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Existence of solutions for logarithmic Kirchhoff equation without compactness in $\mathbb{R}^3$

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Abstract. In this paper, we investigate the logarithmic Kirchhoff-type equation

$$-\left(a+b\int_{\mathbb{R}^3}|\nabla u|^2\mathrm{d}x\right)\Delta u+V(x)u=|u|^{p-2}u\log|u|,\quad x\in\mathbb{R}^3,$$

where $a,b>0$ are constants, and $V:\mathbb{R}^3\to\mathbb{R}$ is continuous and may change sign. Using Nehari manifold method and the concentration-compactness principle, we prove the existence of nontrivial and nonnegative (weak) solutions under some assumptions on the potential function V without radial symmetry or compactness hypotheses.

Keywords: Kirchhoff-type equation, logarithmic nonlinearity, concentration-compactness principle, Schwarz symmetrization, Nehari manifold, lack of compactness.

2020 Mathematics Subject Classification: 35J20, 35D30.

1 Introduction and main result

In this paper, we consider the existence of nontrivial and nonnegative (weak) solutions for the following Kirchhoff-type equation

$$-\left(a+b\int_{\mathbb{R}^3}|\nabla u|^2\mathrm{d}x\right)\Delta u+V(x)u=|u|^{p-2}u\log|u|,\qquad x\in\mathbb{R}^3,\tag{1.1}$$

where $a,b>0$ are constants, $4<p<6$. Meanwhile, we shall impose the following conditions on potential function V:

(V_1) V is continuous and $\lim_{|x|\to+\infty}V(x)=\sup_{x\in\mathbb{R}^3}V(x)=:\alpha<+\infty$;

(V_2) $\inf\limits_{\substack{u\in H^1(\mathbb{R}^3),\\ \|u\|_{L^2(\mathbb{R}^3)}=1}}\int_{\mathbb{R}^3}\left[a|\nabla u|^2+V(x)u^2\right]\mathrm{d}x>0.$

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The classical Kirchhoff model

$$\rho \frac{\partial^2 u}{\partial t^2} - \left(\frac{P_0}{h} + \frac{E}{2L} \int_0^L |\frac{\partial u}{\partial x}|^2 \,\mathrm{d}x \right) \frac{\partial^2 u}{\partial x^2} = 0,$$

was first proposed by Kirchhoff [8], where ρ means the mass density, P_0 represents the initial tension, h is the area of the cross-section, L means the length of the string and E means the Young modulus of the material. When considering the effect of the transverse vibrations on the length of the string, the model just mentioned is an extension of the D'Alembert wave equation. After that, with the help of the functional analysis method, Lions [11] derived the following Kirchhoff equation

$$u_{tt} - \left(a + b \int_{\mathbb{R}^3} |\nabla u|^2 \mathrm{d}x \right) \Delta u = f(x,u).$$

In recent years, the following stationary Kirchhoff-type problem

$$-\left(a + b \int_{\Omega} |\nabla u|^2 \mathrm{d}x \right) \Delta u + V(x)u = f(u), \tag{1.2}$$

has been widely studied by many authors, where the potential function $V \in C(\mathbb{R}^N, [0,+\infty))$, and the nonlinear term f is subcritical. For instance, when $V(x) = 1$ and the nonlinear term is $Q(x)|u|^{p-2}u$ with $2 < p < 2^*$, Zhang, Sun, Wu [18], and Hu, Lu [6] obtained the multiplicity of positive solutions by using the barycenter map. Sun and Wu [13] got the existence and non-existence results under the following conditions on V:

- (V_3) There exists $c > 0$ such that the set $\{V < c\} = \{x \in \mathbb{R}^N \mid V(x) < c\}$ is nonempty and has finite measure;
- (V_4) $\Omega = \mathrm{int}V^{-1}(0)$ is nonempty and has smooth boundary with $\bar{\Omega} = V^{-1}(0)$.

In 2015, by using Hardy inequality and Pohožaev identity, Guo [4] got a positive ground state solution of equation (1.2) in $\mathbb{R}^3$ when $V \in C^1(\mathbb{R}^3, \mathbb{R})$ and satisfies the followings:

- (V_5) There exists a positive constant $A < a$ such that $|(\nabla V(x), x)| \leq \frac{A}{2|x|^2}$ for all $x \in \mathbb{R}^3 \backslash \{0\}$;
- (V_6) There exists a positive constant V_∞ such that for all $x \in \mathbb{R}^3$, $V(x) \leq \liminf_{|y|\to+\infty} V(y) \triangleq V_\infty < +\infty$.

In 2017, Tang, Chen [14] proved that equation (1.2) has a ground state solution of Nehari–Pohožaev type when V satisfies the followings:

- (V_7) For all $x \in \mathbb{R}^3$, $0 \leq V(x) \leq \liminf_{|y|\to+\infty} V(y) \triangleq V_\infty$;
- (V_8) $V \in C^1(\mathbb{R}^3, \mathbb{R})$ and there exists $\theta \in [0,1)$ such that

$$4t^4[V(x) - V(tx)] - (1-t^4)(\nabla V(x), x) \geq -\frac{\theta a(1-t^2)^2}{2|x|^2}, \quad \forall t \geq 0, \quad x \in \mathbb{R}^3 \backslash \{0\},$$

and equation (1.2) has a least energy solution when V satisfies (V_7) and the following:

- (V_9) $V(x)$ is weakly differentiable and there exists $\theta \in [0,1)$ such that $(\nabla V(x), x) \leq \frac{\theta a}{2|x|^2}$ a.e. $x \in \mathbb{R}^3 \backslash \{0\}$.

In addition, there are some relevant results for the critical or supercritical problems. When $V(x) = V > 0$ and f has critical growth, Xu and Chen [16] obtained a radial ground state solution in $\mathbb{R}^3$. By using a truncation argument, Gao, Chen and Zhu [3] proved the existence of a sign-changing solution in $\mathbb{R}^3$ when $V \in C(\mathbb{R}^3, (0, \infty))$ satisfies (V_3) and the nonlinearity is critical or supercritical. In 2021, Shen [12] studied a N-laplacian equation of Kirchhoff type with critical growth in $\mathbb{R}^N$, and obtained a least energy sign-changing solution with precisely two nodal domains.

Recently, logarithmic nonlinearity frequently appeared in the Kirchhoff-type problem (1.2). When $f(u) = |u|^{p-2}u \log u^2$, Wen, Tang, Chen [15] proved that equation (1.2) in a smooth bounded domain of $\mathbb{R}^3$ owns ground state solutions and ground state sign-changing solutions with precisely two nodal domains by using some estimate inequalities, constrained variational method and topological degree. Applying the truncation argument, Huang, Shang [7] showed that a logarithmic fractional Kirchhoff equation with critical or supercritical nonlinearity has a ground state solution and a sign-changing solution.

Some authors investigate the case where V can change sign. In 2020, by employing variational method and some new analytical techniques, He, Qin, Tang [5] got the existence of ground state solution when $V(x)$ meets (V_2) with (V_{10}) or with (V_{11}):

(V_{10}) $V(x)$ is weakly differentiable, and there exist $\alpha \geq \frac{1}{2}$ and $\theta \in [0, 1)$ such that $(2\alpha - 1)V(x) - \nabla V(x) \cdot x + \frac{\theta\alpha(2\alpha+1)}{4|x|^2} \geq 0$, a.e. $x \in \mathbb{R}^3 \backslash \{0\}$;

(V_{11}) $V(x)$ is weakly differentiable, and $\operatorname{ess\,sup}_{x\in\mathbb{R}^3} \nabla V(x) \cdot x < \infty$.

With the help of a monotonicity trick and a new perspective of global compactness lemma, Li and Ye [10] proved the equation (1.2) admits a positive ground state solution in $\mathbb{R}^3$ when $f(u) = |u|^{p-1}u$ and $V(x)$ satisfies (V_2) and the following (V_{12})–(V_{13}):

(V_{12}) $V(x)$ is weakly differentiable and satisfies $(\nabla V(x), x) \in L^\infty(\mathbb{R}^3) \bigcup L^{\frac{3}{2}}(\mathbb{R}^3)$ and $V(x) - (\nabla V(x), x) \geq 0$ a.e. $x \in \mathbb{R}^3$, where $(\cdot, \cdot)$ is the usual inner product in $\mathbb{R}^3$;

(V_{13}) For almost every $x \in \mathbb{R}^3$, $V(x) \leq \liminf_{|y|\to+\infty} V(y) \triangleq V_\infty < +\infty$ and the inequality is strict in a subset of positive Lebesgue measure.

Inspired mainly by the literature mentioned above, we consider the existence of solutions for the logarithmic Kirchhoff equation without compactness condition in $\mathbb{R}^3$. We believe that there are at least two fundamental difficulties that need to be addressed regarding equation (1.1). The first one is caused by the space $\mathbb{R}^3$ and the potential function V, since the embedding $H^1(\mathbb{R}^3) \hookrightarrow L^q(\mathbb{R}^3)$, $q \in [2, 6)$ is not compact, and we cannot use the compact embedding of $X = \{u \in H^1(\mathbb{R}^N) | \int_{\mathbb{R}^3} V(x)u^2 dx < +\infty\}$ into $L^p(\mathbb{R}^3)$ anymore. The second difficulty arises from the logarithmic nonlinear term, which does not satisfy both the monotonicity condition and the Ambrosetti–Rabinowitz condition. And we cannot use Schwarz symmetrization directly when facing the logarithmic nonlinear term. We will attempt to categorize the possible behaviors of minimizing sequences. From this, we will exclude some possibilities and then indicate that there is enough compactness in the remaining cases to derive the proof. Moreover, we note here that the potential function V in our results may change sign.

The main result of our work is the following.

Theorem 1.1. *Let* $p \in (4, 6)$, *and assume that* (V_1) *and* (V_2) *hold. The Problem* (1.1) *admits at least one nontrivial and nonnegative (weak) solution.*

Remark 1.2. Under the above assumptions with condition (V_2) replaced by $\inf_{x\in\mathbb{R}^3} V(x) > 0$, Theorem 1.1 also holds.

Notations. We use the following notations:

- Denote
$$\|u\|_k = \left(\int_{\mathbb{R}^3} |u|^k dx\right)^{\frac{1}{k}}$$
the norm of $u \in L^k(\mathbb{R}^3)$ for $1 \le k < +\infty$.
- Define
$$(u,v) = \int_{\mathbb{R}^3} a\nabla u \cdot \nabla v \mathrm{d}x + \int_{\mathbb{R}^3} V(x)uv\mathrm{d}x,$$
$$\|u\|^2 = (u,u) = \int_{\mathbb{R}^3} a|\nabla u|^2\mathrm{d}x + \int_{\mathbb{R}^3} V(x)u^2\mathrm{d}x.$$
Thanks to (V_1) and (V_2), these are equivalent to the standard scalar product and norm of $H^1(\mathbb{R}^3)$.
- Let $D(\mathbb{R}^3)$ be the set of infinitely differentiable functions whose support is compact in $\mathbb{R}^3$. Denote $\mathcal{E}_{p,q}$ the closure of $D(\mathbb{R}^3)$ in the norm
$$\|\nabla u\|_2 + \|u\|_p + \|u\|_q, \quad 1 \le p \le q < \infty,$$
$\mathcal{E}_{p,p} = \mathcal{E}_p$, in particular, $\mathcal{E}_2 = H^1(\mathbb{R}^3)$.
- $C, C_1, C_2, \ldots$ represent several different positive constants.

2 Preliminaries

A weak solution of Equation (1.1) is a function $u \in H^1(\mathbb{R}^3)$ such that
$$a\int_{\mathbb{R}^3} \nabla u\nabla\varphi\mathrm{d}x + b\int_{\mathbb{R}^3}|\nabla u|^2\int_{\mathbb{R}^3}\nabla u\nabla\varphi\mathrm{d}x + \int_{\mathbb{R}^3} V(x)u\varphi\mathrm{d}x$$
$$= \int_{\mathbb{R}^3}|u|^{p-2}u\varphi\log|u|\mathrm{d}x, \quad \forall\varphi \in H^1(\mathbb{R}^3).$$

Define functional $I : H^1(\mathbb{R}^3) \to \mathbb{R}$ by
$$\begin{aligned} I(u) &= \frac{a}{2}\int_{\mathbb{R}^3}|\nabla u|^2\mathrm{d}x + \frac{b}{4}\left(\int_{\mathbb{R}^3}|\nabla u|^2\mathrm{d}x\right)^2 + \frac{1}{2}\int_{\mathbb{R}^3}V(x)u^2\mathrm{d}x \\ &\quad - \frac{1}{p}\int_{\mathbb{R}^3}|u|^p\log|u|\mathrm{d}x + \frac{1}{p^2}\int_{\mathbb{R}^3}|u|^p\mathrm{d}x \\ &= \frac{1}{2}\|u\|^2 + \frac{b}{4}\|\nabla u\|_2^4 - \frac{1}{p}\int_{\mathbb{R}^3}|u|^p\log|u|\mathrm{d}x + \frac{1}{p^2}\|u\|_p^p. \end{aligned} \tag{2.1}$$

Due to the fact that for $4 < p < q < 6$ and arbitrarily $\varepsilon > 0$, there exists $C_\varepsilon > 0$ such that
$$|t^{p-1}\log|t|| \le \varepsilon|t|^2 + C_\varepsilon|t|^{q-1}, \quad \forall t \in \mathbb{R}\backslash\{0\}, \tag{2.2}$$
we can derive that $I \in C^1(H^1(\mathbb{R}^3))$ and a critical point of I is a weak solution of Equation (1.1).

We will consider the following minimization problem:

$$m = \inf_{u \in \mathcal{N}} I(u), \tag{2.3}$$

where the Nehari manifold

$$\begin{aligned}\mathcal{N} &= \{u \in H^1(\mathbb{R}^3) \mid u \neq 0, I'(u)u = 0\} \\ &= \left\{ u \in H^1(\mathbb{R}^3) \mid u \neq 0, \|u\|^2 + b\|\nabla u\|_2^4 = \int_{\mathbb{R}^3} |u|^p \log|u| \mathrm{d}x \right\}.\end{aligned}$$

Furthermore, if $u \in \mathcal{N}$,

$$\begin{aligned} I(u) &= \frac{1}{2}\|u\|^2 + \frac{b}{4}\|\nabla u\|_2^4 - \frac{1}{p}\int_{\mathbb{R}^3} |u|^p \log|u| \mathrm{d}x + \frac{1}{p^2}\|u\|_p^p \\ &= \frac{1}{2}\|u\|^2 + \frac{b}{4}\|\nabla u\|_2^4 - \frac{1}{p}(\|u\|^2 + b\|\nabla u\|_2^4) + \frac{1}{p^2}\|u\|_p^p \\ &= \left(\frac{1}{2} - \frac{1}{p}\right)\|u\|^2 + \left(\frac{1}{4} - \frac{1}{p}\right) b\|\nabla u\|_2^4 + \frac{1}{p^2}\|u\|_p^p, \end{aligned} \tag{2.4}$$

and since $4 < p < 6$, then

$$I(u) > \left(\frac{1}{2} - \frac{1}{p}\right)\|u\|^2 + \left(\frac{1}{4} - \frac{1}{p}\right) b\|\nabla u\|_2^4 > 0$$

for all $u \in \mathcal{N}$.

Lemma 2.1 ([17, Lemma 2.1])**.** *The following inequalities hold:*

$$(1 - x^s) + sx^s \log x > 0, \quad \forall x \in (0,1) \cup (1, +\infty), s > 0; \tag{2.5}$$

$$\log x \leq \frac{1}{e\sigma} x^\sigma, \quad \forall x \in (0, +\infty), \sigma > 0. \tag{2.6}$$

3 Proof of the main result

Using Nehari manifold method and concentration-compactness principle, this section is devoted to the proof of Theorem 1.1, and the proof is composed of the following series of lemmas.

Lemma 3.1. *The Nehari manifold $\mathcal{N}$ is not empty,* $\inf_{u \in \mathcal{N}} \|u\| > 0$ *and* $m > 0$.

Proof. We fix $u \in H^1(\mathbb{R}^3)$ with $u \neq 0$, and for $t \in \mathbb{R}$ we define the function

$$\Gamma_1(t) = I'(tu)tu = t^2\|u\|^2 + bt^4\|\nabla u\|_2^4 - \int_{\mathbb{R}^3} |tu|^p \log|tu| \mathrm{d}x.$$

From (2.2) and $4 < p < q < 6$ we know that

$$\begin{aligned} \Gamma_1(t) &\geq t^2\|u\|^2 + bt^4\|\nabla u\|_2^4 - \int_{\mathbb{R}^3} (\varepsilon|tu|^3 + C_\varepsilon|tu|^q) \mathrm{d}x \\ &= t^2\|u\|^2 + bt^4\|\nabla u\|_2^4 - \varepsilon t^3\|u\|_3^3 - C_\varepsilon t^q\|u\|_q^q > 0 \end{aligned}$$

for small $t > 0$. From (2.5), we obtain that

$$|tu|^2 \log |tu| \geq \frac{1}{2}(|tu|^2 - 1)$$

and

$$|tu|^p \log |tu| \geq \frac{1}{2}(|tu|^p - |tu|^{p-2}).$$

Therefore,

$$\begin{aligned}\lim_{t\to+\infty} \Gamma_1(t) &\leq \lim_{t\to+\infty} \left(t^2\|u\|^2 + bt^4\|\nabla u\|_2^4 - \int_{\mathbb{R}^3} \frac{1}{2}(|tu|^p - |tu|^{p-2})\mathrm{d}x \right) \\ &= \lim_{t\to+\infty} \left(t^2\|u\|^2 + bt^4\|\nabla u\|_2^4 - \frac{1}{2}t^p\|u\|_p^p + \frac{1}{2}t^{p-2}\|u\|_{p-2}^{p-2} \right) = -\infty.\end{aligned}$$

Then there exists $t > 0$ such that $I'(tu)tu = 0$. Hence $tu \in \mathcal{N}$ and $\mathcal{N}$ is not empty.

If $u \in \mathcal{N}$, from (2.6) and $H^1(\mathbb{R}^3) \hookrightarrow L^q(\mathbb{R}^3)$, we obtain

$$\|u\|^2 + b\|\nabla u\|_2^4 = \int_{\mathbb{R}^3} |u|^p \log|u|\mathrm{d}x \leq \int_{\mathbb{R}^3} |u|^p \frac{1}{\mathrm{e}(q-p)}|u|^{q-p}\mathrm{d}x = \frac{1}{\mathrm{e}(q-p)}\|u\|_q^q \leq C\|u\|^q,$$

namely

$$1 \leq C\|u\|^{q-2}.$$

Hence $\inf_{u\in\mathcal{N}} \|u\| > 0$.

Therefore, if $u \in \mathcal{N}$, from (2.4), we have

$$I(u) = \left(\frac{1}{2} - \frac{1}{p}\right)\|u\|^2 + \left(\frac{1}{4} - \frac{1}{p}\right) b\|\nabla u\|_2^4 + \frac{1}{p^2}\|u\|_p^p > \left(\frac{1}{2} - \frac{1}{p}\right)\|u\|^2 \geq \left(\frac{1}{2} - \frac{1}{p}\right) C^{-\frac{2}{q-2}},$$

and the lemma is proved. □

Now, we consider a minimizing sequence $\{u_k\} \subset \mathcal{N}$. Of course, we can assume $u_k(x) \geq 0$ almost everywhere in $\mathbb{R}^3$. From (2.4), the sequence $\{u_k\}$ is bounded in $H^1(\mathbb{R}^3)$, and therefore, up to subsequences, there exists $u \in H^1(\mathbb{R}^3)$ such that

$$\begin{cases} u_k \rightharpoonup u & \text{in } H^1(\mathbb{R}^3),\ L^p(\mathbb{R}^3),\ L^q(\mathbb{R}^3), \\ u_k \to u & \text{in } L^q_{\mathrm{loc}}(\mathbb{R}^3), \\ u_k \to u & \text{a.e. } x \in \mathbb{R}^3. \end{cases}$$

The last property tells us that $u \geq 0$. By extracting a further subsequence, if necessary, we can define $\beta, \xi \geq 0$ as

$$\beta = \lim_{k\to+\infty} \int_{\mathbb{R}^3} |u_k|^q \mathrm{d}x,$$
$$\xi = \lim_{k\to+\infty} \int_{\mathbb{R}^3} |u_k|^p \mathrm{d}x.$$

We let

$$l = \int_{\mathbb{R}^3} |u|^q \mathrm{d}x,$$
$$l_1 = \int_{\mathbb{R}^3} |u|^p \mathrm{d}x.$$

By weak convergence, it is obvious that $l \in [0, \beta]$.

Lemma 3.2. *It holds that* $\beta > 0$.

Proof. If $\beta = 0$, we have

$$\lim_{k\to+\infty} \|u_k\|_q = 0.$$

Since $u_k \in \mathcal{N}$, it follows from (2.6) that

$$\|u_k\|^2 + b\|\nabla u_k\|_2^4 = \int_{\mathbb{R}^3} |u_k|^p \log |u_k| \mathrm{d}x \le \int_{\mathbb{R}^3} |u_k|^p \frac{1}{\mathrm{e}(q-p)} |u_k|^{q-p} \mathrm{d}x = \frac{1}{\mathrm{e}(q-p)} \|u_k\|_q^q,$$

and then we obtain $u_k \to 0$ in $H^1(R^3)$ and $m = 0$, a contradiction. □

Lemma 3.3. *If* $l = \beta$, *then* $u \in \mathcal{N}$ *and* $I(u) = m$.

Proof. If $l = \beta$, we have $\|u_k\|_q \to \|u\|_q$, this fact together with $u_k \rightharpoonup u$ in $L^q(\mathbb{R}^3)$ implies $u_k \to u$ in $L^q(\mathbb{R}^3)$ (see [2, Proposition 3.32]). Let $t \in (0,1)$ be such that $p = 2t + q(1-t)$. Then by the Hölder's inequality,

$$\int_{\mathbb{R}^3} |u_k - u|^p \mathrm{d}x = \int_{\mathbb{R}^3} |u_k - u|^{2t} |u_k - u|^{q(1-t)} \mathrm{d}x \le \|u_k - u\|_2^{2t} \|u_k - u\|_q^{q(1-t)}.$$

As $\{\|u_k - u\|_2\}_k$ is a bounded sequence, we see that

$$\lim_{k\to\infty} \|u_k - u\|_p = 0.$$

This yields also $u_k \to u$ in $L^p(\mathbb{R}^3)$.

Since $u_k \to u$ in $L^q(\mathbb{R}^3)$, using Theorem 1.2.7 in [1], there exists a subsequence $\{u_{k_j}\}_j$ and a function $v \in L^q(\mathbb{R}^3)$ such that

$$\begin{aligned} u_{k_j}(x) &\to u(x) \quad \text{a.e. in } \mathbb{R}^3 \text{ as } j \to \infty; \\ |u_{k_j}(x)| &\le v(x) \quad \text{a.e. in } \mathbb{R}^3 \text{ for all } j. \end{aligned}$$

We take $\{u_{k_j}\}$ as a new minimizing sequence still named by $\{u_k\}$. It is easy to know that $|u_k|^p \log |u_k| \to |u|^p \log |u|$ a.e. in $\mathbb{R}^3$ as $k \to \infty$. Meanwhile, from (2.6), there exists $\frac{1}{\mathrm{e}(q-p)}|v|^q \in L^1(\mathbb{R}^3)$ such that

$$|u_k|^p \log |u_k| \le |u_k|^p \frac{1}{\mathrm{e}(q-p)} |u_k|^{q-p} = \frac{1}{\mathrm{e}(q-p)} |u_k|^q \le \frac{1}{\mathrm{e}(q-p)} |v|^q$$

a.e. in $\mathbb{R}^3$ for all k. By the Dominated convergence theorem, we have

$$\lim_{k\to+\infty} \int_{\mathbb{R}^3} |u_k|^p \log |u_k| \mathrm{d}x = \int_{\mathbb{R}^3} |u|^p \log |u| \mathrm{d}x.$$

Hence, by weak convergence (see [9, Lemma 2.4]), we obtain

$$I(u) = \frac{1}{2}\|u\|^2 + \frac{b}{4}\|\nabla u\|_2^4 - \frac{1}{p}\int_{\mathbb{R}^3} |u|^p \log |u| \mathrm{d}x + \frac{1}{p^2}\|u\|_p^p \le \liminf_k I(u_k) = m,$$

while the relation

$$\|u_k\|^2 + b\|\nabla u_k\|_2^4 = \int_{\mathbb{R}^3} |u_k|^p \log |u_k| \mathrm{d}x$$

implies

$$\begin{aligned}\|u\|^2 + b\|\nabla u\|_2^4 &\le \liminf_{k\to+\infty}(\|u_k\|^2 + b\|\nabla u_k\|_2^4)\\ &= \liminf_{k\to+\infty}\int_{\mathbb{R}^3}|u_k|^p\log|u_k|\mathrm{d}x = \int_{\mathbb{R}^3}|u|^p\log|u|\mathrm{d}x.\end{aligned}$$

If equality holds, then $u \in \mathcal{N}$ (recall that $l = \beta > 0$, so $u \neq 0$) and the lemma is proved. Next, we prove that the case

$$\|u\|^2 + b\|\nabla u\|_2^4 < \int_{\mathbb{R}^3}|u|^p\log|u|\mathrm{d}x$$

cannot occur. Arguing as Lemma 3.1, $\Gamma_1(t) > 0$ for small $t > 0$. Assume by contradiction that

$$\Gamma_1(1) = \|u\|^2 + b\|\nabla u\|_2^4 - \int_{\mathbb{R}^3}|u|^p\log|u|\mathrm{d}x < 0.$$

Then there exists $t \in (0,1)$ such that $tu \in \mathcal{N}$, it follows from (2.4) that

$$\begin{aligned}m \le I(tu) &= \frac{1}{2}t^2\|u\|^2 + \frac{b}{4}t^4\|\nabla u\|_2^4 - \frac{1}{p}\int_{\mathbb{R}^3}|tu|^p\log|tu|\mathrm{d}x + \frac{1}{p^2}\|tu\|_p^p\\ &= \frac{1}{2}t^2\|u\|^2 + \frac{b}{4}t^4\|\nabla u\|_2^4 - \frac{1}{p}(t^2\|u\|^2 + bt^4\|\nabla u\|_2^4) + \frac{1}{p^2}\|tu\|_p^p\\ &= \left(\frac{1}{2}-\frac{1}{p}\right)t^2\|u\|^2 + \left(\frac{1}{4}-\frac{1}{p}\right)bt^4\|\nabla u\|_2^4 + \frac{1}{p^2}t^p\|u\|_p^p\\ &< \left(\frac{1}{2}-\frac{1}{p}\right)\|u\|^2 + \left(\frac{1}{4}-\frac{1}{p}\right)b\|\nabla u\|_2^4 + \frac{1}{p^2}\|u\|_p^p\\ &\le \liminf_{k\to+\infty} I(u_k) = m,\end{aligned}$$

a contradiction. □

Define

$$\|u\|_\alpha^2 = \int_{\mathbb{R}^3}a|\nabla u|^2\mathrm{d}x + \alpha\int_{\mathbb{R}^3}|u|^2\mathrm{d}x,$$

where α is the positive number defined in (V_1). Of course $\|u\|_\alpha$ is an Hilbertian norm equivalent to $\|u\|$. We define a functional $I_\alpha : H^1(\mathbb{R}^3) \to \mathbb{R}$ as

$$I_\alpha(u) = \frac{1}{2}\|u\|_\alpha^2 + \frac{b}{4}\|\nabla u\|_2^4 - \frac{1}{p}\int_{\mathbb{R}^3}|u|^p\log|u|\mathrm{d}x + \frac{1}{p^2}\|u\|_p^p,$$

and the associated Nehari manifold

$$\begin{aligned}\mathcal{N}_\alpha &= \{u \in H^1(\mathbb{R}^3) \mid u \neq 0, I_\alpha'(u)u = 0\}\\ &= \left\{u \in H^1(\mathbb{R}^3) \mid u \neq 0, \|u\|_\alpha^2 + b\|\nabla u\|_2^4 = \int_{\mathbb{R}^3}|u|^p\log|u|\mathrm{d}x\right\},\\ m_\alpha &= \inf_{u\in\mathcal{N}_\alpha} I_\alpha(u).\end{aligned}$$

Let $1 < p \le q < +\infty$, denote Carathéodory function $f\colon \Omega\times\mathbb{R}\to\mathbb{R}$ belongs to the class $K_{p,q}(\Omega)$ if there exist a positive constant C and functions $f_1 \in L^{\frac{p}{p-1}}(\Omega)$, $f_2 \in L^{\frac{q}{q-1}}(\Omega)$ such that

$$|f(x,t)| \le C(|t|^{p-1} + |t|^{q-1}) + f_1(x) + f_2(x)$$

holds for almost all $x \in \Omega$ and for all $t \in \mathbb{R}$. Denote $K_p(\Omega) = K_{p,p}(\Omega)$. If $\Omega = \mathbb{R}^3$, we omit it in the notation.

Next, we will work in the spaces of radial functions, define

$$D^{1,2}(\mathbb{R}^3) = \left\{ u \in L^6(\mathbb{R}^3) \mid \frac{\partial u}{\partial x_i} \in L^2(\mathbb{R}^3),\ i = 1,2,3 \right\},$$
$$D_r = \{u \in D^{1,2}(\mathbb{R}^3) \mid u \text{ is radial}\},$$
$$H_r = \{u \in H^1(\mathbb{R}^3) \mid u \text{ is radial}\}.$$

According to Schwarz symmetrization, we can find a way to pass from functions in $H^1(\mathbb{R}^3)$ to functions in H_r. Let $u \in D^{1,2}(\mathbb{R}^3)$ be such that $u(x) \geq 0$ a.e. in $\mathbb{R}^3$. We denote, for $t > 0$,

$$\{u > t\} = \{x \in \mathbb{R}^3 \mid u(x) > t\}.$$

Notice that since $u \in L^6(\mathbb{R}^3)$, we have $|\{u > t\}| < +\infty$ for all $t > 0$. The Schwarz symmetrization constructs a radial function $u^* : \mathbb{R}^3 \to \mathbb{R}$ such that

$$|\{u^* > t\}| = |\{u > t\}|.$$

Lemma 3.4. *For every* $u \in H^1(\mathbb{R}^3)$, $u \geq 0$, *it holds that* $u^* \in H_r$, $u^* \geq 0$,

$$\int_{\mathbb{R}^3} |u^*|^p \log |u^*| \mathrm{d}x = \int_{\mathbb{R}^3} |u|^p \log |u| \mathrm{d}x.$$

Proof. Define

$$G(x) = |x|^p \log |x|,$$
$$G^+(x) := \max\{G(x), 0\},$$
$$G^-(x) := \max\{-G(x), 0\},$$

then

$$G(x) = G^+(x) - G^-(x).$$

Since

$$\lim_{t\to 0} \frac{t^{p-1} \log |t|}{t} = 0,$$
$$\lim_{t\to \infty} \frac{t^{p-1} \log |t|}{t^{q-1}} = 0,$$

where $4 < p < q < 6$. Therefore, for arbitrarily $\varepsilon_1 > 0$, there exists $C_{\varepsilon_1} > 0$ such that

$$|t^{p-1} \log |t|| \leq \varepsilon_1 |t| + C_{\varepsilon_1} |t|^{q-1}, \quad \forall t \in \mathbb{R} \backslash \{0\}$$

and

$$|G(x)| = ||x|^p \log |x|| \leq \varepsilon_1 |x|^2 + C_{\varepsilon_1} |x|^q \leq \max\{\varepsilon_1, C_{\varepsilon_1}\}(|x|^2 + |x|^q).$$

It is easy to see that $G^+, G^- : \mathbb{R} \to \mathbb{R}$ is a continuous even nonnegative function, $G^+, G^- \in K_{3,q+1}$ with $4 < q < 6$. Since $u \in H^1(\mathbb{R}^3) (= \mathcal{E}_2)$, according to the embedding relationship in

[9], it can be inferred that $u \in \mathcal{E}_{2,q}$. Then from Theorem D.1. in [9], there exists $u^* \in H_r$ such that

$$\int_{|u^*|>1} |u^*|^p \log |u^*| \mathrm{d}x = \int_{\mathbb{R}^3} G^+(u^*) \mathrm{d}x = \int_{\mathbb{R}^3} G^+(u) \mathrm{d}x = \int_{|u|>1} |u|^p \log |u| \mathrm{d}x$$

and

$$\int_{|u^*|\leq 1} -|u^*|^p \log |u^*| \mathrm{d}x = \int_{\mathbb{R}^3} G^-(u^*) \mathrm{d}x = \int_{\mathbb{R}^3} G^-(u) \mathrm{d}x = \int_{|u|\leq 1} -|u|^p \log |u| \mathrm{d}x.$$

Therefore,

$$\begin{aligned}
\int_{\mathbb{R}^3} |u^*|^p \log |u^*| \mathrm{d}x &= \int_{\mathbb{R}^3} G(u^*) \mathrm{d}x = \int_{\mathbb{R}^3} (G^+(u^*) - G^-(u^*)) \mathrm{d}x \\
&= \int_{\mathbb{R}^3} G^+(u^*) \mathrm{d}x - \int_{\mathbb{R}^3} G^-(u^*) \mathrm{d}x \\
&= \int_{|u^*|>1} |u^*|^p \log |u^*| \mathrm{d}x - \int_{|u^*|\leq 1} -|u^*|^p \log |u^*| \mathrm{d}x \\
&= \int_{|u|>1} |u|^p \log |u| \mathrm{d}x - \int_{|u|\leq 1} -|u|^p \log |u| \mathrm{d}x \\
&= \int_{\mathbb{R}^3} |u|^p \log |u| \mathrm{d}x.
\end{aligned}$$

□

Lemma 3.5. *There exists $u \geq 0$, $u \in \mathcal{N}_\alpha$ such that $I_\alpha(u) = m_\alpha$.*

Proof. We first show that we can take a minimizing sequence for m_α in $\mathcal{N}_\alpha \bigcap H_r$. As the same argument as Lemma 3.1, we know that $\mathcal{N}_\alpha \neq \emptyset$, $m_\alpha > 0$. Let $\{v_k\} \subseteq \mathcal{N}_\alpha$ be a minimizing sequence. As usual, we can assume $v_k \geq 0$ and let $\omega_k = v_k^* \in H_r$ be the nonnegative radial function.

From Lemma 3.4 and Theorem 3.1.5 in [1], we have

$$\begin{aligned}
\|\omega_k\|_\alpha^2 &= \int_{\mathbb{R}^3} a|\nabla v_k^*|^2 \mathrm{d}x + \alpha \int_{\mathbb{R}^3} |v_k^*|^2 \mathrm{d}x \leq \int_{\mathbb{R}^3} a|\nabla v_k|^2 \mathrm{d}x + \alpha \int_{\mathbb{R}^3} |v_k|^2 \mathrm{d}x \\
&= \int_{\mathbb{R}^3} |v_k|^p \log |v_k| \mathrm{d}x - b\|\nabla v_k\|_2^4 \leq \int_{\mathbb{R}^3} |v_k^*|^p \log |v_k^*| \mathrm{d}x - b\|\nabla v_k^*\|_2^4 \\
&= \int_{\mathbb{R}^3} |\omega_k|^p \log |\omega_k| \mathrm{d}x - b\|\nabla \omega_k\|_2^4.
\end{aligned}$$

Therefore,

$$\|\omega_k\|_\alpha^2 + b\|\nabla \omega_k\|_2^4 - \int_{\mathbb{R}^3} |\omega_k|^p \log |\omega_k| \mathrm{d}x \leq 0.$$

Set

$$\Gamma_2(t) = I_\alpha'(t\omega_k)t\omega_k = t^2\|\omega_k\|_\alpha^2 + bt^4\|\nabla \omega_k\|_2^4 - \int_{\mathbb{R}^3} |t\omega_k|^p \log |t\omega_k| \mathrm{d}x.$$

From (2.2), we obtain

$$\Gamma_2(t) \geq t^2\|\omega_k\|_\alpha^2 + bt^4\|\nabla \omega_k\|_2^4 - \varepsilon t^3\|\omega_k\|_3^3 - C_\varepsilon t^q\|\omega_k\|_q^q > 0$$

for small $t > 0$. Meanwhile,

$$\Gamma_2(1) = \|\omega_k\|_\alpha^2 + b\|\nabla \omega_k\|_2^4 - \int_{\mathbb{R}^3} |\omega_k|^p \log |\omega_k| \mathrm{d}x \leq 0.$$

Therefore there exists $t_k \in (0,1]$ such that $\Gamma_2(t_k) = 0$, that is, $t_k\omega_k \in \mathcal{N}_\alpha$. Hence from Theorem 3.1.5 in [1], we obtain

$$\begin{aligned}
m_\alpha \leq I_\alpha(t_k\omega_k) &= \left(\frac{1}{2}-\frac{1}{p}\right) t_k^2\|\omega_k\|_\alpha^2 + \left(\frac{1}{4}-\frac{1}{p}\right) bt_k^4\|\nabla\omega_k\|_2^4 + \frac{1}{p^2}t_k^p\|\omega_k\|_p^p \\
&\leq \left(\frac{1}{2}-\frac{1}{p}\right)\|\omega_k\|_\alpha^2 + \left(\frac{1}{4}-\frac{1}{p}\right) b\|\nabla\omega_k\|_2^4 + \frac{1}{p^2}\|\omega_k\|_p^p \\
&= \left(\frac{1}{2}-\frac{1}{p}\right)\left(\int_{\mathbb{R}^3} a|\nabla v_k^*|^2\mathrm{d}x + \alpha\int_{\mathbb{R}^3}|v_k^*|^2\mathrm{d}x\right) \\
&\quad + \left(\frac{1}{4}-\frac{1}{p}\right) b\left(\int_{\mathbb{R}^3}|\nabla v_k^*|^2\mathrm{d}x\right)^2 + \frac{1}{p^2}\int_{\mathbb{R}^3}|v_k^*|^p\mathrm{d}x \\
&\leq \left(\frac{1}{2}-\frac{1}{p}\right)\left(\int_{\mathbb{R}^3} a|\nabla v_k|^2\mathrm{d}x + \alpha\int_{\mathbb{R}^3}|v_k|^2\mathrm{d}x\right) \\
&\quad + \left(\frac{1}{4}-\frac{1}{p}\right) b\left(\int_{\mathbb{R}^3}|\nabla v_k|^2\mathrm{d}x\right)^2 + \frac{1}{p^2}\int_{\mathbb{R}^3}|v_k|^p\mathrm{d}x \\
&= \left(\frac{1}{2}-\frac{1}{p}\right)\|v_k\|_\alpha^2 + \left(\frac{1}{4}-\frac{1}{p}\right) b\|\nabla v_k\|_2^4 + \frac{1}{p^2}\|v_k\|_p^p \\
&= I_\alpha(v_k).
\end{aligned}$$

This implies that $\{t_k\omega_k\}_k \subset H_r$ is a minimizing sequence for m_α, as we had claimed. Next, we set $u_k = t_k\omega_k$. Of course, $u_k \geq 0$, and we can assume that, up to subsequences, $u_k \rightharpoonup u$ in $H^1(\mathbb{R}^3)$. By Lemma 3.1.4 in [1], we obtain

$$u_k \to u \quad \text{in } L^p(\mathbb{R}^3) \text{ and in } L^q(\mathbb{R}^3).$$

Again up to subsequences, $u_k(x) \to u(x)$ almost everywhere, so that $u(x) \geq 0$ a.e. in $\mathbb{R}^3$ and $u \in H_r$. We now prove that the weak limit u belongs to $\mathcal{N}_\alpha$ and $I_\alpha(u) = m_\alpha$. Let us first check that $u \in \mathcal{N}_\alpha$. Since

$$0 < C \leq \|u_k\|_\alpha^2 + b\|\nabla u_k\|_2^4 = \int_{\mathbb{R}^3}|u_k|^p\log|u_k|\mathrm{d}x, \tag{3.1}$$

passing to the limit,

$$0 < C \leq \int_{\mathbb{R}^3}|u|^p\log|u|\mathrm{d}x,$$

this implies $u \neq 0$. From Lemma 2.4 in [9] and $u_k \in \mathcal{N}_\alpha$, we also get

$$\begin{aligned}
\|u\|_\alpha^2 + b\|\nabla u\|_2^4 &\leq \liminf_{k\to\infty}(\|u_k\|_\alpha^2 + b\|\nabla u_k\|_2^4) \\
&= \lim_{k\to\infty}\int_{\mathbb{R}^3}|u_k|^p\log|u_k|\mathrm{d}x = \int_{\mathbb{R}^3}|u|^p\log|u|\mathrm{d}x.
\end{aligned}$$

If

$$\|u\|_\alpha^2 + b\|\nabla u\|_2^4 = \int_{\mathbb{R}^3}|u|^p\log|u|\mathrm{d}x,$$

then $u \in \mathcal{N}_\alpha$. Arguing by contradiction, we assume that

$$\|u\|_\alpha^2 + b\|\nabla u\|_2^4 < \int_{\mathbb{R}^3}|u|^p\log|u|\mathrm{d}x.$$

Then for $t > 0$,

$$\Gamma_2(t) = I'_\alpha(tu)tu = t^2\|u\|_\alpha^2 + bt^4\|\nabla u\|_2^4 - \int_{\mathbb{R}^3} |tu|^p \log|tu| \mathrm{d}x.$$

From (2.2), we see that

$$\Gamma_2(t) \geq t^2\|u\|_\alpha^2 + bt^4\|\nabla u\|_2^4 - \varepsilon t^3\|u\|_3^3 - C_\varepsilon t^q\|u\|_q^q > 0$$

for small $t > 0$, while

$$\Gamma_2(1) = \|u\|_\alpha^2 + b\|\nabla u\|_2^4 - \int_{\mathbb{R}^3} |u|^p \log|u| \mathrm{d}x < 0.$$

So there exists $t \in (0,1)$ such that $tu \in \mathcal{N}_\alpha$. Hence,

$$\begin{aligned}
0 < m_\alpha \leq I_\alpha(tu) &= \left(\frac{1}{2} - \frac{1}{p}\right)\|tu\|_\alpha^2 + \left(\frac{1}{4} - \frac{1}{p}\right) b\|\nabla(tu)\|_2^4 + \frac{1}{p^2}\|tu\|_p^p \\
&= \left(\frac{1}{2} - \frac{1}{p}\right) t^2\|u\|_\alpha^2 + \left(\frac{1}{4} - \frac{1}{p}\right) bt^4\|\nabla u\|_2^4 + \frac{1}{p^2} t^p\|u\|_p^p \\
&< \left(\frac{1}{2} - \frac{1}{p}\right)\|u\|_\alpha^2 + \left(\frac{1}{4} - \frac{1}{p}\right) b\|\nabla u\|_2^4 + \frac{1}{p^2}\|u\|_p^p \\
&\leq \liminf_{k\to+\infty}\left(\left(\frac{1}{2} - \frac{1}{p}\right)\|u_k\|_\alpha^2 + \left(\frac{1}{4} - \frac{1}{p}\right) b\|\nabla u_k\|_2^4 + \frac{1}{p^2}\|u_k\|_p^p\right) \\
&= \liminf_{k\to+\infty} I_\alpha(u_k) = m_\alpha.
\end{aligned}$$

This is a contradiction. Hence

$$\|u\|_\alpha^2 + b\|\nabla u\|_2^4 = \int_{\mathbb{R}^3} |u|^p \log|u| \mathrm{d}x,$$

that is, $u \in \mathcal{N}_\alpha$. By the weakly lower semi-continuity of the norm, it is straightforward to deduce that $I_\alpha(u) \leq \liminf_{k\to+\infty} I_\alpha(u_k) = m_\alpha$. Since $u \in \mathcal{N}_\alpha$, we have $m_\alpha \leq I_\alpha(u)$. Therefore, $I_\alpha(u) = m_\alpha$. □

Lemma 3.6. *It holds that $m < m_\alpha$.*

Proof. By the result of Lemma 3.5, we know that there exists $u_0 \in \mathcal{N}_\alpha$ such that $I_\alpha(u_0) = m_\alpha$ and $u_0 \geq 0$.

By (V_1), we infer that there exists $\delta_1 > 0$ and a ball $B_R(x_1)$ such that

$$V(x) \leq \alpha - \delta_1, \quad \forall x \in B_R(x_1).$$

Since u_0 does not vanish identically, there exists $\delta_2 > 0$, a ball $B_R(x_2)$ and a set $A \subseteq B_R(x_2)$ of positive measure such that

$$u_0(x) \geq \delta_2 \quad \text{a.e. in } A.$$

We now define a function $u_1 \in H^1(\mathbb{R}^3)$ as

$$u_1(x) = u_0(x - x_1 + x_2).$$

By the invariance of integrals with respect to translations we have

$$\begin{aligned}
I_\alpha(u_1) &= \frac{\alpha}{2}\int_{\mathbb{R}^3}|u_1|^2\mathrm{d}x + \frac{a}{2}\int_{\mathbb{R}^3}|\nabla u_1|^2\mathrm{d}x + \frac{b}{4}\left(\int_{\mathbb{R}^3}|\nabla u_1|^2\mathrm{d}x\right)^2 \\
&\quad - \frac{1}{p}\int_{\mathbb{R}^3}|u_1|^p\log|u_1|\mathrm{d}x + \frac{1}{p^2}\int_{\mathbb{R}^3}|u_1|^p\mathrm{d}x \\
&= \frac{\alpha}{2}\int_{\mathbb{R}^3}|u_0|^2\mathrm{d}x + \frac{a}{2}\int_{\mathbb{R}^3}|\nabla u_0|^2\mathrm{d}x + \frac{b}{4}\left(\int_{\mathbb{R}^3}|\nabla u_0|^2\mathrm{d}x\right)^2 \\
&\quad - \frac{1}{p}\int_{\mathbb{R}^3}|u_0|^p\log|u_0|\mathrm{d}x + \frac{1}{p^2}\int_{\mathbb{R}^3}|u_0|^p\mathrm{d}x \\
&= I_\alpha(u_0) = m_\alpha
\end{aligned}$$

and

$$\begin{aligned}
I'_\alpha(u_1)u_1 &= \alpha\int_{\mathbb{R}^3}|u_1|^2\mathrm{d}x + \int_{\mathbb{R}^3}a|\nabla u_1|^2\mathrm{d}x + b\left(\int_{\mathbb{R}^3}|\nabla u_1|^2\mathrm{d}x\right)^2 - \int_{\mathbb{R}^3}|u_1|^p\log|u_1|\mathrm{d}x \\
&= \alpha\int_{\mathbb{R}^3}|u_0|^2\mathrm{d}x + \int_{\mathbb{R}^3}a|\nabla u_0|^2\mathrm{d}x + b\left(\int_{\mathbb{R}^3}|\nabla u_0|^2\mathrm{d}x\right)^2 - \int_{\mathbb{R}^3}|u_0|^p\log|u_0|\mathrm{d}x \\
&= I'_\alpha(u_0)u_0 = 0.
\end{aligned}$$

Notice that if $x \in B_R(x_1)$, then $x - x_1 + x_2 \in B_R(x_2)$, therefore $u_1(x) \geq \delta_2$ a.e. in a set $A' \subseteq B_R(x_1)$ of positive measure. Then

$$\int_{B_R(x_1)}(\alpha - V(x))u_1^2\mathrm{d}x \geq \int_{A'}\delta_1\delta_2^2\mathrm{d}x = C\delta_1\delta_2^2,$$

where C is the measure of A'.

Since $V(x) \leq \alpha$ for every x, we obtain the estimate

$$\int_{\mathbb{R}^3}(\alpha - V(x))u_1^2\mathrm{d}x \geq \int_{B_R(x_1)}(\alpha - V(x))u_1^2\mathrm{d}x \geq C\delta_1\delta_2^2 > 0.$$

Then

$$\int_{\mathbb{R}^3}V(x)u_1^2\mathrm{d}x < \alpha\int_{\mathbb{R}^3}u_1^2\mathrm{d}x,$$

which implies

$$\int_{\mathbb{R}^3}a|\nabla u_1|^2\mathrm{d}x + \int_{\mathbb{R}^3}V(x)u_1^2\mathrm{d}x < \int_{\mathbb{R}^3}a|\nabla u_1|^2\mathrm{d}x + \alpha\int_{\mathbb{R}^3}u_1^2\mathrm{d}x,$$

that is,

$$\|u_1\|^2 < \|u_1\|_\alpha^2.$$

So we have

$$\begin{aligned}
I'(u_1)u_1 &= \|u_1\|^2 + b\|\nabla u_1\|_2^4 - \int_{\mathbb{R}^3}|u_1|^p\log|u_1|\mathrm{d}x \\
&< \|u_1\|_\alpha^2 + b\|\nabla u_1\|_2^4 - \int_{\mathbb{R}^3}|u_1|^p\log|u_1|\mathrm{d}x \\
&= I'_\alpha(u_1)u_1 = I'_\alpha(u_0)u_0 = 0.
\end{aligned}$$

Hence, by usual arguments, there exists $t \in (0,1)$ such that $tu_1 \in \mathcal{N}$. Thus from (2.4), we have

$$\begin{aligned} m \le I(tu_1) &= \left(\frac{1}{2}-\frac{1}{p}\right) t^2\|u_1\|^2 + \left(\frac{1}{4}-\frac{1}{p}\right) bt^4\|\nabla u_1\|_2^4 + \frac{1}{p^2}t^p\|u_1\|_p^p \\ &< \left(\frac{1}{2}-\frac{1}{p}\right) \|u_1\|^2 + \left(\frac{1}{4}-\frac{1}{p}\right) b\|\nabla u_1\|_2^4 + \frac{1}{p^2}\|u_1\|_p^p \\ &< \left(\frac{1}{2}-\frac{1}{p}\right) \|u_1\|_\alpha^2 + \left(\frac{1}{4}-\frac{1}{p}\right) b\|\nabla u_1\|_2^4 + \frac{1}{p^2}\|u_1\|_p^p \\ &= \left(\frac{1}{2}-\frac{1}{p}\right) \|u_0\|_\alpha^2 + \left(\frac{1}{4}-\frac{1}{p}\right) b\|\nabla u_0\|_2^4 + \frac{1}{p^2}\|u_0\|_p^p \\ &= I_\alpha(u_0) = m_\alpha. \end{aligned}$$

□

Lemma 3.7. *The case $l = 0$ cannot occur.*

Proof. If $l = 0$, then $u = 0$, which implies in particular that $u_k \to 0$ in $L^2_{\mathrm{loc}}(\mathbb{R}^3)$.

We first prove the following claim:

$$\lim_{k\to+\infty} \int_{\mathbb{R}^3} |V(x)-\alpha|u_k^2 \mathrm{d}x = 0. \tag{3.2}$$

To prove this, we fix $\varepsilon > 0$ and take $R_\varepsilon > 0$ such that

$$|V(x)-\alpha| \le \varepsilon, \quad \forall\ |x| \ge R_\varepsilon,$$

this is possible by (V_1). We can then estimate

$$\begin{aligned} \int_{\mathbb{R}^3} |V(x)-\alpha|u_k^2 \mathrm{d}x &= \int_{|x|\le R_\varepsilon} |V(x)-\alpha|u_k^2 \mathrm{d}x + \int_{|x|> R_\varepsilon} |V(x)-\alpha|u_k^2 \mathrm{d}x \\ &\le C\int_{|x|\le R_\varepsilon} u_k^2 \mathrm{d}x + M\varepsilon, \end{aligned}$$

where

$$C = \sup_{x\in\mathbb{R}^3} |V(x)-\alpha| \quad \text{and} \quad M = \sup_k \int_{\mathbb{R}^3} u_k^2 \mathrm{d}x.$$

When $k \to +\infty$ we obtain

$$\limsup_{k\to+\infty} \int_{\mathbb{R}^3} |V(x)-\alpha|u_k^2 \mathrm{d}x \le \varepsilon M$$

for every $\varepsilon > 0$, because $u_k \to 0$ in $L^2_{\mathrm{loc}}(\mathbb{R}^3)$. As this holds for every $\varepsilon > 0$, (3.2) is proved.

From (3.2), we deduce that

$$\lim_{k\to+\infty} \|u_k\|_\alpha = \lim_{k\to+\infty} \|u_k\|.$$

We know that

$$\|u_k\|_\alpha^2 + b\|\nabla u_k\|_2^4 \ge \|u_k\|^2 + b\|\nabla u_k\|_2^4 = \int_{\mathbb{R}^3} |u_k|^p \log|u_k| \mathrm{d}x.$$

By usual arguments, we can prove that for each k there is $t_k \ge 1$ such that $t_k u_k \in \mathcal{N}_\alpha$, namely

$$t_k^2\|u_k\|_\alpha^2 + bt_k^4\|\nabla u_k\|_2^4 = \int_{\mathbb{R}^3} |t_k u_k|^p \log|t_k u_k| \mathrm{d}x \ge \frac{1}{2}t_k^p\|u_k\|_p^p - \frac{1}{2}t_k^{p-2}\|u_k\|_{p-2}^{p-2}. \tag{3.3}$$

Since $\{u_k\}$ is a minimizing sequence of $\mathcal{N}$, from (2.4) and $H^1(\mathbb{R}^3) \hookrightarrow L^q(\mathbb{R}^3)$, we know that the sequences $\{\|u_k\|\}$, $\{\|\nabla u_k\|_2\}$, $\{\|u_k\|_\alpha\}$ and $\{\|u_k\|_q^q\}$ are bounded.

Therefore from (3.3), we deduce that $\{t_k\}_k$ is bounded and, up to a subsequence, we can assume $t_k \to t_0$, of course, $t_0 \geq 1$. In addition, we have

$$\int_{\mathbb{R}^3} |u_k|^p \log |u_k| \mathrm{d}x = \|u_k\|^2 + b\|\nabla u_k\|_2^4.$$

Substituting in (3.3), we get

$$\begin{aligned} &t_k^2\|u_k\|_\alpha^2 + bt_k^4\|\nabla u_k\|_2^4 - t_k^p\|u_k\|^2 - bt_k^p\|\nabla u_k\|_2^4 \\ &\quad = \int_{\mathbb{R}^3} |t_k u_k|^p \log |t_k u_k| \mathrm{d}x - \int_{\mathbb{R}^3} |t_k u_k|^p \log |u_k| \mathrm{d}x. \end{aligned}$$

We also have

$$\|u_k\|_\alpha^2 = \|u_k\|^2 + \varepsilon_k$$

with $\varepsilon_k \to 0$. Hence,

$$(t_k^2 - t_k^p)\|u_k\|^2 + t_k^2\varepsilon_k + b(t_k^4 - t_k^p)\|\nabla u_k\|_2^4 = \int_{\mathbb{R}^3} |t_k u_k|^p \log |t_k| \mathrm{d}x = t_k^p\|u_k\|_p^p \log t_k.$$

Passing to the limit, and setting $\eta_1 = \lim_{k\to+\infty} \|u_k\|^2 > 0$, $\eta_2 = \lim_{k\to+\infty} \|\nabla u_k\|_2^4 \geq 0$, we obtain

$$(t_0^2 - t_0^p)\eta_1 + b(t_0^4 - t_0^p)\eta_2 = t_0^p \xi \log t_0.$$

So $t_0 > 1$ gives a contraction because $4 < p < 6$, and then necessarily $t_0 = 1$.

Since $t_k u_k \in \mathcal{N}_\alpha$ and $\|u_k\|_\alpha^2 = \|u_k\|^2 + \varepsilon_k$, from (2.4), we have

$$\begin{aligned} m_\alpha \leq I_\alpha(t_k u_k) &= \left(\frac{1}{2} - \frac{1}{p}\right) t_k^2\|u_k\|_\alpha^2 + \left(\frac{1}{4} - \frac{1}{p}\right) bt_k^4\|\nabla u_k\|_2^4 + \frac{1}{p^2}t_k^p\|u_k\|_p^p \\ &= \left(\frac{1}{2} - \frac{1}{p}\right) t_k^2\|u_k\|^2 + \left(\frac{1}{2} - \frac{1}{p}\right) t_k^2\varepsilon_k + \left(\frac{1}{4} - \frac{1}{p}\right) bt_k^2\|\nabla u_k\|_2^4 \\ &\quad + \left(\frac{1}{4} - \frac{1}{p}\right) b(t_k^4 - t_k^2)\|\nabla u_k\|_2^4 + \frac{1}{p^2}t_k^2\|u_k\|_p^p + \frac{1}{p^2}(t_k^p - t_k^2)\|u_k\|_p^p \\ &= \left(\frac{1}{2} - \frac{1}{p}\right) t_k^2\|u_k\|^2 + \left(\frac{1}{4} - \frac{1}{p}\right) bt_k^2\|\nabla u_k\|_2^4 + \frac{1}{p^2}t_k^2\|u_k\|_p^p \\ &\quad + \left(\frac{1}{2} - \frac{1}{p}\right) t_k^2\varepsilon_k + \left(\frac{1}{4} - \frac{1}{p}\right) b(t_k^4 - t_k^2)\|\nabla u_k\|_2^4 + \frac{1}{p^2}(t_k^p - t_k^2)\|u_k\|_p^p \\ &= t_k^2 I(u_k) + o(1). \end{aligned}$$

Passing to the limit we obtain

$$m_\alpha \leq m,$$

a contradiction of Lemma 3.6, so we conclude the proof. □

Now, we choose an increasing sequence $\{R_j\}_j$ of positive numbers such that $R_{j+1} > R_j + 1$, so that in particular $R_j \to +\infty$, and the following properties hold:

$$\begin{cases} l - \dfrac{1}{j} \leq \displaystyle\int_{B_{R_j}} |u|^q \mathrm{d}x \leq l, \text{ hence also } \displaystyle\int_{B_{R_j}^c} |u|^q \mathrm{d}x \leq \dfrac{1}{j}, \\ l_1 - \dfrac{1}{j} \leq \displaystyle\int_{B_{R_j}} |u|^p \mathrm{d}x \leq l_1, \text{ hence also } \displaystyle\int_{B_{R_j}^c} |u|^p \mathrm{d}x \leq \dfrac{1}{j}, \\ \displaystyle\int_{B_{R_j}^c} u^2 \mathrm{d}x \leq \dfrac{1}{j}. \end{cases}$$

Define

$$C_j = B_{R_j+1} \backslash B_{R_j} = \{x \in \mathbb{R}^3 | R_j \le |x| < R_j + 1\}.$$

Noticing that B_{R_j} and C_j are bounded in $\mathbb{R}^3$, using the Rellich–Kondrachov Theorem, for every j, it holds that

$$\lim_{k\to+\infty}\int_{B_{R_j}}|u_k|^q\mathrm{d}x = \int_{B_{R_j}}|u|^q\mathrm{d}x, \qquad \lim_{k\to+\infty}\int_{C_j}|u_k|^q\mathrm{d}x = \int_{C_j}|u|^q\mathrm{d}x,$$
$$\lim_{k\to+\infty}\int_{B_{R_j}}|u_k|^p\mathrm{d}x = \int_{B_{R_j}}|u|^p\mathrm{d}x, \qquad \lim_{k\to+\infty}\int_{C_j}|u_k|^p\mathrm{d}x = \int_{C_j}|u|^p\mathrm{d}x,$$
$$\lim_{k\to+\infty}\int_{C_j}|u_k|^2\mathrm{d}x = \int_{C_j}|u|^2\mathrm{d}x.$$

Moreover, since

$$\int_{C_j}|u|^q\mathrm{d}x \le \int_{B^C_{R_j}}|u|^q\mathrm{d}x \le \frac{1}{j},$$
$$\int_{C_j}|u|^p\mathrm{d}x \le \int_{B^C_{R_j}}|u|^p\mathrm{d}x \le \frac{1}{j},$$
$$\int_{C_j}|u|^2\mathrm{d}x \le \int_{B^C_{R_j}}|u|^2\mathrm{d}x \le \frac{1}{j},$$

we can extract a subsequence $\{u_{k_j}\}$ such that, for every $j \in \mathbb{N}$,

$$l - \frac{2}{j} \le \int_{B_{R_j}}|u_{k_j}|^q\mathrm{d}x \le l + \frac{1}{j}, \qquad \int_{C_j}|u_{k_j}|^q\mathrm{d}x \le \frac{2}{j},$$
$$l_1 - \frac{2}{j} \le \int_{B_{R_j}}|u_{k_j}|^p\mathrm{d}x \le l_1 + \frac{1}{j}, \qquad \int_{C_j}|u_{k_j}|^p\mathrm{d}x \le \frac{2}{j},$$
$$\int_{C_j}|u_{k_j}|^2\mathrm{d}x \le \frac{2}{j}.$$

We take $\{u_{k_j}\}$ as a new minimizing sequence renaming it $\{u_j\}_j$.

We consider, for every j, a function $\psi_j \in C^\infty(\mathbb{R}^3)$ such that

$$\begin{cases} 0 \le \psi_j(x) \le 1 & \text{for every } x, \\ \psi_j(x) = 1 & \text{if } |x| \le R_j, \\ \psi_j(x) = 0 & \text{if } |x| \ge R_j + 1, \\ |\nabla\psi_j(x)| \le C & \text{for every } x, \end{cases}$$

and define auxiliary functions

$$u'_j = \psi_j u_j \quad \text{and} \quad u''_j = (1-\psi_j)u_j.$$

Of course, u'_j, $u''_j \ge 0$ and $u_j = u'_j + u''_j$ for every j.

Lemma 3.8. *The following properties hold as $j \to \infty$:*

(1) *u'_j tends to u weakly in $H^1(\mathbb{R}^3)$ and strongly in $L^q(\mathbb{R}^3)$ and in $L^p(\mathbb{R}^3)$, while u''_j tends to 0 weakly in $H^1(\mathbb{R}^3)$.*

(2) $\int_{\mathbb{R}^3} |u_j|^q dx = \int_{\mathbb{R}^3} |u_j'|^q dx + \int_{\mathbb{R}^3} |u_j''|^q dx + o(1)$.

(3) $\int_{\mathbb{R}^3} |u_j|^p dx = \int_{\mathbb{R}^3} |u_j'|^p dx + \int_{\mathbb{R}^3} |u_j''|^p dx + o(1)$.

(4) $\|u_j\|^2 \geq \|u_j'\|^2 + \|u_j''\|^2 + o(1)$.

(5) $\|\nabla u_j\|_2^4 \geq \|\nabla u_j'\|_2^4 + \|\nabla u_j''\|_2^4 + o(1)$.

Proof. The proof of (1)–(4) is similar to [1, Lemma 3.3.9], we omit it here. (5) follows from the fact that

$$\int_{\mathbb{R}^3} |\nabla u_j|^2 dx \geq \int_{\mathbb{R}^3} |\nabla u_j'|^2 dx + \int_{\mathbb{R}^3} |\nabla u_j''|^2 dx + o(1). \qquad \square$$

Lemma 3.9. *It holds that*

$$\left(\frac{1}{2} - \frac{1}{p}\right) \|u\|^2 + \left(\frac{1}{4} - \frac{1}{p}\right) b\|\nabla u\|_2^4 + \frac{1}{p^2}\|u\|_p^p \leq m.$$

Proof. Since $u_k \rightharpoonup u$ in $H^1(\mathbb{R}^3)$ and in $L^p(\mathbb{R}^3)$, by weak convergence (see [9, Lemma 2.4]), from (2.4), we obtain

$$\begin{aligned}
&\left(\frac{1}{2} - \frac{1}{p}\right) \|u\|^2 + \left(\frac{1}{4} - \frac{1}{p}\right) b\|\nabla u\|_2^4 + \frac{1}{p^2}\|u\|_p^p \\
&\quad \leq \left(\frac{1}{2} - \frac{1}{p}\right) \liminf_{k\to+\infty} \|u\|^2 + \left(\frac{1}{4} - \frac{1}{p}\right) b \liminf_{k\to+\infty} \|\nabla u\|_2^4 + \frac{1}{p^2} \liminf_{k\to+\infty} \|u\|_p^p \\
&\quad \leq \liminf_{k\to+\infty} I(u_k) = m.
\end{aligned} \qquad \square$$

Next, we will prove $u \in \mathcal{N}$. From Lemma 3.7, $u \neq 0$, so we only need to check $\|u\|^2 + b\|\nabla u\|_2^4 = \int_{\mathbb{R}^3} |u|^p \log|u| dx$.

Lemma 3.10. *It cannot be*

$$\|u\|^2 + b\|\nabla u\|_2^4 < \int_{\mathbb{R}^3} |u|^p \log|u| dx.$$

Proof. If $\|u\|^2 + b\|\nabla u\|_2^4 < \int_{\mathbb{R}^3} |u|^p \log|u| dx$, then, by usual arguments, there is some $t \in (0,1)$ such that $tu \in \mathcal{N}$, then

$$\begin{aligned}
m \leq I(tu) &= t^2 \left(\frac{1}{2} - \frac{1}{p}\right) \|u\|^2 + bt^4 \left(\frac{1}{4} - \frac{1}{p}\right) \|\nabla u\|_2^4 + t^p \frac{1}{p^2}\|u\|_p^p \\
&< \left(\frac{1}{2} - \frac{1}{p}\right) \|u\|^2 + b\left(\frac{1}{4} - \frac{1}{p}\right) \|\nabla u\|_2^4 + \frac{1}{p^2}\|u\|_p^p \leq m,
\end{aligned}$$

a contradiction. $\square$

Lemma 3.11. *Relation*

$$\|u\|^2 + b\|\nabla u\|_2^4 > \int_{\mathbb{R}^3} |u|^p \log|u| dx$$

cannot hold.

Proof. Using (2.6) and Lemma 3.8, we have

$$\begin{aligned}
&\|u_j'\|^2+\|u_j''\|^2+b\|\nabla u_j'\|_2^4+b\|\nabla u_j''\|_2^4 \le \|u_j\|^2+b\|\nabla u_j\|_2^4+o(1)\\
&\quad=\int_{\mathbb{R}^3}|u_j|^p\log|u_j|\mathrm{d}x+o(1)=\int_{\mathbb{R}^3}|u_j'+u_j''|^p\log|u_j'+u_j''|\mathrm{d}x+o(1)\\
&\quad=\int_{B_{R_j}}|u_j'|^p\log|u_j'|\mathrm{d}x+\int_{C_j}|u_j'+u_j''|^p\log|u_j'+u_j''|\mathrm{d}x+\int_{B^c_{R_{j+1}}}|u_j''|^p\log|u_j''|\mathrm{d}x+o(1)\\
&\quad=\int_{\mathbb{R}^3}|u_j'|^p\log|u_j'|\mathrm{d}x+\int_{\mathbb{R}^3}|u_j''|^p\log|u_j''|\mathrm{d}x+\int_{C_j}|u_j|^p\log|u_j|\mathrm{d}x\\
&\qquad-\int_{C_j}|u_j'|^p\log|u_j'|\mathrm{d}x-\int_{C_j}|u_j''|^p\log|u_j''|\mathrm{d}x+o(1)\\
&\quad\le\int_{\mathbb{R}^3}|u_j'|^p\log|u_j'|\mathrm{d}x+\int_{\mathbb{R}^3}|u_j''|^p\log|u_j''|\mathrm{d}x+\int_{C_j}|u_j|^p\log|u_j|\mathrm{d}x\\
&\qquad+\int_{C_j}\big||u_j'|^p\log|u_j'|\big|\mathrm{d}x+\int_{C_j}\big||u_j''|^p\log|u_j''|\big|\mathrm{d}x+o(1)\\
&\quad\le\int_{\mathbb{R}^3}|u_j'|^p\log|u_j'|\mathrm{d}x+\int_{\mathbb{R}^3}|u_j''|^p\log|u_j''|\mathrm{d}x+\int_{C_j}|u_j|^p\frac{1}{\mathrm{e}(q-p)}|u_j|^{q-p}\mathrm{d}x\\
&\qquad+\int_{C_j}\left||u_j'|^p\frac{1}{\mathrm{e}(q-p)}|u_j'|^{q-p}\right|\mathrm{d}x+\int_{C_j}\left||u_j''|^p\frac{1}{\mathrm{e}(q-p)}|u_j''|^{q-p}\right|\mathrm{d}x+o(1)\\
&\quad\le\int_{\mathbb{R}^3}|u_j'|^p\log|u_j'|\mathrm{d}x+\int_{\mathbb{R}^3}|u_j''|^p\log|u_j''|\mathrm{d}x+\frac{6}{\mathrm{e}(q-p)j}+o(1)\\
&\quad=\int_{\mathbb{R}^3}|u_j'|^p\log|u_j'|\mathrm{d}x+\int_{\mathbb{R}^3}|u_j''|^p\log|u_j''|\mathrm{d}x+o(1).
\end{aligned}\tag{3.4}$$

Assume for contradiction that

$$\|u\|^2+b\|\nabla u\|_2^4>\int_{\mathbb{R}^3}|u|^p\log|u|\mathrm{d}x$$

and pick $\delta>0$ such that

$$\|u\|^2+b\|\nabla u\|_2^4>\int_{\mathbb{R}^3}|u|^p\log|u|\mathrm{d}x+\delta.$$

Since $u_j'\rightharpoonup u$ in $H^1(\mathbb{R}^3)$,

$$\liminf_{j\to+\infty}\|u_j'\|^2\ge\|u\|^2,$$
$$\liminf_{j\to+\infty}\|\nabla u_j'\|_2\ge\|\nabla u\|_2,$$

while $u_j'\to u$ in $L^p(\mathbb{R}^3)$ and in $L^q(\mathbb{R}^3)$. Using Lebesgue's dominated convergence theorem, we deduce that, for large j's and $\delta_1\in(0,\frac{\delta}{2})$,

$$\|u_j'\|^2+b\|\nabla u_j'\|_2^4>\|u\|^2+b\|\nabla u\|_2^4-\frac{\delta}{2}>\int_{\mathbb{R}^3}|u|^p\log|u|\mathrm{d}x+\frac{\delta}{2}>\int_{\mathbb{R}^3}|u_j'|^p\log|u_j'|\mathrm{d}x+\delta_1.$$

This together with (3.4) implies

$$\|u_j''\|^2+b\|\nabla u_j''\|_2^4\le\int_{\mathbb{R}^3}|u_j''|^p\log|u_j''|\mathrm{d}x-\delta_2,$$

for large j's and some $\delta_2 \in (0, \delta_1)$. Recalling that $u_j'' \rightharpoonup 0$ in $H^1(\mathbb{R}^3)$ and $H^1(\mathbb{R}^3) \hookrightarrow\hookrightarrow L^2_{\text{loc}}(\mathbb{R}^3)$, arguing as in Lemma 3.7, we obtain

$$\lim_{j\to+\infty} \|u_j''\|^2 = \lim_{j\to+\infty} \|u_j''\|_\alpha^2,$$

so

$$\|u_j''\|_\alpha^2 + b\|\nabla u_j''\|_2^4 \le \int_{\mathbb{R}^3} |u_j''|^p \log |u_j''| \mathrm{d}x - \delta_3$$

for large j's and for some $\delta_3 \in (0, \delta_2)$. Hence, by usual arguments, we can find $s_j \in (0, 1)$ such that $s_j u_j'' \in \mathcal{N}_\alpha$, from (2.4), we have

$$\begin{aligned}
m_\alpha \le I_\alpha(s_j u_j'') &= \left(\frac{1}{2} - \frac{1}{p}\right) s_j^2 \|u_j''\|_\alpha^2 + \left(\frac{1}{4} - \frac{1}{p}\right) b s_j^4 \|\nabla u_j''\|_2^4 + \frac{1}{p^2} s_j^p \|u_j''\|_p^p \\
&\le \left(\frac{1}{2} - \frac{1}{p}\right) \|u_j''\|^2 + \left(\frac{1}{4} - \frac{1}{p}\right) b \|\nabla u_j''\|_2^4 + \frac{1}{p^2} \|u_j''\|_p^p + o(1) \\
&\le \left(\frac{1}{2} - \frac{1}{p}\right) (\|u_j''\|^2 + \|u_j'\|^2) + \left(\frac{1}{4} - \frac{1}{p}\right) b(\|\nabla u_j''\|_2^4 + \|\nabla u_j'\|_2^4) \\
&\quad + \frac{1}{p^2}(\|u_j''\|_p^p + \|u_j'\|_p^p) + o(1) \\
&\le \left(\frac{1}{2} - \frac{1}{p}\right) \|u_j\|^2 + \left(\frac{1}{4} - \frac{1}{p}\right) b\|\nabla u_j\|_2^4 + \frac{1}{p^2} \|u_j\|_p^p + o(1) \\
&= I(u_j) + o(1).
\end{aligned}$$

Passing to the limit, we have

$$m_\alpha \le m,$$

a contradiction of Lemma 3.6. □

The previous lemmas say that in any case the weak limit u satisfies $u \neq 0, u \ge 0$ and

$$\|u\|^2 + b\|\nabla u\|_2^4 = \int_{\mathbb{R}^3} |u|^p \log |u| \mathrm{d}x.$$

Therefore $u \in \mathcal{N}$ and

$$I(u) = \left(\frac{1}{2} - \frac{1}{p}\right) \|u\|^2 + \left(\frac{1}{4} - \frac{1}{p}\right) b\|\nabla u\|_2^4 + \frac{1}{p^2} \|u\|_p^p \ge m.$$

From Lemma 3.9, we can get $I(u) = m$. So u is a minimum point of I on $\mathcal{N}$.

Lemma 3.12. *The minimum u is a critical point of I in $H^1(\mathbb{R}^3)$.*

Proof. Fix $v \in H^1(\mathbb{R}^3)$ and $\varepsilon > 0$ such that $u + sv \neq 0$ for all $s \in (-\varepsilon, \varepsilon)$. Define a function $\varphi : (-\varepsilon, \varepsilon) \times (0, +\infty) \to \mathbb{R}$ by

$$\begin{aligned}
\varphi(s, t) &= I'(t(u + sv))t(u + sv) \\
&= \|t(u + sv)\|^2 + b\|\nabla(t(u + sv))\|_2^4 - \int_{\mathbb{R}^3} |t(u + sv)|^p \log |t(u + sv)| \mathrm{d}x \\
&= t^2 \|u + sv\|^2 + bt^4 \|\nabla(u + sv)\|_2^4 - \int_{\mathbb{R}^3} |t(u + sv)|^p \log |t(u + sv)| \mathrm{d}x.
\end{aligned}$$

Then

$$\varphi(0,1) = \|u\|^2 + b\|\nabla u\|_2^4 - \int_{\mathbb{R}^3} |u|^p \log|u| \mathrm{d}x = 0$$

and

$$\frac{\partial \varphi}{\partial t}(s,t) = 2t\|u+sv\|^2 + 4bt^3\|\nabla(u+sv)\|_2^4 \\ - pt^{p-1}\int_{\mathbb{R}^3} |u+sv|^p \log|t(u+sv)| \mathrm{d}x - t^{p-1}\int_{\mathbb{R}^3} |u+sv|^p \mathrm{d}x.$$

Hence, from $u \in \mathcal{N}$, we have

$$\begin{aligned}\frac{\partial \varphi}{\partial t}(0,1) &= 2\|u\|^2 + 4b\|\nabla u\|_2^4 - p\int_{\mathbb{R}^3} |u|^p \log|u| \mathrm{d}x - \int_{\mathbb{R}^3} |u|^p \mathrm{d}x \\ &= 2\|u\|^2 + 4b\|\nabla u\|_2^4 - p(\|u\|^2 + b\|\nabla u\|_2^4) - \int_{\mathbb{R}^3} |u|^p \mathrm{d}x \\ &= (2-p)\|u\|^2 + (4-p)b\|\nabla u\|_2^4 - \int_{\mathbb{R}^3} |u|^p \mathrm{d}x < 0.\end{aligned}$$

By the Implicit Function Theorem, there exists $\varepsilon_0 \in (0,\varepsilon)$ such that C^1 function $t : (-\varepsilon_0, \varepsilon_0) \to \mathbb{R}$ satisfies $t(0) = 1$ and

$$\varphi(s, t(s)) = 0$$

for all $s \in (-\varepsilon_0, \varepsilon_0)$. Defining $\Gamma_3(s) : (-\varepsilon_0, \varepsilon_0) \to \mathbb{R}$ by

$$\Gamma_3(s) = I(t(s)(u+sv)).$$

It is easy to see that the function Γ_3 is differentiable. Noticing that $I'(t(s)(u+sv))t(s)(u+sv) = \varphi(s,t(s)) = 0$, we have $t(s)(u+sv) \in \mathcal{N}$ and u is a minimum point for I on $\mathcal{N}$, so Γ_3 has a minimum point at $s = 0$. Therefore,

$$0 = \Gamma_3'(0) = I'(t(0)u)(t'(0)u + t(0)v) = t'(0)I'(u)u + I'(u)v = I'(u)v$$

for all $v \in H^1(\mathbb{R}^3)$, hence $I'(u) = 0$. □

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References

[1] M. Badiale, E. Serra, *Semilinear elliptic equations for beginners: Existence results via the variational approach*, Springer, London, 2011. https://doi.org/10.1007/978-0-85729-227-8; MR2722059 Zbl 1214.35025

[2] H. Brezis, *Functional analysis, Sobolev spaces and partial differential equations*, Springer, New York, 2011. https://doi.org/10.1007/978-0-387-70914-7; MR2759829; Zbl 1220.46002

[3] L. Gao, C. Chen, C. Zhu, Existence of sign-changing solutions for Kirchhoff equations with critical or supercritical nonlinearity, *Appl. Math. Lett.* **107**(2020), 106424. https://doi.org/10.1016/j.aml.2020.106424; MR4097887; Zbl 1442.35136

[4] Z. Guo, Ground states for Kirchhoff equations without compact condition, *J. Differential Equations* **259**(2015), No. 7, 2884–2902. https://doi.org/10.1016/j.jde.2015.04.005; MR3360660; Zbl 1319.35018

[5] F. He, D. Qin, X. Tang, Existence of ground states for Kirchhoff-type problems with general potentials, *J. Geom. Anal.* **31**(2021), No. 8, 7709–7725. https://doi.org/10.1007/s12220-020-00546-4; MR4293909; Zbl 1473.35265

[6] T. Hu, L. Lu, Multiplicity of positive solutions for Kirchhoff type problems in $\mathbb{R}^3$, *Topol. Methods Nonlinear Anal.* **50**(2017), No. 1, 231–252. https://doi.org//10.12775/TMNA.2017.028; MR3706159; Zbl 1387.35025

[7] T. Huang, Y. Shang, Multiple solutions for the logarithmic fractional Kirchhoff equation with critical or supercritical nonlinearity, *Complex Var. Elliptic Equ.***70**(2025), No. 1, 115–136. https://doi.org/10.1080/17476933.2023.2293800; MR4839849; Zbl 1555.35143

[8] G. Kirchhoff, *Vorlesungen über Mechanik*, Teubner, Leipzig, 1883.

[9] I. Kuzin, S. Pohožaev, *Entire solutions of semilinear elliptic equations*, Progress in Nonlinear Differential Equations and Their Applications, Birkhäuser, Basel, 1997. MR1479168; Zbl 0882.35003

[10] G. Li, H. Ye, Existence of positive ground state solutions for the nonlinear Kirchhoff type equations in $\mathbb{R}^3$, *J. Differential Equations* **257**(2014), No. 2, 566–600. https://doi.org/10.1016/j.jde.2014.04.011; MR3200382; Zbl 1290.35051

[11] J. Lions, On some questions in boundary value problems of mathematical physics, in: *Contemporary developments in continuum mechanics and partial differential equations (Proc. Internat. Sympos., Inst. Mat., Univ. Fed. Rio de Janeiro, Rio de Janeiro, 1977)*, North-Holland Mathematics Studies, Vol. 30, 1978, pp. 284–346. https://doi.org/10.1016/S0304-0208(08)70870-3; MR0519648; Zbl 0404.35002

[12] L. Shen, Sign-Changing solutions to a N-Kirchhoff equation with critical exponential growth in $\mathbb{R}^N$, *Bull. Malays. Math. Sci. Soc.* **44**(2021), No. 6, 3553–3570. https://doi.org//10.1007/s40840-021-01127-6; MR4321723; Zbl 1479.35495

[13] J. Sun, T. Wu, Ground state solutions for an indefinite Kirchhoff type problem with steep potential well, *J. Differential Equations* **256**(2014), No. 4, 1771–1792. https://doi.org/10.1016/j.jde.2013.12.006; MR3145774; Zbl 1288.35219

[14] X. Tang, S. Chen, Ground state solutions of Nehari–Pohožaev type for Kirchhoff-type problems with general potentials, *Calc. Var. Partial Differential Equations* **56**(2017), No. 4, 110. https://doi.org/10.1007/s00526-017-1214-9; MR3669779; Zbl 1376.35056

[15] L. Wen, X. Tang, S. Chen, Ground state sign-changing solutions for Kirchhoff equations with logarithmic nonlinearity, *Electron. J. Qual. Theory Differ. Equ.* **2019**, No. 47, 1–13. https://doi.org/10.14232/ejqtde.2019.1.47; MR3991096; Zbl 1438.35159

[16] L. Xu, H. Chen, Ground state solutions for Kirchhoff-type equations with a general nonlinearity in the critical growth, *Adv. Nonlinear Anal.* **7**(2018), No. 4, 535–546. https://doi.org/10.1515/anona-2016-0073; MR3871420; Zbl 1404.35141

[17] W. Yang, J. Liao, Ground state sign-changing solution for a logarithmic Kirchhoff-type equation in $\mathbb{R}^3$, *Electron. J. Qual. Theory Differ. Equ.* **2024**, No. 42, 1–19. https://doi.org/10.14232/ejqtde.2024.1.42; MR4782776; Zbl 1563.35250

[18] J. Zhang, J. Sun, T. Wu, The number of positive solutions affected by the weight function to Kirchhoff type equations in high dimensions, *Nonlinear Anal.* **196**(2020), 11780. https://doi.org/10.1016/j.na.2020.111780; MR4062338; Zbl 1437.35027