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Existence and nonexistence of solutions for unsteady shrinking disk flow

Joseph E. Paullet ✉

School of Science, Penn State Behrend, Erie, Pennsylvania 16563, USA

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Abstract. We analyze a mathematical model recently proposed by Mehmood *et al.* [*J. Appl. Mech. Tech. Phys.* **63**(2022), 782–789] of unsteady fluid flow over a shrinking porous disk. Mehmood *et al.* conjectured that solutions to the model exist only for sufficiently strong suction of fluid through the disk surface. Here we prove that solutions in fact exist for both suction and injection. However, with injection, the solutions are very difficult to determine numerically. In this case, we derive precise bounds on the skin friction coefficient as a function of injection velocity. These bounds are then used to pinpoint the solutions computationally.

Keywords: existence, nonexistence, boundary-layer flow, nonlinear boundary value problem.

2020 Mathematics Subject Classification: 34B15, 76D10.

1 Introduction

Fluid flow induced by a stretching surface is a classic problem in mechanics, first considered by Crane [3] in 1970 for a two dimensional case in which a horizontal boundary stretches away from a fixed point with a velocity proportional to the distance from the fixed point. Since that time, the analysis has been expanded to include mathematical models of various other geometries and configurations (See e.g. [2,5–7,13–16]). For the case of *stretching* surfaces, solutions generally turn out to be unique. (For Crane's [3] original model, see for example [9] or [12].)

In the case of *shrinking* surfaces, the solution sets turn out to be much more varied, and can include parameter regimes with two [1], three (or more) [10], or even a whole continuum of solutions [11]. There can also be ranges of parameters where no solutions exist [10,11].

Recently, Mehmood *et al.* [8] proposed a model for axisymmetric two-dimensional unsteady flow induced by a shrinking porous disk, with accelerated and decelerated flow situations. Using a dimensionless similarity variable transformation, they reduce the governing partial differential equations to an ordinary differential equation boundary value problem.

✉ Corresponding author. Email: jep7@psu.edu

Using numerical integration, Mehmood *et al.* [8] analyze the model and make several conjectures regarding the nature of the solutions. The purpose of this note is to study the problem analytically and compare the results to the computational results of [8].

The model involves two physical parameters governing injection/suction of fluid through the disk surface and acceleration/deceleration of the radial coordinate of the disk as it shrinks. Based on numerical integration of the differential equation, Mehmood *et al.* [8] find solutions to the problem only in the case of sufficiently strong suction. These solutions are found for both accelerating and decelerating disks. However, we will prove that, at least in the case of an *accelerating* disk, solutions exist for both suction *and* injection. But it should be noted that, in the case of injection, these solutions are very difficult to pinpoint numerically. Thus we can readily understand how Mehmood *et al.* [8] came to the conclusion that they might not exist.

The numerical search for solutions to many fluid flow problems can be aided by a-priori knowledge about the skin friction coefficient, and for an *accelerated* disk we are able to derive precise bounds on this coefficient. These bounds become increasingly more accurate as the fluid injection velocity is increased, and can be used to guide numerical calculations.

Existence or nonexistence of a solution for a *decelerating* disk is much less clear cut. In this case we prove that for a large portion of parameter space, any solution must contain a region of flow reversal in the radial direction. In other regions of parameter space it does appear that solutions without flow reversal exist. Mehmood *et al.* [8], in fact, find dual solutions for a decelerating disk. However, we conjecture that there are actually infinitely many solutions for a decelerating disk with suction.

The plan of the paper is as follows. In Section 2, we state the problem and consider the case of an accelerated disk. Section 3 explores a decelerated disk. In Section 4, we use the results of the previous sections to help pinpoint the solutions computationally. Even with the precise bounds on the skin friction coefficient derived in Section 2, we still find it difficult to track the solutions using standard numerical integration routines. We conclude by discussing these numerical issues and state some open problems regarding the model.

2 Accelerated disk

The model proposed by Mehmood *et al.* [8] is the following: Find $f(\eta)$ satisfying

$$f''' - \left(2f + \frac{\beta\eta}{2}\right) f'' + f'(f' - \beta) = 0, \quad 0 < \eta < \infty, \tag{2.1}$$

subject to

$$f(0) = S, \quad f'(0) = 1, \quad f'(\infty) = 0, \tag{2.2}$$

where $\beta > 0$ denotes an accelerating disk, $\beta < 0$ denotes a decelerating disk, $S > 0$ corresponds to fluid injection through the disk surface, and $S < 0$ corresponds to fluid suction. The radial and axial fluid velocities are proportional to $f'(\eta)$ and $f(\eta)$, respectively, where η is the dimensionless similarity variable.

To explore the existence or nonexistence of solutions to the boundary value problem (BVP) (2.1–2.2) in β-S parameter space, we will find it useful to consider a family of related initial value problems (IVP), namely, equations (2.1), $(2.2)_1$, and $(2.2)_2$ along with a third initial condition:

$$f''(0) = \alpha. \tag{2.3}$$

This IVP will have a unique local solution for any value of α. The question then becomes: Can α be chosen so that the solution of the IVP exists for all $\eta > 0$ *and* satisfies $(2.2)_3$, giving a solution to the BVP? Our first result along these lines is a topological shooting argument to prove existence for $\beta > 1$.

Theorem 2.1. *If $\beta > 1$ and $S \in \mathbb{R}$, then a solution to the BVP exists. Further, this solution satisfies $f' > 0$, $f'' < 0$ $f''' > 0$, $f^{(4)} < 0$, and $f^{(5)} > 0$ for all $\eta > 0$.*

Proof. For the family of initial value problems (2.1), $(2.2)_1$, $(2.2)_2$ and (2.3) defined above, consider the two sets:

$$\mathcal{A} = \{\alpha < 0 : f'' = 0 \text{ strictly before } f' = 0\}, \tag{2.4}$$

$$\mathcal{B} = \{\alpha < 0 : f' = 0 \text{ strictly before } f'' = 0\}. \tag{2.5}$$

By construction, the two sets are disjoint. We next show that they are nonempty and open as well. For $\mathcal{A}$ this follows directly from continuity of solutions of the IVP in its initial conditions. First note that when $f''(0) = \alpha = 0$, we have $f'''(0) = \beta - 1 > 0$. Thus $f'(\eta; \alpha = 0) > 1$ for $\eta \in (0, \varepsilon]$ for some $\varepsilon > 0$. By continuity of the solutions of the IVP in its initial conditions, for small $\alpha < 0$, we can arrange that $f'(\varepsilon; \alpha) > 1$. But since $f''(0) = \alpha < 0$, $f'(\eta)$ initially decreases below 1. Thus, f' must attain a minimum where $f'' = 0$. Also, for small $\alpha < 0$ we can ensure that $f' > 0$ on $[0, \varepsilon]$. Thus $f'' = 0$ strictly before $f' = 0$ for small $\alpha < 0$. Thus $\mathcal{A}$ is nonempty. Further note that we cannot have $f'' = f''' = 0$ simultaneously, since the ODE (2.1) then implies that $f' \equiv 0$ or $f' \equiv \beta$, which cannot be the case since $f''(0) = \alpha < 0$. Thus, when $f'' = 0$, $f''' \neq 0$ and so $\mathcal{A}$ is open.

To prove that $\mathcal{B}$ is nonempty and open we consider large $\alpha < 0$. We claim that for $\alpha < 0$ and $|\alpha|$ sufficiently large, we must have $f' = 0$ before $f'' = 0$ in the interval $[0, 1]$, say. Suppose for contradiction that the assertion is false, then one of following three possibilities must occur: (1) there exists a first $\eta_0 \in (0, 1]$ such that $f''(\eta_0) = 0$ with $0 < f' < 1$ on $(0, \eta_0]$ and $f'' < 0$ on $(0, \eta_0)$, (2) $0 < f' < 1$ and $f'' < 0$ on all of $(0, 1]$, or (3) $f' = f'' = 0$ simultaneously.

First consider possibility (1). Since $0 < f' < 1$ on $(0, \eta_0] \subset (0, 1]$, on integration and using $(2.2)_1$ we have $S < f < S + 1$ on $(0, \eta_0]$. Next, an integration of (2.1) from 0 to $\eta > 0$ results in

$$f'' = \alpha + \left(2f + \frac{\beta\eta}{2}\right) f' - 2S + \frac{\beta}{2}(f - S) - 3\int_0^\eta f'(t)^2\, dt. \tag{2.6}$$

Using the bounds on f and f' on $(0, \eta_0]$ in (2.6) we have

$$f'' < \alpha + \beta + 2, \quad \forall \eta \in (0, \eta_0]. \tag{2.7}$$

Choosing $\alpha < -(\beta + 2)$ in (2.7) implies that $f''(\eta_0) < 0$, contradicting the assumption that $f''(\eta_0) = 0$.

Next, consider possibility (2). In this case the bounds on f, f', and f'' from the previous paragraph hold on the entire interval $(0, 1]$, and choosing $\alpha < -(\beta + 3)$ implies that $f'' < -1$ on $(0, 1]$. On integration we then have that $f'(1) < 0$, contradicting the assumption that $f' > 0$ on $(0, 1]$. Thus possibility (2) cannot occur.

Finally, as we have seen previously, f' and f'' cannot vanish simultaneously, which precludes possibility (3). Thus if $\alpha < -(\beta + 3)$, then $\alpha \in \mathcal{B}$ and $\mathcal{B}$ is nonempty. And since from (2.1) we must have $f'' \neq 0$ when $f' = 0$, we see that $\mathcal{B}$ is open.

Since the interval $(-\infty, 0)$ is connected, it cannot be expressed as the union of $\mathcal{A}$ and $\mathcal{B}$. Thus there exists an α^* with $\alpha^* \notin \mathcal{A}$ and $\alpha^* \notin \mathcal{B}$. Thus for $f(\eta; \alpha^*)$ we cannot have $f'' = 0$

before $f' = 0$ nor can we have $f' = 0$ before $f'' = 0$. Also, as we have seen, $f' = f'' = 0$ simultaneously leads to a contradiction.

Thus we must have $0 < f' < 1$ and $f'' < 0$ for as long as $f(\eta;\alpha^*)$ exists. Let $[0,\overline{\eta})$ denote the maximal interval of existence of $f(\eta;\alpha^*)$. We wish to show that $\overline{\eta} = \infty$. Since $0 < f' < 1$ we have that f' is bounded on $[0,\overline{\eta})$, and on integration we conclude that f is also bounded on $[0,\overline{\eta})$. From (2.6) we then have f'' is bounded on any finite interval as well. Thus the solution can be continued past any finite $\overline{\eta}$ so that $\overline{\eta} = \infty$.

Since $f' > 0$ and $f'' < 0$ for all $\eta > 0$ we have that $f'(\infty) = L$ exists with $0 \le L < 1$. We wish to show that $L = 0$, giving a solution to the BVP. If $L > 0$, then $0 < L < f' < 1$ for all $\eta > 0$ and using (2.1) we then have

$$f''' - \left(2f + \frac{\beta\eta}{2}\right) f'' = f'(\beta - f') > L(\beta - 1) = K > 0, \quad \forall \eta > 0. \tag{2.8}$$

Next note that when $f'' < 0$, f''' cannot increase through zero, since at such a point we would need $f^{(4)} \ge 0$, but a differentiation of (2.1) implies that $f^{(4)} = 3\beta f''/2 < 0$ at a root of f''', giving a contradiction. Thus if f''' were ever to equal zero, it would thereafter remain negative, which would then eventually force f' negative, contradicting the fact that $f' > 0$ for all η. Therefore we need $f''' > 0$ for all $\eta \ge 0$.

Using a similar argument and further differentiations of (2.1) it can be shown that $f^{(4)} < 0$ and $f^{(5)} > 0$ for all $\eta \ge 0$. Therefore, f'' and f''' are both monotonic and bounded, so that their limits as $\eta \to \infty$ exist. However, since $f'(\infty) = L$ exists, we must have $f''(\infty) = f'''(\infty) = 0$.

Next, note that $2L + \beta/2 < 2f' + \beta/2 < 2 + \beta/2$ for all $\eta > 0$ and so on integration from 0 to $\eta > 0$ we have

$$2S + \left(\frac{\beta + 4L}{2}\right)\eta < 2f + \frac{\beta\eta}{2} < 2S + \left(\frac{\beta+4}{2}\right)\eta, \quad \forall \eta > 0. \tag{2.9}$$

Further, note that $2S + (\beta + 4L)\eta/2 > 0$ for η sufficiently large, and so by (2.9), $2f + \beta\eta/2 > 0$ for η sufficiently large.

Using these facts in (2.8), we can find $\hat{\eta}$ sufficiently large so that

$$f'' < \frac{-K}{2\left(2S + \frac{\beta+4}{2}\eta\right)}, \quad \forall \eta \ge \hat{\eta}. \tag{2.10}$$

Integrating (2.10) from $\hat{\eta}$ to $\eta > \hat{\eta}$ we have

$$f' < f'(\hat{\eta}) - \frac{K}{\beta+4}\left[\ln\left(2S + \frac{\beta+4}{2}\eta\right) - \ln\left(2S + \frac{\beta+4}{2}\hat{\eta}\right)\right], \quad \forall \eta \ge \hat{\eta}, \tag{2.11}$$

which tends to $-\infty$ as $\eta \to \infty$, contradicting $f' \to L > 0$. Thus we must have $f'(\infty;\alpha^*) = L = 0$, giving a solution to the BVP and proving the theorem. □

Physically, this theorem states that for a sufficiently accelerated disk ($\beta > 1$) solutions to the problem exist for both injection ($S > 0$) and suction ($S < 0$) of fluid through the disk surface, as well as for an impermeable disk ($S = 0$).

Based on numerical integration of the problem, Mehmood *et al.* conjectured that for any value of $\beta \in \mathbb{R}$, no solution exists when $S > 0$. Theorem 2.1 above shows that, at least for $\beta > 1$, this conjecture is incorrect. However, when $\beta > 1$ and $S > 0$, the solutions are very difficult to pinpoint numerically, so that a conjecture of their nonexistence is understandable.

Theorem 2.1 guarantees at least one value of α which gives a solution to the BVP, but gives little indication of its value other than $-(\beta+3) < \alpha < 0$. In the next theorem we obtain much more precise bounds on α. This, in turn, gives bounds on the skin friction coefficient, which is proportional to α.

Theorem 2.2. *If $\beta > 1$ and $S > 0$, then the solution to the BVP given in Theorem 2.1 must satisfy $f''(0) = \alpha$ where*

$$\frac{2(1-\beta)(2S^2+\beta+1)}{8S^3+7S\beta+4S} < \alpha < \frac{2S(1-\beta)}{4S^2+3\beta/2}. \tag{2.12}$$

Proof. The bounds on α will involve the initial values of several derivatives of f. Using (2.1) and its derivatives we have

$$f'''(0) = 2S\alpha + \beta - 1, \tag{2.13}$$

$$f^{(4)}(0) = \left(4S^2 + \frac{3\beta}{2}\right)\alpha + 2S(\beta-1), \tag{2.14}$$

and

$$f^{(5)}(0) = (8S^3 + 7S\beta + 4S)\alpha + 2(\beta-1)(2S^2+\beta+1). \tag{2.15}$$

Recall that the solution of Theorem 2.1 satisfies $0 < f' < 1$ and $f'' < 0$ for all $\eta > 0$. In the proof of Theorem 2.1 it was further shown that $f''' > 0$, $f^{(4)} < 0$, and $f^{(5)} > 0$ for all $\eta \geq 0$. In particular, $f^{(4)}(0) < 0$, and $f^{(5)}(0) > 0$. Using these inequalities in (2.14) and (2.15) results in (2.12). □

The coefficient of wall skin friction, C_f, of the fluid is given as [8]:

$$C_f = -Re_x^{-1/2} f''(0) = -Re_x^{-1/2}\alpha, \tag{2.16}$$

where Re_x is the Reynolds number. Thus the results of Theorem 2.2 give ever more precise bounds on the skin friction coefficient as the injection rate ($S > 0$) increases.

Remark 2.3. The results of this section do not touch on the question of uniqueness of a solution, but for the $\beta > 1$ and $S > 0$ case, we can conclude that for *any* solution, α must satisfy the bounds given in Theorem 2.2. To see this, first note that when $\beta > 0$, f' cannot attain a minimum at or below zero. Thus any solution must satisfy $f' > 0$ for all $\eta \geq 0$. Next, if $\alpha \geq 0$ and $\beta > 1$, then f' is initially increasing. In order to satisfy the boundary condition at infinity, f' would have to first attain a maximum where $f''' < 0$. However, as in the proof of Theorem 2.2, f''' could not thereafter increase through zero, and so $f'(\infty) = 0$ could not be achieved.

So far we have been concerned with existence of a solution and its properties, however there are parameter values for which no solution exists. Specifically:

Theorem 2.4. *For $0 < \beta \leq 1$ and $S \geq 0$, the BVP has no solution.*

Proof. Recall that for $\beta > 0$, f' cannot have a minimum at or below 0, thus any solution must satisfy $f' > 0$ for all $\eta \geq 0$. Also, when $\beta > 0$, f''' cannot increase through zero when $f'' < 0$, so that $f''' > 0$ for all $\eta \geq 0$. However $f'''(0) = 2S\alpha + \beta - 1 > 0$ implies that $\alpha > (1-\beta)/2S \geq 0$. This then implies that $\alpha > 0$ and arguing as in the Remark 2.3, f' could not then satisfy the boundary condition at infinity. □

The results of Theorems 2.1 and 2.4 imply that in the case of fluid injection ($S > 0$), solutions to the problem only exist for a sufficiently accelerating disk ($\beta > 1$) whereas no solution exists for insufficient acceleration ($0 < \beta \leq 1$).

3 Decelerated disk

The analysis regarding a decelerating disk is much less comprehensive than the accelerating case. For $\beta < 0$ and $S > 0$, we conjecture that no solution exists, but have only been able to prove this for a relatively small portion of parameter space (see Theorem 3.2 below). However, in the next theorem we show that for a decelerating disk ($\beta < 0$) and sufficient fluid injection ($S > 0$ sufficiently large) there is no solution to the BVP with the property $f' > 0$ for all $\eta \geq 0$. Thus, if a solution does exist in this case, f' must at some point become negative. Physically, this implies a region of flow reversal in the radial direction. This implication is discussed further in Section 4.

Theorem 3.1. *If $-4 \leq \beta < 0$ and $S > 0$, then the BVP has no solution that satisfies $f' > 0$ for all $\eta \geq 0$. Further, if $\beta < -4$ and*

$$S \geq \frac{1}{8}\left[25\beta^4 + \left(\frac{28\beta^3}{3} - 128\beta + \frac{256}{3}\right)(\beta+4)\right]^{\frac{1}{4}} \equiv p(\beta), \tag{3.1}$$

then the BVP has no solution that satisfies $f' > 0$ for all $\eta \geq 0$.

Proof. We will assume the existence of a solution satisfying $f' > 0$ for all $\eta \geq 0$ and derive a contradiction. We first list some properties that such a solution must possess. When $\beta < 0$, from (2.1) we see that f' cannot have a positive minimum. Also note that when $\beta < 0$, f''' cannot decrease through zero when $f'' < 0$. Thus as f' approaches zero from above, we must have $f'' < 0$ and $f''' > 0$ for η large. Thus $f''(\infty) = 0$. Finally, note that the quantity $u = 2f + \beta\eta/2$ is negative for large η since $\beta < 0$ and $f > 0$ grows sublinearly.

If some value of $f''(0) = \alpha \leq 0$ gives a solution to the BVP, then using (2.6) we have

$$f'' = \alpha + \left(2f + \frac{\beta\eta}{2}\right)f' - 2S + \frac{\beta}{2}(f - S) - 3\int_0^\eta f'(t)^2\,dt. \tag{3.2}$$

As η tends to infinity, the left side of (3.2) tends to zero while the right side is strictly negative, giving a contradiction.

If some value of $f''(0) = \alpha > 0$ gives a solution to the BVP, then f' must attain a positive maximum, say at η_2, where $f''(\eta_2) = 0$. Integrating the ODE (2.1) from η_2 to $\eta > \eta_2$ we have

$$f'' + \left(2f(\eta_2) + \frac{\beta\eta_2}{2}\right)f'(\eta_2) = \left(2f + \frac{\beta\eta}{2}\right)f' + \frac{\beta}{2}(f - f(\eta_2)) - 3\int_{\eta_2}^\eta f'(t)^2\,dt. \tag{3.3}$$

As $\eta \to \infty$ and $f'' \to 0$, the right side of (3.3) is strictly negative, implying that

$$2f(\eta_2) + \frac{\beta\eta_2}{2} < 0. \tag{3.4}$$

But the quantity

$$u(\eta) = 2f(\eta) + \frac{\beta\eta}{2}, \tag{3.5}$$

is initially positive since $u(0) = 2S > 0$. Thus a root of $u = 2f + \beta\eta/2$ must occur as f' increases towards its maximum. However, $f' > f'(0) = 1$ as it increases to its max. But if $-4 \leq \beta < 0$, then

$$u'(\eta) = 2f'(\eta) + \frac{\beta}{2} \geq 0, \quad \forall \eta \in [0, \eta_2]. \tag{3.6}$$

Thus $u = 2f + \beta\eta/2$ cannot decrease through zero, giving a contradiction. Thus if $-4 \leq \beta < 0$ and $S > 0$, the BVP cannot have a solution that satisfies $f' > 0$ for all $\eta > 0$.

If $\beta < -4$, we need additional conditions on S in order to obtain our final contradiction regarding the root of $u = 2f + \beta\eta/2$. Recall that $f''(\eta_2) = 0$ at the maximum of f' and let $\eta_1 < \eta_2$ be the point where $u(\eta_1) = 2f(\eta_1) + \beta\eta_1/2 = 0$ first decreases through zero.

Integrating the ODE (2.1) from zero to η_1 we have

$$\alpha - 2S + \frac{\beta}{2}(f(\eta_1) - S) - 3\int_0^{\eta_1} f'(t)^2\,dt = f''(\eta_1) > 0 \tag{3.7}$$

from which we conclude that

$$\alpha > 2S > 0, \tag{3.8}$$

since $\beta(f(\eta_1) - S)/2$ and $-3\int_0^{\eta_1} f'^2\,dt$ are both strictly negative.

Next, note from (3.8) and the assumption $S \geq p(\beta) > \sqrt{1-\beta}/2$ we have that

$$f'''(0) = 2S\alpha + \beta - 1 > 0, \tag{3.9}$$

so that f'' initially increases above its starting value of α. As f' increases towards its max, f''' must at some first point decrease through zero (and cannot subsequently increase through zero for $\eta \leq \eta_2$). Thus, there must exist a point η_0 such that

$$f''(\eta_0) = \alpha \quad \text{with} \quad f''(\eta) > \alpha \quad \forall\, \eta \in (0, \eta_0) \quad \text{and} \quad f''(\eta) < \alpha \quad \forall\, \eta \in (\eta_0, \eta_2]. \tag{3.10}$$

Note that $\eta_0 < \eta_1$. To see this, note that (2.6) evaluated at η_0 gives

$$\left(2f(\eta_0) - \frac{\beta\eta_0}{2}\right) f'(\eta_0) = 2S - \frac{\beta}{2}(f(\eta_0) - S) + 3\int_0^{\eta_0} f'(t)^2\,dt > 0, \tag{3.11}$$

which implies that $u(\eta_0) = 2f(\eta_0) + \beta\eta_0 > 0$. But $u = 2f + \beta\eta/2$ first decreases through zero at η_1. If $\eta_0 > \eta_1$, then there would have to exist a intermediate point, $\eta^* \in (\eta_1, \eta_0)$, where $u = 2f + \beta\eta/2$ increases back through zero with $f''(\eta^*) > \alpha$ by (3.10). Evaluating (2.6) at such a point gives

$$f''(\eta^*) - \alpha = -2S + \frac{\beta}{2}(f(\eta^*) - S) - 3\int_0^{\eta^*} f'(t)^2\,dt < 0, \tag{3.12}$$

contradicting $f''(\eta^*) > \alpha$. Thus $\eta_0 < \eta_1$.

Next note that when $u(\eta_1) = 2f(\eta_1) + \beta\eta_1/2 = 0$, we need $u'(\eta_1) = 2f'(\eta_1) + \beta/2 \leq 0$, thus $f'(\eta_1) \leq -\beta/4$. Since f' is increasing on $(0, \eta_1)$ we then have that

$$1 < f'(\eta) \leq -\frac{\beta}{4}, \quad \forall\, \eta \in (0, \eta_1). \tag{3.13}$$

Subsequently,

$$\frac{\beta + 4}{2} \leq 2f' + \frac{\beta}{2}, \quad \forall\, \eta \in (0, \eta_1), \tag{3.14}$$

and on integration from zero to $\eta > 0$ we have

$$2S + \frac{\beta + 4}{2}\eta < u(\eta) = 2f(\eta) + \frac{\beta\eta}{2}, \quad \forall\, \eta \in (0, \eta_1). \tag{3.15}$$

Further, note that when $f'' = \alpha$ at $\eta_0 < \eta_1$ we have from (2.1) that

$$\left(2f(\eta_0) + \frac{\beta\eta_0}{2}\right) f''(\eta_0) = f'(\eta_0)(f'(\eta_0) - \beta) + f'''(\eta_0) < f'(\eta_0)(f'(\eta_0) - \beta), \tag{3.16}$$

since $f'''(\eta_0) < 0$. Next, using (3.8) and (3.13) in (3.16) we obtain

$$2f(\eta_0) + \frac{\beta\eta_0}{2} < \frac{5\beta^2}{32S}. \tag{3.17}$$

Combining (3.15) and (3.17) we conclude that

$$2S + \frac{\beta+4}{2}\eta_0 < 2f(\eta_0) + \frac{\beta\eta_0}{2} < \frac{5\beta^2}{32S}, \tag{3.18}$$

or (using $S \geq p(\beta) > \sqrt{5}|\beta|/8$)

$$\eta_0 > \frac{5\beta^2 - 64S^2}{16S(\beta+4)} > 0. \tag{3.19}$$

Next, multiply the ODE (2.1) by f'' and integrate from zero to η_1 to obtain

$$\int_0^{\eta_0} \left(2f(t) + \frac{\beta t}{2}\right) f''(t)^2\, dt = \frac{1}{3}(f'(\eta_0)^3 - 1) - \frac{\beta}{2}(f'(\eta_0)^2 - 1). \tag{3.20}$$

Using (3.13) to bound the right hand side of (3.20) we have

$$\int_0^{\eta_0} \left(2f(t) + \frac{\beta t}{2}\right) f''(t)^2\, dt < \frac{-7\beta^3}{192} - \frac{1}{3} + \frac{\beta}{2}. \tag{3.21}$$

However, using (3.8), (3.15) and (3.19) we also have

$$\frac{25\beta^4 - 4096S^4}{256(\beta+4)} < \int_0^{\frac{5\beta^2-64S^2}{16S(\beta+4)}} \left(2S + \frac{\beta+4}{2}t\right) 4S^2\, dt < \int_0^{\eta_0} \left(2f(t) + \frac{\beta t}{2}\right) f''(t)^2\, dt. \tag{3.22}$$

Combing (3.21) and (3.22) gives

$$\frac{25\beta^4 - 4096S^4}{256(\beta+4)} < \frac{-7\beta^3}{192} - \frac{1}{3} + \frac{\beta}{2} \tag{3.23}$$

which is contradicted if

$$S \geq \frac{1}{8}\left[25\beta^4 + \left(\frac{28\beta^3}{3} - 128\beta + \frac{256}{3}\right)(\beta+4)\right]^{\frac{1}{4}}. \tag{3.24}$$

Thus if $\beta < -4$ and (3.24) holds, then the BVP has no solution with the property $f' > 0$ for all $\eta > 0$. □

Physically, the results of this Theorem 3.1 imply that for a decelerating disk ($\beta < 0$) and sufficiently strong fluid injection ($S > 0$ if $-4 \leq \beta < 0$ and $S > p(\beta)$ if $\beta < -4$), any solution to the BVP must contain a region where $f'(\eta)$ becomes negative. As $f'(\eta)$ is proportional to the fluid velocity in the radial direction, this implies a region of flow reversal in the boundary layer, and such solutions are unlikely to be stable in the full governing PDE and thus are not physically realizable. Thus this theorem gives strong evidence that no solution exists for the parameter values given in the theorem statement. In fact, we can eliminate the possibility of these non-physical solutions mathematically, but only for a small range of parameter space (see Theorem 3.2 below).

Theorem 3.2. *If $-4 \leq \beta < 0$ and $S \geq 3\beta^2/2 - 5\beta/8$, then the BVP has no solution.*

Proof. We will assume a solution exists and derive a contradiction. By Theorem 3.1, for $-4 \leq \beta < 0$, if a solution exists, then f' at some point must become negative. Let η_A be the first point at which f' decreases through zero, $f'(\eta_A) = 0$. Let η_B be the first point, if ever, that the quantity $u = 2f + \beta\eta/2$ decreases through zero, $u(\eta_B) = 2f(\eta_B) + \beta\eta_B/2 = 0$. Let

$$\eta_1 = \min\{\eta_A, \eta_B\}. \tag{3.25}$$

Note that for $\beta < 0$, f' cannot have a minimum at or below $f' = \beta$. Also note that $u' = 2f' + \beta/2 \geq 0$ for $f' \geq -\beta/4$. Thus if a root of $u = 2f + \beta\eta/2$ occurs, it must occur with f' in the range $\beta < f' < -\beta/4$.

With these preliminaries out of the way, we will show that for $-4 \leq \beta < 0$ and $S > 0$ sufficiently large, f' must fall below $f' = \beta$ at some point in the interval $[\eta_1, \eta_1 + 1]$, and thus cannot be a solution to the BVP.

To see this, first consider a value $f''(0) = \alpha \leq 0$ that gives a solution to the BVP. An integration of the ODE (2.1) evaluated at η_1 gives

$$f''(\eta_1) = \alpha - 2S + \frac{\beta}{2}(f(\eta_1) - S) - 3\int_0^{\eta_1} f'(t)^2\, dt < -2S, \tag{3.26}$$

since each term of the right hand side of (3.26) is nonpositive.

In order to satisfy the boundary condition $f'(\infty) = 0$, f' must attain a first negative minimum at some point $\eta_2 > \eta_1$ with

$$\beta < f', \quad \forall \eta \in [\eta_1, \eta_2]. \tag{3.27}$$

Integrating (3.27) from η_1 to $\eta > \eta_1$ and multiplying by $\beta/2 < 0$ we have

$$\frac{\beta}{2}(f - f(\eta_1)) < \frac{\beta^2}{2}(\eta - \eta_1), \quad \forall \eta \in [\eta_1, \eta_2]. \tag{3.28}$$

Again using (3.27), we also have

$$\frac{5\beta}{2} < u' = 2f' + \frac{\beta}{2}, \quad \forall \eta \in [\eta_1, \eta_2]. \tag{3.29}$$

Integrating (3.29) from η_1 to $\eta > \eta_1$ we have

$$\frac{5\beta}{2}(\eta - \eta_1) < 2f(\eta_1) + \frac{\beta\eta_1}{2} + \frac{5\beta}{2}(\eta - \eta_1) < 2f + \frac{\beta\eta}{2}, \quad \forall \eta \in [\eta_1, \eta_2]. \tag{3.30}$$

Taking the worst case scenario where $2f + \beta\eta/2 < 0$ and $f' < 0$ on $(\eta_1, \eta_2]$ in (3.30) we derive the bound

$$\left(2f + \frac{\beta\eta}{2}\right) f' < \frac{5\beta^2}{2}(\eta - \eta_1), \quad \forall \eta \in (\eta_1, \eta_2]. \tag{3.31}$$

Next, integrating the ODE (2.1) from η_1 to $\eta > \eta_1$ we have

$$f'' = f''(\eta_1) + \left(2f + \frac{\beta\eta}{2}\right) f' + \frac{\beta}{2}(f - f'(\eta_1)) - 3\int_{\eta_1}^{\eta} f'(t)^2\, dt. \tag{3.32}$$

Using (3.26), (3.28), and (3.31) in (3.32) we then have

$$f'' < -2S + \frac{5\beta^2}{2}(\eta - \eta_1) + \frac{\beta^2}{2}(\eta - \eta_1), \quad \forall \eta \in (\eta_1, \eta_2]. \tag{3.33}$$

Next, consider the line through the points $(\eta_1, -\beta/4)$ and (η_1+1, β), which has slope $m = 5\beta/4$. Choosing $S > 0$ sufficiently large in (3.33) we can ensure that $f'' < 5\beta/4$ on the interval $(\eta_1, \eta_1+1]$, thus forcing the minimum of f' to occur at $\eta_2 > \eta_1 + 1$. However, this then implies that

$$f'' < -2S + 3\beta^2 \leq \frac{5\beta}{4}, \quad \forall \eta \in (\eta_1, \eta_1+1], \tag{3.34}$$

which, on integration from η_1 to $\eta > \eta_1$, then implies that

$$f' < f'(\eta_1) + \frac{5\beta}{4}(\eta - \eta_1) \leq \frac{5\beta}{4}(\eta - \eta_1) - \frac{\beta}{4}, \quad \forall \eta \in (\eta_1, \eta_1+1], \tag{3.35}$$

since $f'(\eta_1) \leq -\beta/4$. Evaluating at $\eta_1 + 1$ we have

$$f'(\eta_1 + 1) < \beta, \tag{3.36}$$

at which point f' can no longer attain a minimum and therefore cannot be a solution to the BVP.

Inequality (3.34) is satisfied if

$$S \geq \frac{3\beta^2}{2} - \frac{5\beta}{8}. \tag{3.37}$$

Thus if $-4 \leq \beta < 0$ and (3.37) is satisfied, then no value of $f''(0) = \alpha \leq 0$ gives a solution to the BVP.

Next consider the case that a value $f''(0) = \alpha > 0$ gives a solution to the BVP. Then f' is initially increasing and must at some first point, η_0, attain a maximum above $f'(0) = 1$ where $f''(\eta_0) = 0$. Again let η_A be the first subsequent point where f' vanishes, $f'(\eta_A) = 0$. Integrating the ODE (2.1) from η_0 to η_A we have

$$f''(\eta_A) = -\left(2f(\eta_0) + \frac{\beta\eta_0}{2}\right) f'(\eta_0) + \frac{\beta}{2}(f(\eta_A) - f(\eta_0)) - 3\int_{\eta_0}^{\eta_A} f'(t)^2\,dt. \tag{3.38}$$

Note that all of $u = 2f + \beta\eta/2$, f', and f are positive and increasing on $(0, \eta_0)$. Thus $(2f(\eta_0) + \beta\eta_0/2)f'(\eta_0) > 2S$. Thus from (3.38) we have

$$f''(\eta_A) < -2S, \tag{3.39}$$

and the proof proceeds as in the case $f''(0) = \alpha \leq 0$. Thus if $-4 \leq \beta < 0$ and $S \geq 3\beta^2/2 - 5\beta/8$, then the BVP has no solution. □

4 Discussion and numerical results

In this section we discuss the mathematical and physical implications of the results obtained in Sections 2 and 3. These results are summarized graphically in Figure 4.1. The region shaded in green indicates parameter values for which at least one solution to the BVP exists. This solution satisfies $f' > 0$, $f'' < 0$, $f''' > 0$, $f^{(4)} < 0$, and $f^{(5)} > 0$ for all $\eta > 0$. The area shaded in red denotes parameter values for which no solution exists. Finally, the region shaded in orange indicates parameter values for which no solution exists that satisfies $f' > 0$ for all $\eta > 0$. Thus, if a solution to the BVP exists for parameter values in this range, f' must at some point become negative. Physically, this represents a region of flow reversal in the radial direction, and such solutions are unlikely to be stable in the full governing PDEs, and thus are not physically realizable.

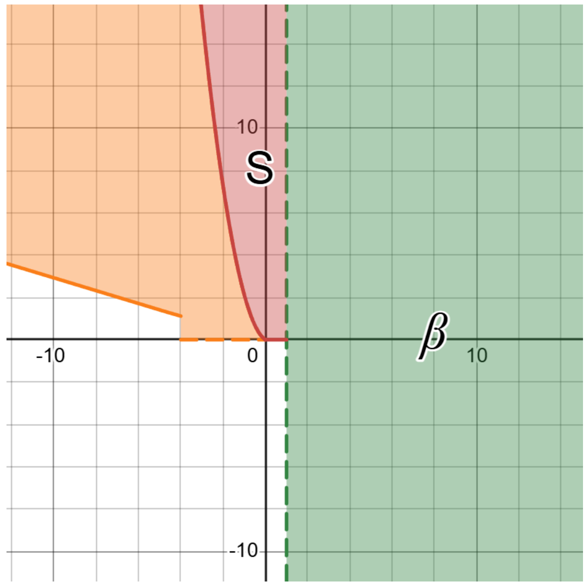

Figure 4.1: β-S parameter space showing regions of existence (green) and nonexistence (red) of solutions to the BVP (2.1–2.2). The region shaded in orange shows parameter values for which no solution exists that satisfies $f' > 0$ for all $\eta > 0$.

To study the problem numerically, we began by replicating the numerical results given in Mehmood *et al.* [8]. Using the same parameter values we found that we could not improve on the results given in [8]. These values include various levels of deceleration ($\beta < 0$) and fluid injection ($S < 0$). A comparison of our results and the results of [8] are given in Tables 4.1 and 4.2.

S	β	α [8]	α
-1.35	-1.14000950	-1.0990	-1.0990
-1.36	-1.66710229	-0.8947	-0.89475
-1.38	-3.17491743	-0.3142	-0.31428
-1.40	-5.57621580	0.5940	-0.5944

Table 4.1: Comparison of numerical values of $f''(0) = \alpha$ between [8] and current results.

Next, based on numerical integration of the problem, it was conjectured in [8] that solutions to the BVP (2.1–2.2) do not exist in the case of fluid injection ($S > 0$). The results given here indicate that, at least for a sufficiently accelerated disk ($\beta > 1$), this conjecture is incorrect. Theorem 2.1 establishes existence of a solution for all $\beta > 1$ and any $S \in \mathbb{R}$.

However, for $\beta > 1$ and $S > 0$, we had difficulty finding these solutions numerically. We employed the numerical integration software XPP-Aut [4]. Using a fourth order Runge–Kutta scheme with a stepsize $d\eta = .0001$ and an interval length $\eta_{\max} = 5$, the automated boundary value solver was unable to converge to a solution.

Consequently, we attempted to pinpoint the solution with manual numerical shooting. We used the bounds on the shooting parameter α given in Theorem 2.2 as a guide. In Figure 4.2,

β	S	α [8]	α
−1	−1.34688680700	−1.1532	−1.1532
−2	−1.36528410900	−0.7659	−0.76596
−3	−1.37810278220	−0.3812	−0.38124
−4	−1.38796975330	0	0
−5	−1.39598577367	0.3779	0.37796

Table 4.2: Comparison of numerical values of $f''(0) = \alpha$ between [8] and current results.

a graph of these bounds on α are given as a function of S for $\beta = 8$. Such bounds greatly narrowed the search for the appropriate value of α, but we were still unable to obtain convergence to a solution.

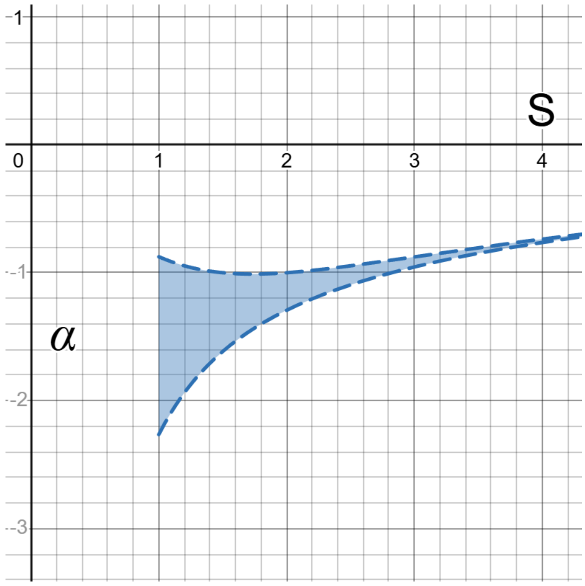

Figure 4.2: Bounds from Theorem 2.2 on $f''(0) = \alpha$ as a function of S for $\beta = 8$. A solution to the BVP (2.1–2.2) must have $f''(0) = \alpha$, where α lies is in this range.

As an example, consider parameter values $\beta = 2$ and $S = 2$. The bounds given in Theorem 2.2 then indicate that a value of α giving a solution to the boundary value problem must lie in the range $-.22 \approx -\frac{11}{50} < \alpha < -\frac{4}{19} \approx -.2105263$. We numerically integrated the differential equation using fourth order Runge–Kutta with a stepsize of $d\eta = .0001$ and an integration length of $\eta_{\max} = 5$. Our manual search resulted in a value of $\alpha \approx -0.217830769818888$ which does lie in the range given by Theorem 2.2. However, the convergence of the solution to the boundary condition at infinity is very slow, and our numerical software lost track of the solution even before $\eta = 4$. Figure 4.3 graphs two numerically integrated solutions to the initial value problem for values of α near the suspected true value. Note the quick divergence at a relatively small value of the independent variable η. Reducing the stepsize from $d\eta = .0001$ to $d\eta = .00001$ did not materially effect the results.

This numerical difficulty can arise from several sources. First, for $\beta > 0$ and $S > 0$, the coefficient $-(2f + \beta\eta/2)$ of f'' in the ODE (2.1) can lead to rapid growth of the solution of the IVP (2.1), $(2.2)_1$, $(2.2)_2$, and (2.3) away from the solution of the BVP (2.1–2.2) during numerical shooting. Also, for positive β and S, the convergence of the solution is algebraic, not exponential, as η tends to infinity, adding further difficulty.

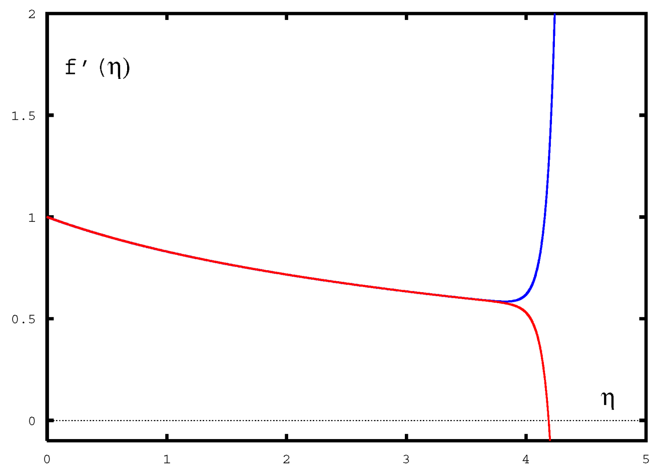

Figure 4.3: Two numerically integrated solutions to the IVP for $\beta = 2$ and $S = 2$ using fourth order Runge–Kutta with $d\eta = .0001$. Upper curve: $\alpha = -.2178307698188879$; lower curve: $\alpha = -.217830769818888$. Theorem 2.2 indicates that the value of α must satisfy $-.22 < \alpha < -.2105263$.

Finally, we note that for $\beta < 0$ and $S < 0$ we were unable to attain any analytical results. In this parameter range, Mehmood *et al.* [8] report dual solutions. However, we conjecture that for $\beta < 0$ and $S < 0$ there exists a continuum of infinitely many solutions. In Figure 4.4 we graph several solutions to the boundary value problem for $\beta = -2$ and $S = -2$. For these parameter values the solutions converge to the boundary condition at infinity very quickly, thus we increased the stepsize to $d\eta = .001$ and increased $\eta_{\text{max}} = 10$. In Figure 4.4, the values of α ranged from 0 up to 5 in increments of 0.5. All solutions still converged to zero as η_{max} was increased. The graph shows the solutions on the range $0 \leq \eta \leq 10$, however, the interval of integration was increased and $f'(\eta)$ continued to show convergence to zero on any interval examined.

A similar graph can be produced using values of α ranging from -5 to 0. The maximal range of α which produces solutions to the BVP could not be determined exactly since for larger values of $|\alpha|$, the convergence of the solution became very slow.

Using a combination of direct analysis of the BVP (2.1–2.2) and numerical integration of the problem, we were able to extend and refine the results initiated in [8]. However, questions remain regarding the solution set of (2.1–2.2). Specifically, we end this note with two open conjectures:

Conjecture 1. For $\beta < 0$ and $S \geq 0$, no solution to the BVP (2.1–2.2) exists. (Theorems 3.1 and 3.2 above give partial results regarding this conjecture.)

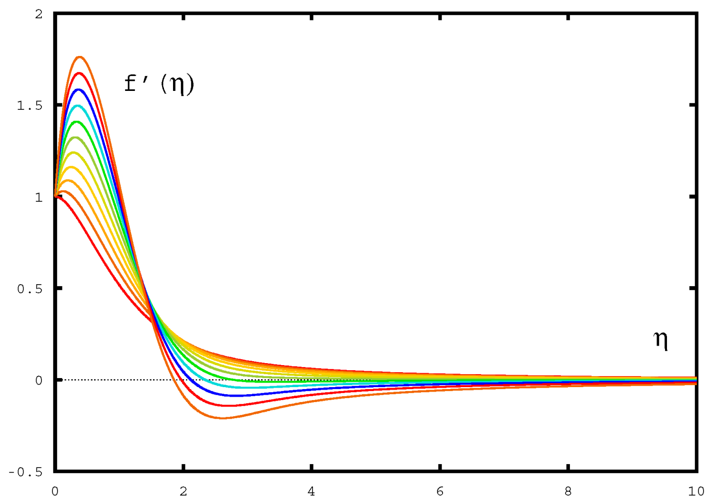

Figure 4.4: Graph of continuum of solutions to the BVP (2.1–2.2) for $\beta = -2$ and $S = -2$. Initial conditions on $f''(0) = \alpha$ range from 0 to 5 in increments of 0.5.

This would imply that for a decelerating disk ($\beta < 0$) and for any amount of fluid injection ($S > 0$), or even an impermeable disk ($S = 0$), no solution exists. However, numerically, it does appear that for a decelerating disk ($\beta < 0$) solutions do exist for sufficiently strong suction (S sufficiently negative). This implies that there exists a critical value for suction, $S_0 < 0$, below which solutions will exist. This critical value is likely to depend on β. Thus we are led to the following conjecture:

Conjecture 2. For $\beta < 0$, there exists an $S_0(\beta) < 0$ such that for all $S < S_0(\beta)$, the BVP (2.1–2.2) possesses a continuum of infinitely many solutions.

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