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New nonlinear Henry–Gronwall type inequality with time-varying delay and applications to fractional differential equations

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Abstract. In this paper, four types of new nonlinear Henry–Gronwall type integral inequalities have been established. As for the first type, by employing inequality techniques, we overcome the limitation of traditional methods in which dividing the range of parameter $\beta \in (0,1)$ into two parts is needed. For the second type, we derive a bound that is more precise than previous studies by comparative analysis. Regarding the third type and the fourth type, they are new models that are studied in our work. Specifically, the third type extends our proposed inequality to case when $\beta \geq 1$, and the fourth type constitutes a new variant of the nonlinear Bihari-type inequality with time-varying delay that offers greater generality. As applications of the derived results, the existence of solutions to the fractional differential equations has been discussed by fixed point theorems and two examples are provided to illustrate the validity of the theorems.

Keywords: fractional integral inequalities, time-varying delay, fractional differential equations, solutions.

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1 Introduction

In the study of ordinary, partial, and functional differential equations, one has often to deal with certain integral inequalities. The Gronwall–Bellman–Bihari inequality [3, 4] and its various linear and nonlinear generalizations [13, 14, 21, 26–28, 34] play a vital role in the study of existence, uniqueness, boundedness, stability and asymptotic behavior of solutions of differential equations [5, 15, 17, 18, 31, 38]. Among these generalizations, what interests us is that Jiang and Meng [13] considered the following nonlinear integral inequality with time-varying delay:

$$x^p(t) \leq \rho(t) + \pi(t) \int_0^t \left[f(s)x^q(s) + h(s)x^r(\sigma(s)) \right] ds, \tag{1.1}$$

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where $x(t), \rho(t), \pi(t), f(t), h(t) \in C(\mathbb{R}_+, \mathbb{R}_+)$, $\rho(t)$ and $\pi(t)$ are non-decreasing in $\mathbb{R}_+$, $p \neq 0$, $p \geq q \geq 0$, $p \geq r \geq 0$. The approach primarily relies on an inequality introduced by Li, Han and Meng [16], so that inequality (1.1) can be simplified to a linear case.

With the development of fractional-order differential equations, the study of integral inequalities with weakly singular kernels has gained significant attention, resulting in inspiring theoretical advances. The pioneering work for weakly singular Gronwall-type inequalities was studied by Henry [12] who proved, by an iterative process, some L^1 bounds given by series related to the Mittag-Leffler function. The Henry version is as follows:

$$u(t) \leq a(t) + b \int_0^t (t-s)^{\beta-1} u(s)\, ds, \tag{1.2}$$

where $u(t)$ and $a(t)$ are non-negative and locally integrable when $0 \leq t < T \leq +\infty$, $b \geq 0$, $\beta > 0$. In [12], Henry first defined $Bu(t)$ as the integral on the right-hand side of inequality (1.2). Then, through an iterative process, inequality (1.2) was transformed into $u \leq \sum_{k=0}^{n-1} B^k a + B^n u$, where $B^n u$ is convergent. The desired bound was obtained as $n \to \infty$.

Henry's pioneering study stimulated extensive follow-up research, with numerous scholars proposing improved approaches [6, 9–11, 19, 30, 36]. Ye, Gao and Ding [36] used the same method as in [12] to extend this result, allowing the constant b to be a continuous non-decreasing function. Webb [30] considered the cases with a double singularity and obtained explicit L^∞ bounds rather than L^1 bounds by a completely different method.

Different from Henry, Medved' [23] presented a new approach to solve integral inequalities of Henry–Gronwall type and their Bihari version. Based on this work, Medved' [24] researched and obtained global solutions of semilinear evolution equations. Medved''s study was also developed by many scholars [20, 25, 35, 37].

In this paper, motivated by [13, 37], we first study the explicit bounds of the following nonlinear Henry–Gronwall type inequality with time-varying delay:

$$x^p(t) \leq \rho(t) + g(t) \int_0^t (t-s)^{\beta-1} [f(s)x^q(s) + h(s)x^r(\sigma(s))]\, ds, \tag{1.3}$$

where $\beta \in (0,1)$. Unlike conventional approaches such as Xu and Meng [33], in which the parameter β was partitioned into intervals $(0, 0.5]$ and $(0.5, 1)$, we employ inequality techniques to allow β to uniformly take values across the entire range $(0,1)$. Then, we consider the nonlinear case of inequality (1.2) which was also studied by Foukrach and Meftah [8]. In contrast to their approach, we employ a different method that utilizes the same inequality. Moreover, as for the nonlinear case of inequality (1.2), we extend it with a time-varying delay term and provide an explicit bound of it when $\beta \geq 1$. Lastly, we consider a new nonlinear Bihari-type inequality with time-varying delay:

$$x^p(t) \leq \rho(t) + \int_0^t (t-s)^{\beta-1} [f(s)x^q(s) + h(s)\phi(x(\sigma(s)))]\, ds, \quad \beta \in (0,1). \tag{1.4}$$

Using the inequality introduced in Li, Han and Meng [16], our bound depends on both the parameter K and the independent variable t. Then we provide an example that demonstrates how to obtain the optimal K, which yields a better bound that depends only on the independent variable t.

As applications, by employing our results and Leray–Schauder alternative fixed point theorem, we ensure the existence of solutions to a certain class of fractional differential equations. At last, we provide two examples to illustrate the validity of our theorems.

2 Preliminaries

In this paper, $\mathbb{R}$ denotes the set of real numbers and $\mathbb{R}_+ = [0, +\infty)$ is the given subset of $\mathbb{R}$. In the following, some basic definitions and useful lemmas are given.

Definition 2.1 ([7])**.** The Riemann–Liouville (R–L) fractional integral of order $\alpha > 0$ of a function $u \in L^1[0, T]$ is defined by

$$I^{\alpha}_{0+}u(t) := \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} u(s)\, ds, \quad \text{a.e. } t \in [0, T],$$

where $\Gamma(\cdot)$ denotes the Gamma function.

Definition 2.2 ([7])**.** For $\alpha \in (0, 1)$ and $u \in L^1[0, T]$, the R–L fractional derivative $D^{\alpha}u$ is defined when $I^{1-\alpha}u \in AC[0, T]$ by

$$D^{\alpha}u(t) := DI^{1-\alpha}u(t), \quad \text{a.e. } t \in [0, T].$$

Definition 2.3 ([7])**.** The Caputo differential operator for order $\alpha \in (0, 1)$ of function $u \in L^1[0, T]$ is defined by

$$D^{\alpha}_{*}u := D^{\alpha}(u - u(0)) = DI^{1-\alpha}(u - u(0)), \quad \text{a.e. } t \in [0, T],$$

whenever this R–L derivative exists, that is when $u(0)$ exists and $I^{1-\alpha}u \in AC$.

Another often used definition of Caputo derivative is when the derivative and fractional integral are taken in the reverse order to that taken in the R–L derivative. But that definition has severe disadvantages when dealing with the equivalence of the solution $u(t)$ of $D^{\alpha}u(t) = f(t), u(0) = u_0$ and $u(t) = u_0 + I^{\alpha}f(t)$. See [15] and [31] for a more detailed explanation.

Lemma 2.4 ([13])**.** *Suppose that $x(t), \rho(t), \pi(t), f(t), h(t) \in C(\mathbb{R}_+, \mathbb{R}_+)$. If $\rho(t)$ and $\pi(t)$ are non-decreasing in $\mathbb{R}_+$, and $x(t)$ satisfies the following form of delay integral inequality:*

$$x^p(t) \leq \rho(t) + \pi(t) \int_0^t \left[f(s)x^q(s) + h(s)x^r(\sigma(s)) \right] ds, \quad t \in \mathbb{R}_+,$$

with the initial condition

$$x(t) = \phi(t), \quad t \in [\alpha, 0],$$

$$\phi(\sigma(t)) \leq (\rho(t))^{1/p} \text{ for } t \in \mathbb{R}_+ \text{ with } \sigma(t) \leq 0,$$

where $p \neq 0$, $p \geq q \geq 0$, $p \geq r \geq 0$, p, q, r be constants, $\sigma(t) \in C(\mathbb{R}_+, \mathbb{R})$, $\sigma(t) \leq t$, $-\infty < \alpha = \inf\{\sigma(t),\, t \in \mathbb{R}_+\} < 0$, and $\phi(t) \in C([\alpha, 0], \mathbb{R}_+)$, then

$$x(t) \leq \left[\rho(t) + \pi(t)A(t) \exp\left(\int_0^t B(s)\, ds \right) \right]^{1/p}, \quad t \in \mathbb{R}_+,$$

for any $K > 0$, where

$$A(t) = \int_0^t \left[f(s) \left(\frac{p-q}{p} K^{q/p} + \frac{q}{p} K^{(q-p)/p} \rho(s) \right) + h(s) \left(\frac{p-r}{p} K^{r/p} + \frac{r}{p} K^{(r-p)/p} \rho(s) \right) \right] ds,$$

$$B(t) = \left[\frac{q}{p} K^{(q-p)/p} f(t) + \frac{r}{p} K^{(r-p)/p} h(t) \right] \pi(t),$$

for $t \in \mathbb{R}_+$.

Lemma 2.5 ([36]). *Suppose $\beta > 0$, $a(t)$ is a non-negative function and locally integrable when $0 \le t < T \le +\infty$, $g(t)$ is a non-negative, non-decreasing continuous function defined on $0 \le t < T$, $g(t) \le M$ (constant), and $u(t)$ is non-negative and locally integrable with*

$$u(t) \le a(t) + g(t) \int_0^t (t-s)^{\beta-1} u(s)\, ds,$$

for a.e. $t \in [0, T)$. Then

$$u(t) \le a(t) + \int_0^t \left[\sum_{n=1}^{\infty} \frac{(g(t)\Gamma(\beta))^n}{\Gamma(n\beta)} (t-s)^{n\beta-1} a(s) \right] ds,$$

for a.e. $t \in [0, T)$. Moreover, if $a(t)$ is non-decreasing on $[0, T)$, then $u(t) \le a(t) E_\beta\big(g(t)\Gamma(\beta) t^\beta\big)$, where E_β is the Mittag-Leffler function defined by $E_\beta(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(k\beta+1)}$.

Lemma 2.6. *Let $t > 0$, $0 < \alpha < \beta < 1$, then we have*

$$\int_0^t \left((t-s)^{\beta-1} s^{\alpha-\beta} \right)^{\frac{1}{1-\alpha}} ds = \Gamma\left(\frac{1-\beta}{1-\alpha}\right) \Gamma\left(\frac{\beta-\alpha}{1-\alpha}\right).$$

Proof. Change the variable of integration from s to $x = \frac{s}{t}$ and the integral becomes

$$\begin{aligned} \int_0^1 t^{\frac{\beta-1}{1-\alpha}} (1-x)^{\frac{\beta-1}{1-\alpha}} t^{\frac{\alpha-\beta}{1-\alpha}} x^{\frac{\alpha-\beta}{1-\alpha}} t dx &= \int_0^1 (1-x)^{\frac{\beta-\alpha}{1-\alpha}-1} x^{\frac{1-\beta}{1-\alpha}-1} dx \\ &= B\left(\frac{1-\beta}{1-\alpha}, \frac{\beta-\alpha}{1-\alpha}\right) \\ &= \Gamma\left(\frac{1-\beta}{1-\alpha}\right)\Gamma\left(\frac{\beta-\alpha}{1-\alpha}\right). \end{aligned}$$ □

Lemma 2.7 ([16]). *Assume that $a \ge 0$, $p \ge q \ge 0$ and $p \ne 0$, then*

$$a^{\frac{q}{p}} \le \frac{q}{p} K^{\frac{q-p}{p}} a + \frac{p-q}{p} K^{\frac{q}{p}},$$

for any $K > 0$.

Lemma 2.8 ([1]). *Let E be a Banach space, C a closed, convex subset of E and $0 \in C$. Let $N : C \to C$ be a continuous and completely continuous map, and let the set $\{x \in E : x = \lambda N x \text{ for some } \lambda \in (0,1)\}$ be bounded. Then N has at least one fixed point in E.*

Lemma 2.9 ([31]). *Let f be continuous on $[0, T] \times \mathbb{R}$, and $0 < \alpha < 1$. Then if $u \in C[0, T]$ satisfies $u(t) = u_0 + I^\alpha f$, then $I^{1-\alpha}(u - u_0) \in AC[0, T]$, $D_*^\alpha u$ exists a.e. and satisfies $D_*^\alpha u(t) = f(t)$, a.e. $t \in [0, T]$, $u(0) = u_0$. Conversely, if $u \in C[0, T]$, $I^{1-\alpha}(u - u_0) \in AC[0, T]$ and u satisfies $D_*^\alpha u(t) = f(t)$, a.e. $t \in [0, T]$, $u(0) = u_0$, then u satisfies $u(t) = u_0 + I^\alpha f(t)$ for all $t \in [0, T]$.*

3 Main results

In this section, we begin by considering a weakly singular version of inequality (1.1).

Theorem 3.1. *Suppose that $x(t)$, $\rho(t)$, $g(t)$, $f(t)$, $h(t) \in C(\mathbb{R}_+, \mathbb{R}_+)$. If $\rho(t)$ and $g(t)$ are non-decreasing in $\mathbb{R}_+$, and x(t) satisfies the following form of delay integral inequality:*

$$x^p(t) \le \rho(t) + g(t)\int_0^t (t-s)^{\beta-1}[f(s)x^q(s) + h(s)x^r(\sigma(s))]\,ds, \quad t \in \mathbb{R}_+, \tag{3.1}$$

with the initial condition:

$$\begin{gathered} x(t) = \varphi(t), t \in [a, 0], \\ \varphi(\sigma(t)) \le 3^{\frac{1-\alpha}{p}} \rho^{\frac{1}{p}}(t) \ \text{ for } t \in R_+ \text{ with } \sigma(t) \le 0, \end{gathered} \tag{3.2}$$

where $0 < \alpha < \beta < 1$, $p \neq 0$, $p \ge q \ge 0$, $p \ge r \ge 0$, p, q, r are constants, $\sigma(t) \in C(\mathbb{R}_+, \mathbb{R})$, $\sigma(t) \le t$, $-\infty < a = \inf\{\sigma(t), t \in \mathbb{R}_+\} < 0$, and $\varphi(t) \in C([a, 0], \mathbb{R}_+)$, then

$$x(t) \le \left[a(t) + b(t)C(t)\exp\left(\int_0^t D(s)\,ds\right)\right]^{\frac{\alpha}{p}}, \quad t \in \mathbb{R}_+, \tag{3.3}$$

for any $K > 0$, where

$$\begin{aligned} a(t) &= 3^{\frac{1}{\alpha}-1}\rho^{\frac{1}{\alpha}}(t), \\ b(t) &= 3^{\frac{1}{\alpha}-1}g^{\frac{1}{\alpha}}(t)\left(\Gamma\left(\frac{1-\beta}{1-\alpha}\right)\Gamma\left(\frac{\beta-\alpha}{1-\alpha}\right)\right)^{\frac{1-\alpha}{\alpha}}, \\ c(t) &= t^{\frac{\beta-\alpha}{\alpha}} f^{\frac{1}{\alpha}}(t), \\ d(t) &= t^{\frac{\beta-\alpha}{\alpha}} h^{\frac{1}{\alpha}}(t), \\ C(t) &= \int_0^t \left[c(s)\left(\frac{p-q}{p}K^{\frac{q}{p}} + \frac{q}{p}K^{\frac{q-p}{p}}a(s)\right) + d(s)\left(\frac{p-r}{p}K^{\frac{r}{p}} + \frac{r}{p}K^{\frac{r-p}{p}}a(s)\right)\right] ds, \\ D(t) &= \left[\frac{q}{p}K^{\frac{q-p}{p}}c(t) + \frac{r}{p}K^{\frac{r-p}{p}}d(t)\right] b(t), \end{aligned} \tag{3.4}$$

for $t \in \mathbb{R}_+$.

Proof. From inequality (3.1) and Hölder's inequality [2], we have

$$\begin{aligned} x^p(t) &\le \rho(t) + g(t)\int_0^t (t-s)^{\beta-1}\left[f(s)x^q(s) + h(s)x^r(\sigma(s))\right] ds \\ &= \rho(t) + g(t)\int_0^t (t-s)^{\beta-1}s^{\alpha-\beta}s^{\beta-\alpha}\left[f(s)x^q(s) + h(s)x^r(\sigma(s))\right] ds \\ &\le \rho(t) + g(t)\left(\int_0^t \left((t-s)^{\beta-1}s^{\alpha-\beta}\right)^{\frac{1}{1-\alpha}} ds\right)^{1-\alpha} \\ &\quad \times \left(\int_0^t \left(s^{\beta-\alpha}\left(f(s)x^q(s) + h(s)x^r(\sigma(s))\right)\right)^{\frac{1}{\alpha}} ds\right)^{\alpha}. \end{aligned} \tag{3.5}$$

By Lemma 2.6 and Minkowski's inequality [2], we obtain

$$\begin{aligned} x^p(t) &\le \rho(t) + g(t)\left(\Gamma\left(\frac{1-\beta}{1-\alpha}\right)\Gamma\left(\frac{\beta-\alpha}{1-\alpha}\right)\right)^{1-\alpha} \\ &\quad \times \left(\int_0^t \left(s^{\beta-\alpha}\left(f(s)x^q(s) + h(s)x^r(\sigma(s))\right)\right)^{\frac{1}{\alpha}} ds\right)^{\alpha} \\ &\le \rho(t) + g(t)\left(\Gamma\left(\frac{1-\beta}{1-\alpha}\right)\Gamma\left(\frac{\beta-\alpha}{1-\alpha}\right)\right)^{1-\alpha}\left(\int_0^t \left(s^{\beta-\alpha}f(s)x^q(s)\right)^{\frac{1}{\alpha}} ds\right)^{\alpha} \\ &\quad + g(t)\left(\Gamma\left(\frac{1-\beta}{1-\alpha}\right)\Gamma\left(\frac{\beta-\alpha}{1-\alpha}\right)\right)^{1-\alpha}\left(\int_0^t \left(s^{\beta-\alpha}h(s)x^r(\sigma(s))\right)^{\frac{1}{\alpha}} ds\right)^{\alpha}, \quad t \in \mathbb{R}_+. \end{aligned} \tag{3.6}$$

By Jensen's inequality [22], one has

$$\begin{aligned} x^{\frac{p}{\alpha}}(t) &\le 3^{\frac{1}{\alpha}-1}\rho^{\frac{1}{\alpha}}(t) + 3^{\frac{1}{\alpha}-1}g^{\frac{1}{\alpha}}(t)\left(\Gamma\left(\frac{1-\beta}{1-\alpha}\right)\Gamma\left(\frac{\beta-\alpha}{1-\alpha}\right)\right)^{\frac{1-\alpha}{\alpha}}\int_0^t\left(s^{\beta-\alpha}f(s)x^q(s)\right)^{\frac{1}{\alpha}}ds \\ &\quad + 3^{\frac{1}{\alpha}-1}g^{\frac{1}{\alpha}}(t)\left(\Gamma\left(\frac{1-\beta}{1-\alpha}\right)\Gamma\left(\frac{\beta-\alpha}{1-\alpha}\right)\right)^{\frac{1-\alpha}{\alpha}}\int_0^t\left(s^{\beta-\alpha}h(s)x^r(\sigma(s))\right)^{\frac{1}{\alpha}}ds \\ &\le a(t) + b(t)\int_0^t\left[c(s)x^{\frac{q}{\alpha}}(s) + d(s)x^{\frac{r}{\alpha}}(\sigma(s))\right]ds, \quad t\in\mathbb{R}_+, \end{aligned} \tag{3.7}$$

where $a(t)$, $b(t)$, $c(t)$, $d(t)$ are defined as in condition (3.4), by Lemma 2.4

$$x(t) \le \left[a(t) + b(t)C(t)\exp\left(\int_0^t D(s)\,ds\right)\right]^{\frac{\alpha}{p}}, \quad t\in\mathbb{R}_+, \tag{3.8}$$

where $C(t), D(t)$ are defined as in condition (3.4). □

Remark 3.2. The hypothesis in initial condition (3.2) that $\varphi(\sigma(t)) \le 3^{\frac{1-\alpha}{p}}\rho^{\frac{1}{p}}(t)$ for $t\in\mathbb{R}_+$ with $\sigma(t)\le 0$ is necessary to handle the case when the delay term $\sigma(t)$ falls into the initial interval $[a,0]$. This allows us to control the term $x(\sigma(s))$ in the integral uniformly for all $s\ge 0$, leading to a single bound that works for the entire domain $\mathbb{R}_+$.

Remark 3.3. Different from [33, Theorem 3.1] where the authors employed conventional approaches, our method eliminates the requirement of dividing the range of parameter β into two intervals $(0, 0.5]$ and $(0.5, 1)$, allowing β to directly take values in the entire range $(0,1)$ and also provides an explicit bound.

Then, we consider the nonlinear inequality which is a generalization of inequality (1.2).

Theorem 3.4. *Suppose that $x(t)$, $\rho(t)$, $g(t)$, $f(t)\in C(\mathbb{R}_+,\mathbb{R}_+)$. If $g(t)$ and $f(t)$ are non-decreasing in $\mathbb{R}_+$, and $x(t)$ satisfies the following form of integral inequality:*

$$x^p(t) \le \rho(t) + g(t)\int_0^t (t-s)^{\beta-1}f(s)x^q(s)\,ds, \quad t\in\mathbb{R}_+, \tag{3.9}$$

where $\beta>0$, $p\ne 0$, $p\ge q\ge 0$, p,q are constants, then

$$x(t) \le \left(\rho(t) + g(t)\left(m(t) + \int_0^t\left[\sum_{n=1}^{+\infty}\frac{(n(t)\Gamma(\beta))^n}{\Gamma(n\beta)}(t-s)^{n\beta-1}m(s)\right]ds\right)\right)^{\frac{1}{p}}, \quad t\in\mathbb{R}_+, \tag{3.10}$$

where

$$\begin{aligned} m(t) &= \int_0^t (t-s)^{\beta-1}f(s)\left(\frac{p-q}{p}K^{\frac{q}{p}} + \frac{q}{p}K^{\frac{q-p}{p}}\rho(s)\right)ds, \\ n(t) &= \frac{q}{p}K^{\frac{q-p}{p}}f(t)g(t), \end{aligned} \tag{3.11}$$

for any $K>0$.

Proof. Define a function by

$$u(t) = \int_0^t (t-s)^{\beta-1}f(s)x^q(s)\,ds, \quad t\in\mathbb{R}_+, \tag{3.12}$$

then inequality (3.9) can be restated as

$$x^p(t) \le \rho(t) + g(t)u(t), \quad t \in \mathbb{R}_+. \tag{3.13}$$

By Lemma 2.7, for any $K > 0$, we have

$$\begin{aligned} x^q(t) &\le [\rho(t) + g(t)u(t)]^{\frac{q}{p}} \\ &\le \frac{q}{p} K^{\frac{q-p}{p}} (\rho(t) + g(t)u(t)) + \frac{p-q}{p} K^{\frac{q}{p}}, \quad t \in \mathbb{R}_+. \end{aligned} \tag{3.14}$$

From equation (3.12) and inequality (3.14), we have

$$\begin{aligned} u(t) &\le \int_0^t (t-s)^{\beta-1} f(s) \left[\frac{q}{p} K^{\frac{q-p}{p}} (\rho(s) + g(s)u(s)) + \frac{p-q}{p} K^{\frac{q}{p}} \right] ds \\ &= \int_0^t (t-s)^{\beta-1} f(s) \left(\frac{p-q}{p} K^{\frac{q}{p}} + \frac{q}{p} K^{\frac{q-p}{p}} \rho(s) \right) ds \\ &\quad + \int_0^t (t-s)^{\beta-1} \frac{q}{p} K^{\frac{q-p}{p}} f(s) g(s) u(s)\, ds \\ &= m(t) + \int_0^t (t-s)^{\beta-1} n(s) u(s)\, ds \\ &\le m(t) + n(t) \int_0^t (t-s)^{\beta-1} u(s)\, ds, \quad t \in \mathbb{R}_+, \end{aligned} \tag{3.15}$$

where $m(t), n(t)$ are defined as in condition (3.11). By Lemma 2.5, we obtain

$$u(t) \le m(t) + \int_0^t \left[\sum_{n=1}^{+\infty} \frac{(n(t)\Gamma(\beta))^n}{\Gamma(n\beta)} (t-s)^{n\beta-1} m(s) \right] ds, \quad t \in \mathbb{R}_+. \tag{3.16}$$

From inequality (3.13), for $t \in \mathbb{R}_+$, we have

$$x(t) \le \left(\rho(t) + g(t) \left(m(t) + \int_0^t \left[\sum_{n=1}^{+\infty} \frac{(n(t)\Gamma(\beta))^n}{\Gamma(n\beta)} (t-s)^{n\beta-1} m(s) \right] ds \right) \right)^{\frac{1}{p}}. \tag{3.17}$$

□

Corollary 3.5. *Under the hypothesis of Theorem 3.4, let $\rho(t)$ be a non-decreasing function in $\mathbb{R}_+$, then*

$$x(t) \le \left(\rho(t) + g(t)m(t)E_\beta \left(n(t)\Gamma(\beta)t^\beta \right) \right)^{\frac{1}{p}}, \quad t \in \mathbb{R}_+. \tag{3.18}$$

Proof. Let $h(t) = f(t) \left(\frac{p-q}{p} K^{\frac{q}{p}} + \frac{q}{p} K^{\frac{q-p}{p}} \rho(t) \right)$, so $h(t)$ is a non-decreasing function in $\mathbb{R}_+$. Then $m(t)$ can be restated as $m(t) = \int_0^t (t-s)^{\beta-1} h(s)\, ds$. By changing the variable of integration, we get

$$m(t) = \int_0^t (t-s)^{\beta-1} h(s)\, ds = t^\beta \int_0^1 (1-\sigma)^{\beta-1} h(t\sigma)\, d\sigma, \quad t \in \mathbb{R}_+. \tag{3.19}$$

So $m(t)$ is non-decreasing in $\mathbb{R}_+$. Using Lemma 2.5 and inequality (3.16), we have

$$u(t) \le m(t)E_\beta \left(n(t)\Gamma(\beta)t^\beta \right), \quad t \in \mathbb{R}_+. \tag{3.20}$$

From inequality (3.13), we get

$$x(t) \le \left(\rho(t) + g(t)m(t)E_\beta \left(n(t)\Gamma(\beta)t^\beta \right) \right)^{\frac{1}{p}}, \quad t \in \mathbb{R}_+. \tag{3.21}$$

□

By the similar techniques as in Theorem 3.4, we obtain the following corollary.

Corollary 3.6. *Suppose that $x(t)$, $\rho(t)$, $g(t)$ satisfy the assumptions in Theorem 3.4, $f_i(t) \in C(\mathbb{R}_+, \mathbb{R}_+)$ and $f_i(t)$ are non-decreasing in $\mathbb{R}_+$, $i = 1, 2 \ldots n$, if $x(t)$ satisfies the following form of integral inequality:*

$$x^p(t) \le \rho(t) + g(t) \int_0^t (t-s)^{\beta-1} \left(\sum_{i=1}^n f_i(s) x^{q_i}(s) \right) ds, \quad t \in \mathbb{R}_+, \tag{3.22}$$

where a sequence of non-negative real numbers $q_1, q_2, \ldots, q_n$ satisfying $p \ge q_i \ge 0$, and $p \ne 0$. Then for $t \in \mathbb{R}_+$, we can get

$$x(t) \le \left(\rho(t) + g(t) \left(m^*(t) + \int_0^t \left[\sum_{n=1}^{+\infty} \frac{(n^*(t)\Gamma(\beta))^n}{\Gamma(n\beta)} (t-s)^{n\beta-1} m^*(s) \right] ds \right) \right)^{\frac{1}{p}}, \tag{3.23}$$

where

$$\begin{aligned} m^*(t) &= \int_0^t (t-s)^{\beta-1} \sum_{i=1}^n f_i(s) \left(\frac{p-q_i}{p} K^{\frac{q_i}{p}} + \frac{q_i}{p} K^{\frac{q_i-p}{p}} \rho(s) \right) ds, \\ n^*(t) &= \sum_{i=1}^n \frac{q_i}{p} K^{\frac{q_i-p}{p}} f_i(t) g(t), \end{aligned} \tag{3.24}$$

for any $K > 0$.

Then under the framework of Theorem 3.4, we extend it with a time-varying delay term.

Theorem 3.7. *Suppose that $x(t)$, $\rho(t)$, $g(t)$, $f(t)$, $h(t) \in C(\mathbb{R}_+, \mathbb{R}_+)$. If $\rho(t)$, $g(t)$, $f(t)$, $h(t)$ are non-decreasing in $\mathbb{R}_+$, and $x(t)$ satisfies the following form of delay integral inequality:*

$$x^p(t) \le \rho(t) + g(t) \int_0^t (t-s)^{\beta-1} [f(s) x^q(s) + h(s) x^r(\sigma(s))] \, ds, \quad t \in \mathbb{R}_+, \tag{3.25}$$

with the initial condition:

$$\begin{aligned} x(t) &= \varphi(t), \quad t \in [a, 0], \\ \varphi(\sigma(t)) &\le \rho^{\frac{1}{p}}(t) \quad \text{for } t \in R_+ \text{ with } \sigma(t) \le 0, \end{aligned} \tag{3.26}$$

where $\beta \ge 1$, $p \ne 0$, $p \ge q \ge 0$, $p \ge r \ge 0$, p, q, r are constants, $\sigma(t) \in C(\mathbb{R}_+, \mathbb{R})$, $\sigma(t) \le t$, $-\infty < a = \inf\{\sigma(t), t \in \mathbb{R}_+\} < 0$, and $\varphi(t) \in C([a, 0], \mathbb{R}_+)$, then

$$x(t) \le \left(\rho(t) + g(t) m_1(t) E_\beta \left(n_1(t) \Gamma(\beta) t^\beta \right) \right)^{\frac{1}{p}}, \quad t \in \mathbb{R}_+, \tag{3.27}$$

where

$$\begin{aligned} m_1(t) &= \int_0^t (t-s)^{\beta-1} \left[f(s) \left(\frac{p-q}{p} K^{\frac{q}{p}} + \frac{q}{p} K^{\frac{q-p}{p}} \rho(s) \right) + h(s) \left(\frac{p-r}{p} K^{\frac{r}{p}} + \frac{r}{p} K^{\frac{r-p}{p}} \rho(s) \right) \right] ds, \\ n_1(t) &= \left(\frac{q}{p} K^{\frac{q-p}{p}} f(t) + \frac{r}{p} K^{\frac{r-p}{p}} h(t) \right) g(t), \end{aligned} \tag{3.28}$$

for any $K > 0$.

Proof. Define a function $u(t)$ by

$$u^p(t) = \rho(t) + g(t)\int_0^t (t-s)^{\beta-1}[f(s)x^q(s) + h(s)x^r(\sigma(s))]\,ds, \quad t \in \mathbb{R}_+. \tag{3.29}$$

Since $\rho(t), g(t)$ are non-decreasing in $\mathbb{R}_+$, $\beta - 1 \geq 0$, $f(s)x^q(s) + h(s)x^r(\sigma(s)) > 0$, from the procedure of Theorem 3.4, $u(t)$ is non-decreasing in $\mathbb{R}_+$. By inequality (3.25), we obtain

$$x(t) \leq u(t), \quad t \in \mathbb{R}_+. \tag{3.30}$$

For $t \in \mathbb{R}_+$ with $\sigma(t) > 0$, since $\sigma(t) \leq t$, from inequality (3.30), we have

$$x(\sigma(t)) \leq u(\sigma(t)) \leq u(t). \tag{3.31}$$

For $t \in \mathbb{R}_+$ with $\sigma(t) \leq 0$, by the initial condition (3.26), we have

$$x(\sigma(t)) = \varphi(\sigma(t)) \leq \rho^{\frac{1}{p}}(t) \leq u(t). \tag{3.32}$$

So we obtain

$$u^p(t) \leq \rho(t) + g(t)\int_0^t (t-s)^{\beta-1}[f(s)u^q(s) + h(s)u^r(s)]\,ds, \quad t \in \mathbb{R}_+. \tag{3.33}$$

From Corollary 3.5, we have

$$x(t) \leq u(t) \leq \left(\rho(t) + g(t)m_1(t)E_\beta\left(n_1(t)\Gamma(\beta)t^\beta\right)\right)^{\frac{1}{p}}, \quad t \in \mathbb{R}_+, \tag{3.34}$$

where $m_1(t), n_1(t)$ are defined as in condition (3.28). □

Remark 3.8. When $\beta \geq 1$, the kernel in inequality (3.25) does not possess a singularity, then we get a different bound that is expressed via the Mittag-Leffler function compared to Lemma 2.4.

Remark 3.9. Comparing with Theorem 3.1, which deals with the case when $\beta \in (0,1)$, we have established the bound when $\beta \geq 1$. This extension allows us to address the nonlinear integral inequality with delay (3.1) for all $\beta > 0$.

Lastly, inspired by the technique of Theorem 1 in Medved' [23], we consider the Bihari-type for Theorem 3.1.

Theorem 3.10. *Let $x(t)$, $\rho(t)$, $f(t)$, $h(t)$ be non-negative functions that are continuous on $[0,T)$, $\phi : [0,+\infty) \to (0,+\infty)$ be a continuous, non-decreasing function, if $\rho(t)$ is non-decreasing in $\mathbb{R}_+$ and $x(t)$ satisfies the following form of delay integral inequality:*

$$x^p(t) \leq \rho(t) + \int_0^t (t-s)^{\beta-1}[f(s)x^q(s) + h(s)\phi\left(x(\sigma(s))\right)]\,ds, \quad t \in [0,T), \tag{3.35}$$

with the initial condition:

$$\begin{aligned} x(t) &= \varphi(t), \quad t \in [a,0], \\ \varphi(\sigma(t)) &\leq 3^{\frac{1-\alpha}{p}}\rho^{\frac{1}{p}}(t) \quad \text{for } t \in [0,T) \text{ with } \sigma(t) \leq 0, \end{aligned} \tag{3.36}$$

where $0 < \alpha < \beta < 1$, $p \neq 0$, $p \geq q \geq 0$, p, q are constants, $\sigma(t) \in C([0,T),\mathbb{R})$, $\sigma(t) \leq t$, $-\infty < a = \inf\{\sigma(t), t \in [0,T)\} < 0$, and $\varphi(t) \in C([a,0],\mathbb{R}_+)$, then

$$x(t) \leq \left(\psi^{-1}(\psi(\tilde{a}(t)) + \int_0^t b(s)\,ds)\right)^{\frac{\alpha}{p}}, \quad t \in [0,T_1], \tag{3.37}$$

where $\tilde{a}(t) = 3^{\frac{1}{\alpha}-1}\rho^{\frac{1}{\alpha}}(t) + \int_0^t \frac{p-q}{p}K^{\frac{q}{p}}c(s)\,ds$, $b(t) = \max\left\{\frac{p-q}{p}K^{\frac{q}{p}}t^{\frac{\beta-\alpha}{\alpha}}f^{\frac{1}{\alpha}}(t), t^{\frac{\beta-\alpha}{\alpha}}h^{\frac{1}{\alpha}}(t)\right\}$, $\psi(t) = \int_{t_0}^t \frac{1}{\tilde{\phi}(s)}\,ds$, $\tilde{\phi}(t) = t + \phi^{\frac{1}{\alpha}}(t^{\frac{\alpha}{p}})$, $t_0 > 0$, ψ^{-1} *is the inverse of* ψ, $T_1 \in (0,T)$ *and* $\psi(\tilde{a}(t)) + \int_0^t b(s)\,ds \in \mathrm{Dom}(\psi^{-1})$ *for all* $t \in [0, T_1]$.

Proof. From the procedure of Theorem 3.1, we have

$$x^{\frac{p}{\alpha}}(t) \le a(t) + \int_0^t \left[c(s)x^{\frac{q}{\alpha}}(s) + d(s)\phi^{\frac{1}{\alpha}}(x(\sigma(s)))\right] ds, \quad t \in [0,T), \tag{3.38}$$

where $a(t)$, $c(t)$, $d(t)$ are defined as in condition (3.4). Let $x^{\frac{1}{\alpha}}(t) = u(t)$, then

$$u^p(t) \le a(t) + \int_0^t \left[c(s)u^q(s) + d(s)\phi^{\frac{1}{\alpha}}(u^\alpha(\sigma(s)))\right] ds, \quad t \in [0,T). \tag{3.39}$$

Define a function $y(t)$ by

$$y^p(t) = a(t) + \int_0^t \left[c(s)u^q(s) + d(s)\phi^{\frac{1}{\alpha}}(u^\alpha(\sigma(s)))\right] ds, \quad t \in [0,T). \tag{3.40}$$

From the procedure of Theorem 3.7, let $g(t) = 1$, we have

$$y^p(t) \le a(t) + \int_0^t \left[c(s)y^q(s) + d(s)\phi^{\frac{1}{\alpha}}(y^\alpha(s))\right] ds, \quad t \in [0,T). \tag{3.41}$$

Set $m(t) = a(t) + \int_0^t \left[c(s)y^q(s) + d(s)\phi^{\frac{1}{\alpha}}(y^\alpha(s))\right] ds$, then $y(t) \le m^{\frac{1}{p}}(t)$.

By Lemma 2.7, we have

$$y^q(t) \le m^{\frac{q}{p}}(t) \le \frac{q}{p}K^{\frac{q-p}{p}}m(t) + \frac{p-q}{p}K^{\frac{q}{p}}, \quad t \in [0,T). \tag{3.42}$$

Then

$$\begin{aligned} m(t) &\le a(t) + \int_0^t \left[c(s)\left(\frac{q}{p}K^{\frac{q-p}{p}}m(s) + \frac{p-q}{p}K^{\frac{q}{p}}\right) + d(s)\phi^{\frac{1}{\alpha}}(m^{\frac{\alpha}{p}}(s))\right] ds \\ &= a(t) + \int_0^t \frac{p-q}{p}K^{\frac{q}{p}}c(s)\,ds \\ &\quad + \int_0^t \left[\frac{q}{p}K^{\frac{q-p}{p}}c(s)m(s) + d(s)\phi^{\frac{1}{\alpha}}(m^{\frac{\alpha}{p}}(s))\right] ds, \quad t \in [0,T). \end{aligned} \tag{3.43}$$

Let $\tilde{a}(t) = a(t) + \int_0^t \frac{p-q}{p}K^{\frac{q}{p}}c(s)\,ds$, $b(t) = \max\left\{\frac{q}{p}K^{\frac{q-p}{p}}c(t), d(t)\right\}$, $\tilde{\phi}(t) = t + \phi^{\frac{1}{\alpha}}(t^{\frac{\alpha}{p}})$. So we have

$$m(t) \le \tilde{a}(t) + \int_0^t b(s)\tilde{\phi}(m(s))\,ds, \quad t \in [0,T). \tag{3.44}$$

Then inequality (3.44) is in the form of the known Bihari inequality [4]. So we have

$$m(t) \le \psi^{-1}\left(\psi(\tilde{a}(t)) + \int_0^t b(s)\,ds\right), \quad t \in [0,T_1]. \tag{3.45}$$

At last, by inequalities (3.38), (3.39), (3.41) and (3.45), we get

$$x^{\frac{1}{\alpha}}(t) = u(t) \le y(t) \le m^{\frac{1}{p}}(t), \quad t \in [0,T). \tag{3.46}$$

$$x(t) \le m^{\frac{\alpha}{p}}(t) \le \left(\psi^{-1}(\psi(\tilde{a}(t)) + \int_0^t b(s)\,ds)\right)^{\frac{\alpha}{p}}, \quad t \in [0,T_1]. \tag{3.47}$$

□

Remark 3.11. Looking closely at Theorem 3.10, we establish a new nonlinear Bihari-type inequality with time-varying delay and obtain the explicit bound which is a generalization of theorems in [23] and [37]. Additionally, Lan and Webb [15] generalized the classical Bihari inequality using a novel approach, where the nonlinearity and time-varying delay term addressed in our work are not covered in [15].

Remark 3.12. Hamlat, Graef and Ouahab [9] considered a generalized Bihari inequality that imposes stricter requirements on ϕ. In contrast, our theorem imposes weaker requirements on ϕ. Furthermore, by employing Lemma 2.7 and the initial condition (3.36), we account for both nonlinearity and time-varying delay term in our theorem which are not considered in [9], ultimately yielding a bound of the similar explicit form.

A key novelty of our results, is the introduction of the free parameter $K > 0$, which allows for the optimization of the bound. To clearly show how to choose the optimal K and the improvements offered by our approach, we present the following example with explicit functions satisfying all hypotheses of Theorem 3.1.

Example 3.13. Let $\rho(t) = t$, $g(t) = 1$, $f(t) = t$, $h(t) = t$, $p = 2$, $q = r = 1$, $\beta = \frac{2}{3}$, $\alpha = \frac{1}{3}$, inequality (3.1) becomes

$$x^2 \leq \ t + \int_0^t (t-s)^{-\frac{1}{3}}[sx(s) + sx(\sigma(s))]\,ds, \quad t \in \mathbb{R}_+, \tag{3.48}$$

From Theorem 3.1, we have $a(t) = 9t^3$, $b(t) = 9(\Gamma(1/2)\Gamma(1/2))^2 = 9\pi^2$, $c(t) = t^4$, $d(t) = t^4$, $C(t) = \frac{1}{5}t^5K^{\frac{1}{2}} + \frac{9}{8}t^8K^{-\frac{1}{2}}$, $D(t) = 9\pi^2t^4K^{-\frac{1}{2}}$, $\int_0^t D(s)\,ds = \frac{9}{5}\pi^2t^5K^{-\frac{1}{2}}$, then we have the bound of inequality (3.48)

$$x(t) \leq \left[9t^3 + \left(\frac{9\pi^2t^5}{5}K^{\frac{1}{2}} + \frac{81\pi^2t^8}{8}K^{-\frac{1}{2}}\right)\exp\left(\frac{9\pi^2t^5}{5}K^{-\frac{1}{2}}\right)\right]^{\frac{1}{6}}, \quad t \in \mathbb{R}_+. \tag{3.49}$$

Define a function $F(t,K)$ by

$$F(t,K) = 9t^3 + \frac{9\pi^2}{5}K^{\frac{1}{2}}t^5\exp\left(\frac{9\pi^2}{5}K^{-\frac{1}{2}}t^5\right) + \frac{81\pi^2}{8}K^{-\frac{1}{2}}t^8\exp\left(\frac{9\pi^2}{5}K^{-\frac{1}{2}}t^5\right). \tag{3.50}$$

We now only need to minimize $F(t,K)$ to find the optimal K. Let $z = At^5K^{-\frac{1}{2}}$, $A = \frac{9\pi^2}{5}$, $B = \frac{81\pi^2}{8}$, we have

$$F(t,z) = 9t^3 + \frac{A^2t^{10}}{z}e^z + \frac{B}{A}t^3ze^z = 9t^3 + \frac{A^2t^{10}}{z}e^z + \frac{45}{8}t^3ze^z. \tag{3.51}$$

Taking the partial derivative with respect to z,

$$\begin{aligned}\frac{\partial F}{\partial z} &= -\frac{A^2t^{10}}{z^2}e^z + \frac{A^2t^{10}}{z}e^z + \frac{45}{8}t^3e^z + \frac{45}{8}t^3ze^z \\ &= \frac{t^3e^z}{z^2}\left(\frac{45}{8}z^3 + \frac{45}{8}z^2 + A^2t^7z - A^2t^7\right).\end{aligned} \tag{3.52}$$

Let $P(z) = \frac{45}{8}z^3 + \frac{45}{8}z^2 + A^2t^7z - A^2t^7$, $t > 0$. By Vièta's formulas, for the three roots z_1, z_2, z_3 of the polynomial $P(z)$, we have $z_1 + z_2 + z_3 = -1 < 0$, $z_1z_2z_3 = \frac{8A^2t^7}{45} > 0$, so when $z > 0$, $P(z)$ has only one real root. This means that the function $F(t,z)$ first decreases and then increases with respect to z. To find the optimal K, we now only need to find the positive root of the polynomial $P(z)$. By usual techniques, we can get the positive root

$$\begin{aligned} z_1 = &\sqrt[3]{\frac{2k}{3} - \frac{1}{27} + \sqrt{\left(\frac{1}{27} - \frac{2k}{3}\right)^2 + \left(\frac{k}{3} - \frac{1}{9}\right)^3}} \\ &+ \sqrt[3]{\frac{2k}{3} - \frac{1}{27} - \sqrt{\left(\frac{1}{27} - \frac{2k}{3}\right)^2 + \left(\frac{k}{3} - \frac{1}{9}\right)^3}} - \frac{1}{3}, \end{aligned} \tag{3.53}$$

where $k = \frac{8A^2t^7}{45}$. Finally, we can get the optimal $K = \frac{A^2t^{10}}{z_1^2}$, A and z_1 are determined as described in the preceding analysis. With this K (depending on t) being optimal for each t, we obtain the corresponding best bound for any given t.

Remark 3.14. Generally, it can be observed that for different values of t, the parameter K that minimizes the bound is different. It is challenging to find a constant K, independent of t, that minimizes the bound. In this example, the optimal $K \to +\infty$ as $t \to +\infty$. If $t \in [0, T]$, and the goal is to find the optimal bound which can be used to prove existence (see Theorem 4.1), the best t-independent constant K is given by the function $K(t)$ evaluated at the endpoint $t = T$.

4 Applications

In this section, we will show that our results are useful in proving the existence of solutions to certain fractional differential equations with time-varying delay. We consider the following fractional differential equation:

$$\begin{cases} D_*^\beta x(t) = H(t, x(t), x(\sigma(t))), & \text{a.e. } t \in [0, T], \\ x(t) = \varphi(t), & t \in [a, 0], \end{cases} \tag{4.1}$$

where D_*^β is the Caputo fractional derivative, $H \in C(\mathbb{R}_+ \times \mathbb{R}^2, \mathbb{R})$, $\beta \in (0,1)$, $\sigma(t) \in C(\mathbb{R}_+, \mathbb{R})$, $\sigma(t) \le t$, $-\infty < a = \inf\{\sigma(t), t \in \mathbb{R}_+\} < 0$, and $\varphi(t) \in C([a, 0], \mathbb{R}_+)$.

Theorem 4.1. *If there exist non-negative continuous functions $k(t)$, $f(t)$ and $h(t)$ such that:*

$$|H(t, x, y)| \le k(t) + f(t)|x|^q + h(t)|y|^r, \quad (t, x, y) \in \mathbb{R}_+ \times \mathbb{R}^2, \tag{4.2}$$

where $0 \le q, r \le 1$. Then equation (4.1) has at least one solution on the interval $[a, T]$ for arbitrarily large T when $\beta \in (0, 1)$.

Proof. Transform the problem into a fixed point problem. Let $N : C[a, T] \to C[a, T]$ be the operator defined by

$$N(x)(t) = \begin{cases} \varphi(0) + \frac{1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} H(s, x(s), x(\sigma(s)))\, ds, & t \in [0, T], \\ \varphi(t), & t \in [a, 0]. \end{cases} \tag{4.3}$$

By Lemma 2.9, it is easy to know that the fixed points of operator N are solutions of equation (4.1). We can show that the operator N is continuous and completely continuous by usual techniques, see [32] and [29]. By Lemma 2.8, we now only need to prove the set $\{x \in E : x =$

λNx for some $\lambda \in (0,1)\}$ is bounded. Let $x(t) \in C[a,T]$ and $x = \lambda N(x)$ for some $\lambda \in (0,1)$, then we have

$$x(t) = \begin{cases} \lambda \left[\varphi(0) + \frac{1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} H(s, x(s), x(\sigma(s)))\, ds\right], & t \in [0,T], \\ \lambda \varphi(t), & t \in [a,0]. \end{cases} \tag{4.4}$$

When $t \in [a,0]$, since $\varphi(t)$ is continuous, let $M_1 = \sup\{|\varphi(t)| : t \in [a,0]\}$, we have

$$|x(t)| \leq |\varphi(t)| \leq M_1. \tag{4.5}$$

When $t \in [0,T]$,

$$\begin{aligned} |x(t)| &\leq |\varphi(0)| + \frac{1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} |H(s, x(s), x(\sigma(s)))|\, ds \\ &\leq |\varphi(0)| + \frac{1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} k(s)\, ds \\ &\quad + \frac{1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} \left[f(s)|x(s)|^q + h(s)|(x(\sigma(s))|^r\right] ds. \end{aligned} \tag{4.6}$$

If $\beta \in (0,1)$, let $0 < \gamma < \beta < 1$, by Hölder inequality, we get

$$\begin{aligned} |x(t)| &\leq |\varphi(0)| + \frac{t^{\beta-\gamma}}{\Gamma(\beta) \left(\frac{\beta-\gamma}{1-\gamma}\right)^{1-\gamma}} \left(\int_0^t k^{\frac{1}{\gamma}}(s)\, ds\right)^{\gamma} ds \\ &\quad + \frac{1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} \left[f(s)|x(s)|^q + h(s)|(x(\sigma(s))|^r\right] ds. \end{aligned} \tag{4.7}$$

Let $\rho(t) = M_1 + |\varphi(0)| + \frac{t^{\beta-\gamma}}{\Gamma(\beta)\left(\frac{\beta-\gamma}{1-\gamma}\right)^{1-\gamma}} \left(\int_0^t k^{\frac{1}{\gamma}}(s)\, ds\right)^{\gamma} ds$, $g(t) = \frac{1}{\Gamma(\beta)}$, then inequality (4.7) can be restated as

$$|x(t)| \leq \rho(t) + g(t) \int_0^t (t-s)^{\beta-1} [f(s)|x(s)|^q + h(s)|x(\sigma(s))|^r]\, ds. \tag{4.8}$$

By Theorem 3.1, we can get

$$\begin{aligned} |x(t)| &\leq \left[a(t) + b(t)C(t) \exp\left(\int_0^t D(s)\, ds\right)\right]^{\alpha} \\ &\leq \left[a(T) + b(T)C(T) \exp\left(\int_0^T D(s)\, ds\right)\right]^{\alpha}. \end{aligned} \tag{4.9}$$

Consequently, from inequalities (4.5) and (4.9), the set $\{x \in E : x = \lambda Nx$ for some $\lambda \in (0,1)\}$ is bounded. Thus, the proof is completed. □

Theorem 4.2. *If there exist non-negative continuous functions $k(t)$, $f(t)$ and $h(t)$ such that:*

$$|H(t,x,y)| \leq k(t) + f(t)|x|^q + h(t)\phi(|y|), \ (t,x,y) \in \mathbb{R}_+ \times \mathbb{R}^2, \tag{4.10}$$

where $0 \leq q \leq 1$, $\phi : [0,+\infty) \to (0,+\infty)$ be a continuous, non-decreasing function. Then (4.1) has at least one solution on $[a, T_1]$ when $\beta \in (0,1)$, $T_1 \in (0,T)$ and $\psi(\tilde{a}(t)) + \int_0^t b(s)\, ds \in \mathrm{Dom}(\psi^{-1})$ for all $t \in [0,T_1]$, $\tilde{a}(t)$, $b(t)$, $\psi(t)$ can be defined as in Theorem 3.10.

Proof. By similar techniques in Theorem 4.1, we should only prove that the solution of equation (4.4) is bounded. When $t \in [a, 0]$, since $\varphi(t)$ is continuous, let $M_1 = \sup\{|\varphi(t)| : t \in [a, 0]\}$, we have

$$|x(t)| \le |\varphi(t)| \le M_1. \tag{4.11}$$

Let $0 < \gamma < \beta < 1$, when $t \in [0, T)$,

$$\begin{aligned}
|x(t)| &\le |\varphi(0)| + \frac{1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} |H(s, x(s), x(\sigma(s)))| \, ds \\
&\le |\varphi(0)| + \frac{1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} k(s) \, ds \\
&\quad + \frac{1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} \left[f(s)|x(s)|^q + h(s)\phi\left(|x(\sigma(s))|\right) \right] ds \\
&\le |\varphi(0)| + \frac{t^{\beta-\gamma}}{\Gamma(\beta)\left(\frac{\beta-\gamma}{1-\gamma}\right)^{1-\gamma}} \left(\int_0^t k^{\frac{1}{\gamma}}(s) \, ds \right)^{\gamma} ds \\
&\quad + \frac{1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} \left[f(s)|x(s)|^q + h(s)\phi\left(|x(\sigma(s))|\right) \right] ds.
\end{aligned} \tag{4.12}$$

Let $\rho(t) = M_1 + |\varphi(0)| + \frac{t^{\beta-\gamma}}{\Gamma(\beta)\left(\frac{\beta-\gamma}{1-\gamma}\right)^{1-\gamma}} \left(\int_0^t k^{\frac{1}{\gamma}}(s) \, ds \right)^{\gamma} ds$, $f_1(t) = \frac{1}{\Gamma(\beta)} f(t)$, $h_1(t) = \frac{1}{\Gamma(\beta)} h(t)$, then inequality (4.12) can be restated as

$$|x(t)| \le \rho(t) + \int_0^t (t-s)^{\beta-1} [f_1(s)|x(s)|^q + h_1(s)\phi\left(|x(\sigma(s))|\right)] \, ds. \tag{4.13}$$

By Theorem 3.10, when $t \in [0, T_1]$, we can get

$$x(t) \le \left(\psi^{-1}(\psi(\tilde{a}(t)) + \int_0^t b(s) \, ds) \right)^{\alpha} \le \left(\psi^{-1}(\psi(\tilde{a}(T_1)) + \int_0^{T_1} b(s) \, ds) \right)^{\alpha}. \tag{4.14}$$

Consequently, from inequalities (4.11) and (4.14), we know the solutions of equation (4.4) is bounded. Thus the proof is completed. □

Corollary 4.3. *Since $\psi(t) = \int_{t_0}^t \frac{1}{\tilde{\phi}(s)} ds$ is a strictly increasing continuous function, if $\lim_{t\to\infty} \psi(t) = \infty$, we can let $T_1 = T$. Then from Theorem 4.2, the solutions exist on the interval $[a, T]$ for arbitrarily large T.*

Example 4.4.

$$\begin{cases} D_*^{\frac{1}{2}} x(t) = t + e^t \frac{x^m(t) + x^n(\sigma(t))}{x^{2c}(t) + x^{2d}(\sigma(t)) + 1}, & \text{a.e. } t \in [0, T], \\ x(t) = \varphi(t), & t \in [a, 0], \end{cases} \tag{4.15}$$

where $\sigma(t) \in C(\mathbb{R}_+, \mathbb{R})$, $\sigma(t) \le t$, $-\infty < a = \inf\{\sigma(t), t \in \mathbb{R}_+\} < 0$, and $\varphi(t) \in C([a, 0], \mathbb{R}_+)$, $0 < m - 2c < 1$, $0 < n - 2d < 1$, $\beta = 1/2$. Since

$$|t + e^t \frac{x^m(t) + x^n(\sigma(t))}{x^{2c}(t) + x^{2d}(\sigma(t))}| \le |t| + e^t |x^{m-2c}(t)| + e^t |x^{n-2d}(\sigma(t))|, \tag{4.16}$$

by Theorem 4.1, equation (4.15) has at least one solution on $[a, T]$ for arbitrarily large T.

Example 4.5.

$$\begin{cases} D_*^{\frac{2}{3}}x(t) = t + e^t \frac{x^m(t)+\sqrt{(x^2(\sigma(t))+k)(\ln(x^2(\sigma(t))+k)-1)}}{x^{2c}(t)+1}, & \text{a.e. } t \in [0,T], \\ x(t) = \varphi(t), & t \in [a,0], \end{cases} \tag{4.17}$$

where $\sigma(t) \in C(\mathbb{R}_+, \mathbb{R})$, $\sigma(t) \le t$, $-\infty < a = \inf\{\sigma(t), t \in \mathbb{R}_+\} < 0$, and $\varphi(t) \in C([a,0], \mathbb{R}_+)$, $0 < m - 2c < 1$, $k \ge \exp(1)$, $\beta = 2/3$, $\alpha = 1/2$. Since

$$\begin{aligned} |D_*^{\frac{2}{3}}x(t)| = |t + e^t \frac{x^m(t)+\sqrt{(x^2(\sigma(t))+k)(\ln(x^2(\sigma(t))+k)-1)}}{x^{2c}(t)+1}| \\ \le |t| + e^t|x^{m-2c}(t)| \\ + e^t\sqrt{(x^2(\sigma(t))+k)(\ln(x^2(\sigma(t))+k)-1)}, \end{aligned} \tag{4.18}$$

we have $\phi(y) = \sqrt{(y^2+k)(\ln(y^2+k)-1)}$. $\tilde{\phi}(t) = t + \phi^{\frac{1}{\alpha}}(t^{\frac{\alpha}{p}}) = (t+k)(\ln(t+k)-1)+t$, where $t \ge 0$. So we have

$$\begin{aligned} \psi(t) &= \int_{t_0}^{t} \frac{1}{\tilde{\phi}(s)}\,ds \\ &= \int_{t_0}^{t} \frac{1}{(s+k)(\ln(s+k)-1)+s}\,ds \\ &\ge \int_{t_0}^{t} \frac{1}{(s+k)\ln(s+k)}\,ds \\ &= \ln(\ln(t+k)) - \ln(\ln(t_0+k)). \end{aligned} \tag{4.19}$$

Then $\lim_{t\to\infty}\psi(t) = +\infty$, by Corollary 4.3, equation (4.17) has at least one solution on $[a,T]$ for arbitrarily large T.

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