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Continuum of limit cycles in a class of three-dimensional quadratic polynomial systems

Alexander Grin ✉ [1] and **Klaus R. Schneider** [2]

[1] Yanka Kupala State University of Grodno, Ozheshko Street 22, Grodno, 230023, Belarus
[2] Weierstrass Institute for Applied Analysis and Stochastics, Mohrenstr. 39, Berlin, 10117, Germany

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Abstract. It is well-known that planar autonomous polynomial dynamical systems have only a finite number of limit cycles. We present a class of parameter depending three-dimensional autonomous systems whose right-hand side consists of quadratic polynomials and establish the bifurcation of a continuum of limit cycles from a period annulus and from an equilibrium point. A limit cycle Γ is understood as a closed orbit without equilibrium point which is the limit set of a trajectory different from Γ. The systems under consideration are distinguished by the existence of two straight lines of equilibria and the existence of a family of invariant manifolds foliating $\mathbb{R}^3 \setminus \{0\}$.

Keywords: autonomous three-dimensional quadratic polynomial systems, invariant manifolds, center, bifurcation, period annulus, continuum of limit cycles.

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1 Introduction

Limit cycles as limit sets of trajectories of systems of autonomous ordinary differential equations play a crucial role in determining the topological structure of the orbits [2, 5, 8, 15, 18]. There is a lot of methods to establish the existence of limit cycles, especially for planar systems [6]. One approach to prove the existence of limit cycles, to estimate their location and to determine their number at least locally consists in the application of bifurcation theory [3, 12, 14]. The bifurcation of limit cycles from an equilibrium point and from a period annulus are well-known and frequently used in applications. The famous van der Pol oscillator belongs to the class of systems where the occurrence of self-sustained oscillations is related to the bifurcation of a unique limit cycle from a period annulus [5]. To any given natural number N we can construct planar polynomial systems with a period annulus such that N limit cycles bifurcate from it, but it is not possible to construct planar polynomial systems with a period annulus from that infinitely many limit cycles bifurcate. This is due to the individual finiteness of limit cycles for planar polynomial systems [7, 9, 13].

✉ Corresponding author. Email: grin@grsu.by

It is well-known that already systems consisting of three autonomous differential equations whose right hand sides are polynomials of second degree exhibit new bifurcation scenarios and new dynamical behaviors compared with planar systems [1,11,14,17]. The question on the existence of infinitely many limit cycles in three-dimensional polynomial systems has been investigated in the paper [10]. The goal of this note is to present three-dimensional polynomial systems having a continuum of limit cycles bifurcating from a continuum of closed orbits (period annulus). The motivation to study this problem comes from the paper [4] where another bifurcation scenario has been considered.

2 Preliminaries

We consider real three-dimensional autonomous systems depending on a scalar bifurcation parameter ε

$$\frac{du}{dt} = A(\varepsilon)u + f(u) \tag{2.1}$$

where the matrix A depends smoothly on ε and the components of f are homogeneous quadratic polynomials. We note that the origin is an equilibrium point of system (2.1).

Definition 2.1. A closed orbit Γ_ε of system (2.1) which does not contain an equilibrium point and which is the α- or ω-limit set of some trajectory of system (2.1) different from Γ_ε is called a limit cycle. If Γ_ε is the α- (ω-) limit set of all trajectories of some neighborhood of Γ_ε, then it is called an unstable (stable) limit cycle.

Remark 2.2. There are different definitions of a limit cycle which are not all equivalent. Often there is required that Γ_ε is an isolated closed orbit. We note that in case of planar polynomial systems each limit cycle is an isolated closed orbit [5].

Definition 2.3. We say that in system (2.1) Hopf bifurcation takes place at the origin for $\varepsilon = \varepsilon_0$ if

(i). there is a neighborhood $\mathcal{N}$ of the origin containing no periodic solution of system (2.1) for $\varepsilon = \varepsilon_0$,

(ii). the stability of the origin changes when ε passes ε_0,

(iii). there is a sufficiently small one-sided neighborhood $\mathcal{N}_\varepsilon$ of ε_0 such that for $\varepsilon \in \mathcal{N}_\varepsilon$ there exists a family of periodic solutions $p(t,\varepsilon)$ of system (2.1) in $\mathcal{N}$ whose norm $\|p(t,\varepsilon)\|$ tends to zero as $\varepsilon \to 0$.

Remark 2.4. The condition (i) is fulfilled generically and does not appear in the usual definition of Hopf bifurcation.

Definition 2.5. We say that system (2.1) has a period annulus (continuum of periodic solutions) $\mathcal{A}$ for $\varepsilon = \varepsilon_0$ if there exists a two-dimensional invariant manifold $\mathcal{M}$ of system (2.1) containing the origin such that there is a region $\mathcal{G}$ in $\mathcal{M}$ containing the origin as unique equilibrium point and filled with closed orbits corresponding to periodic solutions of system (2.1) for $\varepsilon = \varepsilon_0$ which are no limit cycles.

Definition 2.6. We say that in system (2.1) for $\varepsilon = \varepsilon_0$ a continuum of limit cycles bifurcates from a period annulus if system (2.1) has for $\varepsilon = \varepsilon_0$ a period annulus $\mathcal{A}$ and if there is a sufficiently small neighborhood $\mathcal{N}_\varepsilon$ of ε_0 such that for $\varepsilon \in \mathcal{N}_\varepsilon$ system (2.1) has a continuum of limit cycles $\mathcal{C}_\varepsilon$ such that each limit cycle Γ_ε in $\mathcal{C}_\varepsilon$ approaches a closed orbit in $\mathcal{A}$ as $\varepsilon \to 0$.

3 Class of three-dimensional systems

We consider the class of three-dimensional quadratic autonomous systems

$$\begin{aligned} \frac{dx}{dt} &= \varepsilon x - \lambda y + a_2 xy + a_3 y^2 + a_4 xz + a_5 yz, \\ \frac{dy}{dt} &= \lambda x + \varepsilon y - a_2 x^2 - a_3 xy + a_4 yz - a_5 xz, \\ \frac{dz}{dt} &= 2z\left(\varepsilon + a_4 z\right) \end{aligned} \tag{3.1}$$

under the condition

(A_1). λ and a_2, a_3, a_4, a_5 are given real parameters satisfying $\lambda > 0, a_4 \neq 0, a_2^2 + a_3^2 > 0$, the real parameter ε is considered as a bifurcation parameter.

This system has been introduced in the paper [4]. The focus of that paper is on Hopf bifurcation of a two-dimensional system which describes the projection of the orbits of the three-dimensional system (3.1) on an invariant manifold in the (x, y)-plane. The goal of our investigation is to prove that a continuum of limit cycles of system (3.1) bifurcates from a period annulus when ε passes zero.

With respect to the existence of periodic solutions we get from (3.1) the result

Lemma 3.1. *If system (3.1) has periodic solutions, then they are located on the invariant manifolds $\mathcal{I}_0 := \{(x, y, z) \in \mathbb{R}^3 : z = 0\}$ and $\mathcal{M}_\varepsilon := \{(x, y, z) \in \mathbb{R}^3 : z = -\varepsilon/a_4\}$.*

Remark 3.2. We note that the invariant manifolds $\mathcal{I}_0$ and $\mathcal{M}_0$ coincide. Thus, the invariant manifold $\mathcal{M}_\varepsilon$ bifurcates from the invariant manifold $\mathcal{I}_0$ if ε passes zero.

First we consider the special case of system (3.1) defined by $a_5 = 0$.

$$\begin{aligned} \frac{dx}{dt} &= x(\varepsilon + a_4 z) - y(\lambda - a_2 x - a_3 y), \\ \frac{dy}{dt} &= y(\varepsilon + a_4 z) + x(\lambda - a_2 x - a_3 y), \\ \frac{dz}{dt} &= 2z\left(\varepsilon + a_4 z\right). \end{aligned} \tag{3.2}$$

4 Limit cycles of system (3.2)

Taking into account Lemma 3.1 we study system (3.2) on the manifolds $\mathcal{I}_0$ and $\mathcal{M}_\varepsilon$.

The phase portrait of system (3.2) on the manifold $\mathcal{I}_0$ is defined by the two-dimensional system

$$\begin{aligned} \frac{dx}{dt} &= \varepsilon x - y(\lambda - a_2 x - a_3 y), \\ \frac{dy}{dt} &= \varepsilon y + x(\lambda - a_2 x - a_3 y). \end{aligned} \tag{4.1}$$

Concerning the existence of equilibria we obtain from system (4.1)

Lemma 4.1. *In case $\varepsilon \neq 0$, system (4.1) has the origin as unique equilibrium point. In case $\varepsilon = 0$, system (4.1) has the origin as equilibrium point and additionally the straight line $\gamma := \{(x, y) \in \mathbb{R}^2 : \lambda - a_2 x - a_3 y = 0\}$ as a continuum of equilibria.*

Remark 4.2. We note that the location of the straight line γ does not depend on ε.

To study the phase portrait of system (4.1) we introduce polar coordinates (r, φ) by $x = r\cos\varphi, y = r\sin\varphi$. System (4.1) reads in polar coordinates

$$\begin{aligned}\frac{dr}{dt} &= \varepsilon r,\\ \frac{d\varphi}{dt} &= \lambda - r(a_2\cos\varphi + a_3\sin\varphi) = \lambda - a_2 x - a_3 y.\end{aligned} \tag{4.2}$$

Remark 4.3. We note that $\frac{d\varphi}{dt}$ vanishes when the corresponding point (x_0, y_0) in the phase plane is located on the straight line γ, that is, (x_0, y_0) is an equilibrium point of system (4.1).

From the first equation in (4.2) we get

Proposition 4.4. *In case $\varepsilon \neq 0$, system (3.2) has no periodic solution on the invariant manifold $\mathcal{I}_0$.*

In case $\varepsilon = 0$, it follows from the same equation that the phase portrait of system (4.2) consists of circles surrounding the origin. We denote by C_r a circle with radius r centered at the origin of the (x, y)-plane. If the circle C_r does not touch or intersect the straight line γ, then by Lemma 4.1 no equilibrium point is located on C_r and $d\varphi/dt$ is positive. Thus, each such circle C_r corresponds to a periodic solution of system (4.2) and also of system (3.2). If we denote by d the distance of γ from the origin which is defined by

$$d = \frac{\lambda}{\sqrt{a_2^2 + a_3^2}}, \tag{4.3}$$

then we have the result

Proposition 4.5. *Under the assumption (A_1) in the case $\varepsilon = 0$ any circle C_r with $0 < r < d$ represents a periodic solution of system (3.2) on the manifold $\mathcal{I}_0$, in case $r = d$ the circle C_d is a homoclinic orbit, and in case $r > d$ the circles C_r for $r > d$ consist of two heteroclinic orbits.*

Figure 4.1 shows three orbits (dashed circles) of system (3.2) on the invariant manifold $\mathcal{I}_0$ for $\varepsilon = 0$, $\lambda = 3$, $a_2 = 1/2$, $a_3 = 1$ and the corresponding straight line γ of equilibria.

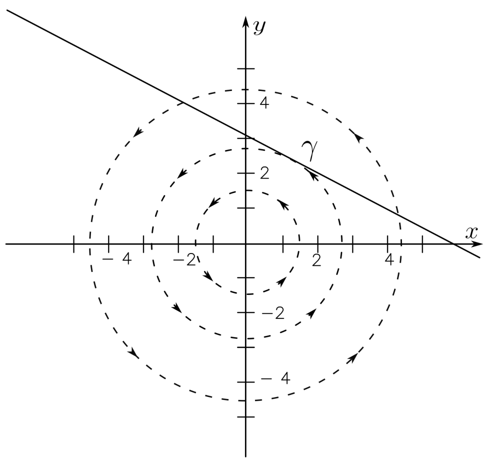

Figure 4.1: Orbits of system (3.2) on $\mathcal{I}_0$ and the straight line γ of equilibria.

To study the problem whether the periodic solutions of system (3.2) for $\varepsilon = 0$ on the manifold $\mathcal{I}_0$ are limit cycles we have to investigate whether to each circle C_r with $0 < r < d$ there is a trajectory Γ of system (3.2) whose α- or ω- limit set is C_r. According to Proposition 4.5 Γ is not located in $\mathcal{I}_0$. To solve this problem we exploit a special property of system (3.2) which has been established in [4].

Proposition 4.6. *System (3.1) has the family of invariant manifolds*

$$\mathcal{P}_s := \{(x,y,z) \in \mathbb{R}^3 : z = s(x^2+y^2)\} \tag{4.4}$$

where s is any real number.

Remark 4.7. The family of invariant manifolds $\mathcal{P}_s$ represents a foliation of $\mathbb{R}^3 \setminus \{0\}$.

Proposition 4.8. *Let Γ be a trajectory of system (3.2) which is not located on $\mathcal{I}_0$ and whose ω-limit set $\omega(\Gamma)$ is contained in the manifold $\mathcal{I}_0$. Then $\omega(\Gamma)$ is the origin.*

Proof. By Remark 4.7 there is a unique invariant manifold $\mathcal{P}_s$ containing Γ. Since $\mathcal{P}_s \cap \mathcal{I}_0$ consists of the origin, $\omega(\Gamma)$ coincides with the origin. □

This result implies immediately

Proposition 4.9. *The circles C_r of system (3.2) located on the invariant manifold $\mathcal{I}_0$ represent for $0 < r < d$ a continuum of periodic solutions which are no limit cycles; they form a period annulus $\mathcal{A}$ for system (3.2). The circles C_r with $r > d$ represent a heteroclinic annulus $\mathcal{H}$.*

Next we study system (3.2) on the manifold $\mathcal{M}_\varepsilon$ with $\varepsilon \neq 0$. It reads

$$\begin{aligned}\frac{dx}{dt} &= -y(\lambda - a_2x - a_3y),\\ \frac{dy}{dt} &= x(\lambda - a_2x - a_3y).\end{aligned} \tag{4.5}$$

We note immediately that this system does not depend on ε, that means, system (3.2) has on all manifolds $\mathcal{M}_\varepsilon$ the same phase portrait. Therefore, Proposition 4.5 which describes the phase portrait of system (3.2) on the manifold $\mathcal{M}_0$ describes also the phase portrait on any invariant manifold $\mathcal{M}_\varepsilon$.

By Proposition 4.9, the circles C_r on $\mathcal{M}_0$ are periodic solutions of system (3.2) for $0 < r < d$ but no limit cycles. To answer the question whether the circles C_r for $0 < r < d$ on $\mathcal{M}_\varepsilon$ with $\varepsilon \neq 0$ are limit cycles, we use again the family of invariant manifolds $\mathcal{P}_s$.

Theorem 4.10. *In case $\varepsilon \neq 0$, the circles C_r on $\mathcal{M}_\varepsilon$ with $0 < r < d$ are limit cycles of system (3.2).*

Proof. The circle C_r with $0 < r < d$ can be considered as the intersection of the invariant manifolds $\mathcal{M}_\varepsilon$ and $\mathcal{P}_s$ with $s = \varepsilon/r^2$. A trajectory Γ of system (3.2) starting at a point on $\mathcal{P}_{\varepsilon/r^2}$ between the manifolds $\mathcal{I}_0$ and $\mathcal{M}_\varepsilon$ has the property that its z-component for increasing t is either increasing or decreasing which depends on the sign of ε and a_4. Hence, the circle C_r is either the ω-limit set or the α-limit set of Γ. Therefore, the circle C_r on $\mathcal{M}_\varepsilon$ represents a limit cycle. □

Since we have the same phase portrait of system (3.2) on all manifolds $\mathcal{M}_\varepsilon$ for any real ε, where only on the manifold $\mathcal{I}_0$ the circles C_r with $0 < r < d$ are no limit cycles, we have the result

Theorem 4.11. *If the parameter ε in system (3.2) passes the origin, a continuum $\mathcal{L}$ of limit cycles bifurcates from a period annulus $\mathcal{A}$.*

5 Limit cycles of system (3.1)

Now we study system (3.1) under the assumption

(A_2). $a_5 \neq 0$.

The procedure used to study system (3.2) will also be applied to system (3.1). First we investigate system (3.1) on the invariant manifold $\mathcal{I}_0$. It is easy to see that the two-dimensional system describing the dynamics of system (3.1) on the manifold $\mathcal{I}_0$ coincides with system (4.1) describing the dynamics of system (3.2) on $\mathcal{I}_0$. Thus, system (3.2) and system (3.1) have the same phase portrait on the manifold $\mathcal{I}_0$ described by Proposition 4.5 and Proposition 4.9.

Now we consider system (3.1) on the manifold $\mathcal{M}_\varepsilon$ with $\varepsilon \neq 0$. If we substitute $z = -\varepsilon/a_4$ into system (3.1) we get the two-dimensional system

$$\begin{aligned} \frac{dx}{dt} &= -y(\lambda + \varepsilon a_5/a_4 - a_2 x - a_3 y), \\ \frac{dy}{dt} &= x(\lambda + \varepsilon a_5/a_4 - a_2 x - a_3 y) \end{aligned} \tag{5.1}$$

describing the dynamics of system (3.1) on $\mathcal{M}_\varepsilon$. If we compare it with system (4.5) we note that both systems have the same structure but differ in the fact that the right hand side of system (5.1) depends on ε. The same structure implies that the phase portrait of system (5.1) consists of circles surrounding the origin and of a straight line γ_ε of equilibria

$$\gamma_\varepsilon := \{(x,y) \in \mathbb{R}^2 : \lambda + \varepsilon/a_4 - a_2 x - a_3 y = 0\}. \tag{5.2}$$

It differs from system (4.5) in the fact that the position of the straight line γ_ε depends on ε. Its distance d_ε from the origin is defined by

$$d_\varepsilon = \frac{|\lambda + \varepsilon a_5/a_4|}{\sqrt{a_2^2 + a_3^2}}. \tag{5.3}$$

For $\varepsilon = 0$ we have $d_0 = d > 0$, for $\varepsilon = \varepsilon_0 := -\lambda a_4/a_5$ the distance d_{ε_0} vanishes. In analogy to Proposition 4.5 it holds

Theorem 5.1. *Assume (A_1) and (A_2) to be valid. In case $\varepsilon \neq \varepsilon_0$ we have $d_\varepsilon > 0$. The corresponding phase portrait of system (3.1) on $\mathcal{M}_\varepsilon$ consists of circles C_r, $r > 0$, surrounding the origin which represent periodic solutions for $0 < r < d_\varepsilon$, C_{d_ε} is a homoclinic orbit, the circles C_r with $r > d_\varepsilon$ consist of two heteroclinic orbits forming a heteroclinic annulus $\mathcal{H}_\varepsilon$. In case $\varepsilon = \varepsilon_0$ we have $d_{\varepsilon_0} = 0$. The corresponding phase portrait consists of circles C_r, $r > 0$, surrounding the origin and consisting of two heteroclinic orbits forming a heteroclinic annulus $\mathcal{H}_{\varepsilon_0}$.*

Corollary 5.2. *In case $\varepsilon = 0$, the circles C_r with $0 < r < d_0 = d$ define a continuum of periodic solutions of system (3.1) representing a period annulus $\mathcal{A}_0$.*

Since each circle C_r is located on an invariant manifold $\mathcal{P}_s$ we can conclude as in the section before and have the result

Theorem 5.3. *Under the assumptions (A_1), (A_2), $\varepsilon \neq 0$ and $\varepsilon \neq \varepsilon_0$ the circles C_r of system (3.1) on the manifold $\mathcal{M}_\varepsilon$ with $0 < r < d_\varepsilon$ represent a continuum $\mathcal{L}_\varepsilon$ of limit cycles.*

Since d_0 is positive we have in analogy to Theorem 4.11

Theorem 5.4. *If the parameter ε in system (3.1) passes the origin, a continuum $\mathcal{L}_\varepsilon$ of limit cycles bifurcates from a period annulus $\mathcal{A}_\varepsilon$.*

Different to system (3.2) where the continuum $\mathcal{L}$ of limit cycles exists for all $\varepsilon \neq 0$, in case of system (3.1) the continuum of limit cycles $\mathcal{L}_\varepsilon$ depends on ε. Since d_{ε_0} vanishes, the continuum $\mathcal{L}_\varepsilon$ exists not for $\varepsilon = \varepsilon_0$. From Theorem 5.1 we obtain

Theorem 5.5. *If the parameter ε in system (3.1) passes ε_0, a continuum $\mathcal{L}_\varepsilon$ of limit cycles bifurcates from the origin.*

Figure 5.1 shows the bifurcation curves of system (3.1) on the invariant manifold $\mathcal{M}_\varepsilon$ for $\lambda = 4$, $a_2 = \sqrt{3}$, $a_3 = a_4 = 1$, $a_5 = -4$. Crossing the piecewise straight line d_ε from $\mathcal{H}_\varepsilon$ to $\mathcal{L}_\varepsilon$ is connected with the bifurcation of a continuum of limit cycles $\mathcal{L}_\varepsilon$ from a continuum of heteroclinic cycles $\mathcal{H}_\varepsilon$. Crossing the finite section $\mathcal{A}_0$ is connected with the bifurcation of a continuum of limit cycles from a period annulus.

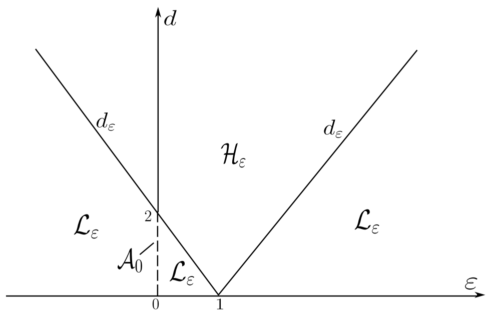

Figure 5.1: Bifurcation curves of system (3.1) on $\mathcal{M}_\varepsilon$ in the (ε, d)-plane.

6 Center conditions

As a byproduct of our approach we are able to derive conditions on the coefficients of a quadratic polynomial system to have a center at the origin. The peculiarity of the system under consideration is to have a continuum of equilibria.

Consider the general planar differential system whose right hand side consists of polynomials with maximum degree two

$$\begin{aligned} \frac{dx}{dt} &= a_{10}x + a_{01}y + a_{20}x^2 + a_{11}xy + a_{02}y^2, \\ \frac{dy}{dt} &= b_{10}x + b_{01}y + b_{20}x^2 + b_{11}xy + b_{02}y^2. \end{aligned} \tag{6.1}$$

System (6.1) has the origin as equilibrium point. The problem under which condition about the coefficients a_{ij}, b_{ij} the origin is a center has been investigated in numerous publications. We note that the following result is not contained in the monograph [16].

Proposition 6.1. *Under the assumptions*

$$a_{20} = b_{02} = 0, a_{10} = b_{01} = 0, a_{01} = -b_{10}, a_{11} = -b_{20}, a_{02} = -b_{11}, b_{11}^2 + b_{20}^2 > 0$$

the origin of system (6.1) is a center.

Proof. Under the conditions above system (6.1) can be written in the form

$$\begin{aligned}\frac{dx}{dt} &= -y(b_{10} + b_{20}x + b_{11}y),\\ \frac{dy}{dt} &= x(b_{10} + b_{20}x + b_{11}y)\end{aligned} \tag{6.2}$$

which has the same structure as system (4.1) in case $\varepsilon = 0$. By Proposition 4.5 the origin is a center. □

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