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Infinitely many solutions for nonlinear elliptic problems in the whole space

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Abstract. This paper investigates the existence of infinitely many weak solutions for a nonlinear elliptic p-Laplacian equation defined in the whole space $\mathbb{R}^N$. In particular, the equation is parameter-dependent and a range of positive parameters in which the equation admits such solutions is provided. Moreover, some particular cases in which the solutions turn out to be nonnegative are presented. The study is conducted via variational methods and critical points theory.

Keywords: nonlinear p-Laplacian equation, unbounded domain, critical points theory, infinitely many solutions.

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1 Introduction

The p-Laplacian differential operator, denoted by $\Delta_p(u) = \operatorname{div}(|\nabla u|^{p-2}\nabla u)$, has been studied extensively by many authors in the latest years with various techniques. Precisely, there is a wide literature on elliptic p-Laplacian problems in bounded domains, while the results are fewer when the problems are defined in unbounded domains or in the whole space. Indeed, considering elliptic equations in the whole space $\mathbb{R}^N$ increases the technical difficulties of the study and, among the results, we mention Ambrosetti–García Azorero–Peral [1], Barletta–Candito–Marano–Perera [3], Drábek–Huang [6], Guarnotta–Livrea–Marano [7] and the references therein. Inspired by this issues, the authors of the present paper considered in [2] the following nonlinear elliptic problem

$$\begin{cases} -\Delta_p u + a(x)|u|^{p-2}u = \lambda f(x,u) & \text{in } \mathbb{R}^N, \\ \lim_{|x|\to\infty} u(x) = 0, \end{cases} \tag{1.1}$$

where $p > N$, $\lambda > 0$ is a parameter, $a\colon \mathbb{R}^N \to \mathbb{R}$ is a $L^\infty(\mathbb{R}^N)$-function with $\operatorname{ess\,inf}_{\mathbb{R}^N} a > 0$ and $f : \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}$ is a L^1-Carathéodory function. Despite the lack of compactness of the embedding of Sobolev spaces in appropriate function spaces, the authors are able to obtain

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results in line with the bounded domain case; precisely, some results on the existence of one and two non-zero solutions are presented. As far as we know, [2] is the first note for the case $p > N$ in the whole space.

The present paper is intended to continue the research carried out in [2], studying the existence of infinitely many solutions for problem (1.1) through variational methods and critical points theory. In particular, the main tool of our investigation is a critical points theorem due to Bonanno [4, Theorem 7.4]. By requiring a suitable oscillation of the primitive of the nonlinear term, we get the existence a sequence of pairwise distinct weak solutions that turn out to be nonnegative with an additional assumption on the nonlinearity. To give an idea of the results that we obtain, we present here the following particular case.

Theorem 1.1. *Let $g\colon \mathbb{R} \to \mathbb{R}$ be a continuous nonnegative function and set $G(t) = \int_0^t g(\xi)\mathrm{d}\xi$ for any $t \in \mathbb{R}$. Given $p > N$, assume that*

$$\liminf_{\xi\to+\infty} \frac{G(\xi)}{\xi^p} = 0 \quad \textit{and} \quad \limsup_{\xi\to+\infty} \frac{G(\xi)}{\xi^p} = +\infty.$$

Then, for any $a \in L^\infty(\mathbb{R}^N)$ with $\operatorname{ess\,inf}_{\mathbb{R}^N} a > 0$ and for any non-null and nonnegative function $h \in L^1(\mathbb{R}^N)$, the following problem

$$\begin{cases} -\Delta_p u + a(x)|u|^{p-2}u = h(x)g(u), & x \in \mathbb{R}^N, \\ \lim_{|x|\to\infty} u(x) = 0, \end{cases}$$

admits a sequence of pairwise distinct nonnegative weak solutions.

The paper is organized as follows. In Section 2, we describe the variational setting of the problem and in Theorem 2.2 we recall the abstract theorem that we use in the proof of our main results, which are presented in Section 3. In particular, Theorem 3.1 and Theorem 3.3 ensure the existence of infinitely many weak solutions for problem (1.1) requiring two different behavior on the nonlinearity. Finally, Section 4 deals with the particular case where the nonlinear term is of separable variable type and also provides an example.

2 Variational framework

Given $p > N$, we consider the Sobolev space $W^{1,p}(\mathbb{R}^N)$ endowed with the usual norm

$$\|u\|_{1,p} = \|u\|_p + \|\nabla u\|_p,$$

where $\|\cdot\|_p$ denotes the classical norm in the Lebesgue space $L^p(\mathbb{R}^N)$. It is well known that $W^{1,p}(\mathbb{R}^N)$ is continuously embedded in $L^\infty(\mathbb{R}^N)$ (see Morrey theorem in Brezis [5, Theorem 9.12]). Hence, there exists a constant $c > 0$ such that

$$\|u\|_\infty \le c\|u\|_{1,p} \quad \text{for all } u \in W^{1,p}(\mathbb{R}^N),$$

and an optimal numerical estimate of the embedding constant c can be found in Proposition 2.2 in Amoroso–Bonanno–Perera [2], see also Remark 2.1. However, for our investigation we equip the space with the following equivalent norm

$$\|u\| = \left(\int_{\mathbb{R}^N} |\nabla u(x)|^p dx + \int_{\mathbb{R}^N} a(x)|u(x)|^p dx\right)^{\frac{1}{p}},$$

where $a\colon \mathbb{R}^N \to \mathbb{R}$ is such that

$$a \in L^\infty(\mathbb{R}^N) \quad \text{and} \quad a_- = \operatorname*{ess\,inf}_{\mathbb{R}^N} a > 0.$$

By Amoroso–Bonanno–Perera [2, Lemma 2.1] one has

$$\|u\|_\infty \le C\|u\| \quad \text{for all } u \in W^{1,p}(\mathbb{R}^N), \tag{2.1}$$

where

$$C = \left(\frac{1}{a_-}\right)^{\frac{p-N}{p^2}} 2^{\frac{p-1}{p}} \left(\frac{\Gamma\left(1+\frac{N}{2}\right)}{\pi^{\frac{N}{2}}}\right)^{\frac{1}{p}} \frac{1}{N^{\frac{N}{p^2}}} \left(\frac{p-1}{p-N}\right)^{\frac{N(p-1)}{p^2}}, \tag{2.2}$$

with $\Gamma(s) = \int_0^\infty e^{-t}t^{s-1}\mathrm{d}t$ being the well known gamma function. Note that $W_0^{1,p}(\mathbb{R}^N)$ coincides with $W^{1,p}(\mathbb{R}^N)$, hence if $u \in W^{1,p}(\mathbb{R}^N)$ then it holds that

$$\lim_{|x|\to\infty} u(x) = 0.$$

Therefore, studying problem (1.1) is equivalent to studying the following one

$$-\Delta_p u + a(x)|u|^{p-2}u = \lambda f(x,u) \quad \text{in } \mathbb{R}^N, \qquad u \in W^{1,p}(\mathbb{R}^N), \tag{P$_\lambda$}$$

that means finding $u \in W^{1,p}(\mathbb{R}^N)$ such that

$$\int_{\mathbb{R}^N} \left(|\nabla u|^{p-2}\nabla u \nabla v + a(x)|u|^{p-2}uv\right) dx = \lambda \int_{\mathbb{R}^N} f(x,u)v\,\mathrm{d}x,$$

for all $v \in W^{1,p}(\mathbb{R}^N)$.

To this end, we suppose that $f\colon \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}$ is a L^1-Carathéodory function, i.e.

(i) $f(\cdot,t)$ is measurable for all $t \in \mathbb{R}$,

(ii) $f(x,\cdot)$ is continuous for almost all $x \in \mathbb{R}^N$,

(iii) for all $\sigma > 0$ the function $\sup_{|t|\le\sigma} |f(\cdot,t)|$ belongs to $L^1(\mathbb{R}^N)$.

Hence, summarizing our hypotheses, we assume that

(H) $p > N$, $a \in L^\infty(\mathbb{R}^N)$ such that $a_- = \operatorname{ess\,inf}_{\mathbb{R}^N} a > 0$ and $f : \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}$ is an L^1-Carathéodory function.

Our aim is to find weak solutions for problem (P$_\lambda$), more precisely we say that $u \in W^{1,p}(\mathbb{R}^N)$ is a weak solution of (P$_\lambda$) if

$$\int_{\mathbb{R}^N} \left(|\nabla u|^{p-2}\nabla u \nabla v + a(x)|u|^{p-2}uv\right) \mathrm{d}x = \lambda \int_{\mathbb{R}^N} f(x,u)v\,\mathrm{d}x$$

for any $v \in W^{1,p}(\mathbb{R}^N)$. Consider the functionals $\Phi, \Psi\colon W^{1,p}(\mathbb{R}^N) \to \mathbb{R}$ defined by

$$\Phi(u) = \frac{1}{p}\|u\|^p, \qquad \Psi(u) = \int_{\mathbb{R}^N} F(x,u(x))\,\mathrm{d}x,$$

where $F(x,t) = \int_0^t f(x,\xi)\,\mathrm{d}\xi$ for any $(x,t) \in \mathbb{R}^N \times \mathbb{R}$. Then, the functional $I_\lambda = \Phi - \lambda\Psi$ is the so-called energy functional associated to problem (P$_\lambda$). It is well known that Φ, Ψ, I_λ are Gâteaux differentiable functionals and it holds that

$$\begin{aligned}
\Phi'(u)(v) &= \int_{\mathbb{R}^N} \left(|\nabla u|^{p-2}\nabla u \nabla v + a(x)|u|^{p-2}uv\right) \mathrm{d}x,\\
\Psi'(u)(v) &= \int_{\mathbb{R}^N} f(x,u)v\,\mathrm{d}x,
\end{aligned}$$

for every $u, v \in W^{1,p}(\mathbb{R}^N)$. Hence, $u \in W^{1,p}(\mathbb{R}^N)$ is a weak solution for (P_λ) if and only if u is a critical point of I_λ, i.e. $I'_\lambda(u)(v) = 0$ for all $v \in W^{1,p}(\mathbb{R}^N)$. Therefore, our aim is to find critical points of the energy functional I_λ, for some $\lambda > 0$.

Our approach is based on variational methods and critical point theory. In particular, our main tool is Theorem 7.4 of Bonanno [4], that we recall here together with the definition of $(\mathrm{PS})^{[r]}$-condition.

Definition 2.1. Let $(X, \|\cdot\|)$ be a Banach space, X^* its dual and $I_\lambda : X \to \mathbb{R}$ a Gâteaux differentiable functional, with $\lambda > 0$. Fix $r \in]-\infty, \infty]$. We say that I_λ satisfies the Palais–Smale condition cut-off upper at r (in short, $(\mathrm{PS})^{[r]}$-condition), if any sequence $\{u_n\} \subseteq X$ such that

(P_1) $I_\lambda(u_n)$ is bounded,

(P_2) $\lim_{n\to\infty} \|I_\lambda(u_n)\|_{X^*} = 0$,

(P_3) $\Phi(u_n) < r$,

has a convergent subsequence in X.

Clearly, if I_λ satisfies the classical (PS)-condition, which involves only conditions (P_1)–(P_2), then it satisfies also the $(\mathrm{PS})^{[r]}$-condition.

Now, let $\Phi, \Psi : X \to \mathbb{R}$ be two functions and for all $r > \inf_X \Phi$, set

$$\varphi(r) = \inf_{u\in\Phi^{-1}((-\infty,r))} \frac{\left(\sup\limits_{v\in(\Phi^{-1}(-\infty,r))} \Psi(v)\right) - \Psi(u)}{r - \Phi(u)}, \tag{2.3}$$

and

$$\alpha := \liminf_{r\to+\infty} \varphi(r), \qquad \beta := \liminf_{r\to(\inf_X \Phi)^+} \varphi(r). \tag{2.4}$$

Theorem 2.2. *Let X be a real Banach space and let $\Phi, \Psi\colon X \to \mathbb{R}$ be two continuously Gâteaux differentiable functions with Φ bounded from below.*

(a) *If $\alpha < +\infty$ and for each $\lambda \in \left(0, \frac{1}{\alpha}\right)$ the function $I_\lambda = \Phi - \lambda\Psi$ satisfies $(\mathrm{PS})^{[r]}$-condition for all $r \in \mathbb{R}$, then, for each $\lambda \in \left(0, \frac{1}{\alpha}\right)$, the following alternative holds:*

either

(a_1) *I_λ possesses a global minimum, or*

(a_2) *there is a sequence $\{u_n\}$ of critical points (local minima) of I_λ such that $\lim_{n\to+\infty} \Phi(u_n) = +\infty$.*

(b) *If $\beta < +\infty$ and for each $\lambda \in \left(0, \frac{1}{\beta}\right)$ the function $I_\lambda = \Phi - \lambda\Psi$ satisfies $(\mathrm{PS})^{[r]}$-condition for some $r > \inf_X \Phi$, then, for each $\lambda \in \left(0, \frac{1}{\beta}\right)$, the following alternative holds:*

either

(b_1) *there is a global minimum of Φ which is a local minimum of I_λ, or*

(b_2) *there is a sequence of pairwise distinct critical points (local minima) of I_λ sucht that $\lim_{n\to+\infty} \Phi(u_n) = \inf_X \Phi$.*

3 Main results

In this section, we present our main result on the existence of infinitely many weak solutions for problem (P_λ), by requiring an appropriate oscillation of the primitive of the nonlinearity and providing also a range for the parameter λ for which the problem admits such solutions.

Fix an arbitrary open ball $B(x_0, R)$ of center x_0 and radius $R > 0$ and set

$$K_R = \frac{1}{C^p}\frac{\Gamma\left(1+\frac{N}{2}\right)}{\pi^{\frac{N}{2}}}\left(\frac{R^{p-N}}{2^p - 2^{p-N} + R^p\|a\|_\infty}\right), \tag{3.1}$$

where C is given in (2.2); we underline that K_R does not depend on the choice of x_0. Furthermore, put

$$A = \liminf_{\xi\to+\infty}\frac{\displaystyle\int_{\mathbb{R}^N}\max_{|t|\le\xi}F(x,t)\,\mathrm{d}x}{\xi^p},$$

$$B = \limsup_{\xi\to+\infty}\frac{\displaystyle\int_{B\left(x_0,\frac{R}{2}\right)}F(x,\xi)\,\mathrm{d}x}{\xi^p},$$

and

$$\lambda_1 = \frac{1}{B}\frac{1}{p\,C^p K_R}, \qquad \lambda_2 = \frac{1}{A}\frac{1}{p\,C^p}. \tag{3.2}$$

Theorem 3.1. *Let* (H) *be satisfied and assume that*

(h_1) $F(x,t) \ge 0$ *for all* $x \in \mathbb{R}^N, t \ge 0$;

(h_2) $\liminf_{\xi\to+\infty}\frac{\int_{\mathbb{R}^N}\max_{|t|\le\xi}F(x,t)\,\mathrm{d}x}{\xi^p} < K_R\limsup_{\xi\to+\infty}\frac{\int_{B(x_0,\frac{R}{2})}F(x,\xi)\,\mathrm{d}x}{\xi^p}$.

Then, for each $\lambda \in (\lambda_1, \lambda_2)$ *the problem* (P_λ) *admits a sequence of pairwise distinct weak solutions.*

Proof. Our aim is to find a sequence of weak solutions for problem (P_λ), that are critical points of the energy functional $I_\lambda = \Phi - \lambda\Psi$ defined in Section 2, by using part (a) of Theorem 2.2. The functionals Φ and Ψ satisfy the regularity assumptions requested in Theorem 2.2 and in [2, Lemma 2.3] we already proved that I_λ satisfies the $(\mathrm{PS})^{[r]}$-condition for every $r > 0$. So, first we need to verify that $\alpha < +\infty$, where α is given in (2.4). Fix $\lambda \in (\lambda_1, \lambda_2)$ and let $\{s_n\}_{n\in\mathbb{N}} \subseteq \mathbb{R}$ be a sequence such that $\lim_{n\to+\infty} s_n = +\infty$ and

$$A = \lim_{n\to+\infty}\frac{\displaystyle\int_{\mathbb{R}^N}\max_{|t|\le s_n}F(x,t)\,\mathrm{d}t}{s_n^p}.$$

Set $r_n = \frac{s_n^p}{pC^p}$ for all $n \in \mathbb{N}$. From (2.1) it follows that for any $u \in W^{1,p}(\mathbb{R}^N)$ such that $\Phi(u) = \frac{1}{p}\|u\|^p < r_n$, one has

$$\|u\|_\infty \le C\|u\| \le C(pr_n)^{\frac{1}{p}} = s_n \quad \text{for all } n \in \mathbb{N}.$$

Hence, we have that

$$\Phi^{-1}\left((-\infty, r_n)\right) \subset \left\{u \in W^{1,p}(\mathbb{R}^N) : \|u\|_\infty \le s_n\right\} \quad \text{for all } n \in \mathbb{N}.$$

Therefore, by (2.3) it holds that

$$\begin{aligned}
\varphi(r_n) &= \inf_{u\in\Phi^{-1}((-\infty,r_n))} \frac{\left(\sup_{v\in(\Phi^{-1}(-\infty,r_n))}\Psi(v)\right)-\Psi(u)}{r_n-\Phi(u)} \\
&\leq \frac{\left(\sup_{v\in(\Phi^{-1}(-\infty,r_n))}\Psi(v)\right)-\Psi(0)}{r_n-\Phi(0)} \leq \frac{\sup_{\|v\|_\infty\leq s_n}\int_{\mathbb{R}^N}F(x,v)\,\mathrm{d}x}{r_n} \\
&\leq p\,C^p\,\frac{\int_{\mathbb{R}^N}\max_{|t|\leq s_n}F(x,t)\,\mathrm{d}x}{s_n^p}.
\end{aligned}$$

So, we get

$$\alpha = \liminf_{n\to+\infty}\varphi(r_n) \leq p\,C^p\,A.$$

From assumption (h$_2$), that can be read as $A < K_R B$, it follows that $A < +\infty$ and so $\alpha < +\infty$. Consequently, either (a$_1$) or (a$_2$) holds.

In order to get the existence of a sequence of critical points of I_λ, we prove that it does not posses a global minimum, i.e. that it is unbounded from below. Consider the fixed open ball $B(x_0,R)$ of center x_0 and radius $R>0$ (see (3.1)) and let $\{b_n\}_{n\in\mathbb{N}}\subseteq$ be a sequence such that $\lim_{n\to+\infty}b_n=+\infty$ and

$$B = \lim_{n\to+\infty}\frac{\displaystyle\int_{B\left(x_0,\frac{R}{2}\right)}F(x,b_n)\,\mathrm{d}x}{b_n^p}. \tag{3.3}$$

Set $S = B\left(x_0,R\right)\setminus B\left(x_0,\frac{R}{2}\right)$ and for any $n\in\mathbb{N}$ consider the following function

$$w_n(t) = \begin{cases} 0 & \text{if } x\in\mathbb{R}^N\setminus B\left(x_0,R\right), \\ \dfrac{2b_n}{R}(R-|x-x_0|) & \text{if } x\in S, \\ b_n & \text{if } x\in B\left(x_0,\frac{R}{2}\right). \end{cases}$$

Clearly, for every $n\in\mathbb{N}$ it holds that $w_n\in W^{1,p}(\mathbb{R}^N)$ (see Brezis [5, Remark 4(ii), p. 265] or Motreanu–Motreanu–Papageorgiou[8, Proposition 1.10, p. 3]) and

$$\begin{aligned}
\|w_n\|^p &= \int_{\mathbb{R}^N}\left(|\nabla w_n|^p + a(x)|w_n|^p\right)\mathrm{d}x \\
&= \int_S\left(\left|\frac{2b_n}{R}\right|^p + a(x)\left|\frac{2b_n}{R}(R-|x-x_0|^p)\right|\right)\mathrm{d}x + \int_{B\left(x_0,\frac{R}{2}\right)}a(x)b_n^p\mathrm{d}x \\
&\leq \int_S\left(\left|\frac{2b_n}{R}\right|^p + \|a\|_\infty\left|\frac{2b_n}{R}\frac{R}{2}\right|^p\right)\mathrm{d}x + \|a\|_\infty\int_{B\left(x_0,\frac{R}{2}\right)}b_n^p\mathrm{d}x \\
&\leq b_n^p\left[\left(\frac{2^p}{R^p}+\|a\|_\infty\right)\mathrm{meas}(S) + \|a\|_\infty\,\mathrm{meas}\left(B\left(x_0,\frac{R}{2}\right)\right)\right],
\end{aligned}$$

where

$$\begin{aligned}
\mathrm{meas}(S) &= \mathrm{meas}\left(B\left(x_0,R\right)\right) - \mathrm{meas}\left(B\left(x_0,\frac{R}{2}\right)\right) \\
&= \frac{\pi^{\frac{N}{2}}}{\Gamma\left(1+\frac{N}{2}\right)}R^N - \frac{\pi^{\frac{N}{2}}}{\Gamma\left(1+\frac{N}{2}\right)}\left(\frac{R}{2}\right)^N \\
&= \frac{2^N-1}{2^N}\frac{\pi^{\frac{N}{2}}}{\Gamma\left(1+\frac{N}{2}\right)}R^N.
\end{aligned}$$

Hence, we obtain

$$\begin{aligned}\|w_n\|^p &\leq b_n^p \frac{\pi^{\frac{N}{2}}}{\Gamma\left(1+\frac{N}{2}\right)}\left[\left(\frac{2^p}{R^p}+\|a\|_\infty\right)\frac{2^N-1}{2^N}R^N+\|a\|_\infty\frac{R^N}{2^N}\right]\\ &= b_n^p \frac{\pi^{\frac{N}{2}}}{\Gamma\left(1+\frac{N}{2}\right)}\left[\frac{2^p-2^{p-N}+R^p\|a\|_\infty}{R^{p-N}}\right],\end{aligned}$$

which leads to

$$\Phi(w_n)=\frac{1}{p}\|w_n\|^p \leq \frac{b_n^p}{p}\frac{\pi^{\frac{N}{2}}}{\Gamma\left(1+\frac{N}{2}\right)}\left[\frac{2^p-2^{p-N}+R^p\|a\|_\infty}{R^{p-N}}\right]=\frac{b_n^p}{p}\frac{1}{K_R C^p}, \tag{3.4}$$

see (3.1). On the other hand, by (h_1) we know that

$$\begin{aligned}\Psi(w_n) &= \int_{\mathbb{R}^N} F(x,w_n)\mathrm{d}x\\ &= \int_S F\left(x,\frac{2b_n}{R}(R-|x-x_0|)\right)\mathrm{d}x+\int_{B\left(x_0,\frac{R}{2}\right)} F(x,b_n)\mathrm{d}x\\ &\geq \int_{B\left(x_0,\frac{R}{2}\right)} F(x,b_n)\mathrm{d}x.\end{aligned} \tag{3.5}$$

Combining together (3.4) and (3.5), we get

$$I_\lambda(w_n)=\Phi(w_n)-\lambda\Psi(w_n)\leq \frac{b_n^p}{p}\frac{1}{K_R C^p}-\lambda\int_{B\left(x_0,\frac{R}{2}\right)} F(x,b_n)\mathrm{d}x, \tag{3.6}$$

for all $\lambda>0, n\in\mathbb{N}$. Now, taking (3.3) into account we have two possibilities:

1. $\underline{B=+\infty}$
   In this case, fix $M>\frac{1}{\lambda}\frac{1}{p}\frac{1}{K_R C^p}$; from (3.3) there exists $\nu_M>0$ such that

   $$\int_{B\left(x_0,\frac{R}{2}\right)} F(x,b_n)\mathrm{d}x > M b_n^p \quad \text{for all } n>\nu_M.$$

   Then, from (3.6) it holds that

   $$I_\lambda(w_n)\leq b_n^p\left(\frac{1}{p}\frac{1}{K_R C^p}-\lambda M\right) \quad \text{for all } n>\nu_M.$$

   By the choice of M, one has

   $$\lim_{n\to+\infty} I_\lambda(w_n)=-\infty.$$

2. $\underline{B<+\infty}$
   In this case, fix $\varepsilon\in\left(0, B-\frac{1}{\lambda p}\frac{1}{K_R C^p}\right)$; from (3.3) there exists $\nu_\varepsilon>0$ such that

   $$\int_{B\left(x_0,\frac{R}{2}\right)} F(x,b_n)\mathrm{d}x > (B-\varepsilon) b_n^p \quad \text{for all } n>\nu_\varepsilon.$$

   Therefore, from (3.6) one has

   $$I_\lambda(w_n)\leq b_n^p\left(\frac{1}{p}\frac{1}{K_R C^p}-\lambda(B-\varepsilon)\right) \quad \text{for all } n>\nu_\varepsilon.$$

   By the choice of ε, it follows that

   $$\lim_{n\to+\infty} I_\lambda(w_n)=-\infty.$$

We underline that the choices of M and ε are allowed thanks to assumption (h$_2$), which also implies that (λ_1, λ_2) is nonempty and

$$(\lambda_1, \lambda_2) \subseteq \left(0, \frac{1}{\alpha}\right).$$

So, we get that the energy functional I_λ is unbounded from below and part (a$_2$) of Theorem 2.2 ensures that for each $\lambda \in (\lambda_1, \lambda_2)$ the functional $I_\lambda = \Phi - \lambda\Psi$ admits a sequence $\{u_n\}_{n\in\mathbb{N}} \subset W^{1,p}(\mathbb{R}^N)$ of critical points such that $\lim_{n\to+\infty} \|u_n\| = +\infty$, which are weak solutions of the problem (P$_\lambda$). □

Requiring an additional assumption, we can get the existence of nonnegative weak solutions, as the following result ensures.

Theorem 3.2. *Let* (H) *be satisfied. Suppose that $f(x, 0) \geq 0$ for any $x \in \mathbb{R}^N$ and assume that* (h$_1$)–(h$_2$) *hold. Then, for each $\lambda \in (\lambda_1, \lambda_2)$, where λ_1, λ_2 are defined in* (3.2)*, the problem* (P$_\lambda$) *admits a sequence of pairwise distinct nonnegative weak solutions.*

Proof. By applying Theorem 3.1 we get the existence of a sequence of pairwise distinct weak solutions and [2, Lemma 2.2] ensures that the solutions are nonnegative. □

Clearly, being $\Phi(\cdot) = \frac{1}{p}\|\cdot\|^p$, one has that

$$\inf_{W^{1,p}(\mathbb{R}^N)} \Phi = \min_{W^{1,p}(\mathbb{R}^N)} \Phi = \Phi(0) = 0.$$

Hence, exploiting part (b) of Theorem 2.2 and arguing as in the proof of Theorem 3.1, we can state the following alternative result.

Theorem 3.3. *Let* (H) *be satisfied and assume that*

(h$'_1$) $F(x, t) \geq 0$ *for all* $x \in \mathbb{R}^N, t \geq 0$;

(h$'_2$) $\liminf_{\xi\to 0^+} \frac{\int_{\mathbb{R}^N} \max_{|t|\leq\xi} F(x,t)\,\mathrm{d}x}{\xi^p} < K_R \limsup_{\xi\to 0^+} \frac{\int_{B\left(x_0, \frac{R}{2}\right)} F(x,\xi)\,\mathrm{d}x}{\xi^p}$.

Then, for each $\lambda \in (\lambda'_1, \lambda'_2)$, with

$$\lambda'_1 = \frac{1}{p\,C^p K_R} \limsup_{\xi\to 0^+} \frac{\xi^p}{\int_{B\left(x_0, \frac{R}{2}\right)} F(x, \xi)\,\mathrm{d}x}, \qquad \lambda'_2 = \frac{1}{p\,C^p} \liminf_{\xi\to 0^+} \frac{\xi^p}{\int_{\mathbb{R}^N} \max_{|t|\leq\xi} F(x, t)\,\mathrm{d}x},$$

the problem (P$_\lambda$) *admits a sequence of pairwise distinct weak solutions which uniformly converges to zero.*
In addition, if $f(x, 0) \geq 0$ for any $x \in \mathbb{R}^N$, the solutions are nonnegative.

4 Particular cases

This section deals with some consequences of the main results, depending on particular nonlinear terms. Precisely, when the nonlinear term is with separated variables, i.e. of the type $f(x, u) = h(x)g(u)$, the problem (P$_\lambda$) becomes the following one

$$-\Delta_p u + a(x)|u|^{p-2}u = \lambda h(x)g(u) \quad \text{in}\mathbb{R}^N, \qquad u \in W^{1,p}(\mathbb{R}^N), \tag{$\tilde{\mathrm{P}}_\lambda$}$$

where we suppose that

($\tilde{\mathrm{H}}$) $p > N, a \in L^\infty(\mathbb{R}^N)$ such that $a_- = \operatorname{ess\,inf}_{\mathbb{R}^N} a > 0$, $h \in L^1(\mathbb{R}^N)$ such that $h \not\equiv 0, h \geq 0$ and $g \in C(\mathbb{R}), g \geq 0$.

In this case, the energy functional related to problem ($\tilde{\mathrm{P}}_\lambda$) is

$$\tilde{I}_\lambda(u) = \Phi(u) - \lambda\tilde{\Psi}(u) \quad \text{for all } u \in W^{1,p}(\mathbb{R}^N),$$

where Φ is defined in Section 2 and

$$\tilde{\Psi}(u) = \int_{\mathbb{R}^N} h(x)G(u(x))\mathrm{d}x, \quad \text{with} \quad G(t) = \int_0^t g(\xi)\mathrm{d}\xi \quad \text{for all } t \in \mathbb{R}.$$

Clearly, $u \in W^{1,p}(\mathbb{R}^N)$ is a critical point of $\tilde{I}_\lambda$ if and only if is a weak solution for problem ($\tilde{\mathrm{P}}_\lambda$). Set

$$\tilde{K}_R = K_R \int_{B\left(x_0,\frac{R}{2}\right)} h(x)\mathrm{d}x, \tag{4.1}$$

where K_R is given by (3.1), and put

$$\tilde{\lambda}_1 = \frac{1}{p\,C^p\tilde{K}_R}\limsup_{\xi\to+\infty}\frac{\xi^p}{G(\xi)}, \qquad \tilde{\lambda}_2 = \frac{1}{p\,C^p\|h\|_1}\liminf_{\xi\to+\infty}\frac{\xi^p}{G(\xi)}.$$

Theorem 4.1. *Let ($\tilde{H}$) be satisfied and assume that*

($\tilde{h}_2$) $\liminf_{\xi\to+\infty}\frac{G(\xi)}{\xi^p} < \frac{\tilde{K}_R}{\|h\|_1}\limsup_{\xi\to+\infty}\frac{G(\xi)}{\xi^p}$.

Then, for each $\lambda \in \left(\tilde{\lambda}_1, \tilde{\lambda}_2\right)$ the problem ($\tilde{\mathrm{P}}_\lambda$) admits a sequence of pairwise distinct nonnegative weak solutions.

Proof. Arguing as in the proof of Theorem 3.1, taking into account that g is nonnegative, we get that

$$\alpha = \liminf_{n\to+\infty}\varphi(r_n) \leq p\,C^p\,\|h\|_1\lim_{n\to+\infty}\frac{G(s_n)}{s_n},$$

and $\alpha < +\infty$ from hypothesis ($\tilde{h}_2$). Furthermore, one has that

$$\tilde{I}_\lambda(w_n) \leq \frac{b_n^p}{p}\frac{1}{K_R\,C^p} - \lambda G(b_n)\int_{B\left(x_0,\frac{R}{2}\right)} h(x)\mathrm{d}x,$$

for all $\lambda > 0, n \in \mathbb{N}$. So, it yields to $\tilde{I}_\lambda(w_n) \to -\infty$ as $n \to +\infty$ thanks to the following:

1. choose $M > \frac{1}{\lambda}\frac{1}{p}\frac{1}{\tilde{K}_R C^p}$ if $\limsup_{\xi\to+\infty}\frac{G(\xi)}{\xi^p} = +\infty$;
2. fix $\varepsilon \in \left(0, \limsup_{\xi\to+\infty}\frac{G(\xi)}{\xi^p} - \frac{1}{\lambda p}\frac{1}{\tilde{K}_R C^p}\right)$ if $\limsup_{\xi\to+\infty}\frac{G(\xi)}{\xi^p} < +\infty$.

Then, we obtain that for each $\lambda \in \left(\tilde{\lambda}_1, \tilde{\lambda}_2\right)$ the problem ($\tilde{\mathrm{P}}_\lambda$) admits a sequence $\{u_n\}_{n\in\mathbb{N}} \subset W^{1,p}(\mathbb{R}^N)$ of weak solutions such that $\lim_{n\to+\infty}\|u_n\| = +\infty$. Finally, since h and g are non-negative functions, by [2, Lemma 2.2] we know that the weak solutions are nonnegative. □

Clearly, Theorem 1.1 is a special case of Theorem 4.1. In the following, we give a version of Theorem 3.3.

Theorem 4.2. *Let ($\tilde{H}$) be satisfied and assume that*

($\tilde{h}_2'$) $\liminf_{\xi\to 0^+} \frac{G(\xi)}{\xi^p} < \frac{\tilde{K}_R}{\|h\|_1} \limsup_{\xi\to 0^+} \frac{G(\xi)}{\xi^p}$.

Then, for each $\lambda \in \left(\tilde{\lambda}_1', \tilde{\lambda}_2'\right)$, *with*

$$\tilde{\lambda}_1' = \frac{1}{p\,C^p \tilde{K}_R} \limsup_{\xi\to 0^+} \frac{\xi^p}{G(\xi)}, \qquad \tilde{\lambda}_2' = \frac{1}{p\,C^p \|h\|_1} \liminf_{\xi\to 0^+} \frac{\xi^p}{G(\xi)}.$$

the problem ($\tilde{\mathrm{P}}_\lambda$) *admits a sequence of pairwise distinct nonnegative weak solutions which uniformly converges to zero.*

Finally, we provide an example of applicability of our results.

Example 4.3. Let $k > 1$ be an arbitrary constant and consider

$$h(x) = e^{-|x|^2} \quad \text{for all } x \in \mathbb{R}^N,$$

and

$$g(t) = \begin{cases} p(1+t)^{p-1}\left(1 + k - \cos(k\ln(1+t)) + \frac{k}{p}\sin(k\ln(1+t))\right) & \text{if } t \geq 0, \\ pk & \text{if } t < 0. \end{cases}$$

Clearly, $h\colon \mathbb{R}^N \to \mathbb{R}$ is non-null, positive and $h \in L^1(\mathbb{R}^N)$ with $\|h\|_1 = \pi^{\frac{N}{2}}$, while $g\colon \mathbb{R} \to \mathbb{R}$ is continuous and positive. Moreover, one has

$$G(t) = \begin{cases} (1+t)^p\,(1 + k - \cos(k\ln(1+t))) - k & \text{if } t \geq 0, \\ pkt & \text{if } t < 0. \end{cases}$$

By choosing the arbitrary ball $B(0,1)$, see (3.1) and (4.1), we have that

$$\tilde{K}_1 = \frac{N\,\gamma\left(\frac{N}{2}, \frac{1}{4}\right)}{C^p\left(2^{p+1} - 2^{p+1-N} + 2\|a\|_\infty\right)},$$

where $\gamma(s,x) = \int_0^x e^{-t} t^{s-1} \mathrm{d}t$ is the lower incomplete gamma function. Furthermore, it holds that

$$\liminf_{\xi\to+\infty} \frac{G(\xi)}{\xi^p} = 1 - k, \qquad \limsup_{\xi\to+\infty} \frac{G(\xi)}{\xi^p} = 1 + k.$$

Then, it is easy to verify that hypothesis ($\tilde{h}_2$) holds and Theorem 4.1 ensures that for any $\lambda \in \left(\frac{2^{p+1}-2^{p+1-N}+2\|a\|_\infty}{Np(1+k)\gamma(\frac{N}{2},\frac{1}{4})}, \frac{1}{pC^p\pi^{\frac{N}{2}}(1-k)}\right)$ and for any $a \in L^\infty(\mathbb{R}^N)$ with $\operatorname{ess\,inf}_{\mathbb{R}^N} a > 0$, the following equation

$$-\Delta_p u + a(x)|u|^{p-2}u = \lambda h(x) g(u), \quad x \in \mathbb{R}^N,$$

admits a sequence of pairwise distinct nonnegative weak solutions in $W^{1,p}(\mathbb{R}^N)$.

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References

[1] A. Ambrosetti, J. García Azorero, I. Peral, Elliptic variational problems in $\mathbb{R}^N$ with critical growth, *J. Differential Equations* **168**(2000), 10–32. https://doi.org/10.1006/jdeq.2000.3875

[2] E. Amoroso, G. Bonanno, K. Perera, Nonlinear elliptic p-Laplacian equations in the whole space, *Nonlinear Anal.* **23**(2023), 113364. https://doi.org/10.1016/j.na.2023.113364

[3] G. Barletta, P. Candito, S.A. Marano, K. Perera, Multiplicity results for critical p-Laplacian problems, *Ann. Mat. Pura Appl.* **196**(2017), 1431–1440. https://doi.org/10.1007/s10231-016-0622-3

[4] G. Bonanno, A critical point theorem via the Ekeland variational principle, *Nonlinear Anal.* **75**(2012), 2992–3007. https://doi.org/10.1016/j.na.2011.12.003

[5] H. Brezis, *Functional analysis, Sobolev spaces and partial differential equations*, Universitext, Springer, New York, 2011. https://doi.org/10.1007/978-0-387-70914-7

[6] P. Drábek, Y. X. Huang, Multiplicity of positive solutions for some quasilinear elliptic equation in $\mathbb{R}^N$ with critical Sobolev exponent, *Demonstr. Math.* **55**(2022), 416–428. https://doi.org/10.1006/jdeq.1997.3306

[7] U. Guarnotta, R. Livrea, S. A. Marano, Some recent results on singular p-Laplacian equations, *Demonstr. Math.* **55**(2022), 416–428. https://doi.org/10.1515/dema-2022-0031

[8] D. Motreanu, V. V. Motreanu, N. S. Papageorgiou, *Topological and variational methods with applications to nonlinear boundary value problems*, Springer, New York, 2014. https://doi.org/10.1007/978-1-4614-9323-5