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On the uniqueness of mild solutions and an averaging principle results for partial functional differential equations with infinite delay using the α-norm

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Abstract. In this paper, we study a class of partial functional differential equations with infinite delay within an abstract phase space framework. By employing the α-norm, we establish the existence and uniqueness of mild solutions under a more general condition on the nonlinear term, which is weaker than the classical Lipschitz condition. The linear part is governed by an unbounded operator that generates an analytic semigroup. Furthermore, an averaging principle is established in this case. To illustrate the applicability of the theoretical results, a concrete example is provided.

Keywords: analytic semigroup, infinite delay, alpha-norm, non-Lipschitz condition.

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1 Introduction

This work is concerned with the uniqueness existence of mild solutions and an averaging principle in the case of finite interval, especially $t \in [0, a]$, of the following partial functional differential equations with infinite delay of the form,

$$\begin{cases} x'(t) = -Ax(t) + f\left(\frac{t}{\varepsilon}, x_t\right), & t \geq 0, \\ x_0 = \varphi \in \mathcal{B}_\alpha, \end{cases} \tag{P_ε}$$

where $\varepsilon > 0$ and $-A$ generates an analytic semigroup $(T(t))_{t\geq 0}$ on a Banach space $(X, \|\cdot\|)$. The space $\mathcal{B}$ is a Banach space of functions mapping $(-\infty, 0]$ to X defined axiomatically. For $0 < \alpha < 1$, A^α denotes the fractional power of A. We assume that f is defined on $\mathbb{R}^+ \times \mathcal{B}_\alpha$ with values in X, where $\mathcal{B}_\alpha$ is defined by

$$\mathcal{B}_\alpha = \{\varphi \in \mathcal{B} : \varphi(\theta) \in D(A^\alpha) \text{ for } \theta \leq 0 \text{ and } A^\alpha \varphi \in \mathcal{B}\},$$

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and the function $A^\alpha\varphi$ is defined by

$$(A^\alpha\varphi)(\theta) = A^\alpha(\varphi(\theta)) \quad \text{for } \theta \le 0.$$

For for every $t \ge 0$, the history function $x_t \in \mathcal{B}_\alpha$ is defined by

$$x_t(\theta) = x(t+\theta) \quad \text{for } \theta \le 0.$$

The delay phenomenon is ubiquitous and extremely important in various engineering and dynamic systems, playing a significant role in their evolution. Mathematical models with delays are typically more suitable for practical problems than those without, and they have a stronger application background. The development was initiated for equations with finite delay by Travis and Webb in [23] and [24] and a book of Wu [25]. Concerning the case of infinite delay, an extensive theory has been developed by many authors, including Hino et al. [11], Hale and Kato [9], and Ezzinbi et al. [1] and [2].

However in many practical cases, the source function f may implicitly include spatial derivatives, then the above established results become invalid. In this case, the problem becomes more complicated, and to tackle this difficulty, Ezzinbi et al. in [6] examined the following equation:

$$\begin{cases} x'(t) = -Ax(t) + f(t, x_t), & t \ge 0, \\ x_0 = \varphi \in \mathcal{B}_\alpha. \end{cases}$$

In their work, they proved the existence of a mild solution without proving uniqueness, under the assumption that $-A$ generates an analytic semigroup and that the function f is continuous with respect to both its time variable and its history function. As far as we know, no research has been published on the unique existence of mild solutions and the averaging principle for the problem (P_ε), when the source function involves implicit spatial derivatives, and the the function f satisfies a condition weaker than the ordinary Lipschitz one as we can see in the sequel.

The initial studies on the averaging principle originate from the problem of celestial mechanic, we refer the reader to [4,12,17], for more details about this topic. The authors provide a justification of the averaging principle in the context of finite-dimensional differential equations. Subsequently, in [5,7,8,10,13–15,20,21], the authors extended the averaging principle to both finite- and infinite-dimensional equations as well as to functional differential equations with finite delay.

The approach here is to consider the following partial functional differential equations with infinite delay in X and a small positive parameter ε of the (normal) form

$$\begin{cases} z'(\tau) = \varepsilon[-Az(\tau) + f(\tau, z_\tau)], & \tau \ge 0, \\ z_0 = \psi. \end{cases} \tag{1.1}$$

Under the same hypotheses on the operator $-A$ and the function f, and adding the following condition on the phase space $\mathcal{B}$,

$$\psi\left(\frac{\cdot}{\varepsilon}\right) \in \mathcal{B}, \quad \text{for all } \varepsilon > 0, \tag{C_1}$$

we can prove that problem (1.1) can be equivalently rewritten as (P_ε).

The averaging principle is based on the replacement of the nonautonomous part of a partial differential equation with small parameter $\varepsilon > 0$ by its time average $f_0 : \mathcal{B} \to E$ defined by

$$f_0(v) = \lim_{t\to+\infty} \frac{1}{t}\int_0^t f(s, v)ds, \quad \text{for } v \in \mathcal{B},$$

and which figure in the parallel problem of (1.1) given by

$$\begin{cases} z'(\tau) = \varepsilon[-Az(\tau) + f_0(z_\tau)], & \tau \geq 0, \\ z_0 = \psi. \end{cases} \tag{1.2}$$

Therefore, we conclude that under the additional condition (C_1) on the phase space $\mathcal{B}$ and in order to establish the averaging principle for problem (1.1) in the case of finite time interval, it is necessary to show that, for any fixed time $a > 0$, the (unique) solution of the problem (P_ε) can be approximated (in a certain sense) by the (unique) solution of the parallel problem

$$\begin{cases} x'(t) = -Ax(t) + f_0(x_t), & t \in [0, a], \\ x_0 = \varphi, \end{cases} \tag{P_0}$$

as $\varepsilon \to 0^+$.

The paper is organized as follows: In Section 2, we recall some necessary preliminaries. In Section 3, we establish a uniqueness existence result of mild solutions in a Banach space with infinite delay in an abstract phase space $\mathcal{B}$, where the linear part $-A$ generates an analytic semigroup which will be denoted by $(T(t))_{t\geq 0}$ on a Banach space X, and the nonlinear part f satisfies a condition with respect to the second argument, which is weaker than the ordinary Lipschitz condition. In Section 4, we consider the problem (P_ε) and its corresponding parallel problem (P_0). we establish the averaging principle result for the problem (P_ε). Finally we illustrate our results in the third and the last section by showing how they can be applied to a parabolic partial differential equation with infinite delay.

2 Preliminaries

Consider the following partial functional differential equation in X of the form:

$$\begin{cases} x'(t) = -Ax(t) + f(t, x_t), & t \in [0, a], \\ x_0 = \varphi \in \mathcal{B}_\alpha, \end{cases} \tag{2.1}$$

In the whole of this work, we assume that

($\mathbf{H}_0$) $-A$ is the infinitesimal generator of a compact analytic semigroup $(T(t))_{t\geq 0}$ on a Banach space X and $0 \in \rho(A)$, where $\rho(A)$ is the resolvent set of A.

Then, there exist constants $M \geq 1$ and $\omega \in \mathbb{R}$ such that $\|T(t)\| \leq Me^{\omega t}$ for $t \geq 0$. Without loss of generality, we assume that $\omega > 0$. If the assumption $0 \in \rho(A)$ is not satisfied, one can substitute the operator A by the operator $(A - \sigma I)$ with σ large enough such that $0 \in \rho(A - \sigma I)$ and so we can always assume that $0 \in \rho(A)$.

For the fractional power $(A^\alpha, D(A^\alpha))$, for $0 < \alpha < 1$, and its inverse $A^{-\alpha}$. One has the following known result.

Theorem 2.1 ([16, Theorems 6.8–6.13, pp. 72–74]). *Let $0 < \alpha < 1$ and assume that ($\mathbf{H}_0$) holds. Then*

(i) $D(A^\alpha)$ is a Banach space with the norm $|x|_\alpha = |A^\alpha x|$ for $x \in D(A^\alpha)$,

(ii) $T(t) : X \to D(A^\alpha)$ for $t > 0$,

(iii) $A^{\alpha}T(t)x = T(t)A^{\alpha}x$ *for* $x \in D(A^{\alpha})$ *and* $t \geq 0$,

(iv) *for every* $t > 0$, $A^{\alpha}T(t)$ *is bounded on* X *and there exists* $M_{\alpha} > 0$ *such that*

$$|A^{\alpha}T(t)| \leq M_{\alpha}\frac{\mathrm{e}^{\omega t}}{t^{\alpha}} \text{ for } t > 0, \tag{2.2}$$

(v) $A^{-\alpha}$ *is a bounded linear operator on* X *with* $D(A^{\alpha}) = \mathrm{Im}(A^{-\alpha})$,

(vi) *if* $0 < \alpha < \beta < 1$, *then* $D\big(A^{\beta}\big) \hookrightarrow D(A^{\alpha})$,

(vii) *there exists* $N_{\alpha} > 0$ *such that*

$$\big|(T(t) - I)A^{-\alpha}\big| \leq N_{\alpha}t^{\alpha} \quad \text{for } t > 0. \tag{2.3}$$

In the sequel, we denote by X_{α} *the Banach space* $(D(A^{\alpha}), |\cdot|_{\alpha})$. *Recall that* $A^{-\alpha}$ *is given by the following formulas*

$$A^{-\alpha} = \frac{\sin(\pi\alpha)}{\alpha}\int_0^{\infty} t^{-\alpha}(t + A)^{-1}dt,$$

or

$$A^{-\alpha} = \frac{1}{\Gamma(\alpha)}\int_0^{\infty} t^{\alpha-1}T(t)\mathrm{d}t.$$

We denote by $C_{\alpha} := C_B((-\infty, 0]; X_{\alpha})$ the space of continuous bounded functions from $(-\infty, 0]$ into X_{α} endowed with the norm

$$\|\varphi\|_{\alpha} := \sup_{\theta \leq 0} |\varphi(\theta)|_{\alpha}, \quad \varphi \in C_{\alpha}.$$

Remark that $(C_{\alpha}, \|\cdot\|_{\alpha})$ is also a Banach space.

From now on, we adopt the axiomatic definition of the phase space $\mathcal{B}$, originally introduced by Hale and Kato in [9]. We denote its norm by $\|\cdot\|_{\mathcal{B}}$. The space $\mathcal{B}$ consists of all functions mapping $(-\infty, 0]$ into X and satisfying the following fundamental axioms:

(**A**) there exist a positive constant N, a locally bounded function $M(\cdot)$ on $[0, +\infty)$ and a continuous function $K(\cdot)$ on $[0, +\infty)$, such that if $x : (-\infty, a] \to X$ is continuous on $[\sigma, a]$ with $x_{\sigma} \in \mathcal{B}$, for some $\sigma < a$, then for all $t \in [\sigma, a]$,

- (i) $x_t \in \mathcal{B}$,
- (ii) $t \mapsto x_t$ is continuous with respect to $\|\cdot\|$ on $[\sigma, a]$,
- (iii) $N|x(t)| \leq \|x_t\| \leq K(t - \sigma)\sup_{\sigma \leq s \leq t}|x(s)| + M(t - \sigma)\|x_{\sigma}\|$.

(**B**) $\mathcal{B}$ is a Banach space.

We suppose that

$(\mathbf{H}_1)$ $A^{-\alpha}\varphi \in \mathcal{B}$ for $\varphi \in \mathcal{B}$, where the function $A^{-\alpha}\varphi$ is defined by

$$\big(A^{-\alpha}\varphi\big)(\theta) = A^{-\alpha}(\varphi(\theta)) \text{ for } \theta \leq 0.$$

Consequently, we get the following result.

Proposition 2.2 ([6]). *Assume that* $(\mathbf{H}_0)$ *and* $(\mathbf{H}_1)$ *hold. If* $\mathcal{B}$ *satisfies axioms (***A***) and (***B***). Then* $\mathcal{B}_{\alpha}$ *satisfies axioms (***A***) and (***B***).*

(**A**) *if* $x : (-\infty, a] \to X_\alpha$ *is continuous on* $[\sigma, a]$ *with* $x_\sigma \in \mathcal{B}$, *for some* $\sigma < a$, *then for all* $t \in [\sigma, a]$,

  (i) $x_t \in \mathcal{B}$,

  (ii) $t \mapsto x_t$ *is continuous with respect to* $\|\cdot\|_\alpha$ *on* $[\sigma, a]$,

  (iii) $N|x(t)|_\alpha \le \|x_t\|_\alpha \le K(t-\sigma)\sup_{\sigma\le s\le t}|x(s)|_\alpha + M(t-\sigma)\|x_\sigma\|_\alpha$.

(**B**) $\mathcal{B}_\alpha$ *is a Banach space.*

We have the following definition.

Definition 2.3. Let $\varphi \in \mathcal{B}_\oslash$. We say that $x \in \mathcal{C}(]-\infty, a]; X_\alpha)$ is a mild solution to problem (2.1) if $x_0 = \varphi$ and,

$$x(t) = T(t)\varphi(0) + \int_0^t T(t-s)f(s, x_s)ds \quad \text{for } 0 \le t \le a.$$

3 Existence results

In the whole of this work we suppose that $\mathcal{B}$ satisfies (**A**) and (**B**).

For the existence result we suppose that the function $f : (t, u) \mapsto f(t, u)$ acts from $[0, a] \times \mathcal{B}_\alpha$ into X and continuous with respect to t and u. Then we have the following theorem,

Theorem 3.1 ([6])**.** *Assume that* $(\mathbf{H}_0)$ *and* $(\mathbf{H}_1)$ *hold. Then for* $\varphi \in \mathcal{B}_\alpha$ *problem* (2.1) *has a mild solution on* $[0, a]$, *for a small enough.*

In order to prove a uniqueness result, we suppose that the function f satisfies a condition weaker than the ordinary Lipschitz condition, which is given as follows

$(\mathbf{H}_2)$ For all $(u, v) \in \mathcal{B}_\alpha \times \mathcal{B}_\alpha$, and $t \ge 0$

$$\|f(t, u) - f(t, v)\| \le L(t, \|u - v\|_\alpha),$$

where $L : [0, a] \times [0, +\infty[\to [0, +\infty[$ satisfies:

  (i) continuous, nondecreasing with respect to the first and the second argument, and $L(t, 0) = 0$, for all $t \in [0, a]$,

  (ii) for any $\zeta > 0$ and for all continuous mapping $h : [0, a] \to [0, +\infty[$, such that:

$$h(t) \le \zeta \int_0^t (t-s)^{-\alpha} L(s, h(s))ds, \qquad \text{for all } t \in [0, a],$$

  then $h(t) = 0$ for all $t \in [0, a]$.

We have the following theorem.

Theorem 3.2. *Assume that* $(\mathbf{H}_0)$–$(\mathbf{H}_2)$ *are satisfied, then problem* (2.1) *has a unique mild solution.*

Proof. The existence is ensured by Theorem 3.1. To attend for a uniqueness result we suppose that problem (2.1) has two solutions x_1 and x_2 in X_α. Let $s \in [0, t]$, by using the fact that the mapping L is nondecreasing with respect to the second argument and **A**-(iii), we get

$$\begin{aligned}
|x_1(s) - x_2(s)|_\alpha &= \left| \int_0^s T(s-\tau)\Big(f\big(\tau, x_\tau^1\big) - f\big(\tau, x_\tau^2\big)\Big)d\tau \right|_\alpha \\
&\le M_\alpha e^{\omega a} \int_0^s (s-\tau)^{-\alpha} \Big\| f\big(\tau, x_\tau^1\big) - f\big(\tau, x_\tau^2\big) \Big\| d\tau \\
&\le M_\alpha e^{\omega a} \int_0^s (s-\tau)^{-\alpha} L\left(\tau, K_a \sup_{0\le\theta\le\tau} |x_1(\theta) - x_2(\theta)|_\alpha\right) d\tau.
\end{aligned}$$

We claim that the function $s \mapsto \int_0^s (s-\tau)^{-\alpha} L\big(\tau, K_a \sup_{0\le\theta\le\tau} |x_1(\theta)-x_2(\theta)|_\alpha\big) d\tau$ is nondecreasing on $[0,t]$. In fact let us denote $g(\tau) = L\big(\tau, K_a \sup_{0\le\theta\le\tau} |x_1(\theta)-x_2(\theta)|_\alpha\big)$, which is a nondecreasing function by $(\mathbf{H}_2)$-(i). Then for $s, s' \in [0,t]$ be such that $s < s'$ we have

$$\int_0^s (s-\tau)^{-\alpha} g(\tau) d\tau = \int_0^s \tau^{-\alpha} g(s-\tau) d\tau \le \int_0^{s'} \tau^{-\alpha} g(s'-\tau) d\tau = \int_0^{s'} (s'-\tau)^{-\alpha} g(\tau) d\tau.$$

Then $\sup_{s\in[0,t]} \int_0^s (s-\tau)^{-\alpha} g(\tau) d\tau = \int_0^t (t-\tau)^{-\alpha} g(\tau) d\tau$. Thus,

$$\sup_{0\le s\le t} |x_1(s)-x_2(s)|_\alpha \le M_\alpha e^{\omega a} \int_0^t (t-\tau)^{-\alpha} L\left(\tau, K_a \sup_{0\le\theta\le\tau} |x_1(\theta)-x_2(\theta)|_\alpha\right) d\tau.$$

Consequently,

$$K_a \sup_{0\le s\le t} |x_1(s)-x_2(s)|_\alpha \le \zeta_{\alpha,a} \int_0^t (t-\tau)^{-\alpha} L\left(\tau, K_a \sup_{0\le\theta\le\tau} |x_1(\theta)-x_2(\theta)|_\alpha\right) d\tau.$$

Using $(\mathbf{H}_2)$-(ii), we get

$$\sup_{0\le s\le t} |x_1(s)-x_2(s)|_\alpha = 0 \quad \text{for all } t \in [0,a].$$

Thus, problem (2.1) has a unique mild solution. □

Corollary 3.3. Let k a positive constant. If the mapping $L(t,u) = ku$ for all $t \ge 0$, then our assumption $(\mathbf{H}_2)$ becomes the ordinary Lipschitz condition and $(\mathbf{H}_2)$-(ii) still hold.

Proof. Let $h : [0,a] \to [0,+\infty[$ a continuous function then by taking $L(t,u) = ku$, for all $t \ge 0$, we can see that

$$h(t) \le \zeta \int_0^t (t-s)^{-\alpha} L(s, h(s)) ds, \quad \text{for all } t \in [0,a],$$

becomes

$$h(t) \le \zeta' \int_0^t \frac{h(s)}{(t-s)^\alpha} ds, \quad \text{for all } t \in [0,a], \tag{3.1}$$

for $\zeta' > 0$. To complete to proof we need the following lemma.

Lemma 3.4 ([16, Lemma 6.7, p. 159])**.** *Let $\varphi : [0,a] \to [0,\infty[$ be continuous. If there are positive constants A, B and $0 < \alpha < 1$ such that*

$$\varphi(t) \le A + B \int_0^t \frac{\varphi(s)}{(t-s)^\alpha} ds, \quad \text{for } 0 \le t \le a,$$

then there exist $A' > 0, B' > 0$, such that

$$\varphi(t) \le A' + B' \int_0^t \varphi(s) ds,$$

where $A' = A \sum_{j=0}^{n-1} \left(\frac{Ba^{1-\alpha}}{1-\alpha}\right)^j$, and $B' = \frac{(B\Gamma(1-\alpha))^n}{\Gamma(n(1-\alpha))} a^{n(1-\alpha)-1}$. In particular there exists $C' \ge 0$ such that $\varphi(t) \le C'$, for all $t \ge 0$.

Proof. By taking $A = 0$, $B = \zeta'$ in (3.1), we can deduce that

$$h(t) \le B' \int_0^t h(s) ds.$$

Gronwall's inequality leads to $h(t) = 0$ for all $t \ge 0$. □

Another example of a mapping with the Assumption ($\mathbf{H}_2$), we give the following:

Example 3.5. [22] We take $L(t,u) = q(t)r(u)$, for all $t \in [0,a]$ and $u \in [0,+\infty[$, where $q : [0,a] \to [0,+\infty[$ is a continuous function, and $r : [0,+\infty[\to [0,+\infty[$, is a continuous nondecreasing function with $r(0) = 0$, such that

$$\int_{0^+} \frac{1}{r(u)} du = \infty.$$

Remark 3.6. To establish the existence of global solutions, we assume the same conditions on f as in Theorem 3.2, and we introduce the following additional assumption:

($\mathbf{H}_3$) There exists a continuous function g defined on $\mathbb{R}^+$ such that,

$$\|f(t,u)\| \le g(t)(1+\|u\|_\alpha) \quad \text{for all } t \in \mathbb{R}^+, \ \ u \in \mathcal{B}_\alpha.$$

The proof is similar to that of Corollary 6 in [6].

4 Averaging result

In order to prove an averaging principle result, we choose the phase space $\mathcal{B} = \mathcal{C}_\infty$ (see [11]). The space $\mathcal{C}_\infty$ is formed by all the continuous functions $\varphi : \mathbb{R}^- \to X$, such that $\lim_{\theta\to-\infty}\varphi(\theta)$ exists in X. Endowed with the following norm

$$\|\varphi\|_{\mathcal{B}} = \sup_{\theta\in]-\infty,0]} |\varphi(\theta)|.$$

Consider the partial functional differential equation in X with a small positive parameter ε of the form

$$\begin{cases} x'(t) = Ax(t) + f\left(\frac{t}{\varepsilon}, x_t\right), & t \in [0,a], \\ x_0 = \varphi. \end{cases} \tag{P_ε}$$

Further, parallel to the problem (P_ε), $\varepsilon > 0$, we consider the averaged problem

$$\begin{cases} x'(t) = Ax(t) + f_0(x_t), \\ x_0 = \varphi. \end{cases} \tag{P_0}$$

Remark 4.1. When the delay is infinite the choice of the phase space is crucial to establish the averaging principle result associate to problem (P_ε). To guarantee the equivalence between

$$\begin{cases} w'(\tau) = \varepsilon[Aw(\tau) + f(\tau, w_\tau)], & \tau \in \left[0, \frac{a}{\varepsilon}\right], \\ w_0 = \psi \end{cases} \tag{4.1}$$

and (P_ε). Also between

$$\begin{cases} w'(t) = \varepsilon[Aw(t) + f_0(w_t)] \\ w_0 = \psi \end{cases} \tag{4.2}$$

and (P_0), we have to suppose that the condition (C_1) is satisfied. Clearly, this condition is not satisfied for any choice of the space $\mathcal{B}$ introduced by Hale and Kato [9], as shown by the following phase space examples:

Let $E=\mathcal{C}([0,1];\mathbb{R})$ be the space of continuous functions endowed with the uniform norm topology. Let $\gamma>0$ and $\mathcal{L}_\gamma$ be the space of measurable functions $\varphi:(-\infty,0]\to X$ such that $e^{\gamma\theta}\varphi(\theta)$ is integrable on $(-\infty,0\,]$. We consider the phase space $\mathcal{B}=E\times\mathcal{L}_\gamma$. The space of functions $\varphi\in\mathcal{L}_\gamma$ which is uniquely defined at 0 and endowed with the following norm

$$\|\varphi\|=|\varphi(0)|+\int_{-\infty}^{0}e^{\gamma\theta}|\varphi(\theta)|d\theta \quad \text{for } \varphi\in\mathcal{B}.$$

Let the function φ be defined on $(-\infty,0]$ by $\varphi(\theta)=e^{-\frac{\gamma}{2}\theta}$.
Since, for $\theta=0$, we have $\varphi(0)=1$, and

$$\int_{-\infty}^{0}e^{\gamma\theta}|\varphi(\theta)|d\theta=\int_{-\infty}^{0}e^{\gamma\theta}e^{-\frac{\gamma}{2}\theta}d\theta=\int_{-\infty}^{0}e^{\frac{\gamma}{2}\theta}d\theta=\frac{2}{\gamma},$$

it follows, $\varphi\in\mathcal{B}$.
Now, for $0<\varepsilon<1$, we consider the scaled function $\varphi\left(\frac{\cdot}{\varepsilon}\right)$, given by:

$$\varphi\left(\frac{\theta}{\varepsilon}\right)=e^{-\frac{\gamma}{2\varepsilon}\theta} \text{ for all } \theta\leq 0.$$

We can easily check that the integral condition is not satisfied.

In fact, we have

$$\int_{-\infty}^{0}e^{\gamma\theta}\left|\varphi\left(\frac{\theta}{\varepsilon}\right)\right|d\theta=\int_{-\infty}^{0}e^{\gamma\theta}e^{-\frac{\gamma}{2\varepsilon}\theta}d\theta=\int_{-\infty}^{0}e^{\left(1-\frac{1}{2\varepsilon}\right)\gamma\theta}d\theta.$$

For $\varepsilon\leq\frac{1}{2}$, the integral diverges, and $\varphi\left(\frac{\cdot}{\varepsilon}\right)$ does not belong to $\mathcal{B}$.

Another example of phase space given in [9], where the condition (C_1) fails, is the phase space $\mathcal{B}=C_\gamma$, where C_γ is the space of continuous functions $\psi:\mathbb{R}^-\to\mathbb{R}$ such $e^{\gamma\theta}\psi(\theta)$ has the limit in $\mathbb{R}$ as $\theta\to-\infty$ endowed with the following norm:

$$\|\psi\|_{\mathcal{B}}=\sup\left\{e^{\gamma\theta}|\psi(\theta)|:\theta\in\left]-\infty,0\right]\right\}.$$

It is clear that the function $\psi(\theta)=e^{-\gamma\theta}$ belongs to $\mathcal{B}$ but, for every $0<\varepsilon<1$,

$$\left\|\psi\left(\frac{\cdot}{\varepsilon}\right)\right\|_{\mathcal{B}}=\sup_{\theta\in]-\infty,0]}e^{\gamma\theta}e^{-\gamma(\theta/\varepsilon)}=\sup_{\theta\in]-\infty,0]}e^{\gamma\theta(1-1/\varepsilon)}=+\infty.$$

Now, in problem (4.1), we consider the change of variables

$$\begin{cases}\theta=\dfrac{\theta'}{\varepsilon}, & \tau=\dfrac{t}{\varepsilon},\quad \theta\leq 0,\quad \tau\geq 0,\\[2ex] \psi\left(\dfrac{\theta'}{\varepsilon}\right)=\varphi(\theta), & w\left(\dfrac{t}{\varepsilon}\right)=x(t).\end{cases}$$

Then we have the following proposition.

Proposition 4.2 ([7]). *For the choice $\mathcal{B}=C_\infty$, the condition (C_1) is satisfied and the problems (4.1) and (4.2) can be equivalently rewritten as (P_ε) and (P_0), respectively.*

We suppose the following hypotheses:
Hypotheses. Suppose that the unbounded linear operator A satisfies the condition $(\mathbf{H}_0)$, $(\mathbf{H}_1)$ and the function f acts from $[0,a]\times\mathcal{B}_\alpha$ into X. We consider the following hypotheses:

(f_1) f is continuous with respect to the first and the second argument,

(f_2) for all $(u,v) \in \mathcal{B}_\alpha \times \mathcal{B}_\alpha$,

$$\|f(t,u) - f(t,v)\| \leq \mathbb{L}(\|u-v\|_\alpha) \quad \text{for all} \quad t \in [0,a],$$

where $\mathbb{L} : [0,+\infty[\to [0,+\infty[$ is such that:
(i) $\mathbb{L}(\cdot)$ is continuous, nondecreasing, and $\mathbb{L}(0) = 0$,
(ii) for every nonnegative continuous mapping $h : [0,a] \to [0,+\infty[$ and every constant ζ, we have,

$$h(t) \leq \zeta \int_0^t (t-s)^{-\alpha}\mathbb{L}(h(s))ds \quad \text{for all } t \in [0,a],$$

then $h(t) = 0$.

(f_3) there exists a constant $C > 0$ such that,

$$\|f(t,u)\| \leq C(1 + \|u\|_\alpha) \quad \text{for all } t \in \mathbb{R}^+, \text{ and } u \in \mathcal{B}_\alpha$$

(Af) there exist $\Delta_0 > 0$ and a function $f_0 : \mathcal{B}_\alpha \to X$ satisfying (f_2) and (f_3) and, for all $u \in \mathcal{B}_\alpha$ and $t_1, t_2 \in [0,a]$ with $0 \leq t_1 \leq t_2 \leq t_1 + \Delta_0 \leq a$, we have

$$\lim_{\varepsilon \to 0^+} \int_{t_1}^{t_2} T(t_2 - \theta)\left[f\left(\frac{\theta}{\varepsilon}, u\right) - f_0(u)\right]d\theta = 0.$$

The function $f_0 : u \mapsto \lim_{T\to\infty} \frac{1}{T}\int_0^T f(t,u)dt$ satisfies a condition similar to (f_2) and (f_3), which is

(f_1') for all $(u,v) \in \mathcal{B}_\alpha \times \mathcal{B}_\alpha$,

$$\|f_0(u) - f_0(v)\| \leq \mathbb{L}(\|u-v\|_\alpha),$$

(f_2') there exists a constant $C > 0$ such that, for all $u \in \mathcal{B}_\alpha$,

$$\|f_0(u)\| \leq C(1 + \|u\|_\alpha).$$

In fact, for $(u,v) \in \mathcal{B}^2$ we have

$$\begin{aligned}
\|f_0(u)\| &= \left\|\lim_{T\to\infty} \frac{1}{T}\int_0^T f(t,u)dt\right\| \\
&\leq \lim_{T\to\infty} \frac{1}{T}\int_0^T \|f(t,u)\|dt \\
&\leq \lim_{T\to\infty} \frac{1}{T}\int_0^T C(|1 + \|u\|_\alpha)dt \\
&\leq C(\|1 + \|u\|_\alpha)
\end{aligned}$$

and

$$\begin{aligned}
\|f_0(x) - f_0(y)\| &= \left\|\lim_{T\to\infty} \frac{1}{T}\int_0^T f(t,u) - f(t,v)dt\right\| \\
&\leq \lim_{T\to\infty} \frac{1}{T}\int_0^T \|f(t,u) - f(t,v)\|dt \\
&\leq \lim_{T\to\infty} \frac{1}{T}\int_0^T \mathbb{L}(\|u-v\|_\alpha)dt \\
&\leq \mathbb{L}(\|u-v\|_\alpha).
\end{aligned}$$

By (f_1') we can easily deduce the continuity of the function f_0.

Theorem 4.3. *Assume that the hypotheses* $(\mathbf{H}_0)$, $(\mathbf{H}_1)$, (f_1)–(f_3), *and* (Af) *are satisfied. For* $\varphi \in \mathcal{B}_\alpha$, *the unique mild solution of the problem* (P_ε), *converges uniformly in each set* $[0, a]$ *to the mild solution* x^∞ *of the problem* (P_0), *as* $\varepsilon \to 0^+$.

Proof. According to Theorem 3.2 , for each $\varepsilon > 0$ there exists a unique mild solution x^ε of the problem (P_ε) and there exists a unique mild solution x^∞ of the problem (P_0). Moreover, for each $\varepsilon > 0$, x^ε is defined by

$$x^\varepsilon(t) = T(t)\varphi(0) + \int_0^t T(t-s) f\left(\frac{s}{\varepsilon}, x_s^\varepsilon\right) ds, \quad 0 \le t \le a,$$

and x^∞ is defined by

$$x^\infty(t) = T(t)\varphi(0) + \int_0^t T(t-s) f_0(x_s^\infty) ds, \quad 0 \le t \le a.$$

By axiom $(\mathbf{A})$-(iii) if $x^\varepsilon \to x^\infty$ in $\mathcal{C}([0, a]; X_\alpha)$, then $x_t^\varepsilon \to x_t^\infty$ in $\mathcal{C}(]-\infty, a]; \mathcal{B}_\alpha)$ as $\varepsilon \to 0^+$. Therefore, in order to prove Theorem 4.3, it remains to show that

$$\lim_{\varepsilon \to 0^+} \sup_{t \in [0,a]} |x^\varepsilon(t) - x^\infty(t)|_\alpha = 0.$$

First, we prove that the sequence $(x^\varepsilon)_{\varepsilon > 0}$ is bounded. Let $\varepsilon > 0$, by using (f_3) and Axiom $(\mathbf{A})$-(iii) and let $M := \sup_{0 \le t \le a} \|T(t)\|$. Then for every $t \in [0, a]$, we have

$$\begin{aligned}
|x^\varepsilon(t)|_\alpha &= \left| T(t)\varphi(0) + \int_0^t T(t-s) f\left(\frac{s}{\varepsilon}, x_s\right) ds \right|_\alpha \\
&\le \|T(t)A^\alpha \varphi(0)\| + \left\| \int_0^t A^\alpha T(t-s) f\left(\frac{s}{\varepsilon}, x_s^\varepsilon\right) ds \right\| \\
&\le M|\varphi(0)|_\alpha + M_\alpha \int_0^t \frac{e^{\omega(t-s)}}{(t-s)^\alpha} C(1 + \|x_s^\varepsilon\|_\alpha)) ds. \\
&\le MH\|\varphi\|_\alpha + M_\alpha \int_0^t \frac{e^{\omega(t-s)}}{(t-s)^\alpha} C\left(1 + K_a \sup_{0 \le \theta \le s} |x^\varepsilon(\theta)|_\alpha + N_a \|\varphi\|_\alpha\right) ds.
\end{aligned}$$

Therefore,

$$|x^\varepsilon(t)|_\alpha \le A + B \int_0^t \frac{1}{(t-s)^\alpha} \sup_{0 \le \theta \le s} |x^\varepsilon(\theta)|_\alpha ds, \tag{4.3}$$

where

$$A = MH\|\varphi\|_\alpha + M_\alpha [C(1 + N_a \|\varphi\|_\alpha)] \int_0^a \frac{e^{\omega s}}{s^\alpha} ds,$$

and

$$B = M_\alpha C K_a e^{\omega a}.$$

Let $z(\sigma) = \sup_{0 \le \tau \le \sigma} |x^\varepsilon(\tau)|_\alpha$. We claim that the function $s \mapsto \int_0^s \frac{1}{(s-\sigma)^\alpha} z(\sigma) d\sigma$ is nondecreasing on $[0, t]$.

In fact let $s, s' \in [0, t]$ be such that $s < s'$. Then

$$\int_0^s \frac{1}{(s-\sigma)^\alpha} z(\sigma) d\sigma = \int_0^s \frac{1}{\sigma^\alpha} z(s-\sigma) d\sigma \le \int_0^{s'} \frac{1}{\sigma^\alpha} z(s'-\sigma) d\sigma \le \int_0^{s'} \frac{1}{(s'-\sigma)^\alpha} z(\sigma) d\sigma.$$

Then we obtain

$$\sup_{0 \le s \le t} |x^\varepsilon(s)|_\alpha \le A + B \int_0^t \frac{1}{(t-s)^\alpha} \sup_{0 \le \sigma \le s} |x^\varepsilon(\sigma)|_\alpha ds.$$

Thus, by Lemma 3.4 there is a constant C' such that

$$\sup_{0\le s\le t} |x^{\varepsilon}(s)|_{\alpha} \le C', \quad \text{for all } \varepsilon > 0,$$

Which proves that the sequence $(x^{\varepsilon})_{\varepsilon>0}$ is bounded.

Now by Axiom $(\mathbf{A})$-(iii), the function $t \mapsto x_t^{\infty}$ is continuous on $[0,a]$. Let $\delta > 0$, then we can find a partition $0 = t_0 < t_1 < \ldots < t_m = a$ of $[0,a]$ such that, if $\max_{1\le i\le m}(t_i - t_{i-1}) \le \min\{\delta, \Delta_0\}$, then

$$\left\| x_t^{\infty} - x_{t_i}^{\infty} \right\| \le \delta \quad \text{for all } t \in [t_{i-1}, t_i], \quad i = 1,\ldots,m, \tag{4.4}$$

where Δ_0 is from (Af).

We define a function $t \mapsto \bar{x}_t^{\infty}$by

$$\bar{x}_t^{\infty} = x_{t_{i-1}}^{\infty} \quad \text{for } t \in [t_{i-1}, t_i[, \quad i = 1,\ldots,m. \tag{4.5}$$

Setting $\tau(t) = \max\{i, t_i \le t\}$ and $t' = t_{\varnothing(t)}$, for every $t \in [0,a]$, we get

$$\begin{aligned} x^{\varepsilon}(t) - x^{\infty}(t) = & \int_{t'}^{t} T(t-\theta)\left[f\left(\frac{\theta}{\varepsilon}, x_{\theta}^{\varepsilon}\right) - f_0(x_{\theta}^{\infty})\right] d\theta \\ & + \int_0^{t'} T(t-\theta)\left[f\left(\frac{\theta}{\varepsilon}, x_{\theta}^{\varepsilon}\right) - f\left(\frac{\theta}{\varepsilon}, x_{\theta}^{\infty}\right)\right] d\theta \\ & + \int_0^{t'} T(t-\theta)\left[f\left(\frac{\theta}{\varepsilon}, x_{\theta}^{\infty}\right) - f\left(\frac{\theta}{\varepsilon}, \bar{x}_{\theta}^{\infty}\right)\right] d\theta \\ & + \int_0^{t'} T(t-\theta)\left[f\left(\frac{\theta}{\varepsilon}, \bar{x}_{\theta}^{\infty}\right) - f_0(\bar{x}_{\theta}^{\infty})\right] d\theta \\ & + \int_0^{t'} T(t-\theta)[f_0(\bar{x}_{\theta}^{\infty}) - f_0(x_{\theta}^{\infty})] d\theta \\ = & \; I_1 + I_2 + I_3 + I_4 + I_5. \end{aligned}$$

Since the sequence $(x^{\varepsilon})_{\varepsilon>0}$ is bounded, by Axiom $(\mathbf{A})$-(iii), the sequence $(x_t^{\varepsilon})_{\varepsilon>0}$ is also bounded in $\mathcal{C}([0,a];\mathcal{B}_{\alpha})$, for all $t \in [0,a]$. Thus,

$$\begin{aligned} |I_1|_{\alpha} &= \left| \int_{t'}^{t} T(t-\theta)\left[f\left(\frac{\theta}{\varepsilon}, x_{\theta}^{\varepsilon}\right) - f_0(x_{\theta}^{\infty})\right] d\theta \right|_{\alpha} \\ &\le M_{\alpha} \int_{t'}^{t} \frac{e^{\omega(t-\theta)}}{(t-\theta)^{\alpha}} \left\| f\left(\frac{\theta}{\varepsilon}, x_{\theta}^{\varepsilon}\right) - f_0(x_{\theta}^{\infty}) \right\| d\theta \\ &\le M_{\alpha} \int_{t'}^{t} \frac{e^{\omega(t-\theta)}}{(t-\theta)^{\alpha}} \left[C\left(1 + \sup_{\varepsilon>0} \left\| x_{(.)}^{\varepsilon} \right\|_{\mathcal{C}([0,a];\mathcal{B}_{\alpha})}\right) + C\left(1 + \max_{t\in[0,a]} \|x_t^{\infty}\|_{\alpha}\right) \right] d\theta \\ &\le M_{\alpha} \left[C\left(1 + \sup_{\varepsilon>0} \left\| x_{(.)}^{\varepsilon} \right\|_{\mathcal{C}([0,a];\mathcal{B}_{\alpha})}\right) + C\left(1 + \max_{t\in[0,a]} \|x_t^{\infty}\|_{\alpha}\right) \right] e^{\omega a} \int_{t'}^{t} \frac{1}{(t-\theta)^{\alpha}} d\theta. \end{aligned}$$

Then by a simple change of variable, we get to

$$\int_{t'}^{t} \frac{1}{(t-\theta)^{\alpha}} d\theta = \frac{1}{1-\alpha}(t-t')^{1-\alpha} \le \frac{1}{1-\alpha}\delta^{1-\alpha}.$$

This leads to

$$|I_1|_{\alpha} \le \tilde{\mu}\delta^{1-\alpha},$$

where

$$\tilde{\mu} = M_\alpha C \frac{e^{\omega a}}{1-\alpha}\left[\left(1 + \sup_{\varepsilon>0}\left\|x^{\varepsilon}_{(.)}\right\|_{\mathcal{C}([0,a];\mathcal{B}_\alpha)}\right) + \left(1 + \max_{t\in[0,a]}\|x^{\infty}_t\|_\alpha\right)\right].$$

By axiom $(\mathbf{A})$, and hypothesis (f_2), we get

$$\begin{aligned}
|I_2|_\alpha &\le \left|\int_0^t T(t-\theta)\left[f\left(\frac{\theta}{\varepsilon}, x^{\varepsilon}_\theta\right) - f\left(\frac{\theta}{\varepsilon}, x^{\infty}_\theta\right)\right]d\theta\right|_\alpha \\
&\le M_\alpha e^{\omega a}\int_0^t (t-\theta)^{-\alpha}\mathbb{L}\left(K(\theta)\sup_{0\le s\le\theta}|x^{\varepsilon}(s) - x^{\infty}(s)|_\alpha\right)d\theta.
\end{aligned}$$

Further, let $q = \frac{2-\alpha}{1-\alpha}$, $p = 2-\alpha$. By using (f_2), (4.4), (4.5) and Hölder's inequality we obtain

$$\begin{aligned}
|I_3|_\alpha &\le M_\alpha \int_0^t \frac{e^{\omega(t-\theta)}}{(t-\theta)^\alpha}\left\|f\left(\frac{\theta}{\varepsilon}, x^{\infty}_\theta\right) - f\left(\frac{\theta}{\varepsilon}, \bar{x}^{\infty}_\theta\right)\right\|d\theta \\
&\le M_\alpha\left(\int_0^t \frac{e^{\omega p\theta)}}{\theta^{p\alpha}}d\theta\right)^{\frac{1}{p}}\left(\int_0^t\left\|f\left(\frac{\theta}{\varepsilon}, x^{\infty}_\theta\right) - f\left(\frac{\theta}{\varepsilon}, \bar{x}^{\infty}_\theta\right)\right\|^q d\theta\right)^{\frac{1}{q}} \\
&\le C''\left(\sum_{i=1}^m\int_{t_{i-1}}^{t_i}\left\|f\left(\frac{\theta}{\varepsilon}, x^{\infty}_\theta\right) - f\left(\frac{\theta}{\varepsilon}, \bar{x}^{\infty}_\theta\right)\right\|^q d\theta\right)^{\frac{1}{q}} \\
&\le C''\left(\sum_{i=1}^m\int_{t_{i-1}}^{t_i}\left\|f\left(\frac{\theta}{\varepsilon}, x^{\infty}_\theta\right) - f\left(\frac{\theta}{\varepsilon}, x^{\infty}_{t_{i-1}}\right)\right\|^q d\theta\right)^{\frac{1}{q}} \\
&\le C''\mathbb{L}(\delta)a^{\frac{1}{q}},
\end{aligned}$$

where $C'' = M_\alpha\left(\int_0^t \frac{e^{\omega p\theta)}}{\theta^{p\alpha}}d\theta\right)^{\frac{1}{p}}$.

For the term I_4, we have

$$\begin{aligned}
|I_4|_\alpha &= \left|\int_0^{t'} T(t-\theta)\left[f\left(\frac{\theta}{\varepsilon}, \bar{x}^{\infty}_\theta\right) - f_0(\bar{x}^{\infty}_\theta)\right]d\theta\right|_\alpha \\
&= \left|\sum_{i=1}^{\tau(t)}\int_{t_{i-1}}^{t_i} T(t-t_i)T(t_i-\theta)\left[f\left(\frac{\theta}{\varepsilon}, x^{\infty}_{t_{i-1}}\right) - f_0\left(x^{\infty}_{t_{i-1}}\right)\right]d\theta\right|_\alpha \\
&\le M\sum_{i=1}^{\tau(t)}\left|\int_{t_{i-1}}^{t_i} T(t_i-\theta)\left[f\left(\frac{\theta}{\varepsilon}, x^{\infty}_{t_{i-1}}\right) - f_0\left(x^{\infty}_{t_{i-1}}\right)\right]d\theta\right|_\alpha,
\end{aligned}$$

where $M = \sup_{t\in[0,a]}\|T(t)\|$.

Bearing in mind hypothesis (Af), we can see

$$\max_{1\le i\le m}\lim_{\varepsilon\to 0^+}\int_{t_{i-1}}^{t_i} T(t_i-\theta)\left[f\left(\frac{\theta}{\varepsilon}, x^{\infty}_{t_{i-1}}\right) - f_0\left(x^{\infty}_{t_{i-1}}\right)\right]d\theta = 0.$$

Thus,

$$\max_{1\le i\le m}\left|\int_{t_{i-1}}^{t_i} T(t_i-\theta)\left[f\left(\frac{\theta}{\varepsilon}, x^{\infty}_{t_i-1}\right) - f_0\left(x^{\infty}_{t_{i-1}}\right)\right]d\theta\right|_\alpha \le \frac{\delta}{m} \quad \text{as } \varepsilon\to 0^+.$$

Therefore,

$$|I_4|_\alpha \le M\frac{\tau(t)}{m}\delta \le M\delta \quad \text{as } \varepsilon\to 0^+.$$

Since the function f_0 satisfies a condition similar to (f_2), the term I_5 can be estimated as I_3, thus,

$$|I_5|_\alpha \leq C''\mathbb{L}(\delta)a^{\frac{1}{q}}.$$

From the estimates obtained for the terms I_1 to I_5, for every $t \in [0,a]$, we get

$$\begin{aligned}|x^\varepsilon(t) - x^\infty(t)|_\alpha &\leq \tilde{\mu}\delta^{1-\alpha} + M\delta + 2C''\mathbb{L}(\delta)a^{\frac{1}{q}} \\ &\quad + M_\alpha e^{\omega a}\int_0^t (t-\theta)^\alpha \mathbb{L}\left(K(\theta)\sup_{0\leq s\leq\theta}|x^\varepsilon(s) - x^\infty(s)|_\alpha\right)d\theta, \quad \text{as } \varepsilon \to 0^+.\end{aligned}$$

By the same way as in the proof of Theorem 3.2, the last expression does not decrease, and if we set

$$r(\delta) = \tilde{\mu}\delta^{1-\alpha} + M\delta + 2C''\mathbb{L}(\delta)a^{\frac{1}{q}}, \quad \text{for every } t \in [0,a],$$

we get

$$\begin{aligned}&K_a \sup_{0\leq s\leq t}|x^\varepsilon(s) - x^\infty(s)|_\alpha \\ &\quad\leq K_a r(\delta) + K_a M_\alpha e^{\omega a}\int_0^t (t-\theta)^\alpha \mathbb{L}\left(K_a \sup_{0\leq s\leq\theta}|x^\varepsilon(s) - x^\infty(s)|_\alpha\right)d\theta \quad \text{as } \varepsilon \to 0^+.\end{aligned}$$

In view of (f_2), the function $\delta \to K_a r(\delta)$ is continuous on $[0,+\infty[$ and $K_a r(0) = 0$. Thus, in view of the arbitrariness of δ, we necessarily conclude that

$$K_a \sup_{0\leq s\leq t}|x^\varepsilon(s) - x^\infty(s)|_\alpha \leq M_\alpha e^{\omega a}\int_0^t (t-\theta)^\alpha \mathbb{L}\left(K_a \sup_{0\leq s\leq\theta}|x^\varepsilon(s) - x^\infty(s)|_\alpha\right)d\theta$$

as $\varepsilon \to 0^+$.

By using (f_2) once again, for every $t \in [0,a]$, we can write

$$K_a \sup_{0\leq s\leq t}|x^\varepsilon(s) - x^\infty(s)|_\alpha = 0 \quad \text{as } \varepsilon \to 0^+.$$

Since K_a is a positive constant, we conclude

$$\sup_{0\leq s\leq t}|x^\varepsilon(s) - x^\infty(s)|_\alpha = 0 \quad \text{as } \varepsilon \to 0^+.$$

This completes the proof. □

Now we give the averaging principle with a more general form.

We denote by $(Af)'$ the following hypothesis :

$$\lim_{T\to+\infty}\frac{1}{T}\int_0^T f(s,u)ds = f_0(u), \quad \text{for all } u \in \mathcal{B}_\alpha.$$

We have the following lemma.

Lemma 4.4 ([19]). *Assume that the hypotheses* ($\mathbf{H}_0$), (f_1)-(f_3) *and* $(Af)'$ *are satisfied. Moreover, suppose that* X *is reflexive. Then*

$$\lim_{\varepsilon\to 0^+}\int_{t_1}^{t_2} T(t_2-\theta)\left[f\left(\frac{\theta}{\varepsilon},u\right) - f_0(u)\right]d\theta = 0.$$

Corollary 4.5. *Assume that the hypotheses* $(\mathbf{H}_0)$, $(\mathbf{H}_1)$, (f_1)–(f_3) *and* $(Af)'$ *are satisfied. Moreover, suppose that* X *is reflexive. Then the unique mild solution* x^ε *of the problem* (P_ε) *converges to the unique mild solution* x^∞ *of the problem* (P_0)*, i.e.,*

$$\lim_{\varepsilon\to 0^+} \|x^\varepsilon - x^\infty\|_{\mathcal{C}(]-\infty,a],X_\alpha)} = 0.$$

Proof. According to Lemma 4.4 all the hypotheses of Theorem 4.3 are satisfieded then the result. □

5 Application

Example 5.1 (Existence result)**.** We consider the following partial differential equation:

$$\begin{cases} \dfrac{\partial}{\partial t}v(t,\xi) = \dfrac{\partial^2}{\partial \xi^2}v(t,\xi) + F\left(t, \displaystyle\int_{-\infty}^{0} a_1(\theta) \int_0^\xi \frac{\partial}{\partial\sigma}v(t-r,\sigma)d\sigma d\theta, \int_{-\infty}^{t}\int_0^\xi a_2(\theta-t)v(\theta,\sigma)d\sigma d\theta\right) \\ v(t,0) = v(t,1) = 0, \quad t \geq 0, \\ v(\theta,\xi) = v_0(\theta,\xi), \quad \theta \leq 0, \quad \xi \in [0,1], \end{cases} \tag{5.1}$$

where $F : \mathbb{R}^+ \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$. The function $v_0 : (-\infty,0] \times [0,1] \to \mathbb{R}$ is continuous and it will be described below.

In order to rewrite equation (5.1) in the abstract form, we choose $X = L^2([0,1];\mathbb{R})$ and we define the operator A on X by

$$\begin{cases} D(A) = H^2(0,1) \cap H_0^1(0,1), \\ Ay = -y''. \end{cases}$$

Then, by Theorem 2.7, page 211 in [16], we get that $-A$ generates an analytic compact semigroup $(T(t))_{t\geq 0}$ on X. From [24], the semigroup $T(t)$ is explicitly given by

$$T(t)y = \sum_{n=1}^{\infty} e^{-n^2 t}\langle y, e_n\rangle e_n \quad \text{for } y \in X.$$

If we choose $\alpha = \frac{1}{2}$, then

$$\begin{cases} A^{\frac{1}{2}}T(t)y = \sum_{n=1}^{\infty} n e^{-n^2 t}\langle y, e_n\rangle e_n & \text{for } y \in X, \\ A^{-\frac{1}{2}}y = \sum_{n=1}^{\infty} \frac{1}{n}\langle y, e_n\rangle e_n & \text{for } y \in X, \\ A^{\frac{1}{2}}y = \sum_{n=1}^{\infty} n\langle y, e_n\rangle e_n & \text{for } y \in D\left(A^{\frac{1}{2}}\right). \end{cases} \tag{5.2}$$

Lemma 5.2 ([24])**.** *If* $y \in D\left(A^{\frac{1}{2}}\right)$*, then* y *is absolutely continuous,* $y' \in X$ *and*

$$\|y'\| = \left\|A^{\frac{1}{2}}y\right\|.$$

Let $\gamma > 0$, we consider the following phase space introduced in [9], which satisfies all the axioms given in the first section

$$\mathcal{B} = \left\{\varphi \in C((-\infty,0];X) : \lim_{\theta\to-\infty} e^{\gamma\theta}\varphi(\theta) \text{ exists in } X\right\}.$$

We equip it with the following norm

$$\|\varphi\|_{\mathcal{B}} = \sup_{\theta\leq 0} e^{\gamma\theta}\|\varphi(\theta)\| \quad \text{for } \varphi \in \mathcal{B}.$$

The norm in $\mathcal{B}_{\frac{1}{2}}$ is given by

$$\|\varphi\|_{\frac{1}{2}} = \sup_{\theta\leq 0} e^{\gamma\theta}\left\|A^{\frac{1}{2}}\varphi(\theta)\right\| = \sup_{\theta\leq 0} e^{\gamma\theta}\sqrt{\int_0^\pi \left(\frac{\partial}{\partial\xi}(\varphi)(\theta)(\xi)\right)^2 d\xi}.$$

Assume that

($\mathbf{H}_4$) $e^{-2\gamma\cdot}a_i(\cdot) \in L^2(\mathbb{R}^-)$, for $i \in \{1,2\}$.

Since

$$\begin{aligned}\int_{-\infty}^t a_2(\theta - t)\int_0^\xi v(\theta,\sigma)\mathrm{d}\sigma\mathrm{d}\theta &= \int_{-\infty}^0 a_2(\theta)\int_0^\xi v(t+\theta,\sigma)\mathrm{d}\sigma\mathrm{d}\theta \\ &= \int_{-\infty}^0 a_2(\theta)\int_0^\xi v_t(\theta,\sigma)\mathrm{d}\sigma\mathrm{d}\theta.\end{aligned}$$

We introduce

$$x(t)(\xi) = v(t,\xi), \quad t \geq 0, 0 \leq \xi \leq 1,$$

$$\varphi(\theta)(\xi) = v_0(\theta,\xi), \quad -\infty < \theta \leq 0, 0 \leq \xi \leq 1,$$

$$f(t,\varphi)(\xi) = F\left(t, \int_{-\infty}^0 a_1(\theta)\int_0^\xi \frac{\partial}{\partial\sigma}(\varphi)(-r)(\sigma)d\sigma d\theta, \int_{-\infty}^0 a_2(\theta)\int_0^\xi \varphi(\theta)(\sigma)d\sigma d\theta\right).$$

Then the problem (5.1) can be rewritten as follows

$$\begin{cases} x'(t) = Ax(t) + f(t,x_t) & \text{for } t \geq 0, \\ x_0 = \varphi \in \mathcal{B}_\alpha. \end{cases} \tag{5.3}$$

Let us consider the following hypotheses on the function F.

(a) $F : [0,1] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is continuous.

(b) There exists a constant $C > 0$, such that $|F(t,\mu,y) - F(t,\mu',y')| \leq C\lambda(|\mu - \mu'| + |y - y'|)$, where $\lambda(\cdot) : [0,+\infty[\to [0,+\infty[$ is a continuous, nondecreasing function with $\lambda(0) = 0$.

As example of such function $\lambda, +\infty[\to [0,+\infty[$, one can write

$$\lambda(z) = \begin{cases} 0, & z = 0 \\ -z\ln z, & 0 < z < 1/e, \\ 1/e, & z > 1/e. \end{cases}$$

The function λ is continuous nondecreasing. For more example one can see [22].

Theorem 5.3. *Assume that the hypotheses (a), (b) and* ($\mathbf{H}_4$) *are satisfied. Then the problem* (5.3) *has a unique mild solution in X.*

Proof. According to Theorem 3.2, we have to prove that f satisfies the hypotheses ($\mathbf{H}_2$). Define in the space $\mathcal{B}_{\frac{1}{2}}$, an equivalent norm $|||\cdot|||_{\frac{1}{2}}$ by

$$|||\varphi|||_{\frac{1}{2}} = 2v\|\psi\|_{\frac{1}{2}},$$

with v is a positive constant chosen, such that

$$v \geq \frac{1}{\sqrt{2\gamma}} \max\Big(\|e^{-2\gamma(\cdot)}a_1(\cdot)\|_{L^2(\mathbb{R}^-)}; \|e^{-2\gamma(\cdot)}a_2(\cdot)\|_{L^2(\mathbb{R}^-)}\Big).$$

Using hypothesis (b) and the fact that the function $\lambda(\cdot)$ is nondecreasing, for $\psi_1, \psi_2 \in \mathcal{B}_{\frac{1}{2}}$, and for $\xi \in [0,1]$, we have

$$\begin{aligned}
&| f(t,\psi_1)(\xi) - f(t,\psi_2)(\xi)|^2 \\
&\leq C^2\Bigg[\lambda\Bigg(\Big(\int_{-\infty}^0 e^{-4\gamma\theta}a_1(\theta)^2 d\theta\Big)^{\frac{1}{2}}\Big(\int_{-\infty}^0 e^{4\gamma\theta}\int_0^\xi \frac{\partial}{\partial\sigma}(\psi_1(r)-\psi_2)^2(-r)(\sigma)d\sigma d\theta\Big)^{\frac{1}{2}} \\
&\quad +\Big(\int_{-\infty}^0 e^{-4\gamma\theta}a_2(\theta)^2 d\theta\Big)^{\frac{1}{2}}\Big(\int_{-\infty}^0 e^{4\gamma\theta}\Big(\int_0^\xi\Big(\psi_1(\theta)(\sigma)-\psi_2(\theta)(\sigma))\Big)^2 \mathrm{d}\sigma\mathrm{d}\theta\Big)^{\frac{1}{2}}\Bigg)\Bigg]^2 \\
&\leq C^2\Bigg[\lambda\Bigg(\Big(\int_{-\infty}^0 e^{-4\gamma\theta}a_1(\theta)^2 d\theta\Big)^{\frac{1}{2}}\frac{1}{\sqrt{2\gamma}}\sup_{\theta\leq 0} e^{\gamma\theta}\Big(\int_0^1 \frac{\partial}{\partial\sigma}(\psi_1-\psi_2)^2(-r)(\sigma)d\sigma\Big)^{\frac{1}{2}} \\
&\quad +\Big(\int_{-\infty}^0 e^{-4\gamma\theta}a_2(\theta)^2 d\theta\Big)^{\frac{1}{2}}\frac{1}{\sqrt{2\gamma}}\sup_{\theta\leq 0} e^{\gamma\theta}\Big(\int_0^1\Big(\psi_1(\theta)(\sigma)-\psi_2(\theta)(\sigma))\Big)^2 \mathrm{d}\sigma\Big)^{\frac{1}{2}}\Bigg)\Bigg]^2 \\
&\leq C^2\Bigg[\lambda\Bigg(v\sup_{\theta\leq 0} e^{\gamma\theta}\Big(\|A^{\frac{1}{2}}(\psi_1-\psi_2)(\theta)\| + \Big(\int_0^1(\psi_1(\theta)(\sigma)-\psi_2(\theta)(\sigma))^2\mathrm{d}\sigma\Big)^{\frac{1}{2}}\Bigg)\Bigg]^2.
\end{aligned}$$

Using formulas (5.2), one has

$$\begin{aligned}
\int_0^1\Big(\psi_1(\theta)(\sigma)-\psi_2(\theta)(\sigma)\Big)^2 d\xi &\leq \int_0^1\Big(\frac{\partial}{\partial\sigma}\Big(\psi_1(\theta)(\sigma)-\psi_2(\theta)(\sigma)\Big)\Big)^2 d\sigma \\
&= \Big\|A^{\frac{1}{2}}\Big(\psi_1(\theta)-\psi_2(\theta)\Big)\Big\|^2.
\end{aligned}$$

Then we get

$$| f(t,\psi_1)(\xi) - f(t,\psi_2)(\xi)|^2 \leq C^2\Bigg[\lambda\Bigg(2v\sup_{\theta\leq 0} e^{\gamma\theta}\|\psi_1-\psi_2\|_{\frac{1}{2}}\Bigg)\Bigg]^2,$$

which implies that for all ψ_1, $\psi_2 \in \mathcal{B}_{\frac{1}{2}}$, we have

$$\|f(t,\psi_1) - f(t,\psi_2)\|_{L^2(0,1)} \leq C\lambda(|||\psi_1-\psi_2|||_{\frac{1}{2}}). \qquad \square$$

Example 5.4 (Averaging principle)**.** Now, we present an example about the averaging principle result.

Consider the following partial differential equation

$$\begin{cases} x'(t) = -Ax(t) + f\big(\frac{t}{\varepsilon}, x_t\big), \\ x_0 = \varphi \in \mathcal{B}_\alpha, \end{cases} \tag{5.4}$$

such that, $-A$ generates an analytic semigroup compact on a reflexive space X (as example we can take A and X given in Example 5.1.

Concerning the phase space $\mathcal{B}$ is identically the phase space $\mathcal{C}_\infty$ described in Section 4, and which satisfies the condition (C_1) (for more examples we refer the reader to a book by Hino, Murakami and Naito [11]).

The function $f : \mathbb{R}^+ \times \mathcal{B}_\alpha \to \mathbb{R}$ is defined as follows

$$f\left(\frac{t}{\varepsilon}, u\right) = \frac{\sin\left(\frac{t}{\varepsilon} + u\right)}{e^{\frac{t}{\varepsilon}}} + 1, \quad \text{for all } t > 0,\ \varepsilon > 0 \ \text{ and } u \in \mathcal{B}_\alpha.$$

Since $-A$ generates an analytic semigroup compact, then $(\mathbf{H}_0)$ is satisfied.

Due to the boundedness of f we can easily show that f meets all the conditions (f_1)–(f_3).

Taking $f_0(u) = 1$, then we can verify that the averaging result condition is satisfied, in fact:

$$\begin{aligned}
\left|\frac{1}{T}\int_0^T f(t,u)dt - f_0(u)\right| &= \left|\frac{1}{T}\int_0^T \frac{\sin(t+u)}{e^t} + 1dt \ - 1\right| \\
&= \left|\frac{1}{T}\int_0^T \frac{\sin(t+u)}{e^t} + 1dt \ - \frac{1}{T}\int_0^T dt\right| \\
&\leq \frac{1}{T}\int_0^T \left|\frac{\sin(t+u)}{e^t}\right| dt \\
&\leq \frac{1}{T}\int_0^T e^{-t} dt \\
&= \frac{1-e^{-T}}{T} \to 0, \quad \text{as } T \to \infty.
\end{aligned}$$

Based on the above discussion and the result in Corollary 4.5, we can deduce that the solution of Equation (5.4) can be approximated as $\varepsilon \to 0$, by the solution of the following autonomous partial differential equation,

$$\begin{cases} x'(t) = -Ax(t) + f_0(x_t), \\ x_0 = \varphi \in \mathcal{B}_\alpha. \end{cases} \tag{5.5}$$

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Conflict of interest

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References

[1] M. Adimy, H. Bouzahir, K. Ezzinbi, Existence for a class of partial functional differential equations with infinite delay, *Nonlinear Anal.* **46**(2001), 91–112. https://doi.org/10.1016/S0362-546X(99)00447-2

[2] M. Adimy, H. Bouzahir, K. Ezzinbi, Existence and stability for some partial neutral functional differential equations with infinite delay, *J. Math. Anal. Appl.* **294**(2004), 438–461. https://doi.org/10.1016/j.jmaa.2004.02.033

[3] P. R. Agarwal, V. Lakshmikantham, *Uniqueness and nonuniqueness criteria for ordinary differential equations*, Series in Real Analysis, Vol. 6, World Scientific, River Edge, NJ, 1993. https://doi.org/10.1142/1988; MR1336820; Zbl 0785.34003.

[4] N. N. Bogoliubov, Y. A. Mitropolsky, *Asymptotic methods in the theory of nonlinear oscillations*, Hindustan Publishing Corp., Delhi; Gordon and Breach Science Publishers, New York, 1961. MR0141845;

[5] V. Burd, *Method of averaging for differential equations on an infinite interval: theory and applications*, Lecture Notes in Pure and Applied Mathematics, Vol. 255, Chapman & Hall/CRC, Boca Raton, FL, 2007. https://doi.org/10.1201/9781584888758; MR2329307

[6] K. Ezzinbi, A. Rebey, Existence of solutions in the α-norm for partial functional differential equations with infinite delay, *Electron. J. Qual. Theory Differ. Equ.* **2012**, No. 59, 1–16. https://doi.org/10.14232/ejqtde.2012.1.59

[7] L. Guedda, A. Ouardani, On the averaging principle for semilinear functional differential equations with infinite delay in a Banach space, *J. Math. Sci.* **265**(2022), No. 4, 629–650. https://doi.org/10.1007/s10958-022-06071-9

[8] J. K. Hale, Averaging methods for differential equations with retarded arguments and a small parameter, *J. Differential Equations* **2**(1966), 57–73. https://doi.org/10.1016/0022-0396(66)90063-5

[9] J. K. Hale, J. Kato, Phase space for retarded equations with infinite delay, *Funkcial. Ekvac.* **21**(1978), 11–41. MR492721

[10] J. K. Hale, S. M. Verduyn Lunel, Averaging in infinite dimensions, *J. Integral Equations Appl.* **2**(1990), 463–494. https://doi.org/10.1216/jiea/1181075583

[11] Y. Hino, S. Murakami, T. Naito, *Functional differential equations with unbounded delay*, Lecture Notes in Mathematics, Vol. 1473, Springer-Verlag, Berlin, 1991. https://doi.org/10.1007/BFb0084432

[12] N. Krylov, N. Bogoliubov, *Introduction to nonlinear mechanics*, Annals of Mathematics Studies, No. 11, Princeton Univ. Press, Princeton, NJ, 1943.

[13] M. Lakrib, T. Sari, Results on the averaging of functional differential equations, *Siberian Math. J.* **45**(2004), 311–320. https://doi.org/10.1023/B:SIMJ.0000021286.49651.c9;

[14] B. Lehman, V. Strygin, Partial and generalized averaging of functional differential equations, *Funct. Differ. Equ.* **9**(2002), 165–200. Zbl 1091.34553

[15] B. Lehman, S. P. Weibel, Fundamental theorems of averaging for functional differential equations, *J. Differential Equations* **152**(1999), 160–190. https://doi.org/10.1006/jdeq.1998.3523

[16] A. Pazy, *Semigroups of linear operators and applications to partial differential equations*, Applied Mathematical Sciences, Vol. 44, Springer-Verlag, New York, 1983. https://doi.org/10.1007/978-1-4612-5561-1

[17] A. M. Samoilenko, R. Petryshyn, *Multifrequency oscillations of nonlinear systems*, Mathematics and Its Applications, Vol. 567, Kluwer Academic Publishers / Springer, Dordrecht, 2004. https://doi.org/10.1007/1-4020-2031-7

[18] J. A. Sanders, F. Verhulst, J. Murdock, *Averaging methods in nonlinear dynamical systems*, Applied Mathematical Sciences, Vol. 59, Springer-Verlag, New York, 2007. https://doi.org/10.1007/978-0-387-48918-6

[19] J. Seidler, I. Vrkoč, An averaging principle for stochastic evolution equations, *Časopis Pěst. Mat.* **115**(1990), 240–263. https://doi.org/10.21136/cpm.1990.118403

[20] I. B. Simonenko, A justification of the averaging method for abstract parabolic equations, *Math. USSR-Sb.* **10**(1970), No. 1, 51–59. https://doi.org/10.1070/SM1970v010n01ABEH002152

[21] V. V. Strygin, The averaging principle for equations with heredity, *Ukrainian Mat. Ž.* **22**(1970), No. 4, 503–513. MR288393

[22] T. Taniguchi, Successive approximations to solutions of stochastic differential equations, *J. Differential Equations* **96**(1992), 152–169. https://doi.org/10.1016/0022-0396(92)90148-G

[23] C. C. Travis, G. F. Webb, Existence, stability, and compactness in the α-norm for partial functional differential equations, *Trans. Amer. Math. Soc.* **240**(1978), 129–143. https://doi.org/10.2307/1998809

[24] C. C. Travis, G. F. Webb, Existence and stability for partial functional differential equations, *Trans. Amer. Math. Soc.* **200**(1974), 395–418. https://doi.org/10.1090/S0002-9947-1974-0382808-3

[25] J. Wu, *Theory and applications of partial functional differential equations*, Applied Mathematical Sciences, Vol. 119, Springer-Verlag, New York, 1996. https://doi.org/10.1007/978-1-4612-4050-1