Electronic Journal of Qualitative Theory of Differential Equations
2026, No. **24**, 1–19; https://doi.org/10.14232/ejqtde.2026.1.24 www.math.u-szeged.hu/ejqtde/

Existence of normalized solutions for nonlinear Schrödinger system with potential

Zhen-Zhen Fan and **Gui-Dong Li** ✉

School of Mathematics and Statistics, Guizhou University, Guiyang, 550025, P.R. China

Received 30 June 2025, appeared 3 July 2026
Communicated by Dimitri Mugnai

Abstract. In this paper, we study the following Schrödinger system with coercive potential

$$\begin{cases} -\Delta u + V_1(x)u = \lambda_1 u + F_u(u,v) & \text{in } \mathbb{R}^N, \\ -\Delta v + V_2(x)v = \lambda_2 v + F_v(u,v) & \text{in } \mathbb{R}^N, \\ \int_{\mathbb{R}^N} |u|^2 \, dx = a \quad \text{and} \quad \int_{\mathbb{R}^N} |v|^2 \, dx = b, \end{cases}$$

where $a, b > 0$, $N \geq 3$, $\lambda_1, \lambda_2 \in \mathbb{R}$ are Lagrange mulitipliers. For sufficiently small mass parameters $a, b > 0$, we establish that the system possesses two distinct solutions: a local minimizer and a mountain pass type solution. Furthermore, we prove that the local minimizer is a ground state solution under certain monotonicity conditions.

Keywords: normalized solution, Schrödinger system, local Ambrosetti–Rabinowitz condition, coercive potential.

2020 Mathematics Subject Classification: 35J50, 35A15, 35J60, 35B38, 35D30.

1 Introduction and main results

In this paper, we consider the following system:

$$\begin{cases} -\Delta u + V_1(x)u = \lambda_1 u + F_u(u,v) & \text{in } \mathbb{R}^N, \\ -\Delta v + V_2(x)v = \lambda_2 v + F_v(u,v) & \text{in } \mathbb{R}^N, \\ \int_{\mathbb{R}^N} |u|^2 \, dx = a \quad \text{and} \quad \int_{\mathbb{R}^N} |v|^2 \, dx = b, \end{cases} \tag{1.1}$$

where $a, b > 0$, $N \geq 3$, $V_i (i = 1, 2)$ are external potentials, $(u, v) \in H^1(\mathbb{R}^N) \times H^1(\mathbb{R}^N)$, $\lambda_i (i = 1, 2) \in \mathbb{R}$ appear as Lagrange multipliers, and F_u, F_v denote the partial derivatives of F. The problem (1.1) arises from studying the standing waves solution $\Phi_1(t, x) = e^{-\mathbf{i}\lambda_1 t} u(x)$ and $\Phi_2(t, x) = e^{-\mathbf{i}\lambda_2 t} v(x)$ (where $\mathbf{i}^2 = -1$) for the following nonlinear Schrödinger system

$$\begin{cases} -\mathbf{i}\partial_t \Phi_1 = \Delta\Phi_1 - V_1(x)\Phi_1 + G_1(|\Phi_1|^2, |\Phi_2|^2)\Phi_1 & \text{in } \mathbb{R} \times \mathbb{R}^N, \\ -\mathbf{i}\partial_t \Phi_2 = \Delta\Phi_2 - V_2(x)\Phi_2 + G_2(|\Phi_1|^2, |\Phi_2|^2)\Phi_2 & \text{in } \mathbb{R} \times \mathbb{R}^N, \end{cases} \tag{1.2}$$

✉ Corresponding author. Email: bestdong123@163.com

where G_1, G_2 are partial derivatives of G and $F(u,v) = \frac{1}{2}G(|u|^2, |v|^2)$. This system is motivated by several physical scenarios, including phase-separation in multi-state Bose–Einstein condensates and the propagation of mutually incoherent wave packets in nonlinear optics. It is well known that equation (1.2) is conservation of mass with the L^2-norms, namely

$$\int_{\mathbb{R}^N} |\Phi_1(t,\cdot)|^2 dx = \int_{\mathbb{R}^N} |\Phi_1(0,\cdot)|^2 dx \quad \text{and} \quad \int_{\mathbb{R}^N} |\Phi_2(t,\cdot)|^2 dx = \int_{\mathbb{R}^N} |\Phi_2(0,\cdot)|^2 dx.$$

Consequently, it is natural and meaningful to consider solutions with a L^2-constraint, which are referred to as normalized solutions.

When $V_1(x) = V_2(x)$, $u = v$ and $\lambda_1 = \lambda_2$, system (1.1) transforms to the following equation

$$\begin{cases} -\Delta u + V(x)u = \lambda u + f(u) & \text{in } \mathbb{R}^N, \\ \int_{\mathbb{R}^N} |u|^2 \, dx = a. \end{cases} \tag{1.3}$$

For the case where $V(x) \equiv 0$, we refer to [2,6,11,19,20]. Jeanjean [19] introduced a minimax approach to study equation (1.3), under the following assumptions on f:

$(f0)$ $f : \mathbb{R} \to \mathbb{R}$ is continuous and odd;

$(f1)$ there exists $(\alpha, \beta) \in \mathbb{R} \times \mathbb{R}$ satisfying $2 + \frac{4}{N} < \alpha \leq \beta < 2^*$ such that $\alpha F(t) \leq f(t)t \leq \beta F(t)$, where $2^* = \frac{2N}{N-2}$ if $N \geq 3$ and $2^* = \infty$ if $N = 1, 2$;

$(f2)$ let $\tilde{F} : \mathbb{R} \to \mathbb{R}, \tilde{F}(t) = f(t)t - 2F(t)$. Then $\tilde{F}'(t)$ exists and $\tilde{F}'(t)t > \frac{2N+4}{N}\tilde{F}(t)$.

Here, $(f1)$ is referred to as the Ambrosetti–Rabinowitz condition, which is often used to address the boundedness of Palais–Smale sequences. By constructing a mountain pass geometry, Jeanjean proved that equation (1.3) admits ground state solutions for $N \geq 1$. Subsequently, when $N \geq 2$ and f satisfies $(f0)$–$(f1)$, Bartsch and de Valeriola [2] obtained infinitely many solutions to equation (1.3) from a fountain theorem. In [6], Bartsch and Soave obtained infinitely many solutions to equation (1.3) by using a natural constraint approach, but they required f to additionally satisfy condition $(f2)$. Moreover, we note that [20] and [11] weaken $(f1)$-$(f2)$ from different perspectives. In [20], Jeanjean and Lu considered:

$(f3)$ $\lim_{t\to\infty} F(t)/|t|^{2+4/N} = +\infty$;

$(f4)$ $t \to \tilde{F}(t)/|t|^{2+4/N}$ is strictly decreasing on $(-\infty, 0)$ and strictly increasing on $(0, \infty)$;

$(f5)$ when $N \geq 3, f(t)t < 2^*F(t)$ for all $t \in \mathbb{R} \setminus \{0\}$.

In particular, the condition $(f4)$ is a weaker version of $(f2)$. Under condition $(f4)$ that $(f3)$ and $(f5)$ can be used to replace $(f1)$. Bieganowski and Mederski in [11] weakened $(f1)$ to:

$(f6)$ $\frac{4}{N}F(t) \preceq \tilde{F}(t) \preceq (2^* - 2)F(t)$ for $t \in \mathbb{R}$,

where $f_1(s) \preceq f_2(s)$ for $s \in \mathbb{R}$ denotes that $f_1(s) \leq f_2(s)$ for all $s \in \mathbb{R}$ and for any $\gamma > 0$ there is $|s| < \gamma$ such that $f_1(s) < f_2(s)$. In [14], Ding and Zhong introduced an external potential V, under the same assumptions on f as in [19], then they obtained the existence of ground state solutions by applying Pohožaev manifold, where V satisfies:

$(A1)$ $\lim_{|x|\to\infty} V(x) = \sup_{x\in\mathbb{R}^N} V(x) = 0$ and there exists $\sigma_1 \in \left[0, \frac{N(\alpha-2)-4}{N(\alpha-2)}\right)$ such that

$$\left|\int_{\mathbb{R}^N} V(x)u^2 \mathrm{d}x\right| \le \sigma_1 |\nabla u|_2^2, \quad \forall u \in H^1(\mathbb{R}^N);$$

$(A2)$ $\nabla V(x)$ exists for a.e. $x \in \mathbb{R}^N$, put $W(x) := \frac{1}{2}(x, \nabla V(x))$. There exists $0 \le \sigma_2 < \min\{\frac{N(\alpha-2)(1-\sigma_1)}{4} - 1, \frac{N}{\beta} - \frac{N-2}{2}\}$ such that

$$\left|\int_{\mathbb{R}^N} W(x)u^2 \mathrm{d}x\right| \le \sigma_2 |\nabla u|_2^2, \quad \forall u \in H^1(\mathbb{R}^N);$$

$(A3)$ $\nabla W(x)$ exists for a.e. $x \in \mathbb{R}^N$, put $Y(x) := (\frac{N}{2}\alpha - N)W(x) + (x, \nabla W(x))$. There exists $\sigma_3 \in [0, \frac{N}{2}\alpha - N - 2)$ such that

$$\left|\int_{\mathbb{R}^N} Y_+(x)u^2 \mathrm{d}x\right| \le \sigma_3 |\nabla u|_2^2, \quad \forall u \in H^1(\mathbb{R}^N).$$

For the case where $f(u) = |u|^{p-2}u$ $(2 < p < 2^*)$, we refer to [1,5,30]. Alves and Ji [1] considered that $2 < p < 2 + \frac{4}{N}$, and V represents various types of potentials, including periodic, asymptotically periodic, steep, and asymptotically constant potentials, then obtained the existence of the global minimizer. Bartsch et al. in [5] considered that $2 + \frac{4}{N} < p < 2^*$, and V is positive and vanishing at infinity, including potentials with singularities, then obtained the existence of solutions by a min-max argument. The same nonlinear term was also considered in [30], whose difference is that $N = 2$ and the potential V is coercive, which satisfies

$(A4)$ $\lim_{|x|\to\infty} V(x) = \infty$, $\inf_{x\in\mathbb{R}^2} V(x) = 0$;

$(A5)$ $(p-3)V + x\cdot\nabla V \ge -C_1$ and $|x\cdot\nabla V(x)| \le C_2(V(x)+1)$, where $C_i > 0, i = 1,2$,

and then authors obtained the existence of two solutions, one is a local minimizer, and the other is a mountain pass type solution. In this proof, p needs to be sufficiently close to 4. In addition, the authors also considered the asymptotic behavior of the solution as p tends to 4. For additional related results about equation (1.3), one can refer to [18,23,25,29,31].

Now let us return to system (1.1). For the following special case of system (1.1),

$$\begin{cases} -\Delta u + \lambda_1 u = \mu_1 u^{p-1} + \beta r_1 u^{r_1-1} v^{r_2} & \text{in } \mathbb{R}^N, \\ -\Delta v + \lambda_2 v = \mu_2 v^{q-1} + \beta r_2 u^{r_1} v^{r_2-1} & \text{in } \mathbb{R}^N, \\ \int_{\mathbb{R}^N} |u|^2 \,\mathrm{d}x = a \quad \text{and} \quad \int_{\mathbb{R}^N} |v|^2 \,\mathrm{d}x = b, \end{cases} \tag{1.4}$$

where $\mu_1, \mu_2, \beta > 0$, we refer to [3,4,8,15,21]. In [15], Gou and Jeanjean proved that system (1.4) possesses a local minimizer and a mountain pass type solution when either of the following conditions holds:

$(H1)$ $N \ge 1,\ 2 < p,q < 2 + \frac{4}{N},\ r_1, r_2 > 1,\ 2 + \frac{4}{N} < r_1 + r_2 < 2^*$;

$(H2)$ $N \ge 1,\ 2 + \frac{4}{N} < p,q < 2^*,\ r_1, r_2 > 1,\ r_1 + r_2 < 2 + \frac{4}{N}$.

When $N = 3,\ p = q = 4,\ r_1 = r_2 = 2$, Bartsch, Jeanjean and Soave [4] obtained the existence of solutions for system (1.4) provided $0 < \beta < \beta_1(a,b)$ or $\beta > \beta_2(a,b)$. Bartsch and Jeanjean in [3] considered the more general case of $2 \le N \le 4,\ 2 + \frac{4}{N} < p, q, r_1 + r_2 < 2^*$ and proved the existence of solutions, but it is also necessary that $0 < \beta < \beta_1(a,b)$ or $\beta > \beta_2(a,b)$. The thresholds $\beta_1(a,b)$ and $\beta_2(a,b)$ depend on a, b heavily. Thus it is natural to ask whether there exists some $\beta > 0$ such that the existence of solutions holds for any $a, b > 0$. In fact, this has been posed as an open problem by Bartsch, Jeanjean and Soave (see [4, Remark 1.3-(a) and (d)]). Partial answers were given by [8,21], both of which have expanded the ranges of β. Notably, in [8], Bartsch, Zhong and Zou established that when $N = 3, p = q = 4, r_1 = r_2 = 2$, the existence result holds for all $a, b > 0$ provided $\beta \in (0, \tau_0 \min\{\mu_1, \mu_2\}] \cup (\tau_0 \max\{\mu_1, \mu_2\}, +\infty)$, where $\tau_0 \in (0,1)$ is explicitly determined. Subsequently, Jeanjean, Zhang and Zhong [21] proved that when $N = 3, 4$ and $2 + \frac{4}{N} < p, q, r_1 + r_2 < 2^*$, the existence of solutions holds for all $a, b, \beta > 0$ if $r_1, r_2 \in (1,2)$. When $V_1, V_2 \neq 0$, Chen and Zou [13] studied the following system with linear coupled:

$$
\begin{cases}
-\Delta u + V_1(x)u = \lambda_1 u + \mu_1 |u|^{p-2}u + \beta u & \text{in } \mathbb{R}^N, \\
-\Delta v + V_2(x)v = \lambda_2 v + \mu_2 |v|^{q-2}v + \beta v & \text{in } \mathbb{R}^N, \\
\displaystyle\int_{\mathbb{R}^N} |u|^2 \, dx = a \quad \text{and} \quad \int_{\mathbb{R}^N} |v|^2 \, dx = b,
\end{cases}
\tag{1.5}
$$

where $2 < p, q < 2 + \frac{4}{N}, \mu_1, \mu_2 > 0,\ \beta \in \mathbb{R} \setminus \{0\}$, and under the assumption that $\lim_{|x|\to\infty} V_i(x) = \sup_{x\in\mathbb{R}^N} V_i(x) \in (c_i, +\infty]$ with $c_i = \min_{x\in\mathbb{R}^N} V_i(x) > -\infty (i = 1, 2)$, they demonstrated the existence of ground state solutions for system (1.5). Building on this work, Guo and Liang [17] considered replacing the linear coupled with quadratically coupled, and also derived existence results. About system (1.1), we also mention that Guo, Zeng and Zhou [16] considered the case of $N = 2$ with $F(u,v) = \frac{\mu_1}{4}u^4 + \frac{\mu_2}{4}v^4 + \frac{\beta}{2}u^2v^2$ $(\mu_1, \mu_2, \beta > 0)$, where $V_i(x) (i = 1, 2)$ are trapping potentials of the following type

$$
V_i(x) \in L^\infty_{loc}(\mathbb{R}^2), \quad \lim_{|x|\to\infty} V_i(x) = \infty \quad \text{and} \quad \inf_{x\in\mathbb{R}^2} V_i(x) = 0,
$$

then they established the existence, non-existence and uniqueness of solutions. For more relevant results of system (1.1), please refer to [24,26] and their references.

Inspired by the results about equation (1.3), we naturally consider generalized nonlinear terms in system (1.1), where F satisfies:

(F1) $F(w) \in C^1(\mathbb{R}^2, \mathbb{R})$ and $F(w) \ge 0$, where $w = (t, s)$;

(F2) $\lim_{|w|\to 0} \frac{F(w)}{|w|^{2+\frac{4}{N}}} = 0$ and $\lim_{|w|\to+\infty} \frac{F(w)}{|w|^{2^*}} = 0$;

(F3) there exist $\theta > 2 + \frac{4}{N}$ and $R_0 \ge 0$, such that for any $|w| \ge R_0,\ 0 < \theta F(w) \le w \cdot \nabla F(w)$, where $t, s \neq 0$ and $\nabla F(w) = (F_t(w), F_s(w))$,

and $V_i (i = 1, 2)$ satisfies:

(V1) $V_i \in C^1(\mathbb{R}^N, \mathbb{R})$, $\inf_{x\in\mathbb{R}^N} V_i(x) = 0$ and there exists $d_0 > 0$ such that

$$
\lim_{|y|\to\infty} \text{meas}(\{x \in \mathbb{R}^N : |x - y| \le d_0, V_i(x) \le D\}) = 0 \quad \forall D > 0;
$$

(V2) $2V_i(x) + x \cdot \nabla V_i(x) \ge -\bar{C}_i$ a.e. in $\mathbb{R}^N$, where $\bar{C}_i > 0$.

From a variational point of view, solutions to system (1.1) can be obtained as critical points of the energy functional

$$\mathcal{I}(u,v)=\frac{1}{2}\int_{\mathbb{R}^N}\left(|\nabla u|^2+|\nabla v|^2+V_1(x)|u|^2+V_2(x)|v|^2\right)\mathrm{d}x-\int_{\mathbb{R}^N}F(u,v)\mathrm{d}x$$

on the constraint

$$S(a)\times S(b):=\left\{(u,v)\in\mathcal{H}:\int_{\mathbb{R}^N}|u|^2\mathrm{d}x=a,\int_{\mathbb{R}^N}|v|^2\mathrm{d}x=b\right\},$$

where

$$\mathcal{H}:=\left\{(u,v)\in H^1(\mathbb{R}^N)\times H^1(\mathbb{R}^N):\int_{\mathbb{R}^N}V_1(x)|u|^2\mathrm{d}x<\infty,\int_{\mathbb{R}^N}V_2(x)|v|^2\mathrm{d}x<\infty\right\}.$$

We define the following scalar product on $\mathcal{H}$,

$$\langle(u_1,v_1),(u_2,v_2)\rangle_{\mathcal{H}}=\int_{\mathbb{R}^N}(\nabla u_1\cdot\nabla u_2+V_1(x)u_1u_2)\mathrm{d}x+\int_{\mathbb{R}^N}(\nabla v_1\cdot\nabla v_2+V_2(x)v_1v_2)\mathrm{d}x,$$

and the corresponding norm is given by

$$\|(u,v)\|_{\mathcal{H}}=\langle(u,v),(u,v)\rangle_{\mathcal{H}}^{\frac{1}{2}}.$$

In the sequel, we denote

$$B(k)=\left\{(u,v)\in S(a)\times S(b):\int_{\mathbb{R}^N}|\nabla u|^2+|\nabla v|^2\mathrm{d}x\le k\right\}.$$

Our first main result is stated as follows:

Theorem 1.1. *Suppose that* (V1), (V2) *and* (F1)–(F3) *hold. Then there exist* $a^*,b^*>0$ *and* $\rho>0$ *such that for all* $a\le a^*,b\le b^*$*, the functional* $\mathcal{I}|_{S(a)\times S(b)}$ *possesses two critical points. One is the local minimizer* $(\bar u,\bar v)\in B(\rho)$*, i.e.,*

$$\mathcal{I}(\bar u,\bar v)=\inf_{(u,v)\in B(\rho)}\mathcal{I}(u,v)=:m(\rho)$$

and the other is the mountain pass type critical point $(\widetilde u,\widetilde v)$ *with* $\mathcal{I}(\widetilde u,\widetilde v)>m(\rho)$*.*

Remark 1.2. It is noted that (F3) indicates that our nonlinear term only satisfies the local Ambrosetti–Rabinowitz type condition. Therefore, our results generalize the existing ones. In addition, when our conditions are imposed on a single equation, a local minimum solution and a mountain pass type solution can be obtained in a similar manner. This shows that our results, in a certain sense, generalize the results of [30].

In addition to the existence results established in Theorem 1.1, we further demonstrate that the local minimizer can be characterized as a ground state solution under specific monotonicity conditions on F and V. To formalize this, we recall the definition of the ground state solution:

Definition 1.3. We say that $(\hat u,\hat v)$ is a ground state solution of system (1.1) on $S(a)\times S(b)$ if $\mathcal{I}'|_{S(a)\times S(b)}(\hat u,\hat v)=0$ and

$$\mathcal{I}(\hat u,\hat v)=\inf\{\mathcal{I}(u,v):\mathcal{I}'|_{S(a)\times S(b)}(u,v)=0\text{ and }(u,v)\in S(a)\times S(b)\}.$$

Theorem 1.4. *Suppose that* (V1), (V2) *and* (F1)–(F3) *hold. Then system* (1.1) *has at least one ground state solution. Moreover, if the following conditions hold,*

(F4) *for any fixed $s_1, s_2 \in \mathbb{R}$, $H(y)$ denotes the Hessian matrix of F at y and*

$$g(y) = \left(2(N+1)F(y) - \left(\frac{3N}{2}+1\right)(y \cdot \nabla F(y)) + \frac{N}{2}(y^T H(y) y)\right) |y|^{-(2+\frac{4}{N})},$$

here $y = y(t) = (ts_1, ts_2)$. Then $g(y(t))$ is strictly increasing with respect to t on $(0, +\infty)$;

(V3) *for any fixed $x \in \mathbb{R}^N$, let $L_i(t) = ((tx) \cdot \nabla W_i(tx) + W_i(tx))\, t^2$ is non-decreasing with respect to t on $(0, +\infty)$, where $W_i(x) = x \cdot \nabla V_i(x)$,*

then the local minimizer $(\bar{u}, \bar{v})$ is a ground state solution of system (1.1).

Remark 1.5. There are a large number of nonlinear terms that satisfy the (F1)–(F4). For example, $F(u,v) = |u|^{r_1}|v|^{r_2}$, where $2 + \frac{4}{N} < r_1 + r_2 < 2^*$ and $r_1, r_2 > 1$. At the same time, there are also numerous potential functions that satisfy the (V1)–(V3). A typical example of such potential function is $V_1(x) = V_2(x) = |x|^2$.

The rest of this paper is organized as follows. In Section 2, we collect and introduce some relevant results for future reference. Then, in Section 3, we establish the proofs of Theorem 1.1 and Theorem 1.4.

Notation. Throughout the paper, we use $|\cdot|_p$ to denote the $L^p(\mathbb{R}^N)$ norm. C, C_1, C_2 stand for positive constants possibly different in different places. For $(t,s) \in \mathbb{R}^2$, we denote $|(t,s)| = \sqrt{|t|^2 + |s|^2}$.

2 Preliminary results

In this section, we introduce some preliminary results which will be used in the rest discussion. The following Gagliardo–Nirenberg inequality can be found in [28].

Lemma 2.1. *For any $u \in H^1(\mathbb{R}^N)$, let $2 \le p \le 2^*$ if $N \ge 3$ and $p \ge 2$ if $N = 1, 2$, we have*

$$\int_{\mathbb{R}^N} |u|^p \mathrm{d}x \le C_{N,p} \left(\int_{\mathbb{R}^N} |u|^2 \mathrm{d}x\right)^{\frac{p(1-\gamma_p)}{2}} \left(\int_{\mathbb{R}^N} |\nabla u|^2 \mathrm{d}x\right)^{\frac{p\gamma_p}{2}},$$

where $\gamma_p = \frac{N(p-2)}{2p}$.

To overcome the lack of compactness, we need the following lemma, which can be proved as in [7].

Lemma 2.2. *Suppose that* (V1), (V2) *hold, then the embedding $\mathcal{H} \hookrightarrow L^p(\mathbb{R}^N) \times L^q(\mathbb{R}^N)$ is compact for any $p, q \in [2, 2^*)$.*

In the proof of Theorem 1.1, the most important step is to verify the convergence of Palais–Smale sequences. Here we recall the following lemma without proof, due to the Brézis and Lieb [12].

Lemma 2.3. *Suppose that $u_n \rightharpoonup u$ in $H^1(\mathbb{R}^N)$, then for $2 \le s \le 2^*$, we have*

$$\lim_{n\to\infty} \int_{\mathbb{R}^N} |\nabla u_n|^2 - |\nabla u|^2 - |\nabla(u_n - u)|^2 \mathrm{d}x = 0,$$

$$\lim_{n\to\infty} \int_{\mathbb{R}^N} |u_n|^s - |u|^s - |u_n - u|^s \,\mathrm{d}x = 0.$$

Finally, we have the following Pohožaev identity, which is inspired by [9, Proposition 1] and [22, Proposition 2.3], and interested readers may refer to these literatures for further details.

Lemma 2.4. *Suppose that* (V1), (V2) *and* (F1)–(F3) *hold. If $(u, v) \in H^1(\mathbb{R}^N) \times H^1(\mathbb{R}^N)$ solves*

$$\begin{cases} -\Delta u + V_1(x)u - \lambda_1 u = F_u(u,v) & \text{in } \mathbb{R}^N, \\ -\Delta v + V_2(x)v - \lambda_2 v = F_v(u,v) & \text{in } \mathbb{R}^N, \end{cases} \tag{2.1}$$

then (u, v) satisfies the following Pohožaev type identity

$$\begin{aligned} P(u,v) = \int_{\mathbb{R}^N} |\nabla u|^2 + |\nabla v|^2 \mathrm{d}x - \frac{1}{2}\int_{\mathbb{R}^N} (x\cdot\nabla V_1)|u|^2 + (x\cdot\nabla V_2)|v|^2 \mathrm{d}x \\ - \frac{N}{2}\int_{\mathbb{R}^N} (u,v)\cdot\nabla F(u,v) - 2F(u,v)\mathrm{d}x = 0. \end{aligned}$$

Proof. Let (u, v) be a solution to system (2.1), then rewrite system (2.1) as:

$$\begin{cases} -\Delta u = g_1(x,u), \\ -\Delta v = g_2(x,v), \end{cases} \tag{2.2}$$

where

$$\begin{cases} g_1(x,u) = -V_1(x)u + \lambda_1 u + F_u(u,v), \\ g_2(x,v) = -V_2(x)v + \lambda_2 v + F_v(u,v). \end{cases}$$

Since $V_i \in C^1(\mathbb{R}^N)$ for $i = 1, 2$, it follows that $V_i \in L^{\frac{N}{2}}_{loc}(\mathbb{R}^N)$, combining with (F1)–(F3), there exists $a_i(x) \in L^{\frac{N}{2}}_{loc}(\mathbb{R}^N)$ such that

$$|g_1(x,u)| \le a_1(x)(1+|u|) \quad \text{and} \quad |g_2(x,v)| \le a_2(x)(1+|v|). \tag{2.3}$$

By [27, Lemma B.3], we have $u, v \in L^q_{loc}(\mathbb{R}^N)$ for any $1 \le q < \infty$. In view of (2.3), one can see $-\Delta u = g_1(x,u) \in L^q_{loc}(\mathbb{R}^N)$ and $-\Delta v = g_2(x,v) \in L^q_{loc}(\mathbb{R}^N)$ for any $1 \le q < \infty$. Applying the Calderon–Zygmund inequality of [27, Theorem B.2], we deduce that $u, v \in W^{2,q}_{loc}(\mathbb{R}^N)$ for any $1 \le q < \infty$. Then we define

$$G_1(x,u) := \int_0^u g_1(x,s)\mathrm{d}s, \qquad G_2(x,v) := \int_0^v g_2(x,s)\mathrm{d}s.$$

At first, we multiply the equation $0 = (\Delta u + g_1(x,u))$ by $x \cdot \nabla u$ and integrate over the ball $B_R = \{x \in \mathbb{R}^N : |x| < R\}$. We then show that the boundary term (on ∂B_R) to converge to 0 as $R \to +\infty$. Indeed, one has

$$\int_{B_R} g_1(x,u)x\cdot\nabla u \mathrm{d}x = \int_{B_R} \mathrm{div}(xG_1(x,u)) - NG_1(x,u) - x\cdot\nabla_x G_1(x,u)\mathrm{d}x,$$

and

$$\int_{B_R} \Delta u x \cdot \nabla u \mathrm{d}x = \int_{B_R} \operatorname{div}(\nabla u x \cdot \nabla u) - |\nabla u|^2 - x \cdot \nabla \left(\frac{|\nabla u|^2}{2} \right) \mathrm{d}x$$
$$= \int_{B_R} \operatorname{div} \left(\nabla u x \cdot \nabla u - x \frac{|\nabla u|^2}{2} \right) + \frac{N-2}{2} |\nabla u|^2 \mathrm{d}x,$$

where $\nabla_x G_1$ is the gradient of G_1 with respect to x. By the divergence theorem, we obtain

$$\int_{B_R} \operatorname{div}(x G_1(x,u)) \mathrm{d}x = \int_{\partial B_R} x G_1(x,u) \cdot \mathbf{n} \mathrm{d}S = \int_{\partial B_R} R G_1(x,u) \mathrm{d}S$$

and

$$\int_{B_R} \operatorname{div} \left(\nabla u x \cdot \nabla u - x \frac{|\nabla u|^2}{2} \right) \mathrm{d}x = \int_{\partial B_R} \left(\nabla u x \cdot \nabla u - x \frac{|\nabla u|^2}{2} \right) \cdot \mathbf{n} \mathrm{d}S$$
$$= \int_{\partial B_R} \left(\frac{|x \cdot \nabla u|^2}{R} - \frac{R}{2} |\nabla u|^2 \right) \mathrm{d}S,$$

where $\mathbf{n}$ denotes the unit outward normal vector to ∂B_R. Combining with the Cauchy–Schwarz inequality, we have

$$-\frac{R}{2} \int_{\partial B_R} |\nabla u|^2 \mathrm{d}S \leq \int_{\partial B_R} \left(\frac{|x \cdot \nabla u|^2}{R} - \frac{R}{2} |\nabla u|^2 \right) \mathrm{d}S \leq \frac{R}{2} \int_{\partial B_R} |\nabla u|^2 \mathrm{d}S.$$

Next we claim that there exists a suitable sequence $\{R_n\}$ such that

$$\lim_{n \to \infty} R_n \int_{\partial B_{R_n}} |\nabla u|^2 + 2|G_1(x,u)| \mathrm{d}S = 0,$$

where $R_n \to \infty$ as $n \to \infty$. In fact, if

$$\liminf_{R \to +\infty} R \int_{\partial B_R} |\nabla u|^2 + 2|G_1(x,u)| \mathrm{d}S = \alpha > 0.$$

Then there exists $\bar{R} > 0$, such that for all $R > \bar{R}$,

$$\int_{\partial B_R} |\nabla u|^2 + 2|G_1(x,u)| \mathrm{d}S \geq \frac{\alpha}{2R}.$$

Therefore,

$$\begin{aligned} +\infty &> \int_{\mathbb{R}^N} |\nabla u|^2 + 2|G_1(x,u)| \mathrm{d}x \\ &= \int_0^{+\infty} \int_{\partial B_R} |\nabla u|^2 + 2|G_1(x,u)| \mathrm{d}S \mathrm{d}r \\ &\geq \int_{\bar{R}}^{+\infty} \int_{\partial B_R} |\nabla u|^2 + 2|G_1(x,u)| \mathrm{d}S \mathrm{d}r \\ &= +\infty, \end{aligned}$$

which is a contradiction. Moreover,

$$\int_{B_{R_n}} |\nabla u|^2 \mathrm{d}x \to \int_{\mathbb{R}^N} |\nabla u|^2 \mathrm{d}x \quad \text{and} \quad \int_{B_{R_n}} G_1(x,u) \mathrm{d}x \to \int_{\mathbb{R}^N} G_1(x,u) \mathrm{d}x$$

as $n \to \infty$. Combining these results, one can see

$$\int_{\mathbb{R}^N} NG_1(x,u) + x \cdot \nabla_x G_1(x,u) - \frac{N-2}{2}|\nabla u|^2 \mathrm{d}x = 0.$$

It follows that

$$\begin{aligned}(N-2)\int_{\mathbb{R}^N} |\nabla u|^2 \mathrm{d}x + N\int_{\mathbb{R}^N} (V_1(x)-\lambda_1)|u|^2 \mathrm{d}x + \int_{\mathbb{R}^N} (x\cdot \nabla V_1(x))|u|^2 \mathrm{d}x \\ = 2N\int_{\mathbb{R}^N} F(u,v)\mathrm{d}x. \end{aligned} \tag{2.4}$$

By a similar discussion, we can deduce that

$$\begin{aligned}(N-2)\int_{\mathbb{R}^N} |\nabla v|^2 \mathrm{d}x + N\int_{\mathbb{R}^N} (V_2(x)-\lambda_2)|v|^2 \mathrm{d}x + \int_{\mathbb{R}^N} (x\cdot \nabla V_2(x))|v|^2 \mathrm{d}x \\ = 2N\int_{\mathbb{R}^N} F(u,v)\mathrm{d}x. \end{aligned} \tag{2.5}$$

Add (2.4) and (2.5), eliminating the terms with λ_1, λ_2, we have

$$\begin{aligned}0 = &\int_{\mathbb{R}^N} |\nabla u|^2 + |\nabla v|^2 \mathrm{d}x - \frac{1}{2}\int_{\mathbb{R}^N} (x\cdot\nabla V_1(x))|u|^2 + (x\cdot \nabla V_2(x))|v|^2 \mathrm{d}x \\ &- \frac{N}{2}\int_{\mathbb{R}^N} (u,v)\cdot \nabla F(u,v) - 2F(u,v)\mathrm{d}x.\end{aligned}$$

This completes the proof. □

3 The proofs of theorems

In this section, we mainly show the existence of two critical points of the functional $\mathcal{I}$ on the sphere $S(a) \times S(b)$. Recall

$$B(k) = \left\{(u,v) \in S(a)\times S(b) : \int_{\mathbb{R}^N} |\nabla u|^2 + |\nabla v|^2 \mathrm{d}x \le k\right\}.$$

The first critical point of $\mathcal{I}$ will be found as a minimizer of a minimization problem on $B(k)$, i.e.

$$m(k) = \inf_{(u,v)\in B(k)} \mathcal{I}(u,v), \tag{3.1}$$

and it is necessary to show that the minimizer is not on the boundary of $B(k)$:

$$\partial B(k) = \left\{(u,v) \in S(a)\times S(b) : \int_{\mathbb{R}^N} |\nabla u|^2 + |\nabla v|^2 \mathrm{d}x = k\right\}.$$

In fact, we have the following proposition.

Proposition 3.1. *Suppose that* (V1), (V2) *and* (F1)–(F3) *hold. Then there exist* $a^*, b^* > 0$ *and* $\rho > 0$, *such that if* $a \le a^*, b \le b^*$, *the functional* $\mathcal{I}$ *has a critical point* $(\bar{u}, \bar{v})$, *and* $\mathcal{I}(\bar{u},\bar{v}) = \inf_{(u,v)\in B(\rho)} \mathcal{I}(u,v)$.

Proof. Invoking (F2) and (F3), for arbitrary $\epsilon_1, \epsilon_2 > 0$, there exists a constant $C_{\epsilon_1,\epsilon_2,p} > 0$ such that

$$\int_{\mathbb{R}^N} F(u,v)\mathrm{d}x \le \epsilon_1 \int_{\mathbb{R}^N} |(u,v)|^{2+\frac{4}{N}}\mathrm{d}x + C_{\epsilon_1,\epsilon_2,p}\int_{\mathbb{R}^N} |(u,v)|^p\mathrm{d}x + \epsilon_2 \int_{\mathbb{R}^N} |(u,v)|^{2^*}\mathrm{d}x, \tag{3.2}$$

where $p \in \left(2+\frac{4}{N}, 2^*\right)$. Then we give an estimate of $\int_{\mathbb{R}^N} |(u,v)|^p\mathrm{d}x$. By the Minkowski inequality and the Gagliardo–Nirenberg inequality, we obtain

$$\begin{aligned}
\int_{\mathbb{R}^N} |(u,v)|^p\mathrm{d}x &= \int_{\mathbb{R}^N} |u^2+v^2|^{\frac{p}{2}}\mathrm{d}x \le (|u|_p^2 + |v|_p^2)^{\frac{p}{2}} \\
&\le \left(C_{N,p}|u|_2^{2-2\gamma_p}|\nabla u|_2^{2\gamma_p} + C_{N,p}|v|_2^{2-2\gamma_p}|\nabla v|_2^{2\gamma_p}\right)^{\frac{p}{2}} \\
&\le C\left(\int_{\mathbb{R}^N} |u|^2+|v|^2\mathrm{d}x\right)^{\frac{p(1-\gamma_p)}{2}}\left(\int_{\mathbb{R}^N} |\nabla u|^2+|\nabla v|^2\mathrm{d}x\right)^{\frac{p\gamma_p}{2}}
\end{aligned} \tag{3.3}$$

where $\gamma_p = \frac{N(p-2)}{2p}$. Therefore, for any $(u,v) \in S(1)\times S(1)$, there exists a $\rho > 0$ sufficiently small, such that for $\int_{\mathbb{R}^N} |\nabla u|^2 + |\nabla v|^2\mathrm{d}x = \rho$, then

$$\begin{aligned}
\mathcal{I}(u,v) &= \frac{1}{2}\int_{\mathbb{R}^N} |\nabla u|^2+|\nabla v|^2\mathrm{d}x + \frac{1}{2}\int_{\mathbb{R}^N} V_1(x)|u|^2 + V_2(x)|v|^2\mathrm{d}x - \int_{\mathbb{R}^N} F(u,v)\mathrm{d}x \\
&\ge \frac{1}{2}\rho - \epsilon_1 C_1\rho - C_2\rho^{\frac{p\gamma_p}{2}} - \epsilon_2 C_3 \rho^{\frac{N}{N-2}} \\
&\ge \frac{1}{4}\rho.
\end{aligned}$$

Since the first eigenvalue of the operator $-\Delta + V_i(x)(i=1,2)$ is positive, then there exists $(w_i,\mu_i) \in H^1(\mathbb{R}^N)\times\mathbb{R}^+, i = 1,2$, such that

$$\int_{\mathbb{R}^N} |\nabla w_i|^2 + V_i(x)|w_i|^2\mathrm{d}x = \mu_i\int_{\mathbb{R}^N} |w_i|^2\mathrm{d}x =: \mu_i c_i.$$

Thus, for $t > 0$, we have

$$\mathcal{I}(tw_1, tw_2) \le \frac{t^2}{2}\left(\mu_1\int_{\mathbb{R}^N} |w_1|^2\mathrm{d}x + \mu_2\int_{\mathbb{R}^N} |w_2|^2\mathrm{d}x\right) = \frac{t^2}{2}(\mu_1c_1+\mu_2c_2).$$

Select $t_0 > 0$ small enough so that

$$t_0^2\left(\int_{\mathbb{R}^N} |\nabla w_1|^2 + |\nabla w_2|^2\mathrm{d}x\right) < \frac{1}{2}\rho \quad \text{and} \quad \frac{t_0^2}{2}(\mu_1c_1+\mu_2c_2) < \frac{\rho}{8}.$$

Set

$$a_* = t_0^2\int_{\mathbb{R}^N} |w_1|^2\mathrm{d}x \quad \text{and} \quad b_* = t_0^2\int_{\mathbb{R}^N} |w_2|^2\mathrm{d}x.$$

For any $a \le a^* := \min\{1, a_*\}$ and $b \le b^* := \min\{1, b_*\}$, there exist $s_1, s_2 \in (0, t_0]$ such that $\int_{\mathbb{R}^N} |s_1w_1|^2\mathrm{d}x = a$, $\int_{\mathbb{R}^N} |s_2w_2|^2\mathrm{d}x = b$ and

$$\int_{\mathbb{R}^N} |\nabla(s_1w_1)|^2 + |\nabla(s_2w_2)|^2\mathrm{d}x < \frac{1}{2}\rho.$$

Moreover,

$$\mathcal{I}(s_1w_1, s_2w_2) \le \frac{s_1^2}{2}\mu_1c_1 + \frac{s_2^2}{2}\mu_2c_2 \le \frac{t_0^2}{2}(\mu_1c_1+\mu_2c_2) < \frac{\rho}{8},$$

which implies

$$m(\rho) \leq \mathcal{I}(s_1 w_1, s_2 w_2) < \inf_{(u,v)\in\partial B(\rho)} \mathcal{I}(u,v) \tag{3.4}$$

by (3.1). Next we consider the minimization problem:

$$m(\rho) = \inf_{(u,v)\in B(\rho)} \mathcal{I}(u,v).$$

Let $\{(u_m, v_m)\} \subset B(\rho)$ be a minimizing sequence of $m(\rho)$. Applying (3.2) and (3.3), we derive the lower bound

$$\begin{aligned} m(\rho) + o_m(1) &= \mathcal{I}(u_m, v_m) \\ &\geq \frac{1}{2}\|(u_m, v_m)\|^2_{\mathcal{H}} - C_1(a+b)^{\frac{2}{N}}\rho - C_2\,(a+b)^{\frac{p(1-\gamma_p)}{2}}\rho^{\frac{p\gamma_p}{2}} - C_3\rho^{\frac{N}{N-2}}, \end{aligned}$$

which establishes the boundedness of $\{(u_m, v_m)\}$ in $\mathcal{H}$. So $(u_m, v_m) \rightharpoonup (\bar{u}, \bar{v}) \in \mathcal{H}$. By the compact embedding $\mathcal{H} \hookrightarrow L^p(\mathbb{R}^N) \times L^q(\mathbb{R}^N)(2 \leq p, q < 2^*)$, we can deduce

$$\int_{\mathbb{R}^N} F(u_m, v_m)\mathrm{d}x \to \int_{\mathbb{R}^N} F(\bar{u}, \bar{v})\mathrm{d}x \quad \text{as} \quad m \to \infty.$$

It follows from the lower semi-continuity of the norm that

$$\mathcal{I}(\bar{u}, \bar{v}) \leq \liminf_{m\to\infty} \mathcal{I}(u_m, v_m) = m(\rho).$$

Therefore, $\mathcal{I}(\bar{u}, \bar{v}) = m(\rho)$, which implies that $(\bar{u}, \bar{v})$ is a minimizer of $m(\rho)$. Again due to the compactness of the embedding and (3.4), one can see that $(\bar{u}, \bar{v}) \in B(\rho) \setminus \partial B(\rho)$. Then $(\bar{u}, \bar{v})$ is the critical point of $\mathcal{I}$. By using the method of Lagrange multipliers [10], we know that $(\bar{u}, \bar{v})$ is the solution to system (1.1). □

Subsequently, we employ the variant mountain pass theorem to search for the second critical point of $\mathcal{I}$. We shall prove that $\mathcal{I}$ possesses the mountain pass geometrical structure.

Lemma 3.2. *Suppose that* (V1), (V2) *and* (F1)–(F3) *hold. Then $\mathcal{I}$ has a mountain pass geometry on $S(a) \times S(b)$.*

Proof. According to Proposition 3.1, there exists a $\rho > 0$ such that $\mathcal{I}(u,v) \geq \frac{\rho}{4}$ for any $\int_{\mathbb{R}^N} |\nabla u|^2 + |\nabla v|^2 \mathrm{d}x = \rho$, and there exists $(\bar{u}, \bar{v}) \in B(\rho) \setminus \partial B(\rho)$ such that $\mathcal{I}(\bar{u}, \bar{v}) < \frac{\rho}{8}$.

Let $(\phi_1, \phi_2) \in C_0^\infty(\mathbb{R}^N, \mathbb{R}^2)$ be such that $|\phi_1|_2^2 = a, |\phi_2|_2^2 = b$. For $t > 0$, denote $t \circ \phi_i = t^{\frac{N}{2}}\phi_i(tx), i = 1, 2$, and

$$\begin{aligned} \mathcal{I}(t\circ\phi_1, t\circ\phi_2) = &\frac{t^2}{2}\int_{\mathbb{R}^N} |\nabla\phi_1|^2 + |\nabla\phi_2|^2\mathrm{d}x + \frac{1}{2}\int_{\mathbb{R}^N} V_1\left(\frac{x}{t}\right)|\phi_1|^2 + V_2\left(\frac{x}{t}\right)|\phi_2|^2\mathrm{d}x \\ &- \frac{1}{t^N}\int_{\mathbb{R}^N} F(t^{\frac{N}{2}}\phi_1, t^{\frac{N}{2}}\phi_2)\mathrm{d}x. \end{aligned}$$

By the definition of $(\phi_1, \phi_2) \in C_0^\infty(\mathbb{R}^N, \mathbb{R}^2)$, we have

$$\int_{\mathbb{R}^N} V_1\left(\frac{x}{t}\right)|\phi_1|^2\mathrm{d}x \to aV_1(0), \quad \int_{\mathbb{R}^N} V_2\left(\frac{x}{t}\right)|\phi_2|^2\mathrm{d}x \to bV_2(0) \quad \text{as} \quad t \to +\infty. \tag{3.5}$$

Combining with (F3), we get that

$$\mathcal{I}(t\circ\phi_1, t\circ\phi_2) \to -\infty \quad \text{as} \quad t \to +\infty.$$

Taking a sufficiently large t_1 such that $\int_{\mathbb{R}^N} |\nabla(t_1\circ\phi_1)|^2 + |\nabla(t_1\circ\phi_2)|^2\mathrm{d}x > \rho$ and $\mathcal{I}(t_1\circ\phi_1, t_1\circ\phi_2) < 0$. Therefore, $\mathcal{I}$ satisfies the mountain pass geometric structure. □

For $s \in \mathbb{R}$ and $u \in H^1(\mathbb{R}^N)$, we define the function

$$(s \star u)(x) := e^{\frac{Ns}{2}} u(e^s x).$$

Then by a direct calculation we have

$$|s \star u|_2^2 = |u|_2^2 \quad \text{and} \quad |\nabla(s \star u)|_2^2 = e^{2s}|\nabla u|_2^2.$$

As a consequence, given $(u,v) \in S(a) \times S(b)$, it results that $s \star (u,v) := (s \star u, s \star v) \in S(a) \times S(b)$ as well, for every $s \in \mathbb{R}$. Let $\mathbb{R} \times \mathcal{H}$ be equipped with the scalar product

$$\langle \cdot, \cdot \rangle_{\mathbb{R}\times\mathcal{H}} = \langle \cdot, \cdot \rangle_{\mathcal{H}} + \langle \cdot, \cdot \rangle_{\mathbb{R}}$$

and corresponding norm

$$\| \cdot \|_{\mathbb{R}\times\mathcal{H}}^2 = \| \cdot \|_{\mathbb{R}}^2 + \| \cdot \|_{\mathcal{H}}^2.$$

To abbreviate the notation we set $Y = \mathbb{R} \times \mathcal{H}$, $\langle \cdot, \cdot \rangle_Y = \langle \cdot, \cdot \rangle_{\mathbb{R}\times\mathcal{H}}$ and $\| \cdot \|_Y = \| \cdot \|_{\mathbb{R}\times\mathcal{H}}$. Let $\tilde{\mathcal{I}} : \mathbb{R} \times \mathcal{H} \to \mathbb{R}$ be the functional

$$\begin{aligned} \tilde{\mathcal{I}}(s,u,v) = \mathcal{I}(s \star u, s \star v) = {} & \frac{1}{2} e^{2s} \int_{\mathbb{R}^N} |\nabla u|^2 + |\nabla v|^2 + V_1(e^{-s}x)|u|^2 + V_2(e^{-s}x)|v|^2 \mathrm{d}x \\ & - \frac{1}{e^{sN}} \int_{\mathbb{R}^N} F(e^{\frac{sN}{2}}(u,v)) \mathrm{d}x. \end{aligned} \tag{3.6}$$

We define

$$\tilde{\Gamma} = \{\tilde{\gamma} \in C([0,1], \mathbb{R} \times S(a) \times S(b)) : \tilde{\gamma}(0) = (0, \bar{u}, \bar{v}), \tilde{\gamma}(1) = (0, t_1 \circ \phi_1, t_1 \circ \phi_2)\}$$

and

$$\tilde{c} = \inf_{\tilde{\gamma} \in \tilde{\Gamma}} \max_{t \in [0,1]} \tilde{\mathcal{I}}(\tilde{\gamma}(t)).$$

Let c be the mountain pass level of functional $\mathcal{I}$ defined by

$$c = \inf_{\gamma \in \Gamma} \max_{t \in [0,1]} \mathcal{I}(\gamma(t))$$

where

$$\Gamma = \{\gamma \in C([0,1], S(a) \times S(b)) : \gamma(0) = (\bar{u}, \bar{v}), \gamma(1) = (t_1 \circ \phi_1, t_1 \circ \phi_2)\}.$$

We want to get that there exists a Pohožaev–Palais–Smale sequence for $\mathcal{I}$ restricted to $S(a) \times S(b)$ at the level c. In fact, we have the following lemma, which is motivated by [3,4,19].

Lemma 3.3. *Suppose that* (V1), (V2) *and* (F1)–(F3) *hold. Then there exists a sequence* $\{(u_n, v_n)\} \subset S(a) \times S(b)$ *such that*

$$\mathcal{I}(u_n, v_n) \to c, \quad \|\mathcal{I}'|_{S(a)\times S(b)}(u_n, v_n)\|_{\mathcal{H}^{-1}} \to 0 \quad \text{and} \quad P(u_n, v_n) \to 0 \quad \text{as} \quad n \to \infty.$$

Proof. By the definition of c, there exists a sequence $\{(\hat{u}_n, \hat{v}_n)\} \subset S(a) \times S(b)$ such that

$$\mathcal{I}(\hat{u}_n, \hat{v}_n) \to c.$$

Next we prove $c = \tilde{c}$. In fact, from $\Gamma \subset \tilde{\Gamma}$ we have $\tilde{c} \leq c$. On the other hand, for any

$$\tilde{\gamma}(t) = (s(t), \gamma_1(t), \gamma_2(t)) \in \tilde{\Gamma},$$

we have for all $t \in [0,1]$, $\gamma(t) = s(t) \star (\gamma_1(t), \gamma_2(t)) \in \Gamma$ and

$$\tilde{\mathcal{I}}(\tilde{\gamma}(t)) = \mathcal{I}(s(t) \star (\gamma_1(t), \gamma_2(t))) = \mathcal{I}(\gamma(t)).$$

It follows from the definition of $\tilde{c}$ that $\tilde{c} \geq c$. Hence $\tilde{c} = c$ and $(0, \hat{u}_n, \hat{v}_n)$ is also a minimizing sequence of $\mathcal{I}$. Now Ekeland's variational principle (see also [3, Lemma 5.5]) yields there exists a sequence $\{(s_n, \tilde{u}_n, \tilde{v}_n)\} \subset \mathbb{R} \times S(a) \times S(b)$ such that

(1) $\tilde{\mathcal{I}}(s_n, \tilde{u}_n, \tilde{v}_n) \in [\tilde{c} - \frac{1}{n}, \tilde{c} + \frac{1}{n}]$,

(2) $\|(s_n, \tilde{u}_n, \tilde{v}_n) - (0, \hat{u}_n, \hat{v}_n)\|_Y \leq \frac{1}{\sqrt{n}}$.

(3) $\|\tilde{\mathcal{I}}'|_{\mathbb{R}\times S(a)\times S(b)}(s_n, \tilde{u}_n, \tilde{v}_n)\|_{Y^{-1}} \leq \frac{2}{\sqrt{n}}$.

For the convenience of notation, we denote $w_n := (u_n, v_n) = s_n \star (\tilde{u}_n, \tilde{v}_n) =: s_n \star \tilde{w}_n$. By (3), we have

$$\langle \tilde{\mathcal{I}}'(s_n, \tilde{w}_n), z \rangle_{Y^{-1},Y} = o(\|z\|_Y), \tag{3.7}$$

for all $z \in T_{(s_n,\tilde{u}_n,\tilde{v}_n)} S(a) \times S(b) := \{(z_1, z_2, z_3) \in Y : \langle \tilde{u}_n, z_2 \rangle_2 = \langle \tilde{v}_n, z_3 \rangle_2 = 0\}$.

Choosing $z = (1, 0, 0)$ in (3.7), we get

$$\begin{aligned}
&\langle \tilde{\mathcal{I}}'(s_n, \tilde{w}_n), (1,0,0) \rangle \\
&\quad = \frac{d}{dt}\mathcal{I}((s_n + t) \star \tilde{w}_n)\Big|_{t=0} \\
&\quad = \frac{d}{dt}\mathcal{I}(e^{\frac{(s_n+t)N}{2}} \tilde{w}_n(e^{s_n+t}x))\Big|_{t=0} \\
&\quad = e^{2s_n} \int_{\mathbb{R}^N} |\nabla \tilde{w}_n|^2 \mathrm{d}x - e^{-s_n N} \int_{\mathbb{R}^N} \frac{N}{2} e^{\frac{s_n N}{2}} \tilde{w}_n \cdot \nabla F(e^{\frac{s_n N}{2}} \tilde{w}_n) - NF(e^{\frac{s_n N}{2}} \tilde{w}_n) \mathrm{d}x \\
&\qquad - \frac{1}{2} \int_{\mathbb{R}^N} e^{-s_n} x \cdot \nabla V_1(e^{-s_n} x) |\tilde{u}_n|^2 + e^{-s_n} x \cdot \nabla V_2(e^{-s_n} x) |\tilde{v}_n|^2 \mathrm{d}x \\
&\quad = \int_{\mathbb{R}^N} |\nabla w_n|^2 \mathrm{d}x - \frac{N}{2} \int_{\mathbb{R}^N} w_n \cdot \nabla F(w_n) - 2F(w_n) \mathrm{d}x \\
&\qquad - \frac{1}{2} \int_{\mathbb{R}^N} x \cdot \nabla V_1(x) |u_n|^2 + x \cdot \nabla V_2(x) |v_n|^2 \mathrm{d}x \\
&\quad = P(u_n, v_n) \to 0.
\end{aligned}$$

Then for any $(\varphi_1, \varphi_2) \in T_{(u_n,v_n)} S(a) \times S(b) := \{\langle u_n, \varphi_1 \rangle_2 = \langle v_n, \varphi_2 \rangle_2 = 0\}$, setting $(\varphi_1, \varphi_2) = (s_n \star \psi_1, s_n \star \psi_2)$, we have

$$\begin{aligned}
\langle \mathcal{I}'(u_n, v_n), (\varphi_1, \varphi_2) \rangle &= \int_{\mathbb{R}^N} \nabla u_n \cdot \nabla \varphi_1 + \nabla v_n \cdot \nabla \varphi_2 \mathrm{d}x \\
&\quad + \int_{\mathbb{R}^N} V_1(x) u_n \varphi_1 + V_2(x) v_n \varphi_2 \mathrm{d}x \\
&\quad - \int_{\mathbb{R}^N} (\varphi_1, \varphi_2) \cdot \nabla F(u_n, v_n) \mathrm{d}x \\
&= e^{s_n(\frac{N+2}{2})} \int_{\mathbb{R}^N} \nabla \tilde{u}_n(e^{s_n} x) \cdot \nabla \varphi_1 + \nabla \tilde{v}_n(e^{s_n} x) \cdot \nabla \varphi_2 \mathrm{d}x \\
&\quad + e^{\frac{s_n N}{2}} \int_{\mathbb{R}^N} V_1(x) \tilde{u}_n(e^{s_n} x) \varphi_1 + V_2(x) \tilde{v}_n(e^{s_n} x) \varphi_2 \mathrm{d}x \\
&\quad - \int_{\mathbb{R}^N} (\varphi_1, \varphi_2) \cdot \nabla F(e^{\frac{s_n N}{2}} \tilde{u}_n(e^{s_n} x), e^{\frac{s_n N}{2}} \tilde{v}_n(e^{s_n} x)) \mathrm{d}x
\end{aligned}$$

$$
\begin{aligned}
&= e^{2s_n}\int_{\mathbb{R}^N} \nabla\tilde{u}_n\cdot\nabla\psi_1+\nabla\tilde{v}_n\cdot\nabla\psi_2\mathrm{d}x\\
&\quad+\int_{\mathbb{R}^N} V_1(e^{-s_n}x)\tilde{u}_n\psi_1+V_2(e^{-s_n}x)\tilde{v}_n\psi_2\mathrm{d}x\\
&\quad-e^{-\frac{s_nN}{2}}\int_{\mathbb{R}^N}(\psi_1,\psi_2)\cdot\nabla F(e^{\frac{s_nN}{2}}\tilde{u}_n,e^{\frac{s_nN}{2}}\tilde{v}_n)\mathrm{d}x\\
&=\langle\mathcal{I}'(s_n\star(\tilde{u}_n,\tilde{v}_n)),(\psi_1,\psi_2)\rangle\\
&=\langle\tilde{\mathcal{I}}'(s_n,\tilde{u}_n,\tilde{v}_n),(0,\psi_1,\psi_2)\rangle.
\end{aligned}\tag{3.8}
$$

For $(\varphi_1,\varphi_2)\in T_{(u_n,v_n)}S(a)\times S(b)$ we have

$$
\begin{aligned}
\int_{\mathbb{R}^N}\tilde{u}_n\psi_1\mathrm{d}x&=\int_{\mathbb{R}^N}((-s_n)\star u_n)((-s_n)\star\varphi_1)\mathrm{d}x\\
&=\int_{\mathbb{R}^N}e^{-s_nN}u_n(e^{-s_n}x)\varphi_1(e^{-s_n}x)\mathrm{d}x\\
&=\int_{\mathbb{R}^N}u_n\varphi_1\mathrm{d}x=0.
\end{aligned}
$$

A similar calculation implies that

$$
\int_{\mathbb{R}^N}\tilde{v}_n\psi_2\mathrm{d}x=\int_{\mathbb{R}^N}v_n\varphi_2\mathrm{d}x=0.
$$

As a consequence $(\psi_1,\psi_2)\in T_{(\tilde{u}_n,\tilde{v}_n)}S(a)\times S(b)$. By (3.8), we have

$$
\langle\mathcal{I}'|_{S(a)\times S(b)}(u_n,v_n),(\varphi_1,\varphi_2)\rangle=o(\|(\psi_1,\psi_2)\|_{\mathcal{H}}),\tag{3.9}
$$

and (2) implies that

$$
|s_n|=|s_n-0|\le\|(s_n,\tilde{u}_n,\tilde{v}_n)-(0,\hat{u}_n,\hat{v}_n)\|_Y\to 0\tag{3.10}
$$

as $n\to\infty$. Then for n large enough there exist $A_1,A_2>0$ such that

$$
A_1<\frac{\|(\varphi_1,\varphi_2)\|_{\mathcal{H}}}{\|(\psi_1,\psi_2)\|_{\mathcal{H}}}<A_2,
$$

where $(\varphi_1,\varphi_2)\neq(0,0)$. It results in

$$
\langle\mathcal{I}'|_{S(a)\times S(b)}(u_n,v_n),(\varphi_1,\varphi_2)\rangle=o(\|(\varphi_1,\varphi_2)\|_{\mathcal{H}}).
$$

This completes the proof. □

Lemma 3.4. *Suppose that* (V1), (V2) *and* (F1)–(F3) *hold. Then the sequence* $\{(u_n,v_n)\}$ *in Lemma 3.3 is bounded in* $\mathcal{H}$.

Proof. We assume that $\|(u_n,v_n)\|_{\mathcal{H}}\to\infty$ as $n\to\infty$. Let $z_n=\frac{(u_n,v_n)}{\|(u_n,v_n)\|_{\mathcal{H}}}$, so $\|z_n\|_{\mathcal{H}}=1$, going if necessary to a subsequence, we may assume that

$$
\begin{aligned}
z_n&\rightharpoonup z && \text{in } \mathcal{H},\\
z_n&\to z && \text{in } L^p(\mathbb{R}^N)\times L^q(\mathbb{R}^N)(2\le p,q<2^*),\\
z_n(x)&\to z(x) && a.e.\ x\in\mathbb{R}^N.
\end{aligned}
$$

Set $\Omega = \{x \in \mathbb{R}^N : z(x) \neq (0,0)\}$. We claim that meas$(\Omega) = 0$, if not, then meas$(\Omega) > 0$ and $|(u_n, v_n)| \to \infty$ for a.e. $x \in \Omega$. From (F1) $-$ (F3) and Fatou's lemma, one has

$$\begin{aligned} \liminf_{n\to\infty} \frac{\int_{\mathbb{R}^N} F(u_n,v_n)\mathrm{d}x}{\|(u_n,v_n)\|_{\mathcal{H}}^2} &\geq \liminf_{n\to\infty} \int_\Omega \frac{F(u_n,v_n)|z_n|^2}{|(u_n,v_n)|^2}\mathrm{d}x \\ &\geq \int_\Omega \liminf_{n\to\infty} \frac{F(u_n,v_n)|z_n|^2}{|(u_n,v_n)|^2}\mathrm{d}x = \infty. \end{aligned} \tag{3.11}$$

Since $\mathcal{I}(u_n, v_n) \to c$, we obtain that

$$\begin{aligned} \liminf_{n\to\infty} \frac{\int_{\mathbb{R}^N} F(u_n,v_n)\mathrm{d}x}{\|(u_n,v_n)\|_{\mathcal{H}}^2} &\leq \limsup_{n\to\infty} \frac{\int_{\mathbb{R}^N} F(u_n,v_n)\mathrm{d}x}{\|(u_n,v_n)\|_{\mathcal{H}}^2} \\ &\leq \frac{1}{2} - \liminf_{n\to\infty} \frac{\mathcal{I}(u_n,v_n)}{\|(u_n,v_n)\|_{\mathcal{H}}^2} \to \frac{1}{2}, \end{aligned} \tag{3.12}$$

which contradicts with (3.11). Hence, meas$(\Omega) = 0$, which implies $z(x) = (0,0)$ for a.e. $x \in \mathbb{R}^N$ and $z_n \to (0,0)$ in $L^p(\mathbb{R}^N) \times L^q(\mathbb{R}^N)(2 \leq p,q < 2^*)$.

By (F2) and (F3), we get that there exists $\widetilde{C}_1 > 0$ satisfying

$$|F(w)| \leq \widetilde{C}_1(|w|^{2+\frac{4}{N}} + |w|^{2^*}) \leq 2\widetilde{C}_1|w|^{2+\frac{4}{N}} \leq 2\widetilde{C}_1|w|^2, \quad \forall |w| \in [0,1].$$

Again by (F2) and (F3), there exists $M > 0$ such that

$$|F(w)| \leq \widetilde{C}_1(|w|^{2+\frac{4}{N}} + |w|^{2^*}) \leq M \leq M|w|^2, \quad \forall |w| \in [1, R_0].$$

Hence, we know that

$$|F(w)| \leq (M + 2\widetilde{C}_1)|w|^2, \quad \forall |w| \in [0, R_0]. \tag{3.13}$$

By a similar discussion, we can deduce that there exists $\widetilde{C}_2 > 0$ such that

$$|w \cdot \nabla F(w)| \leq \widetilde{C}_2|w|^2, \quad \forall |w| \in [0, R_0]. \tag{3.14}$$

It follows from (3.13) and (3.14) that one gets

$$|w \cdot \nabla F(w) - \theta F(w)| \leq \widetilde{C}|w|^2, \quad \forall |w| \in [0, R_0], \tag{3.15}$$

where $\widetilde{C} = \widetilde{C}_2 + \theta(M + 2\widetilde{C}_1)$. Combining with (F3), we deduce that

$$w \cdot \nabla F(w) - \theta F(w) \geq -\widetilde{C}|w|^2, \quad \forall |w| \in \mathbb{R}. \tag{3.16}$$

Hence, it follows from (3.16) and (V2) that

$$\begin{aligned} o_n(1) &= \frac{(\theta-2)\mathcal{I}(u_n,v_n) - \frac{2}{N}P(u_n,v_n)}{\|(u_n,v_n)\|_{\mathcal{H}}^2} \\ &= \frac{N\theta - 2N - 4}{2N} + \frac{\int_{\mathbb{R}^N} (u_n,v_n)\cdot\nabla F(u_n,v_n) - \theta F(u_n,v_n)\mathrm{d}x}{\|(u_n,v_n)\|_{\mathcal{H}}^2} \\ &\quad + \frac{1}{N}\frac{\int_{\mathbb{R}^N}(2V_1 + x\cdot\nabla V_1)|u_n|^2 + (2V_2 + x\cdot\nabla V_2)|v_n|^2\mathrm{d}x}{\|(u_n,v_n)\|_{\mathcal{H}}^2} \\ &\geq \frac{N\theta - 2N - 4}{2N} - (\widetilde{C} + \frac{\bar{C}_1 + \bar{C}_2}{N})\int_{\mathbb{R}^N} |z_n|^2\mathrm{d}x \to \frac{N\theta - 2N - 4}{2N} \quad \text{as } n \to \infty, \end{aligned}$$

which implies that $0 \geq \frac{N\theta-2N-4}{2N}$. This is a contradiction with $\theta > 2 + \frac{4}{N}$. Therefore, $\{(u_n, v_n)\}$ is bounded in $\mathcal{H}$. □

Proof of Theorem 1.1. In view of Lemma 3.3 and Lemma 3.4, there exists $(\widetilde{u},\widetilde{v}) \in \mathcal{H}$ such that $(u_n,v_n) \rightharpoonup (\widetilde{u},\widetilde{v})$ in $\mathcal{H}$. Since $\mathcal{I}'|_{S(a)\times S(b)}(u_n,v_n) \to 0$ in $\mathcal{H}^{-1}$, Lagrange multiplier's rule yields that there exists a sequence $\{(\lambda_{1,n},\lambda_{2,n})\} \subset \mathbb{R}\times\mathbb{R}$ such that

$$\mathcal{I}'|_{S(a)\times S(b)}(u_n,v_n) = \mathcal{I}'(u_n,v_n) - \lambda_{1,n}(u_n,0) - \lambda_{2,n}(0,v_n) \to 0 \quad \text{in } \mathcal{H}^{-1}, \tag{3.17}$$

where

$$\begin{cases} \int_{\mathbb{R}^N} |\nabla u_n|^2 + V_1(x)|u_n|^2 - \lambda_{1,n}|u_n|^2 - F_u(u_n,v_n)u_n \mathrm{d}x = o(\|(u_n,v_n)\|_{\mathcal{H}}), \\ \int_{\mathbb{R}^N} |\nabla v_n|^2 + V_2(x)|v_n|^2 - \lambda_{2,n}|v_n|^2 - F_v(u_n,v_n)v_n \mathrm{d}x = o(\|(u_n,v_n)\|_{\mathcal{H}}), \end{cases} \tag{3.18}$$

which implies $(\lambda_{1,n},\lambda_{2,n})$ is bounded. We may assume there exist $\lambda_1,\lambda_2 \in \mathbb{R}$ such that $(\lambda_{1,n},\lambda_{2,n}) \to (\lambda_1,\lambda_2)$ as $n\to\infty$. Passing to the limit in (3.18), then $(\widetilde{u},\widetilde{v})$ satisfies

$$\begin{cases} \int_{\mathbb{R}^N} |\nabla \widetilde{u}|^2 \mathrm{d}x + \int_{\mathbb{R}^N} V_1(x)|\widetilde{u}|^2 \mathrm{d}x - \lambda_1 \int_{\mathbb{R}^N} |\widetilde{u}|^2 \mathrm{d}x = \int_{\mathbb{R}^N} F_u(\widetilde{u},\widetilde{v})\widetilde{u}\mathrm{d}x, \\ \int_{\mathbb{R}^N} |\nabla \widetilde{v}|^2 \mathrm{d}x + \int_{\mathbb{R}^N} V_2(x)|\widetilde{v}|^2 \mathrm{d}x - \lambda_2 \int_{\mathbb{R}^N} |\widetilde{v}|^2 \mathrm{d}x = \int_{\mathbb{R}^N} F_v(\widetilde{u},\widetilde{v})\widetilde{v}\mathrm{d}x. \end{cases} \tag{3.19}$$

By Lemma 2.3 and (3.2), we deduce

$$\begin{cases} \int_{\mathbb{R}^N} |\nabla u_n - \nabla \widetilde{u}|^2 \mathrm{d}x + \int_{\mathbb{R}^N} V_1(x)|u_n - \widetilde{u}|^2 \mathrm{d}x = o(\|(u_n,v_n)\|_{\mathcal{H}}), \\ \int_{\mathbb{R}^N} |\nabla v_n - \nabla \widetilde{v}|^2 \mathrm{d}x + \int_{\mathbb{R}^N} V_2(x)|v_n - \widetilde{v}|^2 \mathrm{d}x = o(\|(u_n,v_n)\|_{\mathcal{H}}). \end{cases}$$

which implies that $(u_n,v_n) \to (\widetilde{u},\widetilde{v})$ in $\mathcal{H}$. □

Proof of Theorem 1.4. Set $\bar{c} = \inf_{\mathcal{K}} \mathcal{I}$, where

$$\mathcal{K} = \{(u,v) \in S(a)\times S(b) : \mathcal{I}'|_{S(a)\times S(b)}(u,v) = 0\}.$$

Let $\{(\bar{u}_n,\bar{v}_n)\} \subset \mathcal{K}$ be a minimizing sequence of $\bar{c}$. By Lemma 2.4, $\{(\bar{u}_n,\bar{v}_n)\}$ is also a Pohožaev–Palais–Smale sequence for $\mathcal{I}$ restricted to $S(a)\times S(b)$ at the level $\bar{c}$. It follows from a similar argument as above, we conduct that there exists $(u,v) \in \mathcal{H}$ such that $(\bar{u}_n,\bar{v}_n) \to (u,v)$ in $\mathcal{H}$. Thus (u,v) is a ground state solution of system (1.1) so that

$$\mathcal{I}(u,v) = \bar{c} \le m(\rho) < \frac{\rho}{8}. \tag{3.20}$$

Next we prove that the ground state is the local minimizer. In fact, for any $t>0$, let

$$\begin{aligned} \Psi_{(u,v)}(t) = \mathcal{I}(t\circ u, t\circ v) &= \frac{t^2}{2}\int_{\mathbb{R}^N} |\nabla u|^2 + |\nabla v|^2 \mathrm{d}x + \frac{1}{2}\int_{\mathbb{R}^N} V_1\left(\frac{x}{t}\right)|u|^2 + V_2\left(\frac{x}{t}\right)|v|^2 \mathrm{d}x \\ &\quad - \frac{1}{t^N}\int_{\mathbb{R}^N} F(t^{\frac{N}{2}}u, t^{\frac{N}{2}}v)\mathrm{d}x. \end{aligned}$$

From Theorem 1.1, we are aware that $\Psi'_{(u,v)}(t)$ possesses at least two zero points. On the other hand, through a standard computation, we find that

$$\begin{aligned} \Psi''_{(u,v)}(t) = \int_{\mathbb{R}^N} |\nabla u|^2 + |\nabla v|^2 \mathrm{d}x &+ \frac{1}{2}\int_{\mathbb{R}^N} L_1\left(\frac{1}{t}\right)|u|^2 + L_2\left(\frac{1}{t}\right)|v|^2 \mathrm{d}x \\ &- \frac{N}{2}\int_{\mathbb{R}^N} g(t^{\frac{N}{2}}u, t^{\frac{N}{2}}v)|(u,v)|^{2+\frac{4}{N}}\mathrm{d}x. \end{aligned}$$

According to (F4) and (V3), $\Psi''_{(u,v)}(t)$ is monotonically decreasing on $(0,+\infty)$. Thus $\Psi'_{(u,v)}(t)$ only has two zero points, denoted as $t^+_{(u,v)}$ and $t^-_{(u,v)}$, with $0 < t^+_{(u,v)} < t^-_{(u,v)}$. Additionally, we have $(t^+_{(u,v)} \circ u, t^+_{(u,v)} \circ v) \in B(\rho)$ and

$$\mathcal{I}(t^+_{(u,v)} \circ u, t^+_{(u,v)} \circ v) < \frac{\rho}{8} < \mathcal{I}(t^-_{(u,v)} \circ u, t^-_{(u,v)} \circ v).$$

Since $(u,v) \in \mathcal{K}$ and (3.20), we have $t^+_{(u,v)} = 1$, which implies $(u,v) \in B(\rho)$. Thus $\bar{c} = \mathcal{I}(u,v) \geq m(\rho) \geq \bar{c}$. It follows that (u,v) is the local minimizer. □

Acknowledgements

This work was supported by National Natural Science Foundation of China (No. 12561019). The authors would also like to thank the editor for careful manuscript handling and the referees for many valuable comments that help to improve the manuscript.

References

[1] C. O. Alves, C. Ji, Normalized solutions for the Schrödinger equations with L^2-subcritical growth and different types of potentials, *J. Geom. Anal.* **32**(2022), No. 5, Paper No. 165, 25 pp. https://doi.org/10.1007/s12220-022-00908-0; MR4396790

[2] T. Bartsch, S. De Valeriola, Normalized solutions of nonlinear Schrödinger equations, *Arch. Math. (Basel)* **100**(2013), No. 1, 75–83. https://doi.org/10.1007/s00013-012-0468-x; MR3009665

[3] T. Bartsch, L. Jeanjean, Normalized solutions for nonlinear Schrödinger systems, *Proc. Roy. Soc. Edinburgh Sect. A* **148**(2018), No. 2, 225–242. https://doi.org/10.1017/S0308210517000087; MR3777573

[4] T. Bartsch, L. Jeanjean, N. Soave, Normalized solutions for a system of coupled cubic Schrödinger equations on $\mathbb{R}^3$, *J. Math. Pures Appl.* **106**(2016), No. 2, 583–614. https://doi.org/10.1016/j.matpur.2016.03.004; MR3539467

[5] T. Bartsch, R. Molle, M. Rizzi, G. Verzini, Normalized solutions of mass supercritical Schrödinger equations with potential, *Comm. Partial Differential Equations* **46**(2021), No. 9, 1729–1756. https://doi.org/10.1080/03605302.2021.1893747; MR4304693

[6] T. Bartsch, N. Soave, A natural constraint approach to normalized solutions of nonlinear Schrödinger equations and systems, *J. Funct. Anal.* **272**(2017), No. 12, 4998–5037. https://doi.org/10.1016/j.jfa.2017.01.025; MR3639521

[7] T. Bartsch, Z. Q. Wang, Existence and multiplicity results for some superlinear elliptic problems on $\mathbb{R}^N$, *Comm. Partial Differential Equations* **20**(1995), No. 9–10, 1725–1741. https://doi.org/10.1080/03605309508821149; MR1349229

[8] T. Bartsch, X. X. Zhong, W. M. Zou, Normalized solutions for a coupled Schrödinger system, *Math. Ann.* **380**(2021), No. 3–4, 1713–1740. https://doi.org/10.1007/s00208-020-02000-w; MR4297197

[9] H. Berestycki, P.-L. Lions, Nonlinear scalar field equations. I. Existence of a ground state, *Arch. Rational Mech. Anal.* **82**(1983), No. 4, 313–345. https://doi.org/10.1007/BF00250555; MR0695535

[10] H. Berestycki, P.-L. Lions, Nonlinear scalar field equations. II. Existence of infinitely many solutions, *Arch. Rational Mech. Anal.* **82**(1983), No. 4, 347–375. https://doi.org/10.1007/BF00250556; MR0695536

[11] B. Bieganowski, J. Mederski, Normalized ground states of the nonlinear Schrödinger equation with at least mass critical growth, *J. Funct. Anal.* **280**(2021), No. 11 Paper No. 108989, 26 pp. https://doi.org/10.1016/j.jfa.2021.108989; MR4232669

[12] H. Brézis, E. Lieb, A relation between pointwise convergence of functions and convergence of functionals, *Proc. Amer. Math. Soc.* **88**(1983), No. 3, 486–490. https://doi.org/10.2307/2044999; MR0699419

[13] Z. Chen, W. Zou, Normalized solutions for nonlinear Schrödinger systems with linear couples, *J. Math. Anal. Appl.* **499**(2021), No. 1, Paper No. 125013, 22 pp. https://doi.org/10.1016/j.jmaa.2021.125013; MR4208029

[14] Y. H. Ding, X. X. Zhong, Normalized solution to the Schrödinger equation with potential and general nonlinear term: mass super-critical case, *J. Differential Equations* **334**(2022), 194–215. https://doi.org/10.1016/j.jde.2022.06.013; MR4445673

[15] T. X. Gou, L. Jeanjean, Multiple positive normalized solutions for nonlinear Schrödinger systems, *Nonlinearity* **31**(2018), No. 5 2319–2345. https://doi.org/10.1088/1361-6544/aab0bf; MR3816676

[16] Y. J. Guo, X. Y. Zeng, H. S. Zhou, Blow-up solutions for two coupled Gross–Pitaevskii equations with attractive interactions, *Discrete Contin. Dyn. Syst.* **37**(2017), No. 7, 3749–3786. https://doi.org/10.3934/dcds.2017159; MR3639439

[17] Z. Y. Guo, F. Y. Liang, Normalized solutions for coupled Schrödinger systems with potentials and nonhomogeneous nonlinearities, *Commun. Pure Appl. Anal.* **23**(2024), No. 6, 808–828. https://doi.org/10.3934/cpaa.2024036; MR4762523

[18] N. Ikoma, Y. Miyamoto, Stable standing waves of nonlinear Schrödinger equations with potentials and general nonlinearities, *Calc. Var. Partial Differential Equations* **59**(2020), No. 2, Paper No. 48, 20 pp. https://doi.org/10.1007/s00526-020-1703-0; MR4064338

[19] L. Jeanjean, Existence of solutions with prescribed norm for semilinear elliptic equations, *Nonlinear Anal.* **28**(1997), No. 10, 1633–1659. https://doi.org/10.1016/S0362-546X(96)00021-1; MR1430506

[20] L. Jeanjean, S. S. Lu, A mass supercritical problem revisited, *Calc. Var. Partial Differential Equations* **59**(2020), No. 5, Paper No. 174, 43 pp. https://doi.org/10.1007/s00526-020-01828-z; MR4150876

[21] L. Jeanjean, J. J. Zhang, X. X. Zhong, Normalized ground states for a coupled Schrödinger system: mass super-critical case, *NoDEA Nonlinear Differential Equations Appl.* **31**(2024), No. 5, Paper No. 85, 26 pp. https://doi.org/10.1007/s00030-024-00972-1; MR4768739

[22] G. D. Li, Y. Y. Li, C. L. Tang, Existence and asymptotic behavior of ground state solutions for Schrödinger equations with Hardy potential and Berestycki–Lions type conditions, *J. Differential Equations* **275**(2021), 77–115. https://doi.org/10.1016/j.jde.2020.12.007; MR4190578

[23] G. D. Li, J. J. Zhang, Normalized solutions of Schrödinger equations involving Moser–Trudinger critical growth, *Adv. Nonlinear Anal.* **13**(2024), No. 1, Paper No. 20240024, 22 pp. https://doi.org/10.1515/anona-2024-0024; MR4767340

[24] C. Y. Liu, X. L. Yang, Existence of normalized solutions for semilinear elliptic systems with potential, *J. Math. Phys.* **63**(2022), No. 6, Paper No. 061504, 21 pp. https://doi.org/10.1063/5.0077931; MR4434655

[25] Y. Y. Liu, L. G. Zhao, Normalized solutions for Schrödinger equations with potentials and general nonlinearities, *Calc. Var. Partial Differential Equations* **63**(2024), No. 4, Paper No. 99, 37 pp. https://doi.org/10.1007/s00526-024-02699-4; MR4732299

[26] L. Lu, L^2 normalized solutions for nonlinear Schrödinger systems in $\mathbb{R}^3$, *Nonlinear Anal.* **191**(2020), Paper No. 111621, 19 pp. https://doi.org/10.1016/j.na.2019.111621; MR4011113

[27] M. Struwe, *Variational methods*, Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics, Vol. 34, 4th ed., Springer-Verlag, Berlin, 2008. https://doi.org/10.1007/978-3-540-74013-1; MR2431434

[28] M. I. Weinstein, Nonlinear Schrödinger equations and sharp interpolation estimates, *Comm. Math. Phys.* **87**(1982/83), No. 4, 567–576. MR0691044

[29] Q. Xu, Y. C. Lv, G. D. Li, R. D. Carlos, Infinitely many normalized solutions for Schrödinger equations with combined nonlinearities, *Z. Angew. Math. Phys.* **77**(2026), No. 2, Paper No. 50, 25 pp. https://doi.org/10.1007/s00033-025-02684-7; MR5012445

[30] J. F. Yang, J. G. Yang, Normalized solutions and mass concentration for supercritical nonlinear Schrödinger equations, *Sci. China Math.* **65**(2022), No. 7, 1383–1412. https://doi.org/10.1007/s11425-020-1793-9; MR4444235

[31] X. X. Zhong, W. M. Zou, A new deduction of the strict sub-additive inequality and its application: ground state normalized solution to Schrödinger equations with potential, *Differential Integral Equations* **36**(2023), No. 1–2, 133–160. https://doi.org/10.57262/die036-0102-133; MR4484438