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Instability with growth rates of discrete systems in Banach spaces

Carmen-Florinela Popa

West University of Timișoara, Boulevard Vasile Pârvan 4, Timișoara, 300223, Romania

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Abstract. In this paper, we investigate new forms of instability for linear discrete-time dynamical systems in Banach spaces, focusing on characterizations described by non-exponential growth rates. We introduce and analyze the concept of h-instability, where $h = (h_n)_{n\in\mathbb{N}}$ is a non-decreasing sequence that diverges to infinity. This formulation generalizes classical and nonuniform notions of exponential and polynomial instability by allowing for a variety of growth profiles. Our approach studies a linear system. We establish sufficient conditions for h-instability, investigate its connections with known instability concepts, and provide examples illustrating the sharpness of the results.

Keywords: discrete systems, instability, growth rates.

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1 Introduction

In recent years, the study of asymptotic behavior in linear discrete-time systems has seen substantial progress, largely due to their relevance in modeling a broad range of real-world phenomena.

The instability problem has become one of special interests in the field of the asymptotic behavior of linear discrete-time systems. In this context, there are various characterizations of instability have been proposed in papers by van Minh et al. [15], Megan et al. [11,13], Naulin and Vanegas [17], A. L. Sasu [26], Slyusarchuk [28], Wang and Wang [30]. Recently, new concepts of instability have been introduced and studied (see [12] for semigroups of operators, [8] for evolution operators, [9] for families of evolutions, [11] for linear skew-product flow). Among the most studied notions in this field are exponential instability [8,9,11–13] and the polynomial instability [2,7,25].

Besides the concept of instability, presented in various forms, it should be mentioned that we also used the concept of h-decay, introduced by Naulin and Pinto in [16].

This paper aims to introduce and investigate a new type of instability, called h-instability, which is defined with respect to a prescribed sequence (h_n) satisfying $\lim_{n\to\infty} h_n = \infty$. This approach allows for a unified treatment of both regular and irregular growth patterns and offers a flexible tool for describing the asymptotic divergence of solutions. This concept was firstly introduced by M. Pinto [18] in his work with the intention of obtaining results about stability for a weakly stable system under some perturbations. Furthermore, it was studied in the papers [4] and [14].

Let X be a real or complex Banach space and $\mathcal{B}(X)$ the Banach algebra of all bounded linear operators from X into itself. The norms of both spaces will be denoted by $\|\cdot\|$. Let $\mathbb{N}$ be the set of all positive integers and define the following sets:

$$\Delta = \{(m,n) \in \mathbb{N}^2 : m \geq n\}, \qquad T = \{(m,n,p) \in \mathbb{N}^3 : m \geq n \geq p\}.$$

The symbol U denotes the evolution operator of the system, defined on $\mathbb{N}$ and naturally extended to a pairs $(m,n) \in \Delta$ as $\mathcal{U}(m,n)$. It also labels the main equation of the system.

We consider linear discrete-systems of the form

$$(\mathcal{U}) \qquad x_{n+1} = U(n)x_n, \qquad n \in \mathbb{N}$$

where $U : \mathbb{N} \to \mathcal{B}(X)$ is a sequence in $\mathcal{B}(X)$.

Then every solution $x = (x_n)$ of system ($\mathcal{U}$) is given by

$$x_m = \mathcal{U}(m,n)x_n, \quad \text{for all } (m,n) \in \Delta,$$

where

$$\mathcal{U}(m,n) = \begin{cases} U(m-1)U(m-2)\dots U(n), & m > n, \\ I, & m = n \end{cases}$$

and I is the identity operator on X.

In addition, we have that the following properties are verified:

(i) $\mathcal{U}(n+1,n) = \mathcal{U}(n)$, for all $n \in \mathbb{N}$

(ii) $\mathcal{U}(m,n)\mathcal{U}(n,p) = \mathcal{U}(m,p)$, for all $(m,n,p) \in T$.

Our focus lies in the asymptotic behavior of the norms $\|U(m,n)x\|$ for $x \in X$, as $m \to \infty$, in relation to the growth rate h_n.

The notion of h-instability we propose is defined by the existence of constants $N \geq 1$ and $\delta > 0$ such that for all $n \in \mathbb{N}$ and all $x \in X$ with $\|x\| \geq \delta$, there exists $m > n$ satisfying $h_n\|U(m,n)x\| \geq Nh_m\|x\|$.

This paper is structured into three sections. In Sections 2 and 3, we investigate specific properties of uniform and nonuniform h-instability in linear discrete-time systems. Relations between these notions are established, and several theorems are provided to characterize the instability of such systems in Banach spaces.

The study further develops the characterization of instability with prescribed growth rates in discrete-time systems, with emphasis on both the uniform and nonuniform frameworks. By introducing fundamental concepts such as instability with growth rates for linear discrete-time systems, the paper establishes the foundation for [5] Datko-type characterizations and [1], Lyapunov function formulations associated with these instability types.

The results obtained in this work not only contribute to the advancement of the theoretical understanding of discrete dynamical systems but also offer analytical tools that may serve as a basis for future research in the field.

2 Uniform h-instability

The following definition generalizes the classical uniform instability by means of a growth rate.

Definition 2.1. A solution $x = (x_n)$ of the system $(\mathcal{U})$ is uniformly h-unstable (and denote as u.h.is) if there exist $N \geq 1$ and $\nu > 0$ such that

$$h_m^\nu \|x\| \leq N h_n^\nu \|U(m,n)x\|, \quad \text{for all } (m,n,x) \in \Delta \times X.$$

In this remark, we restate the same concept, but in a more general form involving three variables (m,n,p).

Remark 2.2. A solution $x = (x_n)$ of the system $(\mathcal{U})$ is uniformly h-unstable if and only if there exist $N \geq 1$ and $\nu > 0$ such that

$$h_m^\nu \|U(n,p)x\| \leq N h_n^\nu \|U(m,p)x\|, \quad \text{for all } (m,n,p,x) \in T \times X.$$

Proof. Since $\|U(m,p)x\| = \|U(m,n)U(n,p)x\|$, we obtain

$$\|U(m,p)x\| \geq \frac{1}{N}\left(\frac{h_m}{h_n}\right)^\nu \|U(n,p)x\|$$

which is equivalent to the stated inequality. □

The next remark describes the specific cases that arise when the growth rate is changed.

Remark 2.3. As particular cases we have:

1. If $h_n = e^n$, for all $n \in \mathbb{N}$ we obtain the concept of *uniform exponential instability* (u.e.is).
2. If $h_n = n+1$, for all $n \in \mathbb{N}$ we obtain the concept *uniform polynomial instability* (u.p.is).

In the next example, we show a solution of the system that is uniformly h-unstable.

Example 2.4. On $X = \mathbb{R}^2$ endowed with the norm $\|x\| = \|(x_1, x_2)\| = \max\{|x_1|, |x_2|\}$, we consider that a solution $x = (x_n)$ of the systems $(\mathcal{U})$ defined by

$$U(m,n)x = \frac{h_m}{h_n}x$$

is uniformly h-unstable, for all $(m,n,x) \in \Delta \times X$.

Indeed,

$$\|U(m,n)x\| = \frac{h_m}{h_n}\|x\| \geq \frac{1}{N}\left(\frac{h_m}{h_n}\right)^\nu \|x\|$$

which is equivalent to

$$h_m^\nu \|x\| \leq N h_n^\nu \|U(m,n)x\|$$

for $N = 1$, $\nu = 1$ and all $(m,n,x) \in \Delta \times X$.

Thus, a solution $x = (x_n)$ of the system $(\mathcal{U})$ is uniformly h-unstable.

In the following, using the same approach, we show a solution for which the system has uniform h-decay.

Definition 2.5. A solution $x = (x_n)$ of the system ($\mathcal{U}$) has uniform h-decay (and denote as d.h.u) if there exist $M \geq 1$ and $\omega > 0$ such that

$$h_n^\omega \|x\| \leq M h_m^\omega \|U(m,n)x\|, \quad \text{for all } (m,n,x) \in \Delta \times X.$$

Remark 2.6. A solution $x = (x_n)$ of the system ($\mathcal{U}$) has uniform h-decay if only if there exist $M \geq 1$ and $\omega > 0$ such that

$$h_n^\omega \|U(n,p)x\| \leq M h_m^\omega \|U(m,p)x\|, \quad \text{for all } (m,n,,p,x) \in T \times X.$$

Proof. Since, $\|U(m,p)x\| = \|U(m,n)U(n,p)x\|$ we have,

$$\|U(m,p)x\| \geq \frac{1}{M}\left(\frac{h_n}{h_m}\right)^\omega \|U(n,p)x\|$$

which yields the stated relation. □

Remark 2.7. As particular cases we have:

1. If $h_m = e^m$, for all $m \in \mathbb{N}$ we obtain *uniform exponential decay* (d.e.u)
2. If $h_m = m+1$, for all $m \in \mathbb{N}$ we obtain *uniform polynomial decay* (d.p.u)

Example 2.8. On $X = \mathbb{R}^2$ endowed with the norm $\|x\| = \|(x_1,x_2)\| = \max\{|x_1|, |x_2\|\}$, we consider that a solution $x = (x_n)$ of the systems $(\mathcal{U})$ defined by

$$U(m,n)x = \frac{h_n}{h_m}x$$

has uniform h-decay, for all $(m,n,x) \in \Delta \times X$.

Indeed,

$$\|U(m,n)x\| = \frac{h_n}{h_m}\|x\| \geq \frac{1}{N}\left(\frac{h_n}{h_m}\right)^\omega \|x\|$$

which gives

$$h_n^\omega \|x\| \leq M h_m^\omega \|U(m,n)x\|$$

for $M = 1$, $\omega = 1$ and for all $(m,n,x) \in \Delta \times X$.

Thus, a solution $x = (x_n)$ of the system ($\mathcal{U}$) has uniform h-decay.

In what follows, we present the relationship between the concepts of uniformly h-unstable and uniform h-decay, including a example.

Remark 2.9. It is obvious that if a solution $x = (x_n)$ of the system ($\mathcal{U}$) is u.h.is then it has d.h.u. The converse implication, however, is not valid, as shown in the following example.

Example 2.10. On $X = \mathbb{R}^2$ endowed with the norm $\|x\| = \|(x_1,x_2)\| = \max\{|x_1|, |x_2|\}$, we consider a solution $x = (x_n)$ of the system ($\mathcal{U}$) defined by

$$U(m,n)x = \frac{h_n}{h_m}x.$$

Then for $M = 1$ and $\omega = 1$ we have that system has uniform h-decay.

Suppose, however, that a solution $x = (x_n)$ of the system ($\mathcal{U}$) is uniformly h-unstable. Then there would exist constants $N \geq 1$ and $\nu > 0$ such that

$$\|U(m,n)x\| = \frac{h_n}{h_m}\|x\| \geq \frac{1}{N}\left(\frac{h_m}{h_n}\right)^{\nu}\|x\|$$

which is equivalent to

$$h_m^{\nu+1} \leq N h_n^{\nu+1}, \quad \text{for all } (m,n,x) \in \Delta.$$

For $n = 0$ and $m \to \infty$, we obtain a contradiction.

Hence, uniform h-decay does not imply uniform h-instability.

In this paper, we will consider $\mathcal{H}$ the set of growth rates (h_n) that satisfy the following properties:

(0) Let $\mathcal{H}_0$ be the collection of growth rates $h \in \mathcal{H}$ such that $\exists H_0 > 1 : h_m \leq m+1, \forall m \in \mathbb{N}$.

(1) Let $\mathcal{H}_1$ be the collection of growth rates $h \in \mathcal{H}$ such that $\exists H_1 > 1 : h_{m+1} \leq H_1 h_m, \forall m \in \mathbb{N}$.

(2) Let $\mathcal{H}_2$ be the collection of growth rates $h \in \mathcal{H}$ such that $\forall \alpha \in (-1,0), \exists\ H_2 > 1 : \sum_{j=n}^{\infty} h_j^{\alpha} \leq H_2 h_m^{\alpha}, \forall m \in \mathbb{N}$.

(3) Let $\mathcal{H}_3$ be the collection of growth rates $h \in \mathcal{H}$ such that $\forall \alpha \in (0,1), \exists H_3 > 1 : \sum_{j=0}^{m} h_j^{\alpha} \leq H_3 h_m^{\alpha}, \forall m \in \mathbb{N}$.

(4) Let $\mathcal{H}_4$ be the collection of growth rates $h \in \mathcal{H}$ such that $\forall m \geq 0, \exists H_4 \geq 2 : h(H_4 h_m) \leq (H_4)^2 h_m,\ \forall m \in \mathbb{N}$.

(5) Let $\mathcal{H}_5$ be the collection of growth rates $h \in \mathcal{H}$ such that $\exists \alpha < 0, \exists H_5 > 1 : \sum_{j=n}^{\infty} h_j^{\alpha-1} \leq H_5 h_n^{\alpha}, \forall n \in \mathbb{N}$.

The initial characterization of uniform h-instability, based on a finite sum, is provided by

Theorem 2.11. *Let $h \in \mathcal{H}_0 \cap \mathcal{H}_3$ and a solution $x = (x_n)$ of the system ($\mathcal{U}$) with uniform h-decay. A solution $x = (x_n)$ of the system ($\mathcal{U}$) is uniformly h-unstable if and only if exist $D \geq 1$ and $d \in (0,1)$ such that*

$$(uhD_1) \qquad \sum_{j=n}^{m} h_j^{-d}\|U(j,p)x\| \leq D h_m^{-d}\|U(m,p)x\|, \quad \textit{for all } (m,n,p,x) \in T \times X.$$

Proof. Necessity. Assume that a solution $x = (x_n)$ of the system ($\mathcal{U}$) is uniformly h-unstable. Let $d \in (0,\nu)$. Then we have

$$\begin{aligned}\sum_{j=n}^{m} h_j^{-d}\|U(j,p)x\| &\leq N\sum_{j=n}^{m} h_j^{-d}\left(\frac{h_m}{h_n}\right)^{-\nu}\|U(m,p)x\| \\ &= N h_m^{-\nu}\|U(m,p)x\|\sum_{j=n}^{m} h_j^{\nu-d} \\ &\leq N h_m^{-\nu} H_3 h_m^{\nu-d}\|U(m,p)x\| \leq D h_m^{-d}\|U(m,p)x\|\end{aligned}$$

where $D = NH_3 > 1$, for all $(m,n,p,x) \in T \times X$.

Sufficiency. We suppose that there exist $D \geq 1$ and $d \in (0,1)$ satisfying (uhD_1).

Taking $j = n$ in (uhD_1) we obtain

$$h_n^{-d}\|U(n,p)x\| \leq D h_m^{-d}\|U(m,p)x\|.$$

By Remark 2.2, we have that a solution $x = (x_n)$ of the system ($\mathcal{U}$) is uniformly h-unstable. □

The characterization of uniform h-instability via a Datko-type theorem leads to a Lyapunov-type characterization, presented in the following corollary.

Corollary 2.12. *Let $h \in \mathcal{H}_0 \cap \mathcal{H}_3$ and a solution $x = (x_n)$ of the system $(\mathcal{U})$ with uniform h-decay. Then, a solution $x = (x_n)$ of the system $(\mathcal{U})$ is uniformly h-unstable, if only if, there exist $D \geq 1$, $d \in [0, 1)$ and a function $L : \Delta \times X \to \mathbb{R}_+$ such that*

$$(uhL_1) \qquad L(m, p, x) \leq D\|U(m, p)x\|, \qquad \forall (m, p, x) \in \Delta \times X.$$

$$(uhL_2) \qquad L(n, p, x) \leq L(m, p, x) - \sum_{j=n}^{m} \left(\frac{h_j}{h_m}\right)^{-d} \|U(j, p)x\|, \qquad \forall (m, n, p, x) \in T \times X.$$

Proof. Necessity. We suppose that a solution $x = (x_n)$ of the system $(\mathcal{U})$ is uniformly h-unstable and we consider the Lyapunov function $L : \Delta \times X \to \mathbb{R}_+$, defined by

$$L(n, p, x) = \sum_{j=p}^{n} \left(\frac{h_j}{h_m}\right)^{-d} \|U(j, p)x\|.$$

Then using Theorem 2.11, we have

$$(uhL_1) \qquad L(m, p, x) = \sum_{j=p}^{m} \left(\frac{h_j}{h_m}\right)^{-d} \|U(j, p)x\| \leq D\|U(m, p)x\|, \qquad \forall (m, p, x) \in \Delta \times X$$

and

$$(uhL_2) \qquad L(n, p, x) - L(m, p, x) \leq \sum_{j=p}^{n} \left(\frac{h_m}{h_j}\right)^{d} \|U(j, p)x\| - \sum_{j=p}^{m} \left(\frac{h_m}{h_j}\right)^{d} \|U(j, p)x\|$$
$$= -\sum_{j=n}^{m} \left(\frac{h_m}{h_j}\right)^{d} \|U(j, p)x\|, \ \forall (m, n, p, x) \in T \times X.$$

Sufficiency. We suppose that exists $L : \Delta \times C \to \mathbb{R}_+$ and are $D > 1, d \in (0, 1)$ such that the properties (uhL_1) and (uhL_2) are satisfied. Then we have

$$\sum_{j=n}^{m} \left(\frac{h_j}{h_m}\right)^{-d} \|U(j, p)\| \leq L(m, p, x) - L(n, p, x) \leq L(m, p, x).$$

For $p = n$ and using Theorem 2.11, we obtain conclusion. □

Using the same approach as in the previous two results, we now provide a characterization based on an infinite sum, together with a corollary that yields a Lyapunov-type characterization. At the same time, we aim to present the first result, namely the sufficiency theorem, through two different methods.

Theorem 2.13. *Let $h \in \mathcal{H}_0 \cap \mathcal{H}_2$ and suppose that a solution $x = (x_n)$ of the system $(\mathcal{U})$ has uniform h-decay. A solution $x = (x_n)$ of the system $(\mathcal{U})$ is uniformly h-unstable if and only if exist $D \geq 1$ and $d \in (0, 1)$ such that*

$$(uhisD_2) \qquad \sum_{j=n}^{\infty} \frac{h_j^d}{\|U(j, p)x\|} \leq \frac{D h_n^d}{\|U(n, p)x\|}, \quad \textit{for all } (m, n, p, x) \in T \times X, \ U(n, p)x \neq 0.$$

Proof. Necessity. We suppose that a solution $x = (x_n)$ of the system $(\mathcal{U})$ is uniformly h-unstable. Let $d \in (0, \nu)$. Then we have

$$\sum_{j=n}^{\infty} \frac{h_j^d}{\|U(j,p)x\|} \leq N \sum_{j=n}^{\infty} h_j^d \left(\frac{h_j}{h_n}\right)^{-\nu} \frac{1}{\|U(n,p)x\|}$$
$$= \frac{N}{\|U(n,p)x\|} \cdot h_n^{\nu} \cdot \sum_{j=n}^{\infty} h_j^{d-\nu}$$
$$\leq \frac{NH_2 h_n^{\nu} h_n^{d-\nu}}{\|U(n,p)x\|} = \frac{D h_n^d}{\|U(n,p)x\|}$$

where $D = NH_2 > 1$, $U(n,p)x \neq 0$, for all $(m,n,p,x) \in T \times X$.

Sufficiency. **Method I:** We suppose that there are $D \geq 1$ and $d \in (0,1)$ such that (uhD_2) hold.

For $j = m$ in (uhD_2) we have

$$\frac{h_m^d}{\|U(m,p)x\|} \leq \frac{D h_n^d}{\|U(n,p)x\|}.$$

For Remark 2.2, we have that a solution $x = (x_n)$ of the system $(\mathcal{U})$ is uniformly h-unstable for $d = \nu$ and $D = N$.

Method II: Let $(m,n,p) \in T$ and $x \in X/\{0\}$.

If $m \geq n+1$ we have

$$\frac{h_m^d}{\|U(m,p)x\|} = \sum_{j=m-1}^{m} \frac{h_m^d}{\|U(m,j)U(j,p)x\|}$$
$$\leq M \sum_{j=m-1}^{m} \left(\frac{h_m}{h_j}\right)^{\omega} \cdot \frac{h_m^d}{\|U(j,p)x\|} = M \sum_{j=m-1}^{m} \left(\frac{h_m}{h_j}\right)^{\omega+d} \cdot \frac{h_j^d}{\|U(j,p)x\|}$$
$$\leq M \sum_{j=m-1}^{m} \left(\frac{h_m}{h_{m-1}}\right)^{\omega+d} \cdot \frac{h_j^d}{\|U(j,p)x\|} \leq M \frac{1}{H_1^{\omega+d}} \sum_{j=n}^{\infty} \frac{h_j^d}{\|U(j,p)x\|}$$
$$\leq \frac{MDH_1^{\omega+d} h_n^d}{\|U(n,p)x\|} \leq \frac{N_1 h_n^d}{\|U(n,p)x\|}$$

where $N_1 = MDH_1^{\omega+d}$, $U(n,p)x \neq 0, (m,n,p,x) \in T \times X$.

If $m = n+1$ we have

$$\frac{h_m^d}{\|U(m,p)x\|} \leq M \left(\frac{h_m}{h_n}\right)^{\omega+d} \cdot \frac{h_n^d}{\|U(n,p)x\|}$$
$$\leq M \left(\frac{h_{n+1}}{h_n}\right)^{\omega+d} \cdot \frac{h_n^d}{\|U(n,p)x\|} \leq \frac{MH_1^{\omega+d} h_n^d}{\|U(n,p)x\|} = \frac{N_2 h_n^d}{\|U(n,p)x\|}$$

where $N_2 = MH_1^{\omega+d}$, $U(n,p)x \neq 0$, for all $(m,n,p,x) \in T \times X$.

Then, for $N = \max\{N_1, N_2\}$, $\nu = d$ and using Remark 2.2, we have that a solution $x = (x_n)$ of the system $(\mathcal{U})$ is uniformly h-unstable. □

Corollary 2.14. *Let $h \in \mathcal{H}_0 \cap \mathcal{H}_2$ and suppose that a solution $x = (x_n)$ of the system $(\mathcal{U})$ has uniform h-decay. A solution $x = (x_n)$ of the system $(\mathcal{U})$ is uniformly h-unstable, if and only if, exist $D \geq 1$, $d \in [0,1)$ and a function $L : \Delta \times X \to \mathbb{R}_+$ such that:*

$$(uhL_1') \qquad L(n,p,x) \leq \frac{D}{\|U(n,p)x\|}, \quad A_n^p x \neq 0, \quad \forall (m,p,x) \in \Delta \times X.$$

$$(uhL_2') \quad L(m,p,x) \leq L(n,p,x) - \sum_{j=n}^{m} \left(\frac{h_j}{h_n}\right)^d \frac{1}{\|U(j,p)x\|}, \quad A_m^n x \neq 0, \quad \forall (m,n,p,x) \in T \times X.$$

Proof. Necessity. We suppose that a solution $x = (x_n)$ of the system ($\mathcal{U}$) is uniformly h-unstable and we consider the Lyapunov function $L : \Delta \times X \to \mathbb{R}_+$, defined by

$$L(n,p,x) = \sum_{j=n}^{\infty} \left(\frac{h_j}{h_n}\right)^d \frac{1}{\|U(j,p)x\|}.$$

Then using Theorem 2.13, we have

$$(uhL_1') \;\; L(n,p,x) = \sum_{j=n}^{\infty} \left(\frac{h_j}{h_n}\right)^d \frac{1}{\|U(j,p)x\|} \leq D\|U(n,p)x\|, \;\; U(n,p)x \neq 0, \;\; \forall (n,p,x) \in \Delta \times X$$

and

$$(uhL_2') \;\; L(m,p,x) - L(n,p,x) \leq \sum_{j=m}^{\infty} \left(\frac{h_j}{h_m}\right)^d \frac{1}{\|U(j,p)x\|} - \sum_{j=n}^{\infty} \left(\frac{h_j}{h_n}\right)^d \frac{1}{\|U(j,p)x\|}$$
$$= - \sum_{j=n}^{m} \left(\frac{h_j}{h_n}\right)^d \frac{1}{\|U(j,p)x\|}, \; U(m,n)x \neq 0, \;\; \forall (m,n,p,x) \in T \times X.$$

Sufficiency. We suppose that exists $L : \Delta \times C \to \mathbb{R}_+$ and $D \geq 1, d \in (0,1)$ such that the properties (uhL_1') and (uhL_2') are satisfied. Then we have

$$\sum_{j=n}^{\infty} \left(\frac{h_j}{h_n}\right)^d \frac{1}{\|U(j,p)x\|} \leq L(n,p,x) - L(m,p,x) \leq L(n,p,x).$$

For $m \to \infty$ and using Theorem 2.13, we obtain the conclusion. □

Using the same framework, the following two results are derived with an infinite sum, differing only in the exponent of h_j in the sum.

Theorem 2.15. *Let $h \in \mathcal{H}_0 \cap \mathcal{H}_2$ and a solution $x = (x_n)$ of the system ($\mathcal{U}$) with uniform h-decay. A solution $x = (x_n)$ of the system ($\mathcal{U}$) is uniformly h-unstable if and only if exists $D \geq 1$ and $d \in (0,1)$ such that*

$$(uhisD_2') \qquad \sum_{j=n}^{\infty} \frac{h_j^{d-1}}{\|U(j,p)x\|} \leq \frac{D h_n^d}{\|U(n,p)x\|}, \quad \text{for all } (m,n,p,x) \in T \times X, \; U(n,p)x \neq 0.$$

Proof. Necessity. We suppose that a solution $x = (x_n)$ of the system ($\mathcal{U}$) is uniformly h-unstable. Let $d \in (0, \nu)$. Then

$$\sum_{j=n}^{\infty} \frac{h_j^{d-1}}{\|U(j,p)x\|} \leq N \sum_{j=n}^{\infty} h_j^{d-1} \left(\frac{h_j}{h_n}\right)^{-\nu} \frac{1}{\|U(n,p)x\|}$$
$$= \frac{N}{\|U(n,p)x\|} \cdot h_n^{\nu} \cdot \sum_{j=n}^{\infty} h_j^{d-\nu-1}.$$

Using property $\mathcal{H}_5$ we have:

$$\sum_{j=n}^{\infty} h_j^{d-\nu-1} \leq H_5 h_n^{d-\nu}.$$

Hence,

$$\sum_{j=n}^{\infty} \frac{h_j^{d-1}}{\|U(j,p)x\|} \leq \frac{NH_5 h_n^{\nu} h_n^{d-\nu}}{\|U(n,p)x\|} = \frac{Dh_n^d}{\|U(n,p)x\|}$$

where $D = NH_5 > 1$, $U(n,p)x \neq 0$, for all $(m,n,p,x) \in T \times X$.

Sufficiency. **Method I:** We suppose that there are $D \geq 1$ and $d \in (0,1)$ such that (uhD_2') hold.
For $j = m$ in $(uhisD_2')$ we have

$$\frac{h_m^{d-1}}{\|U(m,p)x\|} \leq \frac{Dh_n^d}{\|U(n,p)x\|}$$

which is equivalent to

$$h_m^{d-1}\|U(n,p)x\| \leq h_m^d\|U(m,p)x\| \leq Dh_n^d\|U(m,p)x\|.$$

For Remark 2.2, we have a solution $x = (x_n)$ of the system $(\mathcal{U})$ for $d = \nu$ and $D = N$.

Method II: Let $(m,n,p) \in T$ and $x \in X/\{0\}$.
If $h_m \geq 2n$ we have

$$\begin{aligned}
\frac{h_m^d}{\|U(m,p)x\|} &= \frac{2}{h_m} \sum_{j=h_m/2}^{h_m} \frac{h_m^d}{\|U(m,j)U(j,p)x\|} \\
&\leq 2M \sum_{j=h_m/2}^{h_m} \left(\frac{h_m}{h_j}\right)^{\omega} \cdot \frac{h_j}{h_m} \cdot \frac{h_j^{d-1}}{h_j^d} \cdot \frac{h_m^d}{\|U(j,p)x\|} \leq 2M \sum_{j=h_m/2}^{h_m} \left(\frac{h_m}{h_j}\right)^{\omega+d} \cdot \frac{h_j}{h_m} \frac{h_j^{d-1}}{\|U(j,p)x\|} \\
&\leq 2M \sum_{j=h_m/2}^{h_m} \left(\frac{h_m}{j}\right)^{\omega+d} \cdot \frac{h_j}{h_m} \cdot \frac{h_j^{d-1}}{\|U(j,p)x\|} \leq 2M \sum_{j=h_m/2}^{h_m} \left(\frac{h_m}{2^{-1}h_m}\right)^{\omega+d} \cdot \frac{h_j}{h_m} \cdot \frac{h_j^{d-1}}{\|U(j,p)x\|} \\
&\leq 2^{\omega+d+1} M \sum_{j=h_m/2}^{h_m} \frac{h(h_m)}{h_m} \cdot \frac{h_j^{d-1}}{\|U(j,p)x\|} \leq 2^{\omega+d+1} M \sum_{j=h_m/2}^{h_m} \frac{h(H_4 h_m)}{h_m} \cdot \frac{h_j^{d-1}}{\|U(j,p)x\|} \\
&\leq 2^{\omega+d+1} M (H_4)^2 \sum_{j=n}^{\infty} \frac{h_j^{d-1}}{\|U(j,p)x\|} \frac{2^{\omega+d+1} MDH_4^2 h_n^d}{\|U(n,p)x\|} \leq \frac{N_1 h_n^d}{\|U(n,p)x\|}
\end{aligned}$$

where $N_1 = 2^{\omega+d+1}MDH_4^2$, $U(n,p)x \neq 0, (m,n,p,x) \in T \times X$.
If $h_m \leq 2n$ we have

$$\begin{aligned}
\frac{h_m^d}{\|U(m,p)x\|} &\leq M\left(\frac{h_m}{h_n}\right)^{\omega} + d \cdot \frac{h_n^d}{\|U(n,p)x\|} \\
&\leq M\left(\frac{2n}{n+1}\right)^{\omega+d} \cdot \frac{h_n^d}{\|U(n,p)x\|} \leq \frac{2^{\omega+d} M h_n^d}{\|U(n,p)x\|} = \frac{N_2 h_n^d}{\|U(n,p)x\|}
\end{aligned}$$

where $N_2 = 2^{\omega+d}M$, $U(n,p)x \neq 0, (m,n,p,x) \in T \times X$.
Then for $N = \max\{N_1, N_2\}$, $\nu = d$ and using Remark 2.2, we have that a solution $x = (x_n)$ of the system $(\mathcal{U})$ is uniformly h-unstable. □

Building on the linearity property, we remark that in the finite-dimensional case, verifying the instability conditions for a set of basis solutions is sufficient. The next theorem formalizes this result.

Theorem 2.16. *Let $X = \mathbb{R}^n$ and consider the linear discrete-time system*

$$(\mathcal{U}) \qquad x_{n+1} = U(n)x_n, \qquad n \in \mathbb{N},$$

where $U(n) \in \mathcal{B}(X)$ for all n.

Assume that there exists a set of n linearly independent solution $\{x^{(1)}, \dots, x^{(n)}\}$ (a basis of the solution space) such that, for each $i = 1, \dots, n$ there exist constants $N \geq 1$ and $\nu_i > 0$ satisfying

$$h_m^{\nu_i}\|x^{(i)}\| \leq N_i h_n^{\nu_i}\|x_m^{(i)}\|, \quad \textit{for all } (m,n) \in \Delta.$$

Then a solution $x = (x_n)$ of the system ($\mathcal{U}$) is uniformly h-unstable.

Proof. Let $x = (x_n)$ be an arbitrary solution of ($\mathcal{U}$). Since $\{x^{(1)}, \dots, x^{(n)}\}$ is a basis of the solution space, there exist constants $c_1, \dots, c_n \in \mathbb{R}$ such that

$$x_n = \sum_{i=1}^{n} c_i x_n^{(i)}.$$

By the triangle inequality,

$$\|x_n\| \leq \sum_{i=1}^{n} |c_i|\|x_n^{(i)}\| \leq C \cdot \max_{i \leq i \leq n} \|x_n^{(i)}\|,$$

where $C = \sum_{i=1}^{n} |c_1|$ depends only on the initial condition of the solution.

For each i, by hypothesis,

$$h_m^{\nu_i}\|x^{(i)}\| \leq N_i h_n^{\nu_i}\|x_m^{(i)}\|,$$

so

$$\|x_n^{(i)}\| \leq N_i \left(\frac{h_n}{h_m}\right)^{\nu_i} \|x_m^{(i)}\|.$$

Taking the maximum over i, we obtain

$$\max_i \|x_n^{(i)}\| \leq N \left(\frac{h_n}{h_m}\right)^{\nu} \max_i \|x_m^{(i)}\|,$$

where $N = \max_i N_i$ and $\nu = \min_i \nu_i$. Hence,

$$\|x\| \leq CN \left(\frac{h_n}{h_m}\right)^{\nu} \|x_m\|.$$

Rewriting, we have

$$h_m^{\nu}\|x_n\| \leq CN h_n^{\nu}\|x_m\|.$$

Setting $N' = CN$, we obtain

$$h_m^{\nu}\|x_n\| \leq N' h_n^{\nu}\|U(m,n)x_n\|,$$

for all $(m,n) \in \Delta$, which is exactly the definition of a uniformly h-unstable solution. □

Remark 2.17. This theorem shows that in finite-dimensional spaces, the verification of h-instability can be restricted to a basis of solutions. Once the instability inequalities hold for the basis elements, they extend automatically to any linear combination of them. This result generalizes the analogous property known for exponential and polynomial instability.

3 Nonuniform h-instability

This section is devoted to the notion of nonuniform h-instability. Following the approach of the previous section, we start with a definition and a remark; the main difference lies in the number of variables considered, with the definition employing two variables (m,n) and the remark three variables (m,n,p).

Definition 3.1. A solution $x = (x_n)$ of the system $(\mathcal{U})$ is nonuniformly h-unstable (and denote as n.h.is) if there exists $N \geq 1$, $\epsilon \geq 0$ and $\nu > 0$ such that

$$h_m^\nu \|x\| \leq N h_m^\epsilon h_n^\nu \|U(m,n)x\|, \quad \text{for all } (m,n,x) \in \Delta \times X.$$

Remark 3.2. A solution $x = (x_n)$ of the system $(\mathcal{U})$ is nonuniformly h-unstable (and denote as n.h.is) if and only if there exists $N \geq 1$, $\epsilon \geq 0$ and $\nu > 0$ such that

$$h_m^\nu \|U(n,p)x\| \leq N h_m^\epsilon h_n^\nu \|U(m,p)x\|, \qquad \forall\ (m,n,p,x) \in T \times X.$$

Proof.

$$\|U(m,p)x\| = \|U(m,n)U(n,p)x\| \geq \frac{1}{N} h_m^{-\epsilon} \left(\frac{h_m}{h_n}\right)^\nu \|U(n,p)x\|$$

which is equivalent to

$$h_m^\nu \|U(n,p)x\| \leq N h_m^\epsilon h_n^\nu \|U(m,p)x\| \quad \text{for } (m,n,p,x) \in T \times X.$$ □

The next remark describes the specific cases that arise when modifications occur.

Remark 3.3. As particular cases we have:

1. If $\epsilon = 0$, for all $n \in \mathbb{N}$, we obtain the concept of *uniform h-instability* (u.h.is.)
2. If $h_n = e^n$, for all $n \in \mathbb{N}$, we obtain the concept of *nonuniform exponential instability* (n.e.is.)
3. If $h_n = n+1$, for all $n \in \mathbb{N}$, we obtain the concept *nonuniform polynomial instability* (n.p.is.)

In the following, using the same approach, we show a solution for which the system has nonuniform h-decay.

Definition 3.4. A solution $x = (x_n)$ of the system $(\mathcal{U})$ has nonuniform h-decay (and denote as d.h.n) if there exists $M \geq 1$, $\delta \geq 0$ and $\omega > 0$ such that

$$h_n^\omega \|x\| \leq M h_m^\delta h_n^\omega \|U(m,n)x\|, \quad \text{for all } (m,n,x) \in \Delta \times X.$$

Remark 3.5. A solution $x = (x_n)$ of the system $(\mathcal{U})$ has nonuniform h-decay (and denote as d.h.n) if only if there exists $M \geq 1$, $\delta \geq 0$ and $\omega > 0$ such that

$$h_n^\omega \|U(n,p)x\| \leq M h_m^\delta h_n^\omega \|U(m,p)x\|, \quad \text{for all } (m,n,p,x) \in T \times X.$$

Proof. Since

$$\|U(m,p)x\| = \|U(m,n)U(n,p)x\| \geq \frac{1}{M} h_m^{-\delta} \left(\frac{h_n}{h_m}\right)^\omega \|U(n,p)x\|$$

which is equivalent to

$$h_n^\omega \|U(n,p)x\| \leq M h_m^\delta h_m^\omega \|U(m,p)x\|, \quad \text{for all } (m,n,p,x) \in T \times X.$$

□

Remark 3.6. As particular cases we have:

1. If $\delta = 0$, we obtain the concept of *uniform h-decay* (d.h.u).
2. If $h_m = e^m$, for all $m \in \mathbb{N}$ we obtain the concept of *nonuniform exponential decay* (d.e.n).
3. If $h_m = m + 1$, for all $m \in \mathbb{N}$ we obtain the concept *nonuniform polynomial decay* (d.p.n).

In the next example, we show a solution of the system that is nonuniformly h-unstable.

Example 3.7. Let $X = \mathbb{R}^2$ endowed with the norm $\|x\| = \|(x_1, x_2)\| = \max\{|x_1|, |x_2|\}$. Consider a solution $x = (x_n)$ of the system $(\mathcal{U})$ defined by

$$U(m,n)x = \frac{h_m^2}{h_n}x.$$

Then a solution $x = (x_n)$ of the system $(\mathcal{U})$ is nonuniformly h-unstable, for all $(m,n,x) \in \Delta \times X$.

Indeed,

$$\|U(m,n)x\| = \frac{h_m^2}{h_n}\|x\| \geq \frac{1}{N}\left(\frac{h_m}{h_n}\right)^{\nu} h_m^{\epsilon}\|x\|$$

which is equivalent to

$$h_m^{\nu}\|x\| \leq N h_n^{\nu} h_m^{\epsilon}\|U(m,n)\|$$

for $N = 1$, $\nu = 1$, $\epsilon = 1$ and for all $(m,n,x) \in \Delta \times X$.

Hence, a solution $x = (x_n)$ of the system $(\mathcal{U})$ is nonuniformly h-unstable.

In what follows, we present the relationship between the concepts of nonuniformly h-unstable and nonuniform h-decay, including a example.

Remark 3.8. It is obvious that if a solution $x = (x_n)$ of the system $(\mathcal{U})$ is u.h.is, then it is n.h.is. The following shows that the converse implication is not valid.

Example 3.9. A solution $x = (x_n)$ of the system $(\mathcal{U})$ is nonuniformly h-unstable, but not is uniformly h-unstable.

On $X = \mathbb{R}^2$ endowed with the norm $\|x\| = \|(x_1, x_2)\| = \max\{|x_1|, |x_2|\}$, we consider a solution $x = (x_n)$ of systems $(\mathcal{U})$ defined by

$$U(m,n)x = \frac{h_n^2}{h_m}x.$$

Then for $N = 1$, $\epsilon = 1$ and $\omega = 1$ we have that a solution $x = (x_n)$ of the system is nonuniformly h-unstable.

If we suppose that a solution $x = (x_n)$ of the system $(\mathcal{U})$ is uniformly h-unstable, then exist constants $N \geq 1$ and $\nu > 0$ such that

$$\|U(m,n)x\| = \frac{h_n^2}{h_m}\|x\| \geq \frac{1}{N} h_m^{-\epsilon}\left(\frac{h_m}{h_n}\right)^{\nu}\|x\|$$
$$\iff h_m^{\epsilon+\nu+1} \leq N h_n^{\nu+2}, \quad \text{for all } (m,n,x) \in \Delta.$$

For $n = 0$ and $m \to \infty$, we obtain a contradiction.

The initial characterization of nonuniform h-instability, based on a finite sum, is provided by

Theorem 3.10. *A solution $x = (x_n)$ of the system $(\mathcal{U})$ is nonuniformly h-unstable if and only if there exists a constant $d > 0$, $\epsilon \geq 0$ and $N \geq 1$ such that*

$$\sum_{j=n}^{m} \left(\frac{h_m}{h_j}\right)^d \|U(j,p)x\| \leq N h_m^{\epsilon} \|U(m,p)x\| \tag{nhisD}$$

for all $(m, n, p, x) \in T \times X$.

Proof. Necessity. We suppose that a solution $x = (x_n)$ of the system $(\mathcal{U})$ is nonuniformly h-unstable. Let $d \in (0, \nu)$. Then we have

$$\begin{aligned}\sum_{j=n}^{m} h_j^{-d} \|U(j,p)x\| &\leq N h_m^{\epsilon} \sum_{j=n}^{m} h_j^{-d} \left(\frac{h_m}{h_n}\right)^{-\nu} \|U(m,p)x\| \\ &= N h_m^{\epsilon-\nu} \|U(m,p)x\| \sum_{j=n}^{m} h_j^{\nu-d} \\ &\leq N h_m^{\epsilon-\nu} H_3 h_m^{\nu-d} \|U(m,p)x\| \leq D h_m^{\epsilon-d} \|U(m,p)x\|\end{aligned}$$

where $D = NH_3 > 1, \forall (m, n, p, x) \in T \times X$.

Sufficiency. We suppose that there are $D \geq 1$ and $d \in (0, 1)$ such that (nhD_1) hold.
For $j = n$ in (nhD_1) we have

$$h_n^{-d} \|U(n,p)x\| \leq D h_m^{\epsilon-d} \|U(m,p)x\|.$$

For Remark 2.17, we have that a solution $x = (x_n)$ of the system $(\mathcal{U})$ is nonuniformly h-unstable. □

Author contributions: As original contributions, we may consider Theorems 2.11, 2.13, 2.15 and 2.16, which have not been previously established in the context of instability with growth rates for discrete-time systems. In particular, Theorems 2.13 and 2.15 are noteworthy, as their sufficiency parts are demonstrated by two distinct methods.

References

[1] L. Barreira, C. Valls, Quadratic Lyapunov sequences and arbitrary growth rates, *Discrete Contin. Dyn. Syst.* **26**(2010), 63–74. https://doi.org/10.3934/dcds.2010.26.63 MR2552778; Zbl 1186.39015

[2] R. Boruga, M. Megan, D. M.-M. Toth, Integral characterizations for uniform stability with growth rates in Banach spaces, *Axioms* **10**(2021), 235. https://doi.org/10.3390/axioms10030235

[3] R. Boruga (Toma), M. Megan, Datko criteria for uniform instability in Banach spaces, *Stud. Univ. Babeș-Bolyai Math.* **66**(2021), No. 1, 115–122. https://doi.org/10.24193/subbmath.2021.1.10; MR4325618; Zbl 1504.47067

[4] R. Boruga (Toma), M. Megan, D. M.-M. Toth, On uniform instability with growth rates in Banach spaces, *Carpathian J. Math.* **38**(2022), No. 3, 789–796. https://doi.org/10.37193/CJM.2022.03.22; MR4462050; Zbl 1538.34234

[5] R. Datko, Uniform asymptotic stability of evolutionary processes in a Banach space, *SIAM J. Math. Anal.* **3**(1972), 428–445. https://doi.org/10.1137/0503042; MR0320465; Zbl 0241.34071

[6] A. Găină, M. Megan, C.-F. Popa, Uniform dichotomy concepts for discrete-time skew evolution cocycles in Banach spaces, *Mathematics* **9**(2021), No. 17, 2177. https://doi.org/10.3390/math9172177

[7] P. V. Hai, Polynomial stability and polynomial instability for skew-evolution semiflows, *Results Math.* **74**(2019), No. 4, 1–19. https://doi.org/10.1007/s00025-019-1099-3; MR4013763; Zbl 1434.34045

[8] M. Megan, A. L. Sasu, B. Sasu, Nonuniform exponential unstability of evolution operators in Banach spaces, *Glas. Mat. Ser. III.* **36**(2001), No. 56, 287–295. MR1884449; Zbl 1008.34053

[9] M. Megan, A. L. Sasu, B. Sasu, Banach function spaces and exponential instability of evolution families, *Arch. Math. (Brno)* **39**(2003), No. 4, 277–286. MR2028738; Zbl 1116.34328

[10] M. Megan, A. L. Sasu, B. Sasu, *The asymptotic behavior of evolution families*, Mirton Publishing House, 2003.

[11] M. Megan, A. L. Sasu, B. Sasu, Exponential stability and exponential instability for linear skew-product flows, *Math. Bohem.* **129**(2004), No. 3, 225–243. https://doi.org/10.21136/MB.2004.134146; MR2092710; Zbl 1080.34538

[12] M. Megan, A. L. Sasu, B. Sasu, A. Pogan, Exponential stability and unstability of semigroupes of linear operators in Banach spaces, *Math. Inequal. Appl.* **5**(2002), 557–567. https://doi.org/10.7153/mia-05-55

[13] M. Megan, C. Stoica, Exponential instability of skew-evolution semiflows in Banach spaces, *Stud. Univ. Babeș-Bolyai Math.* **53**(2008), No. 1, 17–24. https://doi.org/10.1109/ICSPS.2009.142; MR2442730; Zbl 1199.93240

[14] C. L. Mihiț, On uniform h-stability of evolution operators in Banach spaces, *Theory Appl. Math. Comput. Sci.* **6**(2016), 19–27. MR3459882; Zbl 1401.34069

[15] N. van Minh, F. Räbiger, R. Schnaubelt, Exponential stability, exponential expansiveness, and exponential dichotomy of evolution equations on the half-line, *Integral Equations Operator Theory* **32**(1998), No. 3, 332–353. https://doi.org/10.1007/BF01203774; MR1652689; Zbl 0977.34056

[16] R. Naulin, M. Pinto, Dichotomies and asymptotic solutions of nonliniar differential systems, *Nonlinear Anal.* **23**(1994), No. 7, 871–882. https://doi.org/10.1016/0362-546X(94)90125-2

[17] R. Naulin, C. J. Vanegas, Instability of discrete systems, *Electron. J. Differential Equations* **1998**, No. 33, 1–11. MR1656590; Zbl 0912.39005.

[18] M. Pinto, Perturbation of asymptotically stable differential systems, *Analysis* **4**(1984), 161–175. https://doi.org/10.1524/anly.1984.4.12.161; MR0775553.

[19] A. Pogan, C. Preda, P. Preda, A discrete Lyapunov theorem for the exponential stability of evolution families, *New York J. Math.* **11**(2005), 457–463. MR2188250; Zbl 1097.34045

[20] C.-F. Popa, Discrete characterizations of *h*-dichotomy for linear discrete-time systems in Banach spaces, *An. Univ. Vest Timiș. Ser. Mat.-Inform.* **61**(2025), 41–54. https://doi.org/10.2478/awutm-2025-0004; MR4924057; Zbl 08084189

[21] I.-L. Popa, *Asymptotic behaviours of linear discrete-time systems in Banach spaces*, PhD thesis, West University of Timișoara, 2012.

[22] I.-L. Popa; T. Ceaușu, M. Megan, On exponential stability for linear systems in Banach spaces, *Comput. Math. Appl.* **63**(2021), 1497–1503. https://doi.org/10.1016/j.camwa.2012.01.027; MR2922048; Zbl 1247.39014

[23] P. Preda, M. Megan, On the stability of linear infinite-dimensional discrete-time systems, *Rad. Mat.* **4**(1988), No. 1, 31–46. MR0948630; Zbl 0648.93050

[24] K. M. Przyluski, S. Rolewicz, On stability of linear time-varying infinit-dimensional discrete-time systems, *Syst. Control Lett.* **4**(1984), 307–315. https://doi.org/10.1016/S0167-6911(84)80042-0; MR0753981; Zbl 0543.93057

[25] M. L. Rămneanțu, M. Megan, T. Ceaușu, Polynomial instability of evolution operators in Banach spaces, *Carpathian J. Math.* **29**(2013), No. 1, 77–83. https://doi.org/10.37193/CJM.2013.01.06; MR3099709; Zbl 1299.34188

[26] A. L. Sasu, Exponential instability and complete admissibility for semigroups in Banach spaces, *Rend. Sem. Mat. Univ. Politec. Torino* **63**(2005), No. 2, 141–151. MR2155690

[27] C. M. Silva, Admissibility and generalized nonuniform dichotomies for discrete dynamics, *Commun. Pure Appl. Anal.* **20**(2021), No. 10, 3419–3443. https://doi.org/10.3934/cpaa.2021112; MR4342177; Zbl 1480.39009

[28] V. E. Slyusarchuk, On the instability of difference equations with respect to the first approximation, *Differ. Uravn.* **22**(1986), No. 4, 722–723. MR0843238

[29] T. M. Személy Fülöp, M. Megan, D. I. Borlea, On uniform stability with growth rates of stochastic skew-evolution semiflows in Banach spaces, *Axioms* **10**(2021), No. 3, 182. https://doi.org/10.3390/axioms10030182

[30] B.-G. Wang, Z.-C. Wang, Exponential dichotomy and admissibility of linearized skew-product semiflows defined on a compact positively invariant subset of semiflows, *Nonlinear Anal. Real World Appl.* **10**(2009), 2062–2071. MR2508416; Zbl 1163.34364