag{2.2}$$

$$|\varphi''(r)| \le M_1 \quad \forall\, r \in \mathbb{R}, \tag{2.3}$$

$$\alpha \in L^\infty(\Omega \times (0,T) \times (0,T)), \tag{2.4}$$

$$y_0,\, \hat{y}_0 \in H_0^1(\Omega) \quad \text{and} \quad v \in L^2(\omega \times (0,T)). \tag{2.5}$$

Also, we will consider the energy functional

$$\Phi(y) := \frac{1}{2}\iint_{\mathcal{O}\times(0,T)} |y(x,t;\tau,v)|^2\,dxdt. \tag{2.6}$$

We will denote by

$$X_0 = \{z \in L^2(Q) : z \in L^2(0,T;H^2(\Omega)) \cap L^\infty(0,T;H_0^1(\Omega)),\, z_t \in L^2(Q)\}.$$

The following theorem guarantees the existence and uniqueness of the solution for system (2.1).

Theorem 2.1. *Under the assumptions* (2.2), (2.4), *and* (2.5). *There exists* $r_0 > 0$ *such that for each* $\tau \in \mathbb{R}$, $y_0 \in H_0^1(\Omega)$, $\hat{y}_0 \in H_0^1(\Omega)$, $\xi \in L^2(Q)$, *and* $v \in L^2(\omega \times (0,T))$ *satisfying*

$$\|y_0\|_{H_0^1(\Omega)} + \|\tau\hat{y}_0\|_{H_0^1(\Omega)} + \|\xi\|_{L^2(Q)} + \|v\|_{L^2(\omega\times(0,T))} \le r_0,$$

then the system (2.1) *has a unique strong solution* y *satisfying*

$$\|y\|_{X_0}^2 \le C_1\left(\|y_0\|_{H_0^1(\Omega)}^2 + \tau^2\|\hat{y}_0\|_{H_0^1(\Omega)}^2 + \|\xi\|_{L^2(Q)}^2 + \|v\|_{L^2(\omega\times(0,T))}^2\right), \tag{2.7}$$

where $C_1 := C_1\big(\Omega,\, M_0,\, T,\, a,\, \|\alpha\|_{L^\infty(\Omega\times(0,T);L^2(0,T))}\big)$.

The proof of Theorem 2.1 can be found in the paper [27, Theorem 2.1].

Remark 2.2. Notice that under other hypotheses of the functions α, φ, and the initial data y_0, results of the existence of a solution of system (2.1) have been studied in papers [32–34, 36].

Definition 2.3. For a given function $\xi \in L^2(Q)$ and $y_0 \in H_0^1(\Omega)$, a control $v \in L^2(\omega \times (0,T))$. We say that the control v *insensitizes* $\Phi(y)$ if

$$\frac{\partial \Phi}{\partial \tau}(y[y_0,\, \xi,\, \hat{y},\, \tau,\, v])\big|_{\tau=0} = 0 \quad \forall\, \hat{y} \in H_0^1(\Omega). \tag{2.8}$$

2.2 The main result

The main result in this paper is the following:

Theorem 2.4. *Under the assumptions* (2.2), (2.3), (2.4), $\omega \cap \mathcal{O} \neq \emptyset$, $y_0 = 0$, *and*

$$|\alpha(x,t,r)|^2 \leq K_0(\Omega) \exp\left(\frac{-2s\eta_*}{t(T-t)}\right) \exp\left(\frac{-2s\eta_*}{r(T-r)}\right) \quad \forall\ (x,t,r) \in \Omega \times (0,T) \times (0,T) \quad (2.9)$$

for $s \geq s_2$, *and* $\eta_* = e^{4\lambda\|\eta^0\|_\infty} - e^{2\lambda\|\eta^0\|_\infty}$ *with* s, s_2, *and* λ *the terms of Remark* 3.4. *Then, there exist two positive constants* $\tilde{M}$ *and* δ *depending only on* N, Ω, T, *and* a *such that for any* $\xi \in L^2(Q)$ *satisfying*

$$\left\| \exp\left(\frac{\tilde{M}}{t}\right) \xi \right\|_{L^2(Q)} \leq \delta, \quad (2.10)$$

one can find a control $v \in L^2(\omega \times (0,T))$, *which insensitizes the functional* Φ *in the sense of Definition* 2.3.

Recently, significant advancements have been achieved in the study regarding the existence of solutions for parabolic systems featuring nonlocal terms in space, as illustrated in the papers [8, 9, 20, 37]. As for the investigation into the existence of solutions for parabolic equations with nonlocal terms in time, the following papers can be referenced: [7, 33–35]. Furthermore, recent developments have also been made in the area of null controllability of parabolic equations with nonlocal terms in both space and time; the papers that can be mentioned include [4, 21, 27]. The main novelty is that we deal with the prove the existence of controls that insensitize the functional (2.6).

The paper is organized as follows.

In Section 3 a Carleman inequality is presented for an adjoint system of type (3.3). Then, we prove the controllability of the linear system (3.2) using the techniques of [13, 14, 19]. In Section 4 we prove Theorem 2.4, using the arguments used in [13] and [21], obviously first to prove the local null controllability of optimal system (3.1), and after we prove the Proposition 3.1. In Section 5, we will mention some open problems.

In the Appendix, we will prove some existence and uniqueness results of the solution of systems (3.1), and (4.44).

3 Preliminary results

3.1 Reformulation of insensitizing problem for the functional Φ

Given the functions y and h which belong to the space $L^2(Q)$, we shall introduce

$$B(y,h)(x,t) = \int_0^T \varphi'\left(\int_0^T \alpha(x,r,s) y(x,s)\, ds\right) y(x,r)\alpha(x,r,t)h(x,r)\, dr$$

and

$$\gamma(y)(x,t) = \varphi\left(\int_0^T \alpha(x,t,r)y(x,r)\, dr\right).$$

Then, the special form of Φ allows us to reformulate our insensitizing problem as a controllability problem of a cascade system.

Proposition 3.1. *Under the assumptions* (2.2), (2.3), (2.4), (2.5), $\xi \in L^2(Q)$, *and* $y_0 = 0$. *If the functions* v *and* ξ *are sufficiently small in their spaces respectively, and the corresponding strong solution* $(y, h) \in X_0 \times X_0$ *of the following nonlinear system*

$$\begin{cases} y_t - \Delta y + \gamma(y)y = \xi + v\chi_\omega & \text{in } Q, \\ -h_t - \Delta h + B(y,h) + \gamma(y)h = y\chi_{\mathcal{O}} & \text{in } Q, \\ y(x,t) = 0,\ h(x,t) = 0 & \text{on } \Sigma, \\ y(x,0) = 0,\ h(x,T) = 0 & \text{in } \Omega, \end{cases} \tag{3.1}$$

satisfies $h(x,0) = 0$ *in* Ω, *then the control* v *insensitizes the functional* Φ *(defined by* (2.6)*).*

Proof. The proof of this proposition can be found in Section 4. □

In order to prove that the system (3.1) is null controllable at time $t = 0$, we consider the linearized system for (3.1).

$$\begin{cases} y_t - \Delta y + \varphi(0)y = v\chi_\omega + f_1 & \text{in } Q, \\ -h_t - \Delta h + \varphi(0)h = y\chi_{\mathcal{O}} + f_2 & \text{in } Q, \\ y(x,t) = 0,\ h(x,t) = 0 & \text{on } \Sigma, \\ y(x,0) = 0,\ h(x,T) = 0 & \text{in } \Omega, \end{cases} \tag{3.2}$$

and the adjoint system of (3.2)

$$\begin{cases} -\phi_t - \Delta\phi + \varphi(0)\phi = \psi\chi_{\mathcal{O}} + g_1 & \text{in } Q, \\ \psi_t - \Delta\psi + \varphi(0)\psi = g_2 & \text{in } Q, \\ \phi(x,t) = 0,\ \psi(x,t) = 0 & \text{on } \Sigma, \\ \phi(x,T) = 0,\ \psi(x,0) = \psi_0(x) & \text{in } \Omega. \end{cases} \tag{3.3}$$

3.2 Carleman estimates for the adjoint system (3.3)

In the context of the null controllability analysis of parabolic systems, Carleman estimates are a very powerful tool (see [14,19]). In order to state our Carleman inequality, we need to define some weight functions. Let ω_0 be a non-empty open subset of $\omega \cap \mathcal{O}$, and set

$$\sigma(x,t) = \frac{e^{4\lambda\|\eta^0\|_\infty} - e^{\lambda\left(2\|\eta^0\|_\infty + \eta^0(x)\right)}}{t(T-t)}$$

and

$$\zeta(x,t) = \frac{e^{\lambda\left(2\|\eta^0\|_\infty + \eta^0(x)\right)}}{t(T-t)}$$

for some parameter $\lambda > 0$. Here, $\eta^0 \in C^2(\overline{\Omega})$ stands for a function that satisfies

$$|\nabla\eta^0| \geq k_0 > 0 \quad \text{in } \Omega \setminus \overline{\omega}_0, \quad \eta^0 > 0 \quad \text{in } \Omega \quad \text{and} \quad \eta^0 = 0 \quad \text{on } \partial\Omega.$$

The proof of the existence of such function η^0 can be found in [14]. We introduce the following notation

$$I(s,\lambda;\phi) := \iint_Q e^{-2s\sigma}\left[(s\zeta)^{-1}(|\phi_t|^2 + |\Delta\phi|^2) + \lambda^2 s\zeta|\nabla\phi|^2 + \lambda^4(s\zeta)^3|\phi|^2\right] dxdt.$$

We will deduce a Carleman inequality for the solution to systems of the kind (3.3).

Proposition 3.2. *Let us assume that $g_1, g_2 \in L^2(Q)$. There exist positive constants λ_1, s_1, and C_2 such that, for any $s \geq s_1$ and $\lambda \geq \lambda_1$, any $\psi_0 \in L^2(\Omega)$, the corresponding solution to (3.3) satisfies*

$$\begin{aligned} I(s,\lambda;\varphi) + I(s,\lambda;\psi) \leq C_2 \Bigg(\iint_Q e^{-2s\sigma} \left[\lambda^4 (s\zeta)^3 |g_1|^2 + |g_2|^2 \right] dxdt \\ + \iint_{\omega\times(0,T)} e^{-2s\sigma} \lambda^8 (s\zeta)^7 |\varphi|^2 \, dxdt \Bigg). \end{aligned} \tag{3.4}$$

Furthermore, C_2 and λ_1 only depend on Ω and ω, and s_1 can be chosen of the form $s_1 = C(T+T^2)$, where C only depends on Ω, ω, and $\varphi(0)$.

Proof. The proof of this Proposition is similar to the proof of the Proposition 2.2 in [10], and the Proposition 2.3 in [28]. □

Now, we will deduce a Carleman estimate with functions blowing up only at $t = 0$, which allows us to prove an observability inequality for the system (3.3).

Let us define the new weight functions

$$\begin{aligned} \bar\sigma(x,t) &= \frac{e^{4\lambda\|\eta^0\|_\infty} - e^{\lambda(2\|\eta^0\|_\infty + \eta^0(x))}}{\ell(t)}, & \bar\zeta(x,t) &= \frac{e^{\lambda(2\|\eta^0\|_\infty + \eta^0(x))}}{\ell(t)}, \\ \sigma^*(t) &= \max_{x\in\bar\Omega} \bar\sigma(x,t), & \zeta^*(t) &= \min_{x\in\bar\Omega} \bar\zeta(x,t), \\ \hat\sigma(t) &= \min_{x\in\bar\Omega} \bar\sigma(x,t), & \hat\zeta(t) &= \max_{x\in\bar\Omega} \bar\zeta(x,t), \end{aligned}$$

where the function ℓ is given by

$$\ell(t) = \begin{cases} t(T-t), & \text{if } 0 \leq t \leq \frac{T}{2}, \\ \frac{T^2}{4}, & \text{if } \frac{T}{2} < t \leq T. \end{cases}$$

Note that $\bar\sigma = \sigma$ and $\bar\zeta = \zeta$ in $\left(\frac{T}{2}, T\right)$, and let us introduce the following notation:

$$\bar I(s,\lambda,\phi) = \iint_Q e^{-2s\bar\sigma} \left[(s\bar\zeta)^{-1} (|\phi_t|^2 + |\Delta\phi|^2) + \lambda^2 s\bar\zeta |\nabla\phi|^2 + \lambda^4 (s\bar\zeta)^3 |\phi|^2 \right] dxdt.$$

For all this, we have the following proposition (Carleman inequality with weights of type $e^{-2s\bar\sigma}\lambda^{k+1}(s\bar\zeta)^k$)

Proposition 3.3. *Let us that g_1, $g_2 \in L^2(Q)$. There exist positive constants λ_2, s_2 such that, for any $s \geq s_2$ and $\lambda \geq \lambda_2$, there exists a constant $C_3(s,\lambda) > 0$ with the following property: for any $\psi_0 \in L^2(\Omega)$, the associated solution to (3.3) satisfies*

$$\begin{aligned} &\bar I(s,\lambda,\varphi) + \bar I(s,\lambda,\psi) \\ &\quad \leq C_3 \left(\iint_Q e^{-2s\bar\sigma} \left[\lambda^4 (s\bar\zeta)^3 |g_1|^2 + |g_2|^2 \right] dxdt + \iint_{\omega\times(0,T)} e^{-2s\bar\sigma} \lambda^8 (s\bar\zeta)^7 |\varphi|^2 \, dxdt \right). \end{aligned}$$

Furthermore, λ_2 and s_2 only depend on Ω, ω, T, and $\varphi(0)$.

Proof. The proof of this proposition can be found in [28, Proposition 4], and it is similar to the proof of the Proposition 2.3 in [10]. □

Remark 3.4. For $\lambda > 0$ sufficiently large such that satisfy $\lambda > \frac{ln(4)}{\|\eta^0\|_\infty}$ and $\lambda \geq \lambda_2$, and fix $s > 0$ satisfying $s \geq s_2$ with λ_2 and s_2 the same of Proposition 3.3. If we denote by

$$\beta(x,t) = \frac{2}{5}\bar{\sigma}(x,t), \qquad \beta^*(t) = \max_{t\in\bar{\Omega}} \beta(x,t), \qquad \hat{\beta}(t) = \min_{x\in\bar{\Omega}} \beta(x,t),$$

the following inequality is satisfied

$$\hat{\beta}(t) \leq \beta(x,t) \leq \frac{5}{4}\hat{\beta}(t) \quad \text{and} \quad \frac{4}{5}\beta^*(t) \leq \beta(x,t) \leq \beta^*(t),$$

and by Proposition 3.3, we have the following

$$\begin{aligned} &I_*(s,\lambda,\varphi) + I_*(s,\lambda,\psi) \\ &\leq C_4 \left(\iint_Q e^{-4s\beta^*} \left[\lambda^4 (s\hat{\zeta})^3 |g_1|^2 + |g_2|^2\right] dxdt + \iint_{\omega\times(0,T)} e^{-4s\beta^*} \lambda^8 (s\hat{\zeta})^7 |\varphi|^2 \, dxdt \right) \end{aligned} \tag{3.5}$$

where,

$$I_*(s,\lambda,\phi) = \iint_Q e^{-5s\beta^*} \left[(s\hat{\zeta})^{-1} (|\phi_t|^2 + |\Delta\phi|^2) + \lambda^2 s\zeta^* |\nabla\phi|^2 + \lambda^2 (s\zeta^*)^3 |\phi|^2 \right] dxdt.$$

3.3 Null controllability for the linear system (3.2)

Assume that the notations and hypothesis of Remark 3.4 hold. In order to simplify the notation, we will denote by

$$\begin{aligned} &\rho_0 = e^{2s\beta^*}(\zeta^*)^{-3/2}, \qquad \rho_1 = e^{2s\beta^*}(\zeta^*)^{-7/2}, \qquad \rho = e^{\frac{5}{2}s\beta^*}, \\ &\hat{\rho}_0 = e^{\frac{3}{2}s\beta^*}(\zeta^*)^{-9/2}, \qquad \hat{\rho}_k = e^{\frac{3}{2}s\beta^*}(\zeta^*)^{-\frac{(15+2k)}{2}}, \qquad k \in \mathbb{Z}^+. \end{aligned} \tag{3.6}$$

Thanks to Proposition 3.3 and Remark 3.4, we will be able to prove the null controllability of (3.2) for right-hand f_1 and f_2 that decay sufficiently fast to zero as $t \to 0^+$. Indeed, we have the following result

Proposition 3.5. *Assume the hypothesis of Proposition 3.3, Remark 3.4, and let f_1 and f_2 satisfying ρf_1, $\rho f_2 \in L^2(Q)$. Then, the system (3.2) is null-controllable at time $t = 0$. More precisely, there exists $v \in L^2(\omega \times (0,T))$ such that, if (y,h) is the solution of (3.2), we have*

a)

$$\begin{aligned} &\iint_{\omega\times(0,T)} \rho_1^2 |v|^2 \, dxdt \leq C_5 \iint_Q \left(\rho^2 |f_1|^2 + \rho^2 |f_2|^2\right) dxdt, \\ &\iint_Q \left(\rho_0^2 |y|^2 + \rho_0^2 |h|^2\right) dxdt < +\infty, \end{aligned} \tag{3.7}$$

b)

$$\begin{aligned} &\sup_{t\in[0,T]} \left(\hat{\rho}_0^2 \int_\Omega |y|^2 \, dx\right) + \sup_{t\in[0,T]} \left(\hat{\rho}_0^2 \int_\Omega |h|^2 \, dx\right) + \iint_Q \hat{\rho}_0^2 \left(|\nabla y|^2 + |\nabla h|^2\right) dxdt \\ &\leq C_6 \left(\iint_Q (\rho_0^2 |y|^2 + \rho_0^2 |h|^2) \, dxdt \right. \\ &\qquad \left. + \iint_Q (\rho^2 |f_1|^2 + \rho^2 |f_2|^2) \, dxdt + \iint_{\omega\times(0,T)} \rho_1^2 |v|^2 \, dxdt \right), \end{aligned} \tag{3.8}$$

c)

$$\begin{aligned}
&\sup_{t\in[0,T]}\left(\hat{\rho}_1^2\int_\Omega |\nabla y|^2\,dx\right)+\sup_{t\in[0,T]}\left(\hat{\rho}_1^2\int_\Omega |\nabla h|^2\,dx\right)\\
&\quad+\iint_Q \hat{\rho}_1^2\left(|\Delta y|^2+|\Delta h|^2\right)\,dxdt+\iint_Q \hat{\rho}_1^2\left(|y_t|^2+|h_t|^2\right)\,dxdt\\
&\le C_7\left(\iint_Q(\rho_0^2|y|^2+\rho_0^2|h|^2)\,dxdt+\iint_{\omega\times(0,T)}\rho_1^2|v|^2\,dxdt\right.\\
&\qquad\left.+\iint_Q(\rho^2|f_1|^2+\rho^2|f_2|^2)\,dxdt\right).
\end{aligned}\tag{3.9}$$

Proof. The proof of this proposition is exactly the same as that of Proposition 5 in [28], and similar to the proof of Proposition 2.6 in [10]. □

4 Proof of main theorem

4.1 Local null controllability of optimal system (3.1)

In this subsection we prove the null controllability for the optimal system using Liusternik's Theorem. We use the strategies used in the papers [3, 10, 12, 18, 26, 28, 29], for the desired controllability.

Theorem 4.1 (Liusternik's Theorem)**.** *Let E and F be Banach spaces and let $A : B_E(0,r) \subset E \to F$ be a C^1 mapping. Let us assume that the derivative $A'(0) : E \to F$ is onto and let us denote by $\xi_0 = A(0)$. Then, there exists $\epsilon > 0$, a mapping $W : B_F(\xi_0,\epsilon) \subset F \to E$, and a constant $K > 0$ satisfying*

$$\begin{gathered}
W(z)\in B_E(0,r)\quad \textit{and}\quad A(W(z))=z,\quad \forall\, z\in B_F(\xi_0,\epsilon)\\
\|W(z)\|_E\le K\|z-\xi_0\|_F\quad \forall\, z\in B_F(\xi_0,\epsilon).
\end{gathered}$$

Proof. The proof of this theorem can be found in [1]. □

Now, let us introduce the following spaces

$$\begin{aligned}
E &= \Big\{(y,h,v) : \rho_0 y,\ \rho_0 h,\ y_{x_i},\ h_{x_i},\ y_{x_ix_j},\ h_{x_ix_j}\in L^2(Q),\ \rho_1 v\in L^2(\omega\times(0,T)),\\
&\qquad \rho(\mathcal{L}_1 y - v\chi_\omega),\ \rho(\mathcal{L}_2 h - y\chi_{\mathcal{O}})\in L^2(Q),\ y|_\Sigma=0,\ h|_\Sigma=0,\ h(T)=0\Big\},\\
Z &= \{w\in L^2(Q):\rho w\in L^2(Q)\},\\
F &= Z\times Z
\end{aligned}$$

where $\mathcal{L}_1 y$ and $\mathcal{L}_2 h$ indicate the subsequent expressions:

$$\mathcal{L}_1 y = y_t-\Delta y+\varphi(0)y\quad\text{and}\quad \mathcal{L}_2 h=-h_t-\Delta h+\varphi(0)h$$

and the norms in E and F are respectively:

$$\begin{aligned}
\|(y,h,v)\|_E^2 &= \|\rho_0 y\|_{L^2(Q)}^2+\|\rho_0 h\|_{L^2(Q)}^2+\|\rho_1 v\|_{L^2(\omega\times(0,T))}^2\\
&\quad+\|\rho(\mathcal{L}_1 y - v\chi_\omega)\|_{L^2(Q)}^2+\|\rho(\mathcal{L}_2 h - y\chi_{\mathcal{O}})\|_{L^2(Q)}^2
\end{aligned}$$

and

$$\|(f_1,f_2)\|_F^2 = \iint_Q |\rho f_1|^2\,dxdt + \iint_Q |\rho f_2|^2\,dxdt.$$

It is evident that E and F constitute two Banach spaces with respect to the norms $\|\cdot\|_E$ and $\|\cdot\|_F$.

Remark 4.2. Observe that, considering Proposition 3.5, it follows that we have

$$\|\hat{\rho}_1 h\|_{X_0}^2 + \|\hat{\rho}_1 y\|_{X_0}^2 \le C_8\|(y,h,v)\|_E^2 \quad \forall\ (y,h,v)\in E.$$

Proof that the mapping A meets the conditions of Theorem 4.1

To apply Liusternik's Theorem in demonstrating the null controllability of system (3.1), we will introduce the following mapping

$$A = (A_1,A_2) : E \to F$$

given by

$$\begin{cases} A_1(y,h,v) = y_t - \Delta y + \gamma(y)y - v\chi_\omega, \\ A_2(y,h,v) = -h_t - \Delta h + \displaystyle\int_0^T \theta(y)(x,r)y(x,r)\alpha(x,r,t)h(x,r)\,dr + \gamma(y)h - y\chi_{\mathcal{O}} \end{cases} \tag{4.1}$$

where $\theta(y)(x,r) = \varphi'\big(\int_0^T \alpha(x,r,s)y(x,s)\,ds\big)$.

In order to guarantee that mapping A meets the conditions set forth in Liusternik's Theorem, we will employ the subsequent lemmas.

Lemma 4.3. *Let $A : E \to F$ be the mapping defined by* (4.1)*. Then, we have*

a) A is well defined and continuous.

b) A is continuously differentiable.

Proof. To demonstrate this lemma, we will utilize the findings from Proposition 3.5, consider the observations in Remark 4.2, and take into account the characteristics of the weights outlined in (3.6).

Proof of a): In order to prove this item, we will attempt the following steps

i) A is well defined in E.

ii) A is continuous in E.

Proof of item i): First, let us prove the following

$$A_1(y,h,v)\in Z \quad \text{for all } (y,h,v)\in E \tag{4.2}$$

and

$$A_2(y,h,v)\in Z \quad \text{for all } (y,h,v)\in E. \tag{4.3}$$

Let us prove (4.2). Given $(y,h,v)\in E$, we have

$$
\begin{aligned}
&\iint_Q \rho^2 |A_1(y,h,v)|^2\, dxdt \\
&\quad = \iint_Q \rho^2 |y_t - \Delta y + \varphi\left(\int_0^T \alpha(x,t,s)y(x,s)\,ds\right) - v\chi_\omega|^2\, dxdt \\
&\quad \le 2\iint_Q \rho^2|\mathcal{L}_1 y - v\chi_\omega|\, dxdt \\
&\qquad + 2\iint_Q |\varphi\left(\int_0^T \alpha(x,t,s)y(x,s)\,ds\right) - \varphi(0)|^2|y|^2\, dxdt,
\end{aligned}
$$

$$
\iint_Q \rho^2|A_1(y,h,v)|^2\, dxdt \le 2\|(y,h,v)\|_E^2 + 2M_0^2 \iint_Q \rho^2 |\int_0^T \alpha(x,t,s)y(x,s)\,ds|^2|y|^2\, dxdt, \tag{4.4}
$$

by applying (2.9), Remark 4.2, Hölder's inequality, Sobolev's immersions, and utilizing the properties of weights defined in (3.6), we obtain

$$
\begin{aligned}
&2M_0^2\iint_Q \rho^2 |\int_0^T \alpha(x,t,s)y(x,s)\,ds|^2|y|^2\, dxdt \\
&\quad \le 2M_0^2K_0 \iint_Q \left(e^{-\frac{2s\eta_*}{t(T-t)}}\rho^2\right)\left(\int_0^T \hat\rho_1^2|y|^2\,ds\right)|y|^2\, dxdt \\
&\quad \le 2M_0^2K_0\hat C_1 \iint_Q (\hat\rho_1^2|y|^2)\left(\int_0^T \hat\rho_1^2|y|^2\,ds\right) dxdt \\
&\quad = 2M_0^2K_0\hat C_1 \int_0^T\int_0^T \hat\rho_1^2(t)\hat\rho_1^2(r)\left(\int_\Omega |y(t)|^2|y(r)|^2\,dx\right) dtdr \\
&\quad \le 2M_0^2K_0\hat C_1 \int_0^T\int_0^T \hat\rho_1^2(t)\hat\rho_1^2(r)\|y(\cdot,r)\|_{L^4(\Omega)}^2\|y(\cdot,t)\|_{L^4(\Omega)}^2 dtdr \\
&\quad \le 2M_0^2K_0\hat C_1\overline{C}_1^4 \int_0^T\int_0^T \hat\rho_1^2(t)\hat\rho_1^2(r)\|\nabla y(\cdot,r)\|_{L^2(\Omega)}^2\|\nabla y(\cdot,t)\|_{L^2(\Omega)}^2 dtdr \\
&\quad \le 2M_0^2K_0\hat C_1\overline{C}_1^4 T^2\left(\sup_{t\in[0,T]} (\hat\rho_1^2\int_\Omega |\nabla y|^2\,dx)\right)^2 \\
&\quad \le 2M_0^2K_0\hat C_1\overline{C}_1^4 T^2\|(y,h,v)\|_E^4
\end{aligned}
$$

where $\overline{C_1}$ is the constant such that satisfies $\|\cdot\|_{L^4(\Omega)} \le \overline{C}_1\|\cdot\|_{H_0^1(\Omega)}$, and the other constant $\hat C_1 = \hat C_1(s,\,\lambda,\,\Omega,\,T)$ and by the expression on the right-hand side of (4.4), we obtain

$$
\iint_Q \rho^2|A_1(y,h,v)|^2\, dxdt \le 2\|(y,h,v)\|_E^2 + 2M_0^2K_0\hat C_1\overline{C}_1^4T^2\|(y,h,v)\|_E^4,
$$

then

$$
\iint_Q \rho^2|A_1(y,h,v)|^2\, dxdt \le C\left(\|(y,h,v)\|_E^2 + \|(y,h,v)\|_E^4\right) < +\infty
$$

and consequently, (4.2) has been proven.

Now, let us prove (4.3). Given $(y,h,v)\in E$, we have

$$
\begin{aligned}
A_2(y,h,v) &= (\mathcal{L}_2 h - y\chi_\omega) + \int_0^T \varphi'\left(\int_0^T \alpha(x,r,s)y(x,s)\,ds\right) y(x,r)\alpha(x,r,t)h(x,r)\,dr \\
&\quad + \left(\varphi(\int_0^T \alpha(x,t,r)y(x,r)\,dr) - \varphi(0)\right) h
\end{aligned}
$$

and consequently,

$$\begin{aligned}
&\iint_Q \rho^2|A_2(y,h,v)|^2\,dxdt\\
&\quad\leq 3\iint_Q \rho^2|\mathcal{L}_2 h - y\chi_\omega|^2\,dxdt\\
&\qquad + 3\iint_Q \rho^2\Big|\int_0^T \varphi'\left(\int_0^T \alpha(x,r,s)y(x,s)\,ds\right) y(x,r)\alpha(x,r,t)h(x,r)\,dr\Big|^2\,dxdt\\
&\qquad + 3\iint_Q \rho^2\Big|\varphi\left(\int_0^T \alpha(x,t,r)y(x,r)\,dr\right) - \varphi(0)\Big|^2|h|^2\,dxdt.
\end{aligned}$$

By utilizing (2.9), Hölder's inequality, Sobolev's immersions, in conjunction with Remark 4.2 for the second and third terms on the right-hand side of the previous inequality, we derive

$$\begin{aligned}
&\iint_Q \rho^2|A_2(y,h,v)|^2\,dxdt\\
&\quad\leq 3\|(y,h,v)\|_E^2 + 3M_0^2K_0\overline{C}_1^4T^3\hat{C}_1\left(\sup_{t\in[0,T]}(\hat{\rho}_1^2\int_\Omega|\nabla y|^2\,dx)\right)\left(\iint_Q \hat{\rho}_1^2|\nabla h|^2\,dxdt\right)\\
&\qquad + 3M_0^2K_0\overline{C}_1^4T^2\hat{C}_1\left(\sup_{t\in[0,T]}(\hat{\rho}_1^2\int_\Omega|\nabla y|^2\,dx)\right)\left(\sup_{t\in[0,T]}(\hat{\rho}_1^2\int_\Omega|\nabla h|^2\,dx)\right)
\end{aligned} \tag{4.5}$$

then,

$$\iint_Q \rho^2|A_2(y,h,v)|^2\,dxdt \leq C\left(\|(y,h,v)\|_E^2 + \|(y,h,v)\|_E^4\right) < +\infty$$

and consequently, (4.2) has been proven.

As a result of (4.2) and (4.3), we can assert that A is well-defined.

The item $i)$ has been proven.

Proof of (ii): In order to prove that A is continuous in E, we will show that for each $(y,h,v)\in E$ we have

$$\begin{aligned}
&\text{if } \{(y^n,h^n,v^n)\}_{n=1}^\infty \subset E \text{ is such that } (y^n,h^n,v^n)\to(y,h,v) \text{ in } E,\\
&\qquad\text{then } A(y^n,h^n,v^n)\to A(y,h,v) \text{ in } F.
\end{aligned} \tag{4.6}$$

Given $\{(y^n,h^n,v^n)\}_{n=1}^\infty \subset E$ such that $(y^n,h^n,v^n)\to(y,h,v)$ in E. In order to prove that $A(y^n,h^n,v^n)\to A(y,h,v)$, we are going to calculate

$$\begin{aligned}
&\iint_Q \rho^2|A_1(y^n,h^n,v^n) - A_1(y,h,v)|^2\,dxdt\\
&\quad\leq 3\iint_Q \rho^2|\mathcal{L}_1(y^n-y) - (v^n-v)\chi_\omega|^2\,dxdt\\
&\qquad + 3\iint_Q \rho^2\left|\varphi\left(\int_0^T \alpha(x,t,r)y^n(x,r)\,dr\right)\right|^2|y^n-y|^2\,dxdt\\
&\qquad + \iint_Q \rho^2\left|\varphi\left(\int_0^T \alpha(x,t,r)y^n(x,r)\,dr\right) - \varphi\left(\int_0^T \alpha(x,t,r)y(x,r)\,dr\right)\right|^2|y|^2\,dxdt\\
&\quad= 3I_1^n + 3I_2^n + 3I_3^n.
\end{aligned} \tag{4.7}$$

Continuing, let us estimate the terms I_1^n, I_2^n, and I_3^n. Obviously, by definition of norm in E, we have

$$I_1^n \leq C\|(y^n,h^n,v^n) - (y,h,v)\|_E^2 \tag{4.8}$$

and by Sobolev's immersion and Remark 4.2,

$$I_2^n \leq M_0^2 K_0 C^4 C_8 \|(y^n,h^n,v^n)-(y,h,v)\|_E^2 \|(y,h,v)\|_E^2 \tag{4.9}$$

and similarly,

$$I_3^n \leq M_0^2 K_0 C^4 C_8 \|(y^n,h^n,v^n)-(y,h,v)\|_E^2 \|(y,h,v)\|_E^2. \tag{4.10}$$

Utilizing the estimates (4.8), (4.9), (4.10) in (4.7), it follows that we obtain

$$\begin{aligned} &\iint_Q \rho^2 |A_1(y^n,h^n,v^n)-A_1(y,h,v)|^2\,dxdt \\ &\qquad \leq C(1+\|(y,h,v)\|_E^2)\|(y^n,h^n,v^n)-(y,h,v)\|_E^2. \end{aligned} \tag{4.11}$$

Now, let us calculate an estimate similar to (4.11) for the term

$$\iint_Q \rho^2 |A_2(y^n,h^n,v^n)-A_2(y,h,v)|^2\,dxdt.$$

Indeed, we have

$$\begin{aligned} &\iint_Q \rho^2 |A_2(y^n,h^n,v^n)-A_2(y,h,v)|^2\,dxdt \\ &\quad \leq 6\iint_Q \rho^2 |\mathcal{L}_2(h^n-h)-(y^n-y)\chi_{\mathcal{O}}|^2\,dxdt \\ &\qquad + 6\iint_Q \rho^2 \left|\int_0^T \theta(y^n)\alpha(x,t,r)y^n(h^n-h)\,dr\right|^2\,dxdt \\ &\qquad + 6\iint_Q \rho^2 \left|\int_0^T \theta(y^n)\alpha(x,t,r)(y^n-y)h\,dr\right|^2\,dxdt \\ &\qquad + 6\iint_Q \rho^2 \left|\int_0^T (\theta(y^n)-\theta(y))\,\alpha(x,t,r)yh\,dr\right|^2\,dxdt \\ &\qquad + 6\iint_Q \rho^2 |\gamma(y^n)-\gamma(0)|^2|h^n-h|^2\,dxdt \\ &\qquad + 6\iint_Q \rho^2 |\gamma(y^n)-\gamma(y)|^2|h|^2\,dxdt \\ &\quad = \sum_{i=1}^{6} 6J_i^n. \end{aligned} \tag{4.12}$$

Let us estimate the terms J_i^n for $i=1,2,\ldots,6$. Indeed, by definition of norm in E, we have

$$J_1^n \leq \|(y^n,h^n,v^n)-(y,h,v)\|_E^2 \tag{4.13}$$

and by (2.3) and Remark 4.2, it follows that we obtain

$$J_2^n \leq M_0^2 K_0 C^4 C_8^2 \|(y^n,h^n,v^n)-(y,h,v)\|_E^2 \|(y^n,h^n,v^n)\|_E^2 \tag{4.14}$$

and similarly,

$$J_3^n \leq M_0^2 K_0 C^4 C_8^2 \|(y^n,h^n,v^n)-(y,h,v)\|_E^2 \|(y,h,v)\|_E^2 \tag{4.15}$$

by (2.3), and Remark 4.2, we have

$$J_4^n \leq M_0^2 K_0^2 C^4 \tilde{C}_1 C_8^3 \|(y^n,h^n,v^n)-(y,h,v)\|_E^2 \|(y,h,v)\|_E^4 \tag{4.16}$$

by (2.2) and Remark 4.2, we have

$$J_5^n \leq M_0^2 K_0 C^4 C_8^2 T \|(y^n, h^n, v^n) - (y, h, v)\|_E^2 \|(y^n, h^n, v^n)\|_E^2 \tag{4.17}$$

similarly,

$$J_6^n \leq M_0^2 K_0 C^4 C_8^2 T \|(y^n, h^n, v^n) - (y, h, v)\|_E^2 \|(y, h, v)\|_E^2. \tag{4.18}$$

Utilizing the estimates (4.13)–(4.18) in (4.12), we have

$$\begin{aligned} &\iint_Q \rho^2 |A_2(y^n, h^n, v^n) - A_2(y, h, v)|^2 \, dxdt \\ &\quad \leq C \left(1 + \|(y^n, h^n, v^n)\|_E^2 + \|(y, h, v)\|_E^2 + \|(y, h, v)\|_E^4\right) \|(y^n, h^n, v^n) - (y, h, v)\|_E^2, \end{aligned} \tag{4.19}$$

and consequently,

$$\begin{aligned} &\|A(y^n, h^n, v^n) - A(y, h, v)\|_F^2 \\ &\quad \leq C \left(1 + \|(y^n, h^n, v^n)\|_E^2 + \|(y, h, v)\|_E^2 + \|(y, h, v)\|_E^4\right) \|(y^n, h^n, v^n) - (y, h, v)\|_E^2. \end{aligned}$$

Then, as $n \to +\infty$, we have $A(y^n, h^n, v^n) \to A(y, h, v)$ in F, and consequently (4.6) holds.

Therefore, A is continuous in E, and the item $a)$ has been proven.

Proof of b): To establish that A is continuously differentiable, we will demonstrate that

i) A is Gâteaux differentiable at any $(y, h, v) \in E$ and calculate $A'(y, h, v)$.

ii) The mapping $(y, h, v) \mapsto A'(y, h, v)$ is continuous from E into $\mathcal{L}(E, F)$.

Proof of item i): For each $(y, h, v) \in E$, let us introduce the linear mapping $DA : E \to F$, with $DA(\tilde{y}, \tilde{h}, \tilde{v}) = \big(DA_1(\tilde{y}, \tilde{h}, \tilde{v}), DA_2(\tilde{y}, \tilde{h}, \tilde{v})\big)$ defined by

$$DA_1(\tilde{y}, \tilde{h}, \tilde{v}) = \tilde{y}_t - \Delta \tilde{y} + \gamma(y)\tilde{y} + \theta(y) \left(\int_0^T \alpha(x, t, r)\tilde{y}(x, r) \, dr\right) y - \tilde{v}\chi_\omega,$$

$$\begin{aligned} DA_2(\tilde{y}, \tilde{h}, \tilde{v}) = &- \tilde{h}_t - \Delta \tilde{h} + \int_0^T \theta(y)(x, r)\tilde{y}(x, r)h(x, r) \, dr \\ &+ \int_0^T \theta(y)(x, r)y(x, r)\alpha(x, r, t)\tilde{h}(x, r) \, dr \\ &+ \int_0^T \mu(y)(x, r) \left(\int_0^T \alpha(x, r, s)\tilde{y}(x, s) \, ds\right) y(x, r)\alpha(x, r, t)h(x, r) \, dr \\ &+ \theta(y) \left(\int_0^T \alpha(x, t, r)\tilde{y}(x, r) \, dr\right) h \\ &+ \gamma(y)\tilde{h} - \tilde{y}\chi_{\mathcal{O}} \end{aligned}$$

where $\mu(y)(x, r) = \varphi''\big(\int_0^T \alpha(x, r, s)y(x, s) \, ds\big)$.

In order to establish that DA represents the G-derivative of A at the point (y, h, v), we are going to demonstrate that

$$\begin{gathered} \frac{1}{\lambda}\left[A_i((y, h, v) + \lambda(\tilde{y}, \tilde{h}, \tilde{v})) - A_i(y, h, v)\right] \to DA_i(\tilde{y}, \tilde{h}, \tilde{v}) \\ \text{strongly in } Z \text{ for } i = 1, 2 \text{ as } \lambda \to 0 \end{gathered} \tag{4.20}$$

Certainly, after performing some calculations, we have

$$\frac{1}{\lambda}\left[A_1((y,h,v)+\lambda(\tilde{y},\tilde{h},\tilde{v}))-A_1(y,h,v)\right]-DA_1(\tilde{y},\tilde{h},\tilde{v})=J_1+J_2 \tag{4.21}$$

where

$$J_1=\left[\frac{(\gamma(y+\lambda\tilde{y})-\gamma(y))}{\lambda}-\theta(y)\left(\int_0^T\alpha(x,t,r)\tilde{y}(x,r)\,dr\right)\right]y$$

and

$$J_2=(\gamma(y+\lambda\tilde{y})-\gamma(y))\,\tilde{y},$$

from these two terms, we obtain

$$\|J_i\|_Z\to 0\quad\text{as }\lambda\to 0\text{ for }i=1,2. \tag{4.22}$$

Let us demonstrate (4.22) for the case when $i=1$. Evidently, we observe that

$$\|J_1\|_Z^2=\|\rho\,(\theta(y+\lambda\tau_1\tilde{y})-\theta(y))\left(\int_0^T\alpha(\cdot,\cdot,r)\tilde{y}(\cdot,r)\,dr\right)y\|_{L^2(Q)}^2$$

for some $\tau_1\in(0,1)$.

Utilizing the assumption presented in (2.9) and incorporating the observations from Remark 4.2 in the concluding equality, we have

$$\begin{aligned}\|J_1\|_Z^2&\le C\lambda^2\|(\tilde{y},\tilde{h},\tilde{v})\|_E^4\|(y,h,v)\|_E^2\\&\le\lambda^2M_1^2K_0^2\iint_Q\left(\int_0^T\hat{\rho}_1^2|\tilde{y}|^2\,ds\right)^2\left(e^{-\frac{2s\eta_*}{t(T-t)}}\rho^2\right)|y|^2\,dxdt\\&\le\lambda^2M_1^2K_0^2\hat{C}_1\overline{C}_3^6T^3\left(\sup_{t\in[0,T]}(\hat{\rho}_1^2\int_\Omega|\nabla\tilde{y}|^2\,dx)\right)^2\left(\sup_{t\in[0,T]}(\hat{\rho}_1^2\int_\Omega|\nabla y|^2\,dx)\right)\\&\le\lambda^2M_1^2K_0^2\hat{C}_1\overline{C}_3^6T^3\|(\tilde{y},\tilde{h},\tilde{v})\|_E^4\|(y,h,v)\|_E^2,\end{aligned}$$

where the constant $\overline{C}_3$ is such that $\|\cdot\|_{L^6(\Omega)}\le\overline{C}_3\|\cdot\|_{H_0^1(\Omega)}$, and as $\lambda\to 0$, we have

$$\lim_{\lambda\to 0}\|J_1\|_Z^2=0. \tag{4.23}$$

Continuing, let us demonstrate the validity of (4.22) for the case when $i=2$. Evidently, we observe that

$$\|J_2\|_Z^2=\|\rho\,(\theta(y+\lambda\tau_1\tilde{y})\left(\int_0^T\alpha(\cdot,\cdot,r)\lambda\tilde{y}(\cdot,r)\,dr\right)\tilde{y}\|_{L^2(Q)}^2$$

for a $\tau_1\in(0,1)$.

Applying the assumption given in (2.9) and taking into account the insights from Remark 4.2 in the final equation, we have

$$\begin{aligned}\|J_2\|_Z^2&\le\lambda^2M_0^2\iint_Q\rho^2|\int_0^T\alpha(x,t,r)\tilde{y}(x,r)\,dr|^2|\tilde{y}|^2\,dxdt\\&\le\lambda^2M_0^2\overline{C}_1^4\hat{C}_1T^2\|(\tilde{y},\tilde{h},\tilde{v})\|_E^4\end{aligned}$$

as $\lambda\to 0$, we have

$$\lim_{\lambda\to 0}\|J_2\|_Z^2=0. \tag{4.24}$$

From (4.23) and (4.24), it follows that (4.20) is valid for $i = 1$.

To demonstrate the validity of (4.20) for $i = 2$, we will evaluate the left-hand side of (4.21) specifically for A_2

$$\frac{1}{\lambda}\left[A_2((y,h,v)+\lambda(\tilde{y},\tilde{h},\tilde{v})) - A_2(y,h,v)\right] - DA_2(\tilde{y},\tilde{h},\tilde{v}) = I_1 + \cdots + I_6 \tag{4.25}$$

where

$$\begin{aligned}
I_1 &= \frac{1}{\lambda}\left(\int_0^T \theta(y+\lambda\tilde{y})(x,r)\alpha(x,t,r)y(x,r)h(x,r)\,dr - \int_0^T \theta(y)(x,r)\alpha(x,t,r)y(x,r)h(x,r)\,dr\right)\\
&\quad - \int_0^T \mu(y)(x,r)\left(\int_0^T \alpha(x,r,s)\tilde{y}(x,s)\,ds\right)y(x,r)\alpha(x,r,t)h(x,r)\,dr\\
I_2 &= \left(\int_0^T \theta(y+\lambda\tilde{y})(x,r)y(x,r)\alpha(x,t,r)h(x,r)\,dr - \int_0^T \theta(y)(x,r)y(x,r)\alpha(x,t,r)h(x,r)\,dr\right)\\
I_3 &= \left(\int_0^T \theta(y+\lambda\tilde{y})(x,r)\tilde{y}(x,r)\alpha(x,t,r)h(x,r)\,dr - \int_0^T \theta(y)(x,r)\tilde{y}(x,r)\alpha(x,t,r)h(x,r)\,dr\right)\\
I_4 &= \lambda\int_0^T \theta(y+\lambda\tilde{y})(x,r)\alpha(x,t,r)\tilde{h}(x,r)\tilde{y}(x,r)\,dr\\
I_5 &= \left(\gamma(y+\lambda\tilde{y}) - \gamma(y)\right)\tilde{h}\\
I_6 &= \left[\frac{1}{\lambda}\left(\gamma(y+\lambda\tilde{y}) - \gamma(y)\right) - \theta(y)\int_0^T \alpha(x,t,r)\tilde{y}(x,r)\,dr\right]h.
\end{aligned}$$

Employing Proposition 3.5 alongside Remark 4.2, we are able to conclude that

$$\lim_{\lambda\to 0}\|I_i\|_Z = 0 \quad \text{for } i = 1,2,\ldots,6 \tag{4.26}$$

Based on results (4.24) and (4.26), we conclude that A is Gâteaux differentiable.

And consequently, item i) has been proven.

Proof of item ii): To prove that $A \in C^1(E,F)$, let us verify that the mapping $(y,h,v) \mapsto A'(y,h,v)$ is continuous from E into $\mathcal{L}(E,F)$, and for this we are going to show that

$$\begin{gathered}
\text{If} \quad (y^n,h^n,v^n) \to (y,h,v) \quad \text{in } E, \quad \text{one has}\\
\|\left(DA(y^n,h^n,v^n) - DA(y,h,v)\right)(\tilde{y},\tilde{h},\tilde{v})\|_F^2 \le \epsilon_n\|(\tilde{y},\tilde{h},\tilde{v})\|_E^2 \quad \forall\,(\tilde{y},\tilde{h},\tilde{v}) \in E\\
\text{with } \epsilon_n \to 0
\end{gathered} \tag{4.27}$$

To demonstrate (4.27)), by performing a straightforward calculation for A_1

$$DA_1(y^n,h^n,v^n)(\tilde{y},\tilde{h},\tilde{v}) - DA_1(y,h,v)(\tilde{y},\tilde{h},\tilde{v}) = O_{1,n} + O_{2,n} + O_{3,n}$$

where

$$\begin{aligned}
O_{1,n} &= \left(\gamma(y^n) - \gamma(y)\right)\tilde{y}\\
O_{2,n} &= \theta(y)\left(\int_0^T \alpha(x,t,r)\tilde{y}(x,r)\,dr\right)(y^n - y)\\
O_{3,n} &= \left(\gamma(y^n) - \gamma(y)\right)\left(\int_0^T \alpha(x,t,r)\tilde{y}(x,r)\,dr\right)y^n
\end{aligned}$$

Utilizing the properties of the function $\varphi(\cdot)$, alongside Proposition 3.5 and Remark 4.2, we obtain the following:

$$\|O_{j,n}\|_Z \le \epsilon_{j,n}^1\|(\tilde{y},\tilde{h},\tilde{v})\|_E \quad \text{with} \quad \lim_{n\to 0}\epsilon_{j,n}^1 = 0,\ j = 1,2,3.$$

The previous inequality suggests that

$$\| \left(DA_1(y^n,h^n,v^n) - DA_1(y,h,v)\right)(\tilde{y},\tilde{h},\tilde{v})\|_Z \leq \left(\sum_{j=1}^{3} \epsilon^1_{j,n}\right) \|(\tilde{y},\tilde{h},\tilde{v})\|_E. \tag{4.28}$$

To achieve a result similar to (4.28) for A_2, we will begin by carrying out a series of calculations concerning A_2

$$DA_2(y^n,h^n,v^n)(\tilde{y},\tilde{h},\tilde{v}) - DA_2(y,h,v)(\tilde{y},\tilde{h},\tilde{v}) = P_{1,n} + P_{2,n} + P_{3,n} + P_{4,n} + P_{5,n}$$

where

$$\begin{aligned}
P_{1,n} &= \int_0^T \theta(y^n)(x,r)\tilde{y}(x,r)\alpha(x,r,t)h^n(x,r)\,dr - \int_0^T \theta(y)(x,r)\tilde{y}(x,r)\alpha(x,r,t)h(x,r)\,dr,\\
P_{2,n} &= \int_0^T \theta(y^n)(x,r)y^n(x,r)\alpha(x,r,t)\tilde{h}(x,r)\,dr - \int_0^T \theta(y)(x,r)y(x,r)\alpha(x,r,t)\tilde{h}(x,r)\,dr,\\
P_{3,n} &= \int_0^T \mu(y^n)(x,r)\left(\int_0^T \alpha\tilde{y}\,ds\right) y^n(x,r)\alpha(x,r,t)h^n(x,r)\,dr,\\
&\quad - \int_0^T \mu(y)(x,r)\left(\int_0^T \alpha\tilde{y}\,ds\right) y(x,r)\alpha(x,r,t)h(x,r)\,dr,\\
P_{4,n} &= \left(\gamma(y^n) - \gamma(y)\right)\tilde{h},\\
P_{5,n} &= \theta(y^n)\left(\int_0^T \alpha(x,t,s)\tilde{y}(x,s)\,ds\right)h^n - \theta(y)\left(\int_0^T \alpha(x,t,s)\tilde{y}(x,s)\,ds\right)h.
\end{aligned}$$

By applying Proposition 3.5 together with Remark 4.2, we obtain

$$\|P_{j,n}\|_Z \leq \epsilon^2_{j,n}\|(\tilde{y},\tilde{h},\tilde{v})\|_E \quad \text{with} \quad \lim_{n\to 0} \epsilon^2_{j,n} = 0,\ j = 1,2,3,4,5.$$

Then, the previous inequality suggests that

$$\| \left(DA_2(y^n,h^n,v^n) - DA_2(y,h,v)\right)(\tilde{y},\tilde{h},\tilde{v})\|_Z \leq \left(\sum_{j=1}^{5} \epsilon^2_{j,n}\right) \|(\tilde{y},\tilde{h},\tilde{v})\|_E, \tag{4.29}$$

and based on (4.28) and (4.29), we can conclude that (4.27) is valid. Consequently, item *ii)* has been proven.

Then, since that the Gâteaux derivative of A is continuous from E into $\mathcal{L}(E,F)$. It is well know that A is Fréchet differentiable in E, and the Fréchet derivative of A is continuous from E into $\mathcal{L}(E,F)$ (to see for instance [2]). Therefore $A \in C^1(E,F)$. All this completes the demonstration of item b).

This ends the proof of Lemma 4.3. □

Lemma 4.4. *Let $A : E \to F$ be the mapping defined by (4.1). Then, $A'(0,0,0)$ is onto.*

Proof. Notice that by Lemma 4.3 we have $A'(0,0,0) = (A_1'(0,0,0), A_2'(0,0,0))$, with

$$\begin{cases} A_1'(0,0,0)(\tilde{y},\tilde{h},\tilde{v}) = \tilde{y}_t - \Delta\tilde{y} + \varphi(0)\tilde{y} - \tilde{v}\chi_\omega,\\ A_2'(0,0,0)(\tilde{y},\tilde{h},\tilde{v}) = -\tilde{h}_t - \Delta\tilde{h} + \varphi(0)\tilde{h} - \tilde{y}\chi_{\mathcal{O}}\end{cases}$$

for all $(\tilde{y},\tilde{h},\tilde{v}) \in E$.

Therefore, $A'(0,0,0)$ is onto if and only if for each $(f_1, f_2) \in F$ there is a $(y, h, v) \in E$ satisfying

$$\begin{cases} y_t - \Delta y + \varphi(0)y = v\chi_\omega + f_1 & \text{in } Q, \\ -h_t - \Delta h + \varphi(0)h = y\chi_{\mathcal{O}} + f_2 & \text{in } Q, \\ y(x,t) = 0,\, h(x,t) = 0 & \text{on } \Sigma, \\ y(x,0) = 0,\, h(x,T) = 0 & \text{in } \Omega. \end{cases} \tag{4.30}$$

According to Proposition 3.5 and Lemma 4.2, for any pair (f_1, f_2) within the set F, there is a corresponding triplet (y, h, v) in the set E that fulfills the condition expressed in (4.30). As a result, it follows that $A'(0,0,0)(y,h,v) = (f_1, f_2)$.

This ends the proof of the lemma.

Proof of the null controllability of optimal system (3.1)

In accordance with the Lemmas 4.3 and 4.4, we can apply Liusternik's Theorem (Theorem 4.1) and then, there exists $\epsilon > 0$, a mapping $W : B_F(0,\epsilon) \subset F \to E$ such that,

$$W(f_1, f_2) \in E \quad \text{and} \quad A(W(f_1, f_2)) = (f_1, f_2) \quad \forall\ (f_1, f_2) \in B_F(0, \epsilon) \tag{4.31}$$

and

$$\|W(f_1, f_2)\|_E \le K \|(f_1, f_2)\|_F \quad \forall\ (f_1, f_2) \in B_F(0, \epsilon). \tag{4.32}$$

In particular, if $(\xi, 0) \in B_F(0, \epsilon)$, there exists $(y, h, v) \in E$ such that

$$(y, h, v) = W(\xi, 0) \quad \text{and} \quad A(y, h, v) = (\xi, 0) \tag{4.33}$$

and

$$\|(y, h, v)\|_E \le K \|(\xi, 0)\|_F. \tag{4.34}$$

Thus, we prove that (3.1) is null locally controllable at time $t = 0$. □

Remark 4.5. Notice that the hypothesis (2.9) played a crucial role in establishing the null controllability of system (3.1), particularly in the estimates of the terms (4.4) and (4.5).

4.2 Proof of Proposition 3.1

For any $\hat{y}_0 \in H_0^1(\Omega)$ satisfying $\|\hat{y}_0\|_{H_0^1(\Omega)} = 1$ and $|\tau| < \tau_0$, let us denote by y_τ the solution of equation (2.1) associated to $y_0 = 0$, τ, and v, and (y, h) is the solution of (3.1). To prove (2.8), we prove

$$\frac{\partial \Phi}{\partial \tau}(y_\tau)|_{\tau=0} = \int_\Omega h(0)\hat{y}_0\, dx, \quad \forall\ \hat{y}_0 \in H_0^1(\Omega) \quad \text{with } \|\hat{y}_0\|_{H_0^1(\Omega)} = 1 \tag{4.35}$$

and

$$y_\tau \to y \quad \text{in } L^2(Q) \text{ as } \tau \to 0. \tag{4.36}$$

Let us represent w_τ as $w_\tau = y_\tau - y$. In this case, w_τ fulfills the system

$$\begin{cases} w_{\tau,t} - \Delta w_\tau + \gamma(y_\tau)w_\tau + (\gamma(w_\tau + y) - \gamma(y))\, y = 0 & \text{in } Q, \\ w_\tau(x,t) = 0 & \text{on } \Sigma, \\ w_\tau(x,0) = \tau \hat{y}_0(x) & \text{in } \Omega. \end{cases} \tag{4.37}$$

By multiplying both sides of equation $(4.37)_1$ by w_τ and then performing integration over the domain Ω, we obtain the following expression:

$$\frac{1}{2}\frac{d}{dt}\|w_\tau\|^2_{L^2(\Omega)} + \|\nabla w_\tau\|^2_{L^2(\Omega)} + \int_\Omega \gamma(y_\tau)|w_\tau|^2\,dx + \int_\Omega (\gamma(w_\tau+y)-\gamma(y))\,yw_\tau\,dx = 0.$$

From the properties of functions φ and α, and using the Young's inequality joint with some Sobolev's immersions we have

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|w_\tau\|^2_{L^2(\Omega)} + \|\nabla w_\tau\|^2_{L^2(\Omega)} &= -\int_\Omega (\gamma(w_\tau+y)-\gamma(y))\,yw_\tau\,dx \\ &\leq M_0\int_\Omega \left|\int_0^T \alpha w_\tau\,dr\right|\cdot|y|\cdot|w_\tau|\,dx \\ &\leq \frac{1}{4\overline{C}_2^2}\|w_\tau\|^2_{L^2(\Omega)} + M_0^2\overline{C}_2^2K_0\overline{C}_1^4\|\nabla w_\tau\|^2_{L^2(Q)}\cdot\|\nabla y\|^2_{L^2(\Omega)}\end{aligned}$$

where the constants $\overline{C}_1$ and $\overline{C}_2$ are such that $\|\cdot\|_{L^4(\Omega)} \leq \overline{C}_1\|\cdot\|_{H_0^1(\Omega)}$ and $\|\cdot\|_{L^2(\Omega)} \leq \overline{C}_2\|\cdot\|_{H_0^1(\Omega)}$. Then, we have

$$\frac{1}{2}\frac{d}{dt}\|w_\tau\|^2_{L^2(\Omega)} + \frac{3}{4}\|\nabla w_\tau\|^2_{L^2(\Omega)} \leq M_0^2\overline{C}_2^2K_0\overline{C}_1^4\|\nabla w_\tau\|^2_{L^2(Q)}\cdot\|\nabla y\|^2_{L^2(\Omega)}. \tag{4.38}$$

By Theorem 6.1, we have that there exists $\tilde{\epsilon}_0 > 0$ such that if

$$\|\xi\|_{L^2(Q)} + \|v\|_{L^2(\omega\times(0,T))} \leq \tilde{\epsilon}_0,$$

then, the solution (y,h) of (3.1) satisfies

$$\|(y,h)\|^2_{X_0\times X_0} \leq C_1\left(\|\xi\|^2_{L^2(Q)} + \|v\|^2_{L^2(\omega\times(0,T))}\right).$$

Taking $\tilde{r}_0 < \tilde{\epsilon}_0$ and $\|\xi\|_{L^2(Q)} + \|v\|_{L^2(\omega\times(0,T))} \leq \tilde{r}_0$, consequently in (4.38), we have

$$\frac{1}{2}\frac{d}{dt}\|w_\tau\|^2_{L^2(\Omega)} + \frac{3}{4}\|\nabla w_\tau\|^2_{L^2(\Omega)} \leq CM_0^2\overline{C}_2^2K_0\overline{C}_1^4C_1\tilde{r}_0^2\|\nabla w_\tau\|^2_{L^2(Q)}.$$

Integrating over $(0,T)$, we have

$$\frac{1}{2}\|w_\tau(T)\|^2_{L^2(\Omega)} + \frac{3}{4}\|\nabla w_\tau\|^2_{L^2(Q)} \leq CM_0^2\overline{C}_2^2K_0\overline{C}_1^4C_1\tilde{r}_0^2T\|\nabla w_\tau\|^2_{L^2(Q)} + \frac{1}{2}\|w_\tau(0)\|^2_{L^2(\Omega)}.$$

Taking $\tilde{r}_0$ small enough such that it satisfies

$$CM_0^2\overline{C}_2^2K_0\overline{C}_1^4C_1\tilde{r}_0^2T \leq \frac{1}{4} \tag{4.39}$$

we have

$$\frac{1}{4}\|w_\tau(T)\|^2_{L^2(\Omega)} + \frac{3}{4}\|\nabla w_\tau\|^2_{L^2(Q)} \leq \frac{1}{2}\|w_\tau(0)\|^2_{L^2(\Omega)}. \tag{4.40}$$

Notice that $w_\tau(0) \to 0$ in $L^2(\Omega)$ (as $\tau \to 0$), we find from (4.40) that

$$w_\tau \to 0 \quad \text{in } L^2(0,T;H_0^1(\Omega)). \tag{4.41}$$

Therefore,

$$y_\tau \to y \quad \text{in } L^2(0,T;H_0^1(\Omega)), \quad \text{as } \tau \to 0, \tag{4.42}$$

which yields (4.36).

In order to prove (4.35), we need to prove that

$$\frac{y_\tau - y}{\tau} \to p \quad \text{in } L^2(Q) \text{ as } \tau \to 0 \tag{4.43}$$

where p satisfies the following system

$$\begin{cases} p_t - \Delta p + \gamma(y)p + \theta(y)\left(\int_0^T \alpha(x,t,s)p(x,s)\,ds\right)y = 0 & \text{in } Q, \\ p(x,t) = 0 & \text{on } \Sigma, \\ p(x,0) = \hat{y}_0(x) & \text{in } \Omega. \end{cases} \tag{4.44}$$

We denote by $p_\tau = \frac{y_\tau - y}{\tau} - p$. It is then straightforward to verify that p_τ satisfies the following.:

$$\begin{cases} p_{\tau,t} - \Delta p_\tau + \gamma(y_\tau)p_\tau + (\gamma(y_\tau) - \gamma(y))\,p + H(y_\tau, y, p, \tau)y = 0 & \text{in } Q, \\ p_\tau(x,t) = 0 & \text{on } \Sigma, \\ p_\tau(x,0) = 0 & \text{in } \Omega, \end{cases} \tag{4.45}$$

where

$$H(y_\tau, y, p, \tau) = \frac{\gamma(y_\tau) - \gamma(y)}{\tau} - \theta(y)\left(\int_0^T \alpha(x,t,s)p(x,s)\,ds\right).$$

Multiplying both sides of the first equation of (4.45) by p_τ and then performing integration over Ω, we obtain

$$\frac{1}{2}\frac{d}{dt}\|p_\tau\|^2_{L^2(\Omega)} + \|\nabla p_\tau\|^2_{L^2(\Omega)} + \int_\Omega \gamma(y_\tau)|p_\tau|^2\,dx = B_1 + B_2 \tag{4.46}$$

where the terms B_1 and B_2 are

$$B_1 = -\int_\Omega (\gamma(y_\tau) - \gamma(y))\,pp_\tau\,dx$$

and

$$B_2 = -\int_\Omega H(y_\tau, y, p, \tau)yp_\tau\,dx.$$

Let us proceed to compute a few estimations for the terms B_1 and B_2, using the properties of φ and Young's inequality, we have

$$\begin{aligned} |B_1| &\leq M_0 \int_\Omega \left|\int_0^T \alpha(x,t,s)w_\tau(x,s)\,ds\right| \cdot |p| \cdot |p_\tau|\,dx \\ &\leq \frac{1}{4\overline{C}_2^2}\|p_\tau\|^2_{L^2(\Omega)} + \overline{C}_2^2 M_0^2 \int_\Omega \left|\int_0^T \alpha(x,t,s)w_\tau(x,s)\,ds\right|^2 |p|^2\,dx. \end{aligned} \tag{4.47}$$

Applying Sobolev's immersions alongside Sobolev's inequalities, we have

$$\begin{aligned} |B_1| &\leq \frac{1}{4}\|\nabla p_\tau\|^2_{L^2(\Omega)} + \overline{C}_2^2 M_0^2 \int_\Omega \left|\int_0^T \alpha(x,t,s)w_\tau(x,s)\,ds\right|^2 |p|^2\,dx \\ &\leq \frac{1}{4}\|\nabla p_\tau\|^2_{L^2(\Omega)} + \overline{C}_2^2 M_0^2 K_0 \overline{C}_1^4 \|\nabla w_\tau\|^2_{L^2(Q)} \|\nabla p\|^2_{L^2(\Omega)}. \end{aligned} \tag{4.48}$$

Similarly for B_2, using the Mean Value Theorem we have

$$B_2 = \left[\theta(\lambda_* y_\tau + (1-\lambda_*)y)\left(\int_0^T \alpha(x,t,s)p_\tau(x,s)\,ds\right)\right.$$
$$\left. + (\theta(\lambda_* y_\tau + (1-\lambda_*)y) - \theta(y))\left(\int_0^T \alpha(x,t,s)p(x,s)\,ds\right)\right]\cdot y\cdot p_\tau$$

for some $\lambda_* \in (0,1)$.

Using the triangle inequality for absolute values along with the properties of the function φ, it follows that

$$|B_2| \leq M_0 \int_\Omega |\int_0^T \alpha(x,t,s)p_\tau(x,s)\,ds|\cdot|y|\cdot|p_\tau|\,dx$$
$$+ M_1 \int_\Omega |\int_0^T \alpha(x,t,s)w_\tau(x,s)\,ds|\cdot|\int_0^T \alpha(x,t,s)p(x,s)\,ds|\cdot|y|\cdot|p_\tau|\,dx.$$

Applying Young's inequality, Sobolev's immersions alongside Sobolev's inequalities, we have

$$\begin{aligned}|B_2| &\leq \frac{2}{4}\|\nabla p_\tau\|^2_{L^2(\Omega)} + M_0^2\overline{C}_2^2 K_0 \overline{C}_1^4 \|\nabla p_\tau\|^2_{L^2(Q)}\|\nabla y\|^2_{L^2(\Omega)} \\ &+ M_1^2\overline{C}_2^2 K_0 \overline{C}_1^2 \|p\|^2_{L^2(0,T;H^2(\Omega))}\|\nabla w_\tau\|^2_{L^2(Q)}\|\nabla y\|^2_{L^2(\Omega)}.\end{aligned} \tag{4.49}$$

By performing the integration over the interval $(0,T)$ in equation (4.46), and applying the estimates provided in (4.48) and (4.49), we obtain the following:

$$\begin{aligned}\frac{1}{2}\|p_\tau(T)\|^2_{L^2(\Omega)} + \|\nabla p_\tau\|^2_{L^2(Q)} &\leq \frac{3}{4}\|\nabla p_\tau\|^2_{L^2(Q)} + \overline{C}_2^2 M_0^2 K_0 \overline{C}_1^4 \|\nabla w_\tau\|^2_{L^2(Q)}\|\nabla p\|^2_{L^2(Q)} \\ &+ M_0^2\overline{C}_2^2 K_0 \overline{C}_1^4 \|\nabla p_\tau\|^2_{L^2(Q)}\|\nabla y\|^2_{L^2(Q)} \\ &+ M_1^2\overline{C}_2^2 K_0 \overline{C}_1^2 \|p\|^2_{L^2(0,T;H^2(\Omega))}\|\nabla w_\tau\|^2_{L^2(Q)}\|\nabla y\|^2_{L^2(Q)}.\end{aligned}$$

By Theorem 6.1 $\|y\|_{X_0} \leq C_1\tilde{r}_0$ with $\tilde{r}_0$ sufficiently small, and considering the system (4.44) in conjunction with Theorem 6.2, it follows that we have

$$\|p\|_{X_0} \leq C\|\hat{y}_0\|_{H_0^1(\Omega)}$$

and assuming that

$$M_0^2\overline{C}_2^2 K_0 \overline{C}_1^4 \tilde{r}_0^2 \leq \frac{1}{8}$$

consequently, we have

$$\begin{aligned}\frac{1}{2}\|p_\tau(T)\|^2_{L^2(\Omega)} + \|\nabla p_\tau\|^2_{L^2(Q)} &\leq \frac{3}{4}\|\nabla p_\tau\|^2_{L^2(Q)} + \frac{1}{8}\|\nabla p_\tau\|^2_{L^2(Q)} \\ &+ M_1^2\overline{C}_2^2 K_0 \overline{C}_1^2 C\|\hat{y}_0\|^2_{H^1(\Omega)}\tilde{r}_0^2\|\nabla w_\tau\|^2_{L^2(Q)}.\end{aligned}$$

Therefore,

$$\|\nabla p_\tau\|^2_{L^2(Q)} \leq 8M_1^2\overline{C}_2^2 K_0 \overline{C}_1^2 C\|\hat{y}_0\|^2_{H^1(\Omega)}\tilde{r}_0^2\|\nabla w_\tau\|^2_{L^2(Q)} \tag{4.50}$$

from (4.41), we obtain

$$w_\tau \to 0 \quad \text{in } L^2(0,T;H_0^1(\Omega)) \text{ as } \tau \to 0.$$

By equation (4.41), it can be observed that we have

$$p_\tau \to 0 \quad \text{in } L^2(0,T;H_0^1(\Omega)) \text{ as } \tau \to 0.$$

This implies that,

$$\frac{y_\tau - y}{\tau} \to p \quad \text{in } L^2(0,T;H_0^1(\Omega)) \text{ as } \tau \to 0. \tag{4.51}$$

According to the definition of the functional Φ, we can determine that

$$\begin{aligned} \frac{\partial \Phi}{\partial \tau}(y_\tau)|_{\tau=0} &= \frac{1}{2}\lim_{\tau\to 0}\iint_{\mathcal{O}\times(0,T)}(y_\tau + y)\left(\frac{y_\tau - y}{\tau}\right) dxdt \\ &= \iint_{\mathcal{O}\times(0,T)} yp\, dxdt \end{aligned} \tag{4.52}$$

where p is the solution of (4.44).

Now, let us multiply in $(4.44)_1$ by h (which is the solution of (3.1)) and then perform the integration over the domain Q. We obtain

$$\iint_{\mathcal{O}\times(0,T)} py\, dxdt = \int_\Omega h(x,0)\hat{y}_0(x)\, dx, \quad \forall\ \hat{y}_0 \in H_0^1(\Omega) \quad \text{with} \quad \|\hat{y}_0\|_{H_0^1(\Omega)} = 1. \tag{4.53}$$

According to the calculations in (4.52) and (4.53), we find that if $h(0) = 0,$ then

$$\frac{\partial \Phi}{\partial \tau}(y_\tau)|_{\tau=0} = 0.$$

This completes the proof of Proposition 3.1

4.3 Proof of Theorem 2.4

Due to the locally null controllability of system (3.1), we can assert that there are constants $\delta > 0$ and $\tilde{M} > 0$ such that if

$$\iint_Q e^{\frac{2\tilde{M}}{t}}|\xi|^2\, dxdt < \delta, \tag{4.54}$$

then, one can find a control $v \in L^2(\omega \times (0,T))$ of (3.1) such that the state associated (y,h) satisfies

$$h(x,0) = 0 \quad \text{in } \Omega.$$

Furthermore, owing to (3.7) and (4.34), we deduce that

$$\iint_{\omega\times(0,T)} \rho_1^2|v|^2\, dxdt \le C\iint_Q \rho^2|\xi|^2\, dxdt \tag{4.55}$$

from (4.54) and (4.55), we have that v and ξ are sufficiently small satisfying,

$$\|\xi\|^2_{L^2(Q)} + \|v\|^2_{L^2(\omega\times(0,T))} \le \tilde{\epsilon}_0^2.$$

According to Proposition 3.1, all of these factors suggest that the control v insensitizes the functional Φ in the sense of Definition 2.3.

5 Some open questions

Let us now highlight a few open questions that naturally emerge from the findings presented in this paper:

- **Insensitizing controls for a nonlocal-in-time version of the types (2.1), incorporating a different functional Φ**

  In this work, the existence of insensitive controls for the system (2.1) was proven with a functional Φ of type (2.6), however, we can find in other papers such as [16] and [30] that the existence of insensitive controls for parabolic equations was worked with another functional of the form

$$\Phi(y) := \frac{1}{2}\iint_{\mathcal{O}\times(0,T)} |\nabla y(x,t,\tau,v)|^2\,dxdt. \tag{5.1}$$

  As a result of this paper and the functional (5.1), it is reasonable to pose the following question:

  Will it be possible to prove the existence of controls that insensitizes the system (2.1) with functional (5.1)?

- **Insensitizing controls for a nonlocal-in-time system of type (2.1) when the dimension N is greater than or equal to 4**

  Notice that in the proof of Proposition 3.1, we need to estimate the term:

$$\int_\Omega \left|\int_0^T \alpha(x,t,s) w_\tau(x,s)\,ds\right| \cdot \left|\int_0^T \alpha(x,t,s)p(x,s)\,ds\right| \cdot |y|\cdot|p_\tau|\,dx \tag{5.2}$$

  and to obtain a good estimate of (5.2) we need to use the Sobolev's immersion

$$H^2(\Omega) \hookrightarrow L^\infty(\Omega),$$

  since this immersion is only applicable when $N = 1, 2$, or 3, a similar issue arises in demonstrating the locally null controllability of (3.1). Consequently, proving insensitizing controls for a nonlocal-in-time type of (2.1) when $N \geq 4$ is an open problem.

6 Appendix: Existence and uniqueness results of solutions for the systems (3.1) and (4.44)

In this section, we will ensure both the existence and uniqueness of a broader system compared to the system (3.1). More specifically, we will focus on the following system:

$$\begin{cases} y_t - \Delta y + \gamma(y)y = \xi + v\chi_\omega + f_1 & \text{in } Q,\\ -h_t - \Delta h + B(y,h) + \gamma(y)h = y\chi_{\mathcal{O}} + f_2 & \text{in } Q,\\ y(x,t) = 0,\ h(x,t) = 0 & \text{on } \Sigma,\\ y(x,0) = 0,\ h(x,T) = 0 & \text{in } \Omega. \end{cases} \tag{6.1}$$

Theorem 6.1. *Under the assumptions* (2.2) *and* (2.4)*, there exists* $\tilde{\epsilon}_0 > 0$ *such that for* ξ, f_1, $f_2 \in L^2(Q)$, $v \in L^2(\omega \times (0,T))$ *satisfying*

$$\|f_1\|_{L^2(Q)} + \|f_2\|_{L^2(Q)} + \|\xi\|_{L^2(Q)} + \|v\|_{L^2(\omega\times(0,T))} \leq \tilde{\epsilon}_0,$$

then the system (6.1) *has a unique strong solution* $(y,h) \in X_0 \times X_0$ *satisfying*

$$\|(y,h)\|^2_{X_0\times X_0} \leq C_9 \left(\|f_1\|^2_{L^2(Q)} + \|f_2\|^2_{L^2(Q)} + \|\xi\|^2_{L^2(Q)} + \|v\|^2_{L^2(\omega\times(0,T))} \right)$$

where $C_9 := C_9(\Omega, M_0, T, a, \|\alpha\|_{L^\infty(\Omega\times(0,T);L^2(\Omega))})$.

Proof. Initially, we will demonstrate the existence of a strong solution for the system described by (6.1), and subsequently, we will establish the uniqueness of the solution.

Existence of a strong solution for system (6.1)

Let us define the following sets

$$\begin{cases} X_1 = L^\infty(0,T;L^2(\Omega)), \\ \hat{X}_1 = X_1 \times X_1, \\ \hat{X}_0 = X_0 \times X_0 \end{cases}$$

and for $R_0 > 0$, we define the mapping

$$\begin{aligned} \Lambda : B_{\hat{X}_1}[0,R_0] &\to \hat{X}_1 \\ (\hat{y},\hat{h}) &\mapsto (\tilde{y},\tilde{h}) \end{aligned}$$

where $(\tilde{y},\tilde{h})$ is the solution of system

$$\begin{cases} \tilde{y}_t - \Delta\tilde{y} + \gamma(\hat{y})\tilde{y} = \xi + v\chi_\omega + f_1 & \text{in } Q, \\ -\tilde{h}_t - \Delta\tilde{h} + \gamma(\hat{y})\tilde{h} = \displaystyle\int_0^T \theta(\hat{y})(x,r)\hat{y}(x,r)\alpha(x,r,t)\hat{h}(x,r)\,dr + \tilde{y}\chi_{\mathcal{O}} + f_2 & \text{in } Q, \\ \tilde{y}(x,t) = 0,\ \tilde{h}(x,t) = 0 & \text{on } \Sigma, \\ \tilde{y}(x,0) = 0,\ \tilde{h}(x,T) = 0 & \text{in } \Omega, \end{cases} \tag{6.2}$$

and $B_{\hat{X}_1}[0,R_0] = \{(y,h) : y,h \in X_1,\ \|(y,h)\|_{\hat{X}_1} \leq R_0\}$.

By the theory of parabolic equations (to see [11]), the system (6.2), has a unique strong solution $(\tilde{y},\tilde{h})$ that satisfies

$$\begin{aligned} \|(\tilde{y},\tilde{h})\|^2_{\hat{X}_0} \leq{}& \tilde{C}_1 \left(1 + \|\hat{y}\|^2_{X_1} + \|\hat{h}\|^2_{X_1} + \|\hat{y}\|^4_{X_1} + \|\hat{h}\|^4_{X_1}\right)\hat{R}^2 \\ &+ \tilde{C}_1 \left\| \int_0^T \theta(\hat{y})(\cdot,r)\hat{y}(\cdot,r)\alpha(\cdot,r,\cdot)\hat{h}(\cdot,r)\,dr \right\|^2_{L^2(Q)}, \end{aligned} \tag{6.3}$$

where

$$\hat{R} = \left(\|f_1\|^2_{L^2(Q)} + \|f_2\|^2_{L^2(Q)} + \|\xi\|^2_{L^2(Q)} + \|v\|^2_{L^2(\omega\times(0,T))} \right)^{1/2}.$$

Let us take $R_1 = \sqrt{2\tilde{C}_1}\hat{R}$ such that

$$B_{\hat{X}_0}[0,R_1] \subset B_{\hat{X}_1}[0,R_0]. \tag{6.4}$$

Notice that Λ is continuous. Then for $(\hat{y},\hat{h}) \in B_{\hat{X}_0}[0,R_1]$, by (6.3) and the results obtained in Proposition 3.5 and Remark 4.2 we have

$$
\begin{aligned}
\|\Lambda(\hat{y},\hat{h})\|^2_{\hat{X}_0} &\le \tilde{C}_1\left(1+R_1^2+R_1^4\right)\hat{R}^2+\tilde{C}_1\left\|\int_0^T \theta(\hat{y})(\cdot,r)\hat{y}(\cdot,r)\alpha(\cdot,r,\cdot)\hat{h}(\cdot,r)\,dr\right\|^2_{L^2(Q)}\\
&\le \frac{R_1^2}{2}+\tilde{C}_1\left(R_1^2+R_1^4\right)\hat{R}^2+\tilde{C}_1M_0^2TK_0\overline{C}_3^4R_1^4\\
&\le \frac{R_1^2}{2}+R_1^2\tilde{C}_1\hat{R}^2+R_1^2 2\tilde{C}_1^2\hat{R}^4+M_0^2TK_0\overline{C}_3^4 2\tilde{C}_1^2\hat{R}^2R_1^2.
\end{aligned}
$$

Take $\hat{R}$ small such that

$$
M_0^2TK_0\overline{C}_3^4 2\tilde{C}_1^2\hat{R}^2\le\frac{1}{8},\qquad \tilde{C}_1\hat{R}^2\le\frac{1}{8},\qquad 2\tilde{C}_1^2\hat{R}^4\le\frac{1}{8} \tag{6.5}
$$

By selecting and setting R_1 and $\hat{R}$ to fulfill conditions (6.4) and (6.5), we can ascertain that there is an $\tilde{\epsilon}_0>0$ sufficiently small such that if $\hat{R}<\tilde{\epsilon}_0$ we have

$$
\frac{R_1^2}{2}+R_1^2\tilde{C}_1\tilde{\epsilon}_0^2+R_1^2 2\tilde{C}_1^2\tilde{\epsilon}_0^4+M_0^2TK_0\overline{C}_3^4 2\tilde{C}_1^2\tilde{\epsilon}_0^2R_1^2\le R_1^2. \tag{6.6}
$$

Therefore,

$$
\|\Lambda(\hat{y},\hat{h})\|^2_{\hat{X}_0}\le R_1^2\quad\text{and}\quad \Lambda(B_{\hat{X}_0}[0,R_1])\subset B_{\hat{X}_0}[0,R_1]
$$

and as $\hat{X}_0$ is compact in $\hat{X}_1$ (see [22] and [24]), by Schauder's Fixed Point Theorem (see [15]), there exists $(y,h)\in B_{\hat{X}_0}[0,R_1]$ satisfying $\Lambda(y,h)=(y,h)$ and with (y,h) solves the system (6.2) with

$$
\begin{aligned}
\|(y,h)\|^2_{\hat{X}_0}&\le R_1^2=2\tilde{C}_1\hat{R}^2\\
&=2\tilde{C}_1\left(\|f_1\|^2_{L^2(Q)}+\|f_2\|^2_{L^2(Q)}+\|\xi\|^2_{L^2(Q)}+\|v\|^2_{L^2(\omega\times(0,T))}\right)\\
&\le 2\tilde{C}_1\tilde{\epsilon}_0^2.
\end{aligned}\tag{6.7}
$$

Uniqueness of the solution for system (6.1):

Let (y,h) and $(\tilde{y},\tilde{h})$ two solutions of (6.1), denote by $w=y-\tilde{y}$ and $u=h-\tilde{h}$, then we have

$$
\begin{cases}
w_t-\Delta w+\gamma(y)w+\left(\gamma(y)-\gamma(\tilde{y})\right)\tilde{y}=0 & \text{in } Q,\\
-u_t-\Delta u+\left(B(y,h)-B(\tilde{y},\tilde{h})\right)+\gamma(y)u+\left(\gamma(y)-\gamma(\tilde{y})\right)\tilde{h}=w\chi_{\mathcal{O}} & \text{in } Q,\\
w(x,t)=0,\ u(x,t)=0 & \text{on } \Sigma,\\
w(x,0)=0,\ u(x,T)=0 & \text{in } \Omega.
\end{cases}\tag{6.8}
$$

Multiplying by w in $(6.8)_1$, and integrating over Ω, we obtain

$$
\frac{1}{2}\frac{d}{dt}\|w\|^2_{L^2(\Omega)}+\|\nabla w\|^2_{L^2(\Omega)}+\int_\Omega\gamma(y)|w|^2\,dx=-\int_\Omega\left(\gamma(y)-\gamma(\tilde{y})\right)\tilde{y}w\,dx,
$$

by properties of the function φ,

$$
\frac{1}{2}\frac{d}{dt}\|w\|^2_{L^2(\Omega)}+\|\nabla w\|^2_{L^2(\Omega)}\le M_0\int_\Omega\left|\int_0^T\alpha(x,t,s)w(x,s)\,ds\right|\cdot|\tilde{y}|\cdot|w|\,dx.
$$

Using Sobolev's immersion joint with Hölder's inequalities,

$$
\begin{aligned}
\frac{1}{2}\frac{d}{dt}\|w\|^2_{L^2(\Omega)}+\|\nabla w\|^2_{L^2(\Omega)}&\le\overline{C}_2^2M_0^2\int_\Omega\left|\int_0^T\alpha(x,t,s)w(x,s)\,ds\right|^2|\tilde{y}|^2\,dx+\frac{1}{4\overline{C}_2^2}\|w\|^2_{L^2(\Omega)}\\
&\le\overline{C}_2^2M_0^2\|\alpha\|^2_{L^\infty(\Omega\times(0,T);L^2(0,T))}\int_\Omega\|w\|^2_{L^2(0,T)}|\tilde{y}|^2\,dx+\frac{1}{4}\|\nabla w\|^2_{L^2(\Omega)}.
\end{aligned}
$$

Integrating over $(0,T)$ and by (6.7), we have

$$
\begin{aligned}
&\frac{1}{2}\|w(T)\|^2_{L^2(\Omega)}+\frac{3}{4}\|\nabla w\|^2_{L^2(Q)}\\
&\quad\le M_0^2\overline{C}_2^2\|\alpha\|^2_{L^\infty(\Omega\times(0,T);L^2(0,T))}\int_0^T\int_0^T\int_\Omega|w(x,t)|^2|\tilde{y}(x,r)|^2\,dxdtdr\\
&\quad\le M_0^2\overline{C}_2^2\|\alpha\|^2_{L^\infty(\Omega\times(0,T);L^2(0,T))}\int_0^T\int_0^T\|w(\cdot,t)\|^2_{L^4(\Omega)}\|\tilde{y}(\cdot,r)\|^2_{L^4(\Omega)}\,dtdr\\
&\quad\le M_0^2\overline{C}_2^2\|\alpha\|^2_{L^\infty(\Omega\times(0,T);L^2(0,T))}\overline{C}_1^4T\|\nabla w\|^2_{L^2(Q)}\|\nabla\tilde{y}\|^2_{L^\infty(0,T;L^2(\Omega))}\\
&\quad\le M_0^2\overline{C}_2^2\|\alpha\|^2_{L^\infty(\Omega\times(0,T);L^2(0,T))}\overline{C}_1^4T2\tilde{C}_1\tilde{\epsilon}_0^2\|\nabla w\|^2_{L^2(Q)}.
\end{aligned}
$$

Let us assume that $\tilde{\epsilon}_0$ is sufficiently small, satisfying

$$
M_0^2\overline{C}_2^2\|\alpha\|^2_{L^\infty(\Omega\times(0,T);L^2(0,T))}\overline{C}_1^4T2\tilde{C}_1\tilde{\epsilon}_0^2\le\frac{1}{2}. \tag{6.9}
$$

We have

$$
\frac{1}{2}\|w(T)\|^2_{L^2(\Omega)}+\frac{1}{4}\|\nabla w\|^2_{L^2(Q)}\le 0,
$$

then

$$
w=0 \quad \text{in } L^2(0,T;H_0^1(\Omega)) \quad \text{and} \quad y=\tilde{y} \quad \text{a.e. in } Q. \tag{6.10}
$$

Also, from (6.10) we have that

$$
B(y,h)-B(\tilde{y},\tilde{h})=\int_0^T\theta(y)(x,r)\alpha(x,t,r)y(x,r)u(x,r)\,dr
$$

and the equation (6.8) is now,

$$
-u_t-\Delta u+\int_0^T\theta(y)(x,r)\alpha(x,t,r)y(x,r)u(x,r)\,dr+\gamma(y)u=0. \tag{6.11}
$$

Let us multiply by u in (6.11), and integrate over Ω

$$
\begin{aligned}
&-\frac{1}{2}\frac{d}{dt}\|u\|^2_{L^2(\Omega)}+\|\nabla u\|^2_{L^2(\Omega)}+\int_\Omega\gamma(y)|u|^2\,dx\\
&\qquad=-\int_\Omega\left(\int_0^T\theta(y)(x,r)\alpha(x,t,r)y(x,r)u(x,r)\,dr\right)u\,dx.
\end{aligned}
$$

Using the properties of φ and Young's inequality,

$$
\begin{aligned}
&-\frac{1}{2}\frac{d}{dt}\|u\|^2_{L^2(\Omega)}+\|\nabla u\|^2_{L^2(\Omega)}\\
&\quad\le\frac{1}{4\overline{C}_2^2}\|u\|^2_{L^2(\Omega)}+\overline{C}_2^2\int_\Omega\Big|\int_0^T\theta(y)(x,r)\alpha(x,t,r)y(x,r)u(x,r)\,dr\Big|^2\,dx\\
&\quad\le\frac{1}{4\overline{C}_2^2}\|u\|^2_{L^2(\Omega)}+\overline{C}_2^2M_0^2\|\alpha\|^2_{L^\infty(\Omega\times(0,T);L^2(0,T))}\int_0^T\|y\|^2_{L^4(\Omega)}\|u\|^2_{L^4(\Omega)}\,dt.
\end{aligned}
$$

Utilizing Sobolev's immersions and performing integration over the interval $(0,T)$, we obtain

$$
\begin{aligned}
&\frac{1}{2}\|u(0)\|^2_{L^2(\Omega)}+\|\nabla u\|^2_{L^2(Q)}\\
&\quad\le\frac{1}{4}\|\nabla u\|^2_{L^2(Q)}+M_0^2\overline{C}_2^2\overline{C}_1^4\|\alpha\|^2_{L^\infty(\Omega\times(0,T);L^2(0,T))}T\|\nabla y\|^2_{L^\infty(0,T;L^2(\Omega))}\|\nabla u\|^2_{L^2(Q)}\\
&\quad\le\frac{1}{4}\|\nabla u\|^2_{L^2(Q)}+M_0^2\overline{C}_2^2\overline{C}_1^4\|\alpha\|^2_{L^\infty(\Omega\times(0,T);L^2(0,T))}T2\tilde{C}_1\tilde{\epsilon}_0^2\|\nabla u\|^2_{L^2(Q)}.
\end{aligned}
$$

Let us assume that $\tilde{\epsilon}$ is sufficiently small such that

$$M_0^2\overline{C}_2^{2}\overline{C}_1^{4}\|\alpha\|^2_{L^\infty(\Omega\times(0,T);L^2(0,T))}T2\tilde{C}_1\tilde{\epsilon}_0^2 \leq \frac{1}{2}. \tag{6.12}$$

We have

$$\frac{1}{2}\|u(0)\|^2_{L^2(\Omega)} + \frac{1}{4}\|\nabla u\|^2_{L^2(Q)} \leq 0$$

and consequently,

$$u = 0 \quad \text{in } L^2(0,T;H_0^1(\Omega)) \quad \text{and} \quad h = \tilde{h} \quad \text{a.e. in } Q \tag{6.13}$$

from (6.10) and (6.13), we conclude that

$$(y,h) = (\tilde{y},\tilde{h}) \quad \text{a.e. in } Q.$$

This concludes the proof of the uniqueness result for the system (6.1).

The proof of Theorem 6.1 is thus complete. □

Theorem 6.2. *Let us assume* (2.2) *and* (2.4)*. If $\hat{y}_0 \in H_0^1(\Omega)$, $v \in L^2(\omega\times(0,T))$, and $y \in B_{X_0}[0,\hat{\epsilon}_0]$ with $\hat{\epsilon}_0 > 0$ sufficiently small, then, there exists a unique strong solution p of* (4.44) *satisfying*

$$p \in X_0 \quad \text{and} \quad \|p\|^2_{X_0} \leq C_{10}\|\hat{y}_0\|^2_{H_0^1(\Omega)} \tag{6.14}$$

where $C_{10} := C_{10}(\Omega, M_0, T, a, \|\alpha\|_{L^\infty(\Omega\times(0,T);L^2(\Omega))})$.

Proof. To begin, we shall demonstrate the existence of a strong solution for (4.44), and after the uniqueness of solution.

Existence of a strong solution of (4.44)

In order to prove the existence of strong solution of (4.44), let us consider the following mapping:

$$\Lambda : X_1 \to X_1, \tag{6.15}$$

$$\hat{p} \mapsto \Lambda(\hat{p}) = \tilde{p}, \tag{6.16}$$

where $\tilde{p}$ is the unique strong solution of

$$\begin{cases} \tilde{p}_t - \Delta\tilde{p} + \gamma(y)\tilde{p} = -\theta(y)\left(\displaystyle\int_0^T \alpha(x,t,s)\hat{p}(x,s)\,ds\right)y & \text{in } Q, \\ \tilde{p}(x,t) = 0 & \text{on } \Sigma, \\ \tilde{p}(x,0) = \hat{y}_0(x) & \text{in } \Omega. \end{cases} \tag{6.17}$$

For the theory of parabolic equation (to see [11]), the system (6.17) has a unique strong solution $\tilde{p}$ satisfying

$$\|\tilde{p}\|^2_{X_0} \leq \tilde{C}_1\left(\|\hat{y}_0\|^2_{H_0^1(\Omega)} + \|\theta(y)\left(\int_0^T \alpha(x,t,s)\hat{p}(x,s)\,ds\right)y\|^2_{L^2(Q)}\right). \tag{6.18}$$

Then, taking $\hat{R}_0 = \sqrt{2\tilde{C}_1}\|\hat{y}_0\|_{H_0^1(\Omega)}$ and if $\hat{p} \in B_{X_0}[0,\hat{R}_0]$, by the estimate (6.18) and using Hölder's inequality we have

$$\begin{aligned}\|\tilde{p}\|^2_{X_0} &\leq \frac{\hat{R}^2}{2} + \tilde{C}_1M_0^2\|\alpha\|^2_{L^\infty(\Omega\times(0,T);L^2(0,T))}\int_0^T\int_0^T \|y(\cdot,t)\|^2_{L^4(\Omega)}\|\hat{p}(\cdot,r)\|^2_{L^4(\Omega)}\,dtdr \\ &\leq \frac{\hat{R}^2}{2} + \tilde{C}_1M_0^2\|\alpha\|^2_{L^\infty(\Omega\times(0,T);L^2(0,T))}\overline{C}_1^4\|\nabla y\|^2_{L^\infty(0,T;L^2(\Omega))}\hat{R}_0^2 \\ &\leq \frac{\hat{R}^2}{2} + \tilde{C}_1M_0^2\|\alpha\|^2_{L^\infty(\Omega\times(0,T);L^2(0,T))}\overline{C}_1^4\hat{\epsilon}_0^2\hat{R}_0^2.\end{aligned}$$

Assuming that $\hat{\epsilon}_0$ is small enough such that

$$\tilde{C}_1 M_0^2 \|\alpha\|^2_{L^\infty(\Omega\times(0,T);L^2(0,T))} \overline{C}_1^4 \hat{\epsilon}_0^2 \leq \frac{1}{2} \tag{6.19}$$

we have

$$\|\tilde{p}\|^2_{X_0} \leq \hat{R}_0^2,$$

and consequently,

$$\Lambda(\hat{p}) = \tilde{p} \in B_{X_0}[0,\hat{R}_0] \quad \forall\ \hat{p} \in B_{X_0}[0,\hat{R}_0]. \tag{6.20}$$

Notice that, due to that $\tilde{p}$ is the strong solution of the linear system (6.17) one has that Λ is continuous, and by (6.20) we have

$$\Lambda(B_{X_0}[0,\hat{R}_0]) \subset B_{X_0}[0,\hat{R}_0].$$

As X_0 is compact in X_1. Then, by Schauder's Fixed Point Theorem, there exists $p \in B_{X_0}[0,\hat{R}_0]$ satisfying

$$\Lambda(p) = p$$

and p solves the system (4.44) satisfying

$$\|p\|^2_{X_0} \leq \hat{R}_0^2 = 2\tilde{C}_1 \|\hat{y}_0\|^2_{H_0^1(\Omega)}.$$

Uniqueness of solution for system (4.44)

Let p and $\tilde{p}$ two solutions of (4.44), denote by $u = p - \tilde{p}$, then, we have

$$\begin{cases} u_t - \Delta u + \gamma(y)u + \theta(y)\left(\int_0^T \alpha(x,t,s)u(x,s)\,ds\right)y = 0 & \text{in } Q, \\ u(x,t) = 0 & \text{on } \Sigma, \\ u(x,0) = 0 & \text{in } \Omega. \end{cases} \tag{6.21}$$

Multiplying by u in $(6.21)_1$, and integrating over Ω, we obtain

$$\frac{1}{2}\frac{d}{dt}\|u\|^2_{L^2(\Omega)} + \|\nabla u\|^2_{L^2(\Omega)} + \int_\Omega \gamma(y)|u|^2\,dx = -\int_\Omega \theta(y)\left(\int_0^T \alpha(x,t,r)u(x,r)\,dr\right) yu\,dx$$

from the properties of φ, Sobolev's immersions, Hölder's inequality and Young's inequality, we have

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\|u\|^2_{L^2(\Omega)} + \|\nabla u\|^2_{L^2(\Omega)} \\ &\quad \leq \frac{1}{4\overline{C}_2^2}\|u\|^2_{L^2(\Omega)} + M_0^2\overline{C}_2^2 \int_\Omega |\int_0^T \alpha(x,t,r)u(x,r)\,dr|^2 |y|^2\,dx \\ &\quad \leq \frac{1}{4}\|\nabla u\|^2_{L^2(\Omega)} + M_0^2\overline{C}_2^2\overline{C}_1^4 \|\alpha\|^2_{L^\infty(\Omega\times(0,T);L^2(0,T))} \|\nabla u\|^2_{L^2(Q)} \|\nabla y\|^2_{L^2(\Omega)}. \end{aligned}$$

Integrating over $(0,T)$, we deduce that

$$\frac{1}{2}\|u(T)\|^2_{L^2(\Omega)} + \frac{3}{4}\|\nabla u\|^2_{L^2(Q)} \leq M_0^2\overline{C}_2^2\overline{C}_1^4 \|\alpha\|^2_{L^\infty(\Omega\times(0,T);L^2(0,T))} T\hat{\epsilon}_0^2 \|\nabla u\|^2_{L^2(Q)}.$$

Taking $\hat{\epsilon}_0$ small enough such that

$$M_0^2\overline{C}_2^2\overline{C}_1^4\|\alpha\|^2_{L^\infty(\Omega\times(0,T);L^2(0,T))}T\hat{\epsilon}_0^2 \leq \frac{1}{2} \tag{6.22}$$

we have

$$\frac{1}{2}\|u(T)\|^2_{L^2(\Omega)} + \frac{1}{4}\|\nabla u\|^2_{L^2(Q)} \leq 0$$

and consequently,

$$u = 0 \quad \text{in } L^2(0,T;H_0^1(\Omega)) \quad \text{and} \quad p = \tilde{p} \quad \text{a.e. in } Q.$$

This concludes the proof of the uniqueness result for the system (4.44).

Therefore, the proof of Theorem 6.2 is thus complete. □

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