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Nonexistence and existence of solutions for a Choquard–Kirchhoff type equation involving mixed local and nonlocal operators

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Abstract. In this paper, we study the following Choquard–Kirchhoff type equation

$$\left(a + b\|u\|^{p(\theta-1)}\right)\left(-\Delta_p u + (-\Delta)_p^s u\right) = \left(\int_{\mathbb{R}^N} \frac{|u(y)|^{p_\mu^*}}{|x-y|^\mu} dy\right) |u|^{p_\mu^*-2}u + \lambda f(x) \ \text{ in } \mathbb{R}^N,$$

where $a, b > 0$, $\theta > 1$, $\lambda \geq 0$, $0 < s < 1 < p < N$, and $0 < \mu < N$. Here $p_\mu^* = \frac{p(2N-\mu)}{2(N-p)}$ denotes the critical exponent associated with the Hardy–Littlewood–Sobolev inequality, and the norm

$$\|u\| = \left(\int_{\mathbb{R}^N} |\nabla u|^p \, dx + \iint_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^p}{|x-y|^{N+ps}} \, dx \, dy\right)^{\frac{1}{p}}.$$

The function $f \in L^{\frac{p^*}{p^*-1}}(\mathbb{R}^N) \setminus \{0\}$ is nonnegative, with $p^* = \frac{Np}{N-p}$ being the Sobolev critical exponent. For $\lambda = 0$, we establish the nonexistence of nontrivial solutions. For $\lambda > 0$, by developing a concentration compactness principle suited to the local-nonlocal setting and combining it with Ekeland's variational principle and the mountain pass theorem, we obtain the existence of multiple nonnegative solutions.

Keywords: Kirchhoff-type equation, local-nonlocal operators, critical equation, Choquard nonlinearity, concentration compactness principle.

2020 Mathematics Subject Classification: 35A15, 35M10, 35J92, 35R11.

1 Introduction

In this paper, we investigate the following Choquard–Kirchhoff type problem that combines mixed local and non-local operators

$$\begin{cases} \left(a + b\|u\|^{p(\theta-1)}\right)\left(-\Delta_p u + (-\Delta)_p^s u\right) = \left(\int_{\mathbb{R}^N} \frac{|u(y)|^{p_\mu^*}}{|x-y|^\mu} dy\right) |u|^{p_\mu^*-2}u + \lambda f(x) \ \text{ in } \mathbb{R}^N, \\ u \in X^{1,p}(\mathbb{R}^N), \end{cases} \tag{1.1}$$

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where $a > 0$, $b > 0$, $\theta > 1$, $\lambda \geq 0$ is a parameter. Here, $0 < s < 1 < p < N$, $0 < \mu < N$, and $p_\mu^* = \frac{p(2N-\mu)}{2(N-p)}$ is the critical exponent in the sense of the Hardy–Littlewood–Sobolev inequality. The left hand side of (1.1) involves the classical p-Laplace operator $\Delta_p u = \operatorname{div}(|\nabla u|^{p-2}\nabla u)$, and $(-\Delta)_p^s$ is the fractional p-Laplacian operator. Up to normalization factors, for any $x \in \mathbb{R}^N$ and $u \in C_0^\infty(\mathbb{R}^N)$, it can be defined as

$$(-\Delta)_p^s u(x) = \lim_{\varepsilon \to 0^+} \int_{\mathbb{R}^N \setminus B_\varepsilon(x)} \frac{|u(y)-u(x)|^{p-2}(u(y)-u(x))}{|x-y|^{N+ps}}\, dy,$$

where $B_\varepsilon(x)$ denotes the ball in $\mathbb{R}^N$ centered at x with radius ε. For $s \in (0,1)$ and a measurable function $u : \mathbb{R}^{\mathbb{N}} \longrightarrow \mathbb{R}$, we define the Gagliardo seminorm as

$$[u]_{s,p} = \left(\iint_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^p}{|x-y|^{N+ps}}\, dx\, dy \right)^{\frac{1}{p}}.$$

Moreover, the space $X^{1,p}(\mathbb{R}^N)$ is defined as the completion of $C_0^\infty(\mathbb{R}^N)$ with respect to the norm

$$\|u\| = \left(\int_{\mathbb{R}^N} |\nabla u|^p\, dx + \iint_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^p}{|x-y|^{N+ps}}\, dx\, dy \right)^{\frac{1}{p}}.$$

In recent years, the research on partial differential equations incorporating a blend of local and non-local operators has witnessed substantial advancements in both theoretical exploration and practical implementations. These hybrid operators, constructed through the superposition of the p-Laplacian $(-\Delta)_p$ and the fractional p-Laplacian $(-\Delta)_p^s$, have emerged as significant mathematical models in diverse disciplines. For instance, they play a crucial role in population dynamics, where they can describe the movement and distribution of species, in biomathematics for modeling biological processes, and in optimal foraging strategies to understand how organisms search for food (see [19, 20, 37] and the references therein). Concurrently, these operators have spurred novel developments within the realm of mathematical analysis. This includes the generalization of classical inequalities such as the Hong–Krahn–Szegő inequality [10] and the Faber–Krahn inequality [9], as well as the in-depth investigation of non-local evolution equations, exemplified by the Cahn–Hilliard equation [14] and the porous medium equation [18].

The interplay between local and non-local diffusions has become a focal point of interest for numerous mathematicians. In the specific case of the fractional Laplacian ($p = 2$), variational techniques were employed in [5, 39, 40] to prove the existence and multiplicity of solutions for elliptic equations. When $p \neq 2$, the Brézis–Nirenberg problem associated with fractional p-Laplacian operators was explored in [35]. Moreover, in [30], Morse theory was applied to study the existence of solutions under general reaction terms. Fiscella and Pucci [23] delved into stationary Kirchhoff problems involving fractional operators, Hardy potentials, and critical nonlinearities, deriving results regarding the asymptotic behavior of solutions. The ground state solutions for fractional Choquard equations with sub-critical nonlinearities were also investigated in [7].

Kirchhoff-type equations that combine fractional Laplacians and critical Sobolev exponents have attracted considerable attention from the academic community. These studies cover both bounded domains [22, 36, 42] and the entire space [24, 29, 33], involving Kirchhoff models with critical-growth nonlinearities. Recent contributions in this area include fractional p-Laplacian Choquard–Kirchhoff equations [38], fractional Kirchhoff problems with a combination of Choquard and singular terms [41], and Choquard–Kirchhoff equations featuring Hardy–Littlewood–Sobolev critical exponents [31]. In [43], nonhomogeneous p-Kirchhoff problems with convex-concave nonlinearities were analyzed. Additionally, Xiang et al. [44] studied the following fractional p-Kirchhoff problem:

$$\begin{cases} \left(a + b[u]_{s,p}^{p(\theta-1)}\right)(-\Delta)_p^s u = |u|^{p_s^*-2}u + \lambda f(x) \ \text{ in } \mathbb{R}^N, \\ u \in D^{s,p}(\mathbb{R}^N), \end{cases} \tag{1.2}$$

where $a \geq 0$, $b > 0$, $\theta > 1$, $1 < p < N/s$, $p_s^* = \frac{Np}{N-ps}$, $\lambda \geq 0$ is a parameter, $f \in L^{\frac{p_s^*}{p_s^*-1}}(\mathbb{R}^N) \setminus \{0\}$, and $D^{s,p}(\mathbb{R}^N)$ is defined as the completion of $C_0^\infty(\mathbb{R}^N)$ with respect to the Gagliardo seminorm $[u]_{s,p}$. For the case of $\lambda = 0$, Xiang et al. demonstrated that the non - existence of non-trivial solutions to (1.2) depends on the parameters N, θ, s, p, a and b. When $\lambda > 0$, they utilized Ekeland's variational principle and the mountain pass theorem to prove the existence of at least two non-negative solutions.

Driven by the extensive applications of mixed local-nonlocal operators, considerable research efforts have been devoted to the existence and qualitative analysis of solutions to equations of the form

$$-\Delta_p u + (-\Delta)_p^s u = f(x,u) \quad \text{in } \Omega,$$

where $\Omega \subset \mathbb{R}^N$ is a domain, $p \in (1,\infty)$, $s \in (0,1)$ and f is a given function. For $p = 2$, symmetry properties of solutions to semilinear equations involving such hybrid operators were established in [11]. Fundamental contributions to existence theory, local boundedness, the strong maximum principle, Lipschitz regularity, and interior Sobolev regularity can be found in [1,6,8]. By combining variational techniques with a Pohožaev identity, Anthal et al. [4] derived existence and nonexistence results for Choquard-type equations with subcritical perturbations.

When $p \neq 2$, Garain and Kinnunen [27] investigated regularity properties-including local boundedness, Harnack estimates, local Hölder continuity, and semicontinuity-for weak solutions in the homogeneous case $f = 0$. In the nonhomogeneous case $f \neq 0$, Da Silva et al. [17] recently proved the existence and multiplicity of nontrivial solutions by integrating variational methods with topological tools such as the Krasnoselskii genus and Lusternik–Schnirelman category theories.

Define the optimal constants

$$S_{H,C} = \inf_{u \in C_0^\infty(\mathbb{R}^N)\setminus\{0\}} \frac{\int_{\mathbb{R}^N} |\nabla u|^p \, dx}{\left(\iint_{\mathbb{R}^{2N}} \frac{|u(x)|^{p_\mu^*}|u(y)|^{p_\mu^*}}{|x-y|^\mu} \, dx\, dy \right)^{\frac{p}{2p_\mu^*}}}, \tag{1.3}$$

and

$$S_{H,M} = \inf_{u \in C_0^\infty(\mathbb{R}^N)\setminus\{0\}} \frac{\int_{\mathbb{R}^N} |\nabla u|^p \, dx + \iint_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^p}{|x-y|^{N+ps}} \, dx\, dy}{\left(\iint_{\mathbb{R}^{2N}} \frac{|u(x)|^{p_\mu^*}|u(y)|^{p_\mu^*}}{|x-y|^\mu} \, dx\, dy \right)^{\frac{p}{2p_\mu^*}}}, \tag{1.4}$$

well-posed via the classical Hardy–Littlewood–Sobolev inequality (cf. Proposition 2.1). Leveraging these definitions, we establish the following nonexistence result for trivial solutions when $\lambda = 0$.

Theorem 1.1. *Assume $\lambda = 0$ and $\theta > 1$. Then problem (1.1) admits no nontrivial solution.*

For the case $\lambda \neq 0$, we analyze problem (1.1). Owing to the lack of explicit expressions and asymptotic estimates for minimizers of $S_{H,M}$, we cannot directly characterize the threshold for nontrivial solutions as in [12, 26, 35]. To address the compactness deficit, we develop a specialized concentration-compactness principle for mixed local-nonlocal operators (see Section 4).

Henceforth, unless stated otherwise, we assume f satisfies

(f) $f \geq 0$, $f \not\equiv 0$ and $f \in L^{\frac{p^*}{p^*-1}}(\mathbb{R}^N)$.

Theorem 1.2. *Suppose $a > 0$, $b > 0$, and either $\theta = \frac{3N-\mu-p}{2(N-p)}$ or $a > 0$, $0 < b < S_{H,M}^{-\frac{2p_\mu^*}{p}}$, $\theta = \frac{2N-\mu}{N-p}$. Under assumption (f), there exists $\lambda^* > 0$ such that for all $\lambda \in (0, \lambda^*)$, problem (1.1) possesses a nonnegative weak solution $u_1 \in X^{1,p}(\mathbb{R}^N)$ satisfying $I_\lambda(u_1) < 0$. Moreover, there exists $\lambda^{**} \in (0, \lambda^*]$ such that for all $\lambda \in (0, \lambda^{**})$, problem (1.1) admits another nonnegative weak solution $u_2 \in X^{1,p}(\mathbb{R}^N)$ with $I_\lambda(u_2) > 0$.*

The paper is organized as follows. Section 2 reviews essential preliminaries. In Section 3, we analyze the optimal constant $S_{H,M}$ and prove Theorem 1.1. Section 4 extends Lions' concentration-compactness principle [32] to the mixed operator framework. Finally, Section 5 employs Ekeland's variational principle and the mountain pass theorem to establish the existence of two nonnegative solutions for (1.1) within a suitable parameter range of λ.

Notations

- The standard norm on $L^q(\mathbb{R}^N)$ denotes by $|\cdot|_q$ for $q \in [1, \infty)$.
- We shall also use the notation $\|u\|_{HL}^p = \left(\iint_{\mathbb{R}^{2N}} \frac{|u(x)|^{p_\mu^*}|u(y)|^{p_\mu^*}}{|x-y|^\mu}\, dx\, dy \right)^{\frac{p}{2p_\mu^*}}$ for $u \in X^{1,p}(\mathbb{R}^N)$.
- C denotes positive constant possibly different from line to line.

2 Preliminaries

Proposition 2.1 (Hardy–Littlewood–Sobolev inequality). *Let $r, q > 1$ and $0 < \mu < N$ satisfy $\frac{1}{r} + \frac{1}{q} + \frac{\mu}{N} = 2$. For $g \in L^r(\mathbb{R}^N)$ and $h \in L^q(\mathbb{R}^N)$, there exists a constant $C(r, q, n, \mu)$ independent of g and h such that*

$$\int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{g(x)h(x)}{|x-y|^\mu}\, dx\, dy \leq C(r, q, N, \mu)|g|_r |h|_q.$$

In particular, setting $g = h = |u|^t$, the Hardy–Littlewood–Sobolev inequality ensures the integral

$$\int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|u(x)|^t |u(y)|^t}{|x-y|^\mu}\, dx\, dy$$

is well defined provided $|u|^t \in L^v(\mathbb{R}^N)$ with $v = \frac{2N}{2N-\mu}$. By the Sobolev embedding $X^{1,p}(\mathbb{R}^N) \hookrightarrow L^{p^*}(\mathbb{R}^N)$, we deduce $t = \frac{p(2N-\mu)}{2(N-p)} := p_\mu^*$. Moreover, for $u \in X^{1,p}(\mathbb{R}^N)$, it holds that

$$\int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|u(x)|^{p_\mu^*}|u(y)|^{p_\mu^*}}{|x-y|^\mu}\, dx\, dy \leq C(N, \mu)|u|_{p^*}^{2p_\mu^*}.$$

Next, we introduce the concept of weak solutions for problem (1.1).

Definition 2.2. A function $u \in X^{1,p}(\mathbb{R}^N)$ is a weak solution of (1.1) if for all $\varphi \in X^{1,p}(\mathbb{R}^N)$, it satisfies

$$\begin{aligned}&\left(a + b\|u\|^{(\theta-1)p}\right)\\ &\quad\times\left(\int_{\mathbb{R}^N} |\nabla u|^{p-2}\nabla u \nabla\varphi\,dx + \iint_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^{p-2}(u(x)-u(y))(\varphi(x)-\varphi(y))}{|x-y|^{N+ps}}\,dx\,dy\right)\\ &= \iint_{\mathbb{R}^{2N}} \frac{|u(y)|^{p_\mu^*}|u(x)|^{p_\mu^*-2}u(x)\varphi(x)}{|x-y|^\mu}\,dx\,dy + \lambda\int_{\mathbb{R}^N} f(x)\varphi(x)\,dx \qquad (2.1)\end{aligned}$$

We define the energy functional

$$I_\lambda(u) = \frac{a}{p}\|u\|^p + \frac{b}{\theta p}\|u\|^{\theta p} - \frac{1}{2p_\mu^*}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u^+(y)|^{p_\mu^*}|u^+(x)|^{p_\mu^*}}{|x-y|^\mu}\,dx\,dy - \lambda\int_{\mathbb{R}^N} f(x)u\,dx \qquad (2.2)$$

for $u \in X^{1,p}(\mathbb{R}^N)$, where $u^+ = \max\{u, 0\}$. Since $f \in (X^{1,p}(\mathbb{R}^N))'$, the functional I_λ belongs to $C^1(X^{1,p}(\mathbb{R}^N))$. For all $u, \varphi \in X^{1,p}(\mathbb{R}^N)$, its derivative is given by

$$\begin{aligned}&\langle I_\lambda'(u), \varphi\rangle\\ &\quad= \left(a + b\|u\|^{(\theta-1)p}\right)\\ &\qquad\times\left(\int_{\mathbb{R}^N} |\nabla u|^{p-2}\nabla u\nabla\varphi\,dx + \iint_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^{p-2}(u(x)-u(y))(\varphi(x)-\varphi(y))}{|x-y|^{N+ps}}\,dx\,dy\right)\\ &\qquad- \iint_{\mathbb{R}^{2N}} \frac{|u^+(y)|^{p_\mu^*}|u^+(x)|^{p_\mu^*-2}u^+(x)\varphi(x)}{|x-y|^\mu}\,dx\,dy - \lambda\int_{\mathbb{R}^N} f(x)\varphi(x)\,dx.\end{aligned}$$

Consequently, u is a weak solution of (1.1) if and only if u is a critical point of the functional I_λ.

Remark 2.3. To verify the well-definedness of (2.2), we confirm the finiteness of all integrals in (2.1). For $u, \varphi \in X^{1,p}(\mathbb{R}^N)$, Hölder's inequality yields

$$\begin{aligned}&\int_{\mathbb{R}^N} |\nabla u|^{p-2}\nabla u\nabla\varphi\,dx + \iint_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^{p-2}(u(x)-u(y))(\varphi(x)-\varphi(y))}{|x-y|^{N+ps}}\,dx\,dy\\ &\quad\le |\nabla u|_p^{p-1}|\nabla\varphi|_p + [u]_{s,p}^{p-1}[\varphi]_{s,p} < \infty.\end{aligned}$$

Additionally, leveraging the Sobolev embedding $X^{1,p}(\mathbb{R}^N) \hookrightarrow L^{p^*}(\mathbb{R}^N)$ and the Hardy–Littlewood–Sobolev inequality, we estimate

$$\begin{aligned}&\iint_{\mathbb{R}^{2N}} \frac{|u(y)|^{p_\mu^*}|u(x)|^{p_\mu^*-2}u(x)\varphi(x)}{|x-y|^\mu}\,dx\,dy + \lambda\int_{\mathbb{R}^N} f(x)\varphi(x)\,dx\\ &\quad\le C(N,\mu)|u|_{p^*}^{\frac{p(2N-\mu)+2Np}{2(N-p)} - \frac{2N}{2N-\mu}}|\varphi|_{p^*}^{\frac{2N}{2N-\mu}} + \lambda|f|_{(p^*)'}|\varphi|_{p^*} < \infty.\end{aligned}$$

3 Proof of Theorem 1.1

We dedicate this section to prove Theorem 1.1 in the case $\lambda = 0$.

Lemma 3.1. *Let $s \in (0,1)$. For $S_{H,C}$ and $S_{H,M}$ defined in (1.3) and (1.4), it holds that $S_{H,M} = S_{H,C}$.*

Proof. From the definitions of $S_{H,C}$ and $S_{H,M}$, the inequality $S_{H,M} \geq S_{H,C}$ is immediate. To prove the reverse inequality, consider any $u \in C_0^\infty(\mathbb{R}^N)$. There exists $k_0(u) \in \mathbb{N}$ such that

$$\text{supp}(u) \subseteq B_{kr}(0), \quad \text{for all } k \geq k_0.$$

Define the scaled function $u_k := k^{\frac{N-p}{p}} u(kx)$ for $k \geq k_0$, which satisfies $supp(u_k) \subseteq B_r(0)$. By the definition of $S_{H,M}$, we have for every $k \geq k_0$,

$$S_{H,M} \leq \frac{\|u_k\|^p}{\|u_k\|_{HL}^p} = \frac{|\nabla u|_p^p}{\|u\|_{HL}^p} + k^{ps-p} \frac{[u]_{s,p}^p}{\|u\|_{HL}^p}.$$

Letting $k \to \infty$ and using $0 < s < 1$, the second term vanishes, yielding

$$S_{H,M} \leq \frac{|\nabla u|_p^p}{\|u\|_{HL}^p}.$$

By the arbitrariness of $u \in C_0^\infty(\mathbb{R}^N)$, we conclude

$$S_{H,M} \leq S_{H,C}.$$

Thus $S_{H,M} = S_{H,C}$. □

Lemma 3.2. *The optimal constant $S_{H,M}$ is never achieved.*

Proof. Assume for contradiction that there exists a nonzero function $v \in X^{1,p}(\mathbb{R}^N)$ such that $\|v\|_{HL} = 1$ and $\|v\|^p = S_{H,M} = S_{H,L}$, then

$$S_{H,L} \leq |\nabla v|_p^p \leq |\nabla v|_p^p + [v]_{s,p}^p = S_{H,L}.$$

This implies $[v]_{s,p} = 0$, meaning v is a constant in $\mathbb{R}^N$. However, a constant function v satisfies $\|v\| = 0$, contradicting $\|v\|_{HL} = 1$. Therefore, $S_{H,M}$ can not be achieved. □

Lemma 3.3. *Let $\alpha > 0$. The problem*

$$\begin{cases} -\Delta_p u + (-\Delta)_p^s u = \alpha \left(\int_{\mathbb{R}^N} \frac{|u(y)|^{p_\mu^*}}{|x-y|^\mu} \, dy \right) |u|^{p_\mu^*-2} u & \text{in } \mathbb{R}^N, \\ u \in X^{1,p}(\mathbb{R}^N), \end{cases} \tag{3.1}$$

admits no nontrivial solutions.

Proof. Suppose u is a solution to (3.1). Then u satisfies

$$\|u\|^p = \alpha \left(\int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|u(x)|^{p_\mu^*} |u(y)|^{p_\mu^*}}{|x-y|^\mu} \, dx \, dy \right).$$

Define $w = lu$ for some $l > 0$. Substituting into the equation gives

$$\frac{\|w\|^p}{l^p} = \alpha \cdot \frac{\int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|w(x)|^{p_\mu^*} |w(y)|^{p_\mu^*}}{|x-y|^\mu} \, dx \, dy}{l^{2p_\mu^*}}.$$

Setting $S_{H,M} = \alpha l^{p-2p_\mu^*}$, we obtain

$$\|w\|^p = S_{H,M} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|w(x)|^{p_\mu^*} |w(y)|^{p_\mu^*}}{|x-y|^\mu} \, dx \, dy.$$

This implies w achieves $S_{H,M}$, contradicting Lemma 3.2. Hence, no nontrivial solutions exist. □

Proof of Theorem 1.1. Assume for contradiction that there exists a nonzero solution v to (1.1) with $\lambda = 0$. Then v satisfies

$$\left(a + b\|v\|^{p(\theta-1)}\right)\left(-\Delta_p v + (-\Delta)_p^s v\right) = \left(\int_{\mathbb{R}^N} \frac{|v(y)|^{p_\mu^*}}{|x-y|^\mu} dy\right) |v|^{p_\mu^*-2} v.$$

For any $\psi \in X^{1,p}(\mathbb{R}^N)$, testing the equation with ψ yields

$$\int_{\mathbb{R}^N} |\nabla v|^{p-2} \nabla v \nabla \psi \, dx + \iint_{\mathbb{R}^{2N}} \frac{|v(x)-v(y)|^{p-2}(v(x)-v(y))(\psi(x)-\psi(y))}{|x-y|^{N+ps}} \, dx \, dy$$
$$= \frac{1}{a + b\|v\|^{(\theta-1)p}} \iint_{\mathbb{R}^{2N}} \frac{|v(y)|^{p_\mu^*} |v(x)|^{p_\mu^*-2} v(x)\psi(x)}{|x-y|^\mu} \, dx \, dy.$$

Let $\alpha = \frac{1}{a+b\|v\|^{(\theta-1)p}} > 0$. The equation then reduces to

$$-\Delta_p v + (-\Delta)_p^s v = \alpha \left(\int_{\mathbb{R}^N} \frac{|v(y)|^{p_\mu^*}}{|x-y|^\mu} \, dy\right) |v|^{p_\mu^*-2} v,$$

which contradicts Lemma 3.3. Therefore, no nontrivial solutions exist when $\lambda = 0$. □

4 The principle of concentration compactness

In [32], Lions established the celebrated concentration compactness principle in classical Sobolev spaces, which has since been widely applied to solve elliptic problems involving critical exponents. Xiang et al. [44] later extended this principle to fractional Sobolev spaces $D^{s,p}(\mathbb{R}^N)$, while Gao et al. [25] developed a concentration compactness framework for convolution nonlinearities to study critical Choquard equations. However, to the best of our knowledge, no existing results have formulated the concentration compactness principle for mixed local-nonlocal operators with Choquard-type nonlinearities. Motivated by these contributions, we establish such a principle in $X^{1,p}(\mathbb{R}^N)$, which will be pivotal in Section 5. Define

$$C_c(\mathbb{R}^N) = \{u \in C(\mathbb{R}^N) : \text{supp}(u) \text{ is a compact subset of } \mathbb{R}^N\}$$

and denote by $C_0(\mathbb{R}^N)$ the closure of $C_c(\mathbb{R}^N)$ with respect to the supremum norm $|\eta|_\infty = \sup_{x\in\mathbb{R}^N} |\eta(x)|$. Recall that a finite measure on $\mathbb{R}^N$ corresponds to a continuous linear functional on $C_0(\mathbb{R}^N)$. For a measure ξ, its total variation norm is given by:

$$\|\xi\| = \sup_{\eta \in C_0(\mathbb{R}^N), |\eta|_\infty = 1} |(\xi, \eta)|,$$

where $(\xi, \eta) = \int_{\mathbb{R}^N} \eta \, d\xi$.

Definition 4.1. Let $\mathcal{M}(\mathbb{R}^N)$ denote the space of finite nonnegative Borel measure space on $\mathbb{R}^N$. For $\xi \in \mathcal{M}(\mathbb{R}^N)$, we have $\xi(\mathbb{R}^N) = \|\xi\|$. A sequence $\{\xi_n\} \subset \mathcal{M}(\mathbb{R}^N)$ converges weak-$*$ to $\xi \in \mathcal{M}(\mathbb{R}^N)$ (denoted $\xi_n \rightharpoonup \xi$) if $(\xi_n, \eta) \to (\xi, \eta)$ for all $\eta \in C_0(\mathbb{R}^N)$ as $n \to \infty$.

Theorem 4.2. *Let $\{u_n\}_n \subset X^{1,p}(\mathbb{R}^N)$ be a sequence with $\|u_n\| \le C$ for some constant $C > 0$, and suppose*

$$\begin{cases} u_n \rightharpoonup u & \text{weakly in } X^{1,p}(\mathbb{R}^N), \\ |\nabla u_n|^p + \displaystyle\int_{\mathbb{R}^N} \frac{|u_n(x)-u_n(y)|^p}{|x-y|^{N+ps}} \, dy \rightharpoonup \xi & \text{weak-}* \text{ in } \mathcal{M}(\mathbb{R}^N), \\ \displaystyle\int_{\mathbb{R}^N} \frac{|u_n(y)|^{p_\mu^*}}{|x-y|^\mu} \, dy |u_n(x)|^{p_\mu^*} \rightharpoonup \nu & \text{weak-}* \text{ in } \mathcal{M}(\mathbb{R}^N). \end{cases}$$

Then the following hold

$$\xi(\mathbb{R}^N) \le C^p, \qquad\qquad \nu(\mathbb{R}^N) \le S_{H,M}^{-\frac{2p_\mu^*}{p}} C^{2p_\mu^*}, \tag{4.1}$$

$$\xi = |\nabla u|^p + \int_{\mathbb{R}^N} \frac{|u(x)-u(y)|^p}{|x-y|^{N+ps}}\,dy + \sum_{j\in J} \xi_j \delta_{x_j} + \tilde{\xi}, \tag{4.2}$$

$$\nu = \int_{\mathbb{R}^N} \frac{|u(y)|^{p_\mu^*}}{|x-y|^\mu}\,dy |u(x)|^{p_\mu^*} + \sum_{j\in J} \nu_j \delta_{x_j}. \tag{4.3}$$

where J is at most countable, sequence $\{\xi_j\}_j$, $\{\nu_j\}_j \subset \mathbb{R}_0^+$, $\{x_j\}_j \subset \mathbb{R}^N$, δ_{x_j} is the Dirac mass centered at x_j, $\tilde{\xi}$ is a nonnegative non-atomic measure. Moreover, we have

$$\nu(\mathbb{R}^N) \le S_{H,M}^{-\frac{2p_\mu^*}{p}} \xi(\mathbb{R}^N)^{\frac{2p_\mu^*}{p}}, \qquad \nu_j \le S_{H,M}^{-\frac{2p_\mu^*}{p}} \xi_j^{\frac{2p_\mu^*}{p}}, \quad \text{for all } j \in J, \tag{4.4}$$

where $S_{H,M}$ is the optimal constant defined in (1.4).

Lemma 4.3. *Let $\{u_n\}_n \subset X^{1,p}(\mathbb{R}^N)$ be the sequence given by Theorem 4.2. Fix $x_0 \in \mathbb{R}^N$ and let $\varphi \in C_0^\infty(\mathbb{R}^N)$ satisfy $0 \le \varphi \le 1$, $\varphi \equiv 1$ on $B_1(0)$, $\varphi \equiv 0$ in $\mathbb{R}^N \setminus B_2(0)$, and $|\nabla\varphi| \le 2$. Define $\varphi_\varepsilon = \varphi((x-x_0)/\varepsilon)$ for all $x \in \mathbb{R}^N$. Then the following hold*

$$\lim_{\varepsilon\to 0} \lim_{n\to\infty} \int_{\mathbb{R}^N} |u_n|^p |\nabla \varphi_\varepsilon|^p\,dx = 0, \tag{4.5}$$

$$\lim_{\varepsilon\to 0} \limsup_{n\to\infty} \iint_{\mathbb{R}^{2N}} \frac{|u_n(y)|^p |\varphi_\varepsilon(x)-\varphi_\varepsilon(y)|^p}{|x-y|^{N+ps}}\,dx\,dy = 0. \tag{4.6}$$

Proof. We first prove (4.5). By the definition of φ_ε, we have:

$$\int_{\mathbb{R}^N} |u_n|^p |\nabla\varphi_\varepsilon|^p\,dx \le \int_{B_{2\varepsilon}(x_0)\setminus B_\varepsilon(x_0)} \frac{2^p}{\varepsilon^p} |u_n|^p\,dx.$$

Scaling the variable $z = \frac{x-x_0}{\varepsilon}$, this becomes

$$\int_{B_2(0)\setminus B_1(0)} 2^p \varepsilon^{N-p} |u_n(x_0+\varepsilon z)|^p\,dz.$$

For the bounded domain $\Omega = B_2(0) \setminus B_1(0)$, the Sobolev embedding $X^{1,p}(\Omega) \hookrightarrow L^p(\Omega)$ holds (cf. [13, Lemma 2.1]), where $X^{1,p}(\Omega)$ is the completion of $C_0^\infty(\Omega)$ with respect to the norm

$$\|u\|_{X^{1,p}(\Omega)} = \left(\int_\Omega |\nabla u|^p\,dx + \iint_{\Omega\times\Omega} \frac{|u(x)-u(y)|^p}{|x-y|^{N+ps}}\,dx\,dy\right)^{\frac{1}{p}}.$$

Using the boundedness of $\{u_n\}$ in $X^{1,p}(\mathbb{R}^N)$ and $N > p$, we estimate

$$\int_{\mathbb{R}^N} |u_n|^p |\nabla\varphi_\varepsilon|^p\,dx \le C\varepsilon^{N-p} \|u_n\|^p_{X^{1,p}(\Omega)} \le C\varepsilon^{N-p} \|u_n\|^p \to 0$$

as $n \to \infty$ and $\varepsilon \to 0$, establishing (4.5). For (4.6), the proof follows similarly to [44, Lemma 2.3] by leveraging the fractional Sobolev seminorm properties and the decay of u_n at infinity. Details are omitted here for brevity. □

Proof of Theorem 4.2. We divide the proof into four steps.

Step 1. Proof of (4.1).

Fix $R > 0$ and let $\eta \in C_0^\infty(B_{2R}(0))$ satisfy $0 \le \eta \le 1$ and $\eta \equiv 1$ on $B_R(0)$. By weak-$*$ convergence,

$$\int_{\mathbb{R}^N} \left(|\nabla u_n|^p + \int_{\mathbb{R}^N} \frac{|u_n(x) - u_n(y)|^p}{|x-y|^{N+ps}} \, dy \right) \eta(x) \, dx \to \int_{\mathbb{R}^N} \eta(x) \, d\xi$$

as $n \to \infty$. Since $\|u_n\| \le C$, we have

$$\begin{aligned} &\int_{\mathbb{R}^N} \left(|\nabla u_n|^p + \int_{\mathbb{R}^N} \frac{|u_n(x) - u_n(y)|^p}{|x-y|^{N+ps}} \, dy \right) \eta(x) \, dx \\ &\quad \le \int_{\mathbb{R}^N} |\nabla u_n|^p \, dx + \iint_{\mathbb{R}^{2N}} \frac{|u_n(x) - u_n(y)|^p}{|x-y|^{N+ps}} \, dy \, dx \le C^p. \end{aligned}$$

Taking $R \to \infty$, we conclude $\xi(\mathbb{R}^N) \le C^p$. For $\nu(\mathbb{R}^N)$, by the definition of $S_{H,M}$ and $\|u_n\| \le C$,

$$\iint_{\mathbb{R}^{2N}} \frac{|u_n(x)|^{p_\mu^*} |u_n(y)|^{p_\mu^*}}{|x-y|^\mu} \, dx \, dy \le S_{H,M}^{-\frac{2p_\mu^*}{p}} \|u_n\|^{2p_\mu^*} \le S_{H,M}^{-\frac{2p_\mu^*}{p}} C^{2p_\mu^*},$$

yielding $\nu(\mathbb{R}^N) \le S_{H,M}^{-\frac{2p_\mu^*}{p}} C^{2p_\mu^*}$.

Step 2. Proof of (4.2).

Define the functional

$$K(u) = \int_{\mathbb{R}^N} \left(|\nabla u|^p + \int_{\mathbb{R}^N} \frac{|u(x) - u(y)|^p}{|x-y|^{N+ps}} \, dy \right) \eta(x) \, dx$$

for $\eta \in C_0(\mathbb{R}^N)$. Since K is continuously differentiable and convex on $X^{1,p}(\mathbb{R}^N)$, it is weakly lower semicontinuous. Thus,

$$\begin{aligned} &\liminf_{n\to\infty} \int_{\mathbb{R}^N} \left(|\nabla u_n|^p + \int_{\mathbb{R}^N} \frac{|u_n(x) - u_n(y)|^p}{|x-y|^{N+ps}} \, dy \right) \eta(x) \, dx \\ &\quad \ge \int_{\mathbb{R}^N} \left(|\nabla u|^p + \int_{\mathbb{R}^N} \frac{|u(x) - u(y)|^p}{|x-y|^{N+ps}} \, dy \right) \eta(x) \, dx. \end{aligned}$$

By weak-$*$ convergence, the left-hand side equals $\int_{\mathbb{R}^N} \eta \, d\xi$. Therefore,

$$\int_{\mathbb{R}^N} \eta \, d\xi \ge \int_{\mathbb{R}^N} \left(|\nabla u|^p + \int_{\mathbb{R}^N} \frac{|u(x) - u(y)|^p}{|x-y|^{N+ps}} \, dy \right) \eta(x) \, dx.$$

By the arbitrariness of $\eta \ge 0$ in $C_0(\mathbb{R}^N)$, we deduce

$$\xi \ge |\nabla u|^p + \int_{\mathbb{R}^N} \frac{|u(x) - u(y)|^p}{|x-y|^{N+ps}} \, dy.$$

The decomposition

$$\xi = |\nabla u|^p + \int_{\mathbb{R}^N} \frac{|u(x) - u(y)|^p}{|x-y|^{N+ps}} \, dy + \sum_{j\in J} \xi_j \delta_{x_j} + \tilde{\xi}$$

follows from the structure of nonnegative Radon measures.

Step 3. Proof of (4.3).
For $\eta \in C_0(\mathbb{R}^N)$, using [34, Equation (3.6)], we have

$$\lim_{n\to\infty} \int_{\mathbb{R}^N} \left(\int_{\mathbb{R}^N} \frac{|u_n(y)|^{p_\mu^*}}{|x-y|^\mu} \, dy \, |u_n(x)|^{p_\mu^*} - \int_{\mathbb{R}^N} \frac{|\bar{u}_n(y)|^{p_\mu^*}}{|x-y|^\mu} \, dy \, |\bar{u}_n(x)|^{p_\mu^*} \right) \eta(x) \, dx$$
$$= \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|u(y)|^{p_\mu^*}}{|x-y|^\mu} \, dy \, |u(x)|^{p_\mu^*} \eta(x) \, dx,$$

where $\bar{u}_n = u_n - u$. Define $\bar{\nu} = \nu - \int_{\mathbb{R}^N} \frac{|u(y)|^{p_\mu^*}}{|x-y|^\mu} \, dy \, |u(x)|^{p_\mu^*}$. By weak-$*$ convergence,

$$\int_{\mathbb{R}^N} \eta \, d\bar{\nu} = \lim_{n\to\infty} \iint_{\mathbb{R}^{2N}} \frac{|\bar{u}_n(y)|^{p_\mu^*}}{|x-y|^\mu} \, dy \, |\bar{u}_n(x)|^{p_\mu^*} \eta(x) \, dx.$$

To analyze the atomic part, fix $x_0 \in \mathbb{R}^N$ and let $\varphi \in C_0^\infty(\mathbb{R}^N)$ satisfy $0 \le \varphi \le 1$, $\varphi \equiv 1$ on $B_1(0)$, $\varphi \equiv 0$ outside $B_2(0)$, and $|\nabla \varphi| \le 2$. Define $\varphi_\varepsilon(x) = \varphi((x-x_0)/\varepsilon)$. Using [25, Equation (2.8)], we derive

$$\lim_{\varepsilon\to 0} \lim_{n\to\infty} \iint_{\mathbb{R}^{2N}} \frac{|\varphi_\varepsilon u_n(y)|^{p_\mu^*} |\varphi_\varepsilon u_n(x)|^{p_\mu^*}}{|x-y|^\mu} \, dy \, dx = \nu(\{x_0\})$$

and

$$\lim_{\varepsilon\to 0} \lim_{n\to\infty} \left(\int_{\mathbb{R}^N} |\nabla u_n|^p \varphi_\varepsilon^p \, dx + \iint_{\mathbb{R}^{2N}} \frac{|\varphi_\varepsilon u_n(x) - \varphi_\varepsilon u_n(y)|^p}{|x-y|^{N+ps}} \, dy \, dx \right) = \xi(\{x_0\}).$$

Recall the following Young inequality

$$|\zeta_1 + \zeta_2|^p \le (1+\beta)^{p-1} |\zeta_1|^p + (1+1/\beta)^{p-1} |\zeta_2|^p, \tag{4.7}$$

where $\zeta_1, \zeta_2 \in \mathbb{R}$ and $\beta > 0$. Applying (4.7) and Lemma 4.3, we find

$$\nu(\{x_0\}) \le S_{H,M}^{-\frac{2p_\mu^*}{p}} \xi(\{x_0\})^{\frac{2p_\mu^*}{p}}.$$

Thus, the atoms of ν are contained in those of ξ. For the non-atomic part, we use the Radon–Nikodym theorem and Lebesgue's differentiation theorem to show $\tilde{\nu} = 0$.

Step 4. Proof of (4.4).
Fix $\eta \in C_0^\infty(B_{2R}(0))$ with $0 \le \eta \le 1$, $\eta \equiv 1$ on $B_R(0)$, and $|\nabla \eta| \le 2/R$. By Young's inequality and Lemma 4.3,

$$\iint_{\mathbb{R}^{2N}} \frac{|\eta u_n(y)|^{p_\mu^*} |\eta u_n(x)|^{p_\mu^*}}{|x-y|^\mu} \, dy \, dx$$
$$\le S_{H,M}^{-\frac{2p_\mu^*}{p}} \left((1+\beta)^{p-1} \int_{\mathbb{R}^N} |u_n \nabla \eta|^p \, dx + (1+\beta^{-1})^{p-1} \int_{\mathbb{R}^N} \eta^p \, d\xi \right)^{\frac{2p_\mu^*}{p}}.$$

Letting $R \to \infty$ and $\beta \to \infty$, we obtain

$$\int_{\mathbb{R}^N} \eta^{2p_\mu^*} \, d\nu \le S_{H,M}^{-\frac{2p_\mu^*}{p}} \left(\int_{\mathbb{R}^N} \eta^p \, d\xi \right)^{\frac{2p_\mu^*}{p}}.$$

Taking $R \to \infty$ and using the atomic decomposition, we conclude

$$\nu(\mathbb{R}^N) \le S_{H,M}^{-\frac{2p_\mu^*}{p}} \xi(\mathbb{R}^N)^{\frac{2p_\mu^*}{p}} \quad \text{and} \quad \nu_j \le S_{H,M}^{-\frac{2p_\mu^*}{p}} \xi_j^{\frac{2p_\mu^*}{p}} \quad \text{for all } j \in J.$$

This completes the proof. $\square$

Theorem 4.4. *Let $\{u_n\}_n \subset X^{1,p}(\mathbb{R}^N)$ be a bounded sequence such that*

$$\begin{cases} |\nabla u_n|^p + \int_{\mathbb{R}^N} \frac{|u_n(x)-u_n(y)|^p}{|x-y|^{N+ps}}\,dy \rightharpoonup \xi & \text{weak-}* \text{ in } \mathcal{M}(\mathbb{R}^N), \\ \int_{\mathbb{R}^N} \frac{|u_n(y)|^{p_\mu^*}}{|x-y|^\mu}\,dy|u_n(x)|^{p_\mu^*} \rightharpoonup \nu & \text{weak-}* \text{ in } \mathcal{M}(\mathbb{R}^N), \end{cases}$$

and set

$$\xi_\infty = \lim_{R\to\infty}\limsup_{n\to\infty}\int_{\{x\in\mathbb{R}^N:|x|>R\}}\left(|\nabla u_n|^p + \int_{\mathbb{R}^N}\frac{|u_n(x)-u_n(y)|^p}{|x-y|^{N+ps}}dy\right)dx,$$

$$\nu_\infty = \lim_{R\to\infty}\limsup_{n\to\infty}\int_{\{x\in\mathbb{R}^N:|x|>R\}}\int_{\mathbb{R}^N}\frac{|u_n(y)|^{p_\mu^*}}{|x-y|^\mu}\,dy\,|u_n(x)|^{p_\mu^*}\,dx,$$

where the quantities ξ_∞ and ν_∞ are well defined. Then the following hold

$$\limsup_{n\to\infty}\int_{\mathbb{R}^N}\left(|\nabla u_n|^p + \int_{\mathbb{R}^N}\frac{|u_n(x)-u_n(y)|^p}{|x-y|^{N+ps}}dy\right)dx = \xi(\mathbb{R}^N)+\xi_\infty,$$

$$\limsup_{n\to\infty}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|u_n(y)|^{p_\mu^*}}{|x-y|^\mu}\,dy\,|u_n(x)|^{p_\mu^*}\,dx = \nu(\mathbb{R}^N)+\nu_\infty,$$

$$\nu_\infty \le S_{H,M}^{-\frac{2p_\mu^*}{p}}\,\xi_\infty^{\frac{2p_\mu^*}{p}}.$$

Proof. Fix $\chi \in C^\infty(\mathbb{R}^N)$ with $0\le\chi\le 1$, $\chi = 0$ in $B_1(0)$, $\chi = 1$ in $\mathbb{R}^N\setminus B_2(0)$, and $|\nabla\chi|\le 2$. Define $\chi_R(x) = \chi(x/R)$ for all $x\in\mathbb{R}^N$ and $R>0$. Then

$$\begin{aligned}&\int_{\{x\in\mathbb{R}^N:|x|>2R\}}\left(|\nabla u_n|^p + \int_{\mathbb{R}^N}\frac{|u_n(x)-u_n(y)|^p}{|x-y|^{N+ps}}\,dy\right)dx\\ &\quad\le \int_{\mathbb{R}^N}\left(|\nabla u_n|^p + \int_{\mathbb{R}^N}\frac{|u_n(x)-u_n(y)|^p}{|x-y|^{N+ps}}\,dy\right)\chi_R^p\,dx\\ &\quad\le \int_{\{x\in\mathbb{R}^N:|x|>R\}}\left(|\nabla u_n|^p + \int_{\mathbb{R}^N}\frac{|u_n(x)-u_n(y)|^p}{|x-y|^{N+ps}}\,dy\right)dx,\end{aligned}$$

which means

$$\xi_\infty = \lim_{R\to\infty}\limsup_{n\to\infty}\int_{\mathbb{R}^N}\left(|\nabla u_n|^p + \int_{\mathbb{R}^N}\frac{|u_n(x)-u_n(y)|^p}{|x-y|^{N+ps}}dy\right)\chi_R^p\,dx.$$

Similarly,

$$\nu_\infty = \lim_{R\to\infty}\limsup_{n\to\infty}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|u_n(y)|^{p_\mu^*}}{|x-y|^\mu}\,dy\,|u_n(x)|^{p_\mu^*}\chi_R^{2p_\mu^*}\,dx.$$

By the definition of weak-$*$ convergence, we have

$$\int_{\mathbb{R}^N}\left(|\nabla u_n|^p + \int_{\mathbb{R}^N}\frac{|u_n(x)-u_n(y)|^p}{|x-y|^{N+ps}}dy\right)(1-\chi_R^p)\,dx \to \int_{\mathbb{R}^N}(1-\chi_R^p)\,d\xi$$

as $n\to\infty$. So we get

$$\xi(\mathbb{R}^N) = \lim_{R\to\infty}\lim_{n\to\infty}\int_{\mathbb{R}^N}\left(|\nabla u_n|^p + \int_{\mathbb{R}^N}\frac{|u_n(x)-u_n(y)|^p}{|x-y|^{N+ps}}dy\right)(1-\chi_R^p)\,dx.$$

Then the following decomposition

$$\begin{aligned}&\int_{\mathbb{R}^N}\left(|\nabla u_n|^p+\int_{\mathbb{R}^N}\frac{|u_n(x)-u_n(y)|^p}{|x-y|^{N+ps}}dy\right)dx\\&=\int_{\mathbb{R}^N}\left(|\nabla u_n|^p+\int_{\mathbb{R}^N}\frac{|u_n(x)-u_n(y)|^p}{|x-y|^{N+ps}}dy\right)\chi_R^p\,dx\\&\quad+\int_{\mathbb{R}^N}\left(|\nabla u_n|^p+\int_{\mathbb{R}^N}\frac{|u_n(x)-u_n(y)|^p}{|x-y|^{N+ps}}dy\right)(1-\chi_R^p)\,dx\end{aligned}$$

gives that

$$\limsup_{n\to\infty}\int_{\mathbb{R}^N}\left(|\nabla u_n|^p+\int_{\mathbb{R}^N}\frac{|u_n(x)-u_n(y)|^p}{|x-y|^{N+ps}}dy\right)dx=\xi_\infty+\xi(\mathbb{R}^N).$$

Similarly, we can conclude that

$$\limsup_{n\to\infty}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|u_n(y)|^{p_\mu^*}}{|x-y|^\mu}\,dy\,|u_n(x)|^{p_\mu^*}\,dx=\nu(\mathbb{R}^N)+\nu_\infty.$$

To prove the last inequality

$$\nu_\infty\le S_{H,M}^{-\frac{2p_\mu^*}{p}}\xi_\infty^{\frac{2p_\mu^*}{p}},$$

we first claim that

$$\lim_{R\to\infty}\lim_{n\to\infty}\int_{\mathbb{R}^N}|u_n|^p|\nabla\chi_R|^p\,dx=0, \tag{4.8}$$

$$\lim_{R\to\infty}\limsup_{n\to\infty}\iint_{\mathbb{R}^{2N}}\frac{|u_n(y)|^p|\chi_R(x)-\chi_R(y)|^p}{|x-y|^{N+ps}}\,dx\,dy=0. \tag{4.9}$$

By the same argument as equation (4.5) in Lemma 4.3, we can derive (4.8). For (4.9), the proof follows similarly to [44, Equation (2.15)]. Details are omitted here for brevity.

By Young's inequality, (4.8) and (4.9), we deduce

$$\begin{aligned}&\lim_{n\to\infty}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|u_n(y)|^{p_\mu^*}}{|x-y|^\mu}\,dy\,|u_n(x)|^{p_\mu^*}|\chi_R|^{2p_\mu^*}\,dx\\&=\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|u_n(y)\chi_R|^{p_\mu^*}}{|x-y|^\mu}dy\,|u_n(x)\chi_R|^{p_\mu^*}\,dx\\&\le S_{H,M}^{-\frac{2p_\mu^*}{p}}\left((1+\beta^{-1})^{p-1}\int_{\mathbb{R}^N}|\nabla u_n\chi_R|^p\,dx+(1+\beta^{-1})^{p-1}\iint_{\mathbb{R}^{2N}}\frac{|u_n(x)-u_n(y)|^p|\chi_R(x)|^p}{|x-y|^{N+ps}}\,dx\,dy\right)^{\frac{2p_\mu^*}{p}}\end{aligned}$$

Letting $R\to\infty$ and $\beta\to\infty$, we conclude

$$\nu_\infty\le S_{H,M}^{-\frac{2p_\mu^*}{p}}\xi_\infty^{\frac{2p_\mu^*}{p}}.$$

Therefore, the proof is complete. □

5 Proof of Theorem 1.2

Throughout this section, we consistently assume that $1<\theta\le\frac{2N-\mu}{N-p}$, without additional mention.

Definition 5.1. Let $c \in \mathbb{R}$, suppose $X^{1,p}(\mathbb{R}^N)$ is a Banach space and $I_\lambda \in C^1(X^{1,p}(\mathbb{R}^N), \mathbb{R})$. We say $\{u_n\}_n$ is a $(PS)_c$ sequence in $X^{1,p}(\mathbb{R}^N)$ if $I_\lambda(u_n) \to c$ and $I'_\lambda(u_n) \to 0$.

Lemma 5.2. *If $\{u_n\}_n$ is a $(PS)_c$ sequence, then $\{u_n\}_n$ is bounded in $X^{1,p}(\mathbb{R}^N)$ and $\{u_n^+\}_n$ is also a $(PS)_c$ sequence.*

Proof. Since $\{u_n\}_n$ is a $(PS)_c$ sequence, there exists $C > 0$ such that

$$|I_\lambda(u_n)| \le C, \quad \left| \left\langle I'_\lambda(u_n), \frac{u_n}{\|u_n\|} \right\rangle \right| \le C.$$

Then we obtain

$$I_\lambda(u_n) - \frac{1}{2p^*_\mu}\langle I'_\lambda(u_n), u_n\rangle \le C + C\|u_n\|. \tag{5.1}$$

When $\theta < \frac{2N-\mu}{N-p}$ and $a > 0,\ b > 0$, we have

$$\begin{aligned} I_\lambda(u_n) - \frac{1}{2p^*_\mu}\langle I'_\lambda(u_n), u_n\rangle &= a\left(\frac{1}{p} - \frac{1}{2p^*_\mu}\right)\|u_n\|^p + b\left(\frac{1}{\theta p} - \frac{1}{2p^*_\mu}\right)\|u_n\|^{\theta p} \\ &\quad + \lambda\left(\frac{1}{2p^*_\mu} - 1\right)\int_{\mathbb{R}^N} f(x)u_n\,dx \\ &\ge b\left(\frac{1}{\theta p} - \frac{1}{2p^*_\mu}\right)\|u_n\|^{\theta p} - \lambda\left(1 - \frac{1}{2p^*_\mu}\right)S^{-\frac{1}{p}}|f|_{(p^*)'}\|u_n\|. \end{aligned}$$

When $\theta = \frac{2N-\mu}{N-p}$ and $a > 0,\ b > 0$, we have

$$\begin{aligned} I_\lambda(u_n) - \frac{1}{2p^*_\mu}\langle I'_\lambda(u_n), u_n\rangle &= a\left(\frac{1}{p} - \frac{1}{2p^*_\mu}\right)\|u_n\|^p + b\left(\frac{1}{\theta p} - \frac{1}{2p^*_\mu}\right)\|u_n\|^{\theta p} \\ &\quad + \lambda\left(\frac{1}{2p^*_\mu} - 1\right)\int_{\mathbb{R}^N} f(x)u_n\,dx \\ &\ge a\left(\frac{1}{p} - \frac{1}{2p^*_\mu}\right)\|u_n\|^{p} - \lambda\left(1 - \frac{1}{2p^*_\mu}\right)S^{-\frac{1}{p}}|f|_{(p^*)'}\|u_n\|. \end{aligned}$$

By invoking $1 < p < \theta p$ and (5.1), we conclude that $\{u_n\}_n$ is bounded in $X^{1,p}(\mathbb{R}^N)$.
Next, we aim to demonstrate that $\{u_n^+\}_n$ is also a $(PS)_c$ sequence, the proof is divided into two steps.

Step 1. We prove $I_\lambda(u_n^+) \to c$.
Since $\{u_n\}_n$ is a $(PS)_c$ sequence, we have $I_\lambda(u_n) \to c$ and $\langle I'_\lambda(u_n), u_n\rangle \to 0$. So $\langle I'_\lambda(u_n), u_n^+\rangle \to 0$. From the definition of I_λ, we have

$$\begin{aligned} &\langle I'_\lambda(u_n), u_n\rangle - \langle I'_\lambda(u_n), u_n^+\rangle \\ &= \left(a + b\|u_n\|^{p(\theta-1)}\right)\left(\int_{\mathbb{R}^N} |\nabla u_n|^{p-2}\nabla u_n(-\nabla u_n^-)\,dx \right.\\ &\qquad \left. + \iint_{\mathbb{R}^N} \frac{|u_n(x)-u_n(y)|^{p-2}(u_n(x)-u_n(y))(-u_n^-(x)+u_n^-(y))}{|x-y|^{N+ps}}\,dx\,dy\right) \\ &\qquad - \lambda\int_{\mathbb{R}^N} f(x)u_n\,dx + \lambda\int_{\mathbb{R}^N} f(x)u_n^+\,dx \to 0 \end{aligned} \tag{5.2}$$

as $n \to \infty$. Based on the fact that

$$|\xi^- - \eta^-|^p \leq |\xi - \eta|^{p-2}(\xi - \eta)(-\xi^- + \eta^-),$$

we obtain

$$|u_n(x) - u_n(y)|^{p-2} \left(u_n(x) - u_n(y)\right) \left(-u_n^-(x) + u_n^-(y)\right) \geq |u_n^-(x) - u_n^-(y)|^p \geq 0. \tag{5.3}$$

Besides,

$$\begin{aligned} &\int_{\mathbb{R}^N} |\nabla u_n|^{p-2}\nabla u_n \left(-\nabla u_n^-\right) \, dx \\ &\quad = \int_{\mathbb{R}^N} |\nabla u_n|^{p-2} \left(\nabla u_n^+ - \nabla u_n^-\right) \left(-\nabla u_n^-\right) \, dx = \int_{\mathbb{R}^N} |\nabla u_n^-|^p \, dx \geq 0. \end{aligned} \tag{5.4}$$

Moreover, $f \geq 0$ and $\lambda \geq 0$ implies

$$\lambda \int_{\mathbb{R}^N} f(x) \left(u_n^+ - u_n\right) \, dx = \lambda \int_{\mathbb{R}^N} f(x) u_n^- \, dx \geq 0. \tag{5.5}$$

By (5.2)–(5.5), we derive

$$[u_n^-]_{s,p}^p \to 0, \qquad \int_{\mathbb{R}^N} |\nabla u_n^-|^p \, dx \to 0, \qquad \int_{\mathbb{R}^N} f u_n^- \, dx \to 0 \tag{5.6}$$

as $n \to \infty$. By

$$[u_n^+]_{s,p} - [u_n^-]_{s,p} \leq [u_n]_{s,p} = [u_n^+ - u_n^-]_{s,p} \leq [u_n^+]_{s,p} + [u_n^-]_{s,p},$$

we eventually obtain $[u_n^+]_{s,p}^p \to [u_n]_{s,p}^p$. Similarly, we can get $\int_{\mathbb{R}^N} |\nabla u_n^+|^p \, dx \to \int_{\mathbb{R}^N} |\nabla u_n|^p \, dx$. It is also easy to deduce $\int_{\mathbb{R}^N} f u_n^+ \, dx \to \int_{\mathbb{R}^N} f u_n \, dx$. Hence we conclude $I_\lambda(u_n^+) \to c$ since $I_\lambda(u_n) \to c$.

Step 2. We prove $\langle I'_\lambda(u_n^+), v\rangle \to 0$ for all $v \in X^{1,p}(\mathbb{R}^N)$.

Note that

$$\langle I'_\lambda(u_n), v\rangle - \langle I'_\lambda(u_n^+), v\rangle = \left(a + b\|u_n\|^{p(\theta-1)}\right) (\mathrm{I}_1 + \mathrm{I}_2) + o(1), \tag{5.7}$$

where

$$\mathrm{I}_1 = \int_{\mathbb{R}^N} \left(|\nabla u_n|^{p-2}\nabla u_n - |\nabla u_n^+|^{p-2}\nabla u_n^+\right) v \, dx$$

and

$$\mathrm{I}_2 = \iint_{\mathbb{R}^N} \frac{\left(|u_n(x)-u_n(y)|^{p-2}(u_n(x)-u_n(y)) - |u_n^+(x)-u_n^+(y)|^{p-2}(u_n^+(x)-u_n^+(y))\right)(v(x)-v(y))}{|x-y|^{N+ps}} \, dx \, dy.$$

Next we are going to estimate I_1 and I_2.

For I_1:

Case 1. When $u_n \geq 0$, we have $\mathrm{I}_1 = 0$.

Case 2. When $u_n < 0$,

$$\mathrm{I}_1 = \int_{\{x:\ u_n(x)<0\}} |\nabla u_n|^{p-2}\nabla u_n v \, dx \leq \int_{\mathbb{R}^N} |\nabla u_n^-|^p \, dx \int_{\mathbb{R}^N} |v|^p \, dx \to 0$$

as $n \to \infty$ by (5.6).

For I_2:

Case 1. When $u_n(x) \geq 0,\ u_n(y) \geq 0$, it's obvious that $\mathrm{I}_2 = 0$.

Case 2. When $u_n(x) \geq 0,\ u_n(y) < 0$,

$$
\begin{aligned}
\mathrm{I}_2 &= \int_{\{x:\ u_n(x)\geq 0\}} \int_{\{y:\ u_n(y)<0\}} \frac{\left[(u_n(x)-u_n(y))^{p-1} - (u_n(x))^{p-1}\right](v(x)-v(y))}{|x-y|^{N+ps}}\,dx\,dy.\\
&= \int_{\{x:\ u_n(x)\geq 0\}} \int_{\{y:\ u_n(y)<0\}} \frac{\left(\sum_{i,j\in Z^+,\ i+j=p-1,\ j\geq 1}(u_n(x))^i(-u_n(y))^j\right)(v(x)-v(y))}{|x-y|^{N+ps}}\,dx\,dy\\
&= \sum_{\substack{i,j\in Z^+\\ i+j=p-1\\ j\geq 1}} \int_{\{x:\ u_n(x)\geq 0\}} \int_{\{y:\ u_n(y)<0\}} \frac{(u_n(x))^i(v(x)-v(y))^{\frac{i}{p-1}}(-u_n(y))^j(v(x)-v(y))^{\frac{j}{p-1}}}{|x-y|^{N+ps}}\,dx\,dy\\
&\leq \sum_{\substack{i,j\in Z^+\\ i+j=p-1\\ j\geq 1}} \iint_{\mathbb{R}^{2N}} \frac{|u_n^+(x)-u_n^+(y)|^i|v(x)-v(y)|^{\frac{i}{p-1}}}{|x-y|^{(N+ps)\cdot\frac{i}{p-1}}} \cdot \frac{|u_n^-(x)-u_n^-(y)|^j|v(x)-v(y)|^{\frac{j}{p-1}}}{|x-y|^{(N+ps)\cdot\frac{j}{p-1}}}\,dx\,dy\\
&\leq \sum_{\substack{i,j\in Z^+\\ i+j=p-1\\ j\geq 1}} \left(\iint_{\mathbb{R}^{2N}} \frac{|u_n^+(x)-u_n^+(y)|^{p-1}|v(x)-v(y)|}{|x-y|^{N+ps}}\,dx\,dy\right)^{\frac{i}{p-1}}\\
&\quad \left(\iint_{\mathbb{R}^{2N}} \frac{|u_n^-(x)-u_n^-(y)|^{p-1}|v(x)-v(y)|}{|x-y|^{N+ps}}\,dx\,dy\right)^{\frac{j}{p-1}}\\
&\leq \sum_{\substack{i,j\in Z^+\\ i+j=p-1\\ j\geq 1}} \left(\iint_{\mathbb{R}^{2N}} \frac{|u_n^+(x)-u_n^+(y)|^p}{|x-y|^{N+ps}}\,dx\,dy\right)^{\frac{p-1}{p}} \left(\iint_{\mathbb{R}^{2N}} \frac{|v(x)-v(y)|^p}{|x-y|^{N+ps}}\,dx\,dy\right)^{\frac{1}{p}}\\
&\quad \left(\iint_{\mathbb{R}^{2N}} \frac{|u_n^-(x)-u_n^-(y)|^p}{|x-y|^{N+ps}}\,dx\,dy\right)^{\frac{p-1}{p}} \left(\iint_{\mathbb{R}^{2N}} \frac{|v(x)-v(y)|^p}{|x-y|^{N+ps}}\,dx\,dy\right)^{\frac{1}{p}}\\
&\leq \sum_{\substack{j\in Z^+\\ 1\leq j\leq p-1}} C\|u_n^-\|^{\frac{j}{p}} \to 0
\end{aligned}
$$

as $n \to \infty$ by (5.6).

Case 3. When $u_n(x) < 0,\ u_n(y) \geq 0$, we can obtain $\mathrm{I}_2 \to 0$ as $n \to \infty$ through a discussion similar to Case 2.

Case 4. When $u_n(x) < 0,\ u_n(y) < 0$, we have

$$
\begin{aligned}
\mathrm{I}_2 &= \int_{\{x:\ u_n(x)<0\}} \int_{\{y:\ u_n(y)<0\}} \frac{|u_n(x)-u_n(y)|^{p-2}(u_n(x)-u_n(y))(v(x)-v(y))}{|x-y|^{N+ps}}\,dx\,dy\\
&\leq \iint_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^p}{|x-y|^{N+ps}}\,dx\,dy \iint_{\mathbb{R}^{2N}} \frac{|v(x)-v(y)|^p}{|x-y|^{N+ps}}\,dx\,dy\\
&\leq C\|u_n^-\| \to 0
\end{aligned}
$$

as $n \to \infty$ by (5.6).

By virtue of (5.7) and the fact that $\langle I'_\lambda(u_n), v\rangle \to 0$, we derive that $\langle I'_\lambda(u_n^+), v\rangle \to 0$ for all $v \in X^{1,p}(\mathbb{R}^N)$. Consequently, we conclude that $\{u_n^+\}_n$ is a $(PS)_c$ sequence. □

Lemma 5.3. *Suppose $a>0,\ b>0$ and either $\theta<\frac{2N-\mu}{N-p}$, or $a>0,\ 0<b<S_{H,M}^{-\frac{2p_\mu^*}{p}}$, $\theta=\frac{2N-\mu}{N-p}$. Then there exist $\lambda_0>0$, $\rho>0$ and $\alpha>0$ such that for any $\lambda\in(0,\lambda_0)$, $I_\lambda(u)\geq\alpha$ holds for every $u\in X^{1,p}(\mathbb{R}^N)$ satisfying $\|u\|=\rho$.*

Proof. For any $u\in X^{1,p}(\mathbb{R}^N)$, by the Sobolev inequality, we have

$$\begin{aligned}I_\lambda(u)&=\frac{a}{p}\|u\|^p+\frac{b}{\theta p}\|u\|^{\theta p}-\frac{1}{2p_\mu^*}\iint_{\mathbb{R}^{2N}}\frac{|u^+(x)|^{p_\mu^*}|u^+(y)|^{p_\mu^*}}{|x-y|^\mu}\,dx\,dy-\lambda\int_{\mathbb{R}^N}f(x)u\,dx\\&\geq\frac{a}{p}\|u\|^p+\frac{b}{\theta p}\|u\|^{\theta p}-\frac{1}{2p_\mu^*}S_{H,M}^{-\frac{2p_\mu^*}{p}}\|u\|^{2p_\mu^*}-\lambda S^{-\frac{1}{p}}|f|_{(p^*)'}\|u\|,\end{aligned}$$

where S is the optimal constant for the Sobolev embedding $X^{1,p}(\mathbb{R}^N)\hookrightarrow L^{p^*}(\mathbb{R}^N)$.

Case 1. $a>0,\ b>0$ and $\theta<\frac{2N-\mu}{N-p}$. We have

$$I_\lambda(u)\geq\left(\frac{b}{\theta p}\|u\|^{\theta p-1}-\frac{1}{2p_\mu^*}S_{H,M}^{-\frac{2p_\mu^*}{p}}\|u\|^{2p_\mu^*-1}-\lambda S^{-\frac{1}{p}}|f|_{(p^*)'}\right)\|u\|.$$

Define

$$k(t):=\frac{b}{\theta p}t^{\theta p-1}-\frac{1}{2p_\mu^*}S_{H,M}^{-\frac{2p_\mu^*}{p}}t^{2p_\mu^*-1}$$

for all $t\geq0$. Since $2p_\mu^*>\theta p$ by $\theta<\frac{2N-\mu}{N-p}$, we obtain that $\max_{t\geq0}k(t)=k(\rho)>0$, where

$$\rho=\left(\frac{2bp_\mu^*(\theta p-1)S_{H,M}^{\frac{2p_\mu^*}{p}}}{\theta p(2p_\mu^*-1)}\right)^{\frac{1}{2p_\mu^*-\theta p}}.$$

Set

$$\lambda_0=\frac{k(\rho)}{S^{-\frac{1}{p}}|f|_{(p^*)'}},$$

then for any $\lambda\in(0,\lambda_0)$ we have

$$I_\lambda(u)\geq\left(k(\rho)-\lambda S^{-\frac{1}{p}}|f|_{(p^*)'}\right)\rho=:\alpha>0\quad\text{for all }\|u\|=\rho.$$

Case 2. $a>0,\ 0<b<S_{H,M}^{-\frac{2p_\mu^*}{p}}$ and $\theta=\frac{2N-\mu}{N-p}$. We have

$$I_\lambda(u)\geq\left[\frac{a}{p}\|u\|^{p-1}-\left(\frac{1}{2p_\mu^*}S_{H,M}^{-\frac{2p_\mu^*}{p}}-\frac{b}{\theta p}\right)\|u\|^{\theta p-1}-\lambda S^{-\frac{1}{p}}|f|_{(p^*)'}\right]\|u\|$$

with $2p_\mu^*=\theta p$. Define

$$l(t):=\frac{a}{p}t^{p-1}-\left(\frac{1}{2p_\mu^*}S_{H,M}^{-\frac{2p_\mu^*}{p}}-\frac{b}{\theta p}\right)t^{\theta p-1}$$

for all $t\geq0$. Then $\max_{t\geq0}l(t)=l(\rho)>0$, where

$$\rho=\left(\frac{2a(p-1)p_\mu^*}{p(2p_\mu^*-1)(S_{H,M}^{-\frac{2p_\mu^*}{p}}-b)}\right)^{\frac{1}{2p_\mu^*-p}}.$$

Set

$$\lambda_0 = \frac{l(\rho)}{S^{-\frac{1}{p}}|f|_{(p^*)'}},$$

then for all $\lambda \in (0, \lambda_0)$ we deduce

$$I_\lambda(u) \geq \left(l(\rho) - \lambda S^{-\frac{1}{p}}|f|_{(p^*)'}\right)\rho =: \alpha > 0 \text{ for all } \|u\| = \rho.$$

Thus, the proof is completed. □

Lemma 5.4. *Suppose $a > 0$, $b > 0$, and either $\theta < \frac{2N-\mu}{N-p}$ or $a > 0$, $0 < b < S_{H,M}^{-\frac{2p_\mu^*}{p}}$, $\theta = \frac{2N-\mu}{N-p}$. Then for any $\lambda > 0$, there exists $e \in X^{1,p}(\mathbb{R}^N)$ such that $\|e\| \geq \rho$ and $I_\lambda(e) < 0$, where ρ is the number given in Lemma 5.3.*

Proof. By the definition of $S_{H,M}$ as given in (1.4), for all $\varepsilon > 0$, there exists $u \in X^{1,p}(\mathbb{R}^N) \setminus \{0\}$ such that

$$\frac{\|u\|^p}{\left(\int\int_{\mathbb{R}^{2N}} \frac{|u(x)|^{p_\mu^*}|u(y)|^{p_\mu^*}}{|x-y|^\mu}\,dx\,dy\right)^{\frac{p}{2p_\mu^*}}} \leq S_{H,M} + \varepsilon.$$

Then there exists a positive U satisfying $\|U\|_{HL} = 1$ such that

$$\|U\|^p = S_{H,M} + h(\varepsilon), \tag{5.8}$$

where $h(\varepsilon)$ is a nonnegative constant associated with ε and $h(\varepsilon) \leq \varepsilon$. Then for any $t > 0$, we have

$$\begin{aligned} I_\lambda(tU) &= \frac{a\|U\|^p}{p}t^p + \frac{b\|U\|^{\theta p}}{\theta p}t^{\theta p} - \frac{t^{2p_\mu^*}}{2p_\mu^*} - \lambda t\int_{\mathbb{R}^N} f(x)U\,dx \\ &\leq \frac{a\|U\|^p}{p}t^p + \frac{b\|U\|^{\theta p}}{\theta p}t^{\theta p} - \frac{t^{2p_\mu^*}}{2p_\mu^*}. \end{aligned}$$

When $\theta < \frac{2N-\mu}{N-p}$(i.e. $\theta p < 2p_\mu^*$), there exists a sufficiently large $t > 0$ such that $\|tU\| \geq \rho$ and $I_\lambda(tU) < 0$.

When $\theta = \frac{2N-\mu}{N-p}$(i.e. $\theta p = 2p_\mu^*$), we derive

$$I_\lambda(tU) \leq \frac{a\|U\|^p}{p}t^p - \left(\frac{1}{2p_\mu^*} - \frac{b\|U\|^{\theta p}}{\theta p}\right)t^{2p_\mu^*}.$$

Given $0 < b < S_{H,M}^{-\theta}$, there exists $t > 0$ such that $\|tU\| \geq \rho$ and $I_\lambda(tU) < 0$. Setting $e = tU$ completes the proof. □

From Lemma 5.2, we have that any $(PS)_c$ sequence can be regarded as a sequence of nonnegative functions. Thus, we may assume without loss of generality that all $(PS)_c$ sequences are nonnegative in the subsequent analysis.

Define

$$\Lambda = \begin{cases} \left(\frac{1}{p} - \frac{1}{\theta p}\right) (aS_{H,M})^{\frac{\theta}{\theta-1}} \left(\frac{1}{1-bS_{H,M}^{\theta}}\right)^{\frac{1}{\theta-1}}, & \text{if } \theta = \frac{2N-\mu}{N-p},\ a>0,\ 0<b<S_{H,M}^{-\theta}, \\ \left(\frac{a}{p} - \frac{a}{\theta p}\right) S_{H,M} \left(\frac{bS_{H,M}^{\theta} + \sqrt{b^2 S_{H,M}^{2\theta} + 4aS_{H,M}}}{2}\right)^{\frac{2\theta-1}{\theta-1}\cdot\frac{p}{2p_\mu^*}} & \\ \quad + \left(\frac{1}{\theta p} - \frac{1}{2p_\mu^*}\right) \left(\frac{bS_{H,M}^{\theta} + \sqrt{b^2 S_{H,M}^{2\theta} + 4aS_{H,M}}}{2}\right)^{\frac{2\theta-1}{\theta-1}}, & \text{if } \theta = \frac{3N-\mu-p}{2(N-p)},\ a>0,\ b>0, \end{cases}$$

and

$$\Theta(\lambda) = \frac{p-1}{p} \left[\left(a - \frac{a}{\theta}\right) S\right]^{-\frac{1}{p-1}} \left(1 - \frac{1}{\theta p}\right)^{\frac{p}{p-1}} \lambda^{\frac{p}{p-1}} |f|_{(p^*)'}^{\frac{p}{p-1}},$$

where $(p^*)' = \frac{p^*}{p^*-1}$.

Lemma 5.5. *Assume that $a>0$, $b>0$ and $\theta = \frac{3N-\mu-p}{2(N-p)}$, or $a>0$, $0<b<S_{H,M}^{-\frac{2\cdot p_\mu^*}{p}}$ and $\theta = \frac{2N-\mu}{N-p}$, if $\{u_n\}_n \in X^{1,p}(\mathbb{R}^N)$ is a bounded and nonnegative sequence satisfying $I_\lambda(u_n) \to c < \Lambda - \Theta(\lambda)$ and $I'_\lambda(u_n) \to 0$ in $(X^{1,p}(\mathbb{R}^N))'$ as $n\to\infty$, then there exists a nonnegative function $u \in X^{1,p}(\mathbb{R}^N)$ such that, up to a subsequence, $u_n \to u$ in $X^{1,p}(\mathbb{R}^N)$ as $n\to\infty$.*

Proof. As $\{u_n\}_n$ is bounded and nonnegative, then there exists a subsequence, still noted $\{u_n\}_n$, and a nonnegative function $u \in X^{1,p}(\mathbb{R}^N)$ such that $u_n \rightharpoonup u$ in $X^{1,p}(\mathbb{R}^N)$, $u_n \to u$ in L^r_{loc} for $r \in [1,p^*)$ and $u_n \to u$ *a.e* in $\mathbb{R}^N$. Applying Theorem 4.2, up to a further subsequence, there exists a (at most) countable set J, a non-atomic measure $\tilde{\xi}$ and points $\{x_j\}_{j\in J} \subset \mathbb{R}^N$ and $\{\xi_j\}_{j\in J}$, $\{\nu_j\}_{j\in J} \subset \mathbb{R}^+$ such that as $n\to\infty$ we have

$$|\nabla u_n|^p + \int_{\mathbb{R}^N} \frac{|u_n(x)-u_n(y)|^p}{|x-y|^{N+ps}}\,dy \rightharpoonup \xi = |\nabla u|^p + \int_{\mathbb{R}^N} \frac{|u(x)-u(y)|^p}{|x-y|^{N+ps}}\,dy + \sum_{j\in J} \xi_j \delta_{x_j} + \tilde{\xi}, \tag{5.9}$$

$$\int_{\mathbb{R}^N} \frac{|u_n(y)|^{p_\mu^*}}{|x-y|^\mu}\,dy\,|u_n(x)|^{p_\mu^*} \rightharpoonup \nu = \int_{\mathbb{R}^N} \frac{|u(y)|^{p_\mu^*}}{|x-y|^\mu}\,dy\,|u(x)|^{p_\mu^*} + \sum_{j\in J} \nu_j \delta_{x_j} \tag{5.10}$$

in the measure sense, where δ_{x_j} is the Dirac measure concentrated at x_j. Moreover,

$$\nu_j \leq S_{H,M}^{-\frac{2p_\mu^*}{p}} \xi_j^{\frac{2p_\mu^*}{p}} \tag{5.11}$$

for all $j\in J$, where $S_{H,M}$ is the optimal constant defined in (1.4). We now proceed with the proof in three steps.

Step 1. We claim that $J=\emptyset$. Suppose by contradiction that $J\neq\emptyset$. Fix $j\in J$. For any $\varepsilon>0$, choose $\varphi_{\varepsilon,j} \in C_0^\infty(\mathbb{R}^N)$ such that

$$\varphi_{\varepsilon,j} = 1 \quad \text{in } B_\varepsilon(x_j), \qquad \varphi_{\varepsilon,j} = 0 \quad \text{in } \mathbb{R}^N \setminus B_{2\varepsilon}(x_j), \qquad |\nabla\varphi_{\varepsilon,j}| \leq 2/\varepsilon.$$

Evidently, $\varphi_{\varepsilon,j}u_n \in X^{1,p}(\mathbb{R}^N)$. It follows from $\langle I'_\lambda(u_n), \varphi_{\varepsilon,j}u_n\rangle \to 0$ that

$$\begin{aligned}&\left(a+b\|u_n\|^{(\theta-1)p}\right)\left(\int_{\mathbb{R}^N}|\nabla u_n|^{p-2}\nabla u_n\nabla(\varphi_{\varepsilon,j}u_n)\,dx\right.\\&\quad\left.+\iint_{\mathbb{R}^{2N}}\frac{|u_n(x)-u_n(y)|^{p-2}(u_n(x)-u_n(y))(\varphi_{\varepsilon,j}(x)u_n(x)-\varphi_{\varepsilon,j}(y)u_n(y))}{|x-y|^{N+ps}}\,dx\,dy\right)\\&=\iint_{\mathbb{R}^{2N}}\frac{|u_n(y)|^{p^*_\mu}|u_n(x)|^{p^*_\mu-2}u_n(x)u_n(x)\varphi_{\varepsilon,j}(x)}{|x-y|^\mu}\,dx\,dy+\lambda\int_{\mathbb{R}^N}f(x)\varphi_{\varepsilon,j}(x)u_n\,dx+o(1).\end{aligned}\tag{5.12}$$

For the left-hand side of (5.12), applying Hölder's inequality and Lemma 4.3 yields

$$\begin{aligned}&\lim_{\varepsilon\to0}\lim_{n\to\infty}\int_{\mathbb{R}^N}|\nabla u_n|^{p-2}\nabla u_n\nabla\varphi_{\varepsilon,j}u_n\,dx\\&\le\lim_{\varepsilon\to0}\lim_{n\to\infty}\left(\int_{\mathbb{R}^N}|\nabla u_n|^p\,dx\right)^{\frac{p-1}{p}}\left(\int_{\mathbb{R}^N}|\nabla\varphi_{\varepsilon,j}u_n|^p\,dx\right)^{\frac1p}\\&\le C\lim_{\varepsilon\to0}\lim_{n\to\infty}\left(\int_{\mathbb{R}^N}|\nabla\varphi_{\varepsilon,j}u_n|^p\,dx\right)^{\frac1p}=0,\end{aligned}\tag{5.13}$$

and

$$\begin{aligned}&\lim_{\varepsilon\to0}\lim_{n\to\infty}\left|\iint_{\mathbb{R}^{2N}}\frac{|u_n(x)-u_n(y)|^{p-2}(u_n(x)-u_n(y))(\varphi_{\varepsilon,j}(x)-\varphi_{\varepsilon,j}(y))u_n(y)}{|x-y|^{N+ps}}\,dx\,dy\right|\\&\le C\lim_{\varepsilon\to0}\lim_{n\to\infty}\left(\iint_{\mathbb{R}^{2N}}\frac{|u_n(x)-u_n(y)|^p}{|x-y|^{N+ps}}\,dx\,dy\right)^{\frac{p-1}{p}}\left(\iint_{\mathbb{R}^{2N}}\frac{|(\varphi_{\varepsilon,j}(x)-\varphi_{\varepsilon,j}(y))u_n(y)|^p}{|x-y|^{N+ps}}\,dx\,dy\right)^{\frac1p}\\&\le C\lim_{\varepsilon\to0}\lim_{n\to\infty}\left(\frac{|(\varphi_{\varepsilon,j}(x)-\varphi_{\varepsilon,j}(y))u_n(y)|^p}{|x-y|^{N+ps}}\,dx\,dy\right)^{\frac1p}=0.\end{aligned}\tag{5.14}$$

This together with equations (5.9) leads to

$$\begin{aligned}&\lim_{\varepsilon\to0}\lim_{n\to\infty}\left(a+b\|u_n\|^{(\theta-1)p}\right)\int_{\mathbb{R}^N}\left(|\nabla u_n|^p+\int_{\mathbb{R}^N}\frac{|u_n(x)-u_n(y)|^p}{|x-y|^{N+ps}}dy\right)\varphi_{\varepsilon,j}(x)\,dx\\&\ge\lim_{\varepsilon\to0}\lim_{n\to\infty}a\int_{\mathbb{R}^N}\left(|\nabla u_n|^p+\int_{\mathbb{R}^N}\frac{|u_n(x)-u_n(y)|^p}{|x-y|^{N+ps}}dy\right)\varphi_{\varepsilon,j}(x)\,dx\\&\quad+\lim_{\varepsilon\to0}\lim_{n\to\infty}b\left(\int_{\mathbb{R}^N}\left(|\nabla u_n|^p+\int_{\mathbb{R}^N}\frac{|u_n(x)-u_n(y)|^p}{|x-y|^{N+ps}}\,dy\right)\varphi_{\varepsilon,j}(x)\,dx\right)^\theta\\&=a\xi_j+b\xi_j^\theta.\end{aligned}\tag{5.15}$$

For the first term on the right-hand side of (5.12), it follows from (5.10) that

$$\begin{aligned}&\lim_{\varepsilon\to0}\lim_{n\to\infty}\iint_{\mathbb{R}^{2N}}\frac{|u_n(y)|^{p^*_\mu}}{|x-y|^\mu}\,dy\,|u_n(x)|^{p^*_\mu}\varphi_{\varepsilon,j}\,dx\\&=\lim_{\varepsilon\to0}\left(\iint_{\mathbb{R}^{2N}}\frac{|u(y)|^{p^*_\mu}}{|x-y|^\mu}\,dy\,|u(x)|^{p^*_\mu}\varphi_{\varepsilon,j}\,dx+\sum_{j\in J}\nu_j\delta_{x_j}\right)=\nu_j.\end{aligned}\tag{5.16}$$

Moreover,

$$\lim_{\varepsilon\to0}\lim_{n\to\infty}\int_{\mathbb{R}^N}f\varphi_{\varepsilon,j}u_n\,dx=\lim_{\varepsilon\to0}\int_{\mathbb{R}^N}f\varphi_{\varepsilon,j}u\,dx=0.\tag{5.17}$$

Therefore, by substituting (5.13)–(5.17) into (5.12), we obtain

$$\nu_j \geq a\xi_j + b\xi_j^{\theta}. \tag{5.18}$$

Combining (5.11) with (5.18) implies

$$\nu_j \geq \begin{cases} \left(\dfrac{aS_{H,M}}{1-bS_{H,M}^{\theta}}\right)^{\frac{\theta}{\theta-1}} & \text{if } \theta = \frac{2N-\mu}{N-p},\ a>0,\ 0<b<S_{H,M}^{-\theta}, \\ \left(\dfrac{bS_{H,M}^{\theta}+\sqrt{b^2S_{H,M}^{2\theta}+4aS_{H,M}}}{2}\right)^{\frac{2\theta-1}{\theta-1}} & \text{if } \theta = \frac{3N-\mu-p}{2(N-p)},\ a>0,\ b>0. \end{cases} \tag{5.19}$$

For $R>0$, assume that $\psi_R \in C_0^{\infty}(\mathbb{R}^N)$ satisfies $\psi_R \in [0,1]$, $\psi_R(x)=1$ for $|x-x_j| \leq R$, $\psi_R(x)=0$ for $|x-x_j|>2R$, and $|\nabla\psi_R| \leq \frac{2}{R}$. By (5.9) and (5.10), we have

$$\begin{aligned} c &= \lim_{n\to\infty}\left(I_\lambda(u_n) - \frac{1}{\theta p}\langle I_\lambda'(u_n), u_n\rangle\right) \\ &= \lim_{R\to\infty}\lim_{n\to\infty}\left(\left(\frac{a}{p}-\frac{a}{\theta p}\right)\left(\int_{\mathbb{R}^N}|\nabla u_n|^p\psi_R\,dx + \iint_{\mathbb{R}^{2N}}\frac{|u_n(x)-u_n(y)|^p}{|x-y|^{N+ps}}\psi_R(x)\,dy\,dx\right)\right. \\ &\quad \left.+\left(\frac{1}{\theta p}-\frac{1}{2p_\mu^*}\right)\iint_{\mathbb{R}^{2N}}\frac{|u_n(y)|^{p_\mu^*}|u_n(x)|^{p_\mu^*}\psi_R(x)}{|x-y|^{\mu}}\,dy\,dx - \left(1-\frac{1}{\theta p}\right)\lambda\int_{\mathbb{R}^N} fu_n\,dx\right) \\ &= \lim_{R\to\infty}\left(\left(\frac{a}{p}-\frac{a}{\theta p}\right)\left(\int_{\mathbb{R}^N}|\nabla u|^p\psi_R\,dx + \iint_{\mathbb{R}^{2N}}\frac{|u(x)-u(y)|^p}{|x-y|^{N+ps}}\psi_R(x)\,dy\,dx\right) + \left(\frac{a}{p}-\frac{a}{\theta p}\right)\xi_j\right. \\ &\quad +\left(\frac{1}{\theta p}-\frac{1}{2p_\mu^*}\right)\iint_{\mathbb{R}^{2N}}\frac{|u(y)|^{p_\mu^*}|u(x)|^{p_\mu^*}\psi_R(x)}{|x-y|^{\mu}}\,dy\,dx \\ &\quad \left.+\left(\frac{1}{\theta p}-\frac{1}{2p_\mu^*}\right)\nu_j - \left(1-\frac{1}{\theta p}\right)\lambda\int_{\mathbb{R}^N} fu\,dx\right) \\ &\geq \left(\frac{a}{p}-\frac{a}{\theta p}\right)\left(\int_{\mathbb{R}^N}|\nabla u|^p dx + \iint_{\mathbb{R}^{2N}}\frac{|u(x)-u(y)|^p}{|x-y|^{N+ps}}\,dy\,dx\right) + \left(\frac{a}{p}-\frac{a}{\theta p}\right)\xi_j \\ &\quad +\left(\frac{1}{\theta p}-\frac{1}{2p_\mu^*}\right)\iint_{\mathbb{R}^{2N}}\frac{|u(y)|^{p_\mu^*}|u(x)|^{p_\mu^*}}{|x-y|^{\mu}}\,dy\,dx + \left(\frac{1}{\theta p}-\frac{1}{2p_\mu^*}\right)\nu_j - \left(1-\frac{1}{\theta p}\right)\lambda\int_{\mathbb{R}^N} fu\,dx \\ &\geq \left(\frac{a}{p}-\frac{a}{\theta p}\right)S|u|_{p^*}^p + \left(\frac{a}{p}-\frac{a}{\theta p}\right)\xi_j + \left(\frac{1}{\theta p}-\frac{1}{2p_\mu^*}\right)\iint_{\mathbb{R}^{2N}}\frac{|u(y)|^{p_\mu^*}|u(x)|^{p_\mu^*}}{|x-y|^{\mu}}\,dy\,dx \\ &\quad +\left(\frac{1}{\theta p}-\frac{1}{2p_\mu^*}\right)\nu_j - \left(1-\frac{1}{\theta p}\right)\lambda\int_{\mathbb{R}^N} fu\,dx, \end{aligned}$$

which implies that

$$c \geq \begin{cases} \left(\frac{a}{p}-\frac{a}{\theta p}\right)S|u|_{p^*}^p + \left(\frac{a}{p}-\frac{a}{\theta p}\right)\xi_j - \left(1-\frac{1}{\theta p}\right)\lambda\int_{\mathbb{R}^N} fu\,dx, \\ \qquad\qquad \text{if } \theta = \frac{2N-\mu}{N-p},\ a>0,\ b>0, \\ \left(\frac{a}{p}-\frac{a}{\theta p}\right)S|u|_{p^*}^p + \left(\frac{a}{p}-\frac{a}{\theta p}\right)\xi_j + \left(\frac{1}{\theta p}-\frac{1}{2p_\mu^*}\right)\nu_j - \left(1-\frac{1}{\theta p}\right)\lambda\int_{\mathbb{R}^N} fu\,dx, \\ \qquad\qquad \text{if } \theta = \frac{3N-\mu-p}{2(N-p)},\ a>0,\ b>0. \end{cases}$$

Then we apply the Hölder inequality, the Sobolev embedding, and Young inequality to derive

$$\begin{aligned}\left(1-\frac{1}{\theta p}\right)\lambda\int_{\mathbb{R}^N} fu\,dx &= \left(1-\frac{1}{\theta p}\right)\lambda|f|_{(p^*)'}|u|_{p^*}\\ &\leq \left(\frac{a}{p}-\frac{a}{\theta p}\right)S|u|_{p^*}^p+\frac{p-1}{p}\left[\left(a-\frac{a}{\theta}\right)S\right]^{-\frac{1}{p-1}}\left(1-\frac{1}{\theta p}\right)^{\frac{p}{p-1}}\lambda^{\frac{p}{p-1}}|f|_{(p^*)'}^{\frac{p}{p-1}}.\end{aligned} \tag{5.20}$$

By (5.20) we obtain

$$c\geq\begin{cases}\left(\frac{a}{p}-\frac{a}{\theta p}\right)\xi_j-\frac{p-1}{p}\left[\left(a-\frac{a}{\theta}\right)S\right]^{-\frac{1}{p-1}}\left(1-\frac{1}{\theta p}\right)^{\frac{p}{p-1}}\lambda^{\frac{p}{p-1}}|f|_{(p^*)'}^{\frac{p}{p-1}}\\ \qquad\qquad\text{if } \theta=\frac{2N-\mu}{N-p},\ a>0,\ b>0,\\ \left(\frac{a}{p}-\frac{a}{\theta p}\right)\xi_j+\left(\frac{1}{\theta p}-\frac{1}{2p_\mu^*}\right)\nu_j-\frac{p-1}{p}\left[\left(a-\frac{a}{\theta}\right)S\right]^{-\frac{1}{p-1}}\left(1-\frac{1}{\theta p}\right)^{\frac{p}{p-1}}\lambda^{\frac{p}{p-1}}|f|_{(p^*)'}^{\frac{p}{p-1}}\\ \qquad\qquad\text{if } \theta=\frac{3N-\mu-p}{2(N-p)},\ a>0,\ b>0.\end{cases}$$

Combining (5.11) with (5.19), we arrive at $c\geq\Lambda-\Theta(\lambda)$, which contradicts our initial assumption. Thus the claim $J=\emptyset$ holds.

Step 2. We claim $\nu_\infty=0$. Let $R>0$. Set

$$\xi_\infty=\lim_{R\to\infty}\limsup_{n\to\infty}\int_{\{x\in\mathbb{R}^N:|x|>R\}}\left(|\nabla u_n|^p+\int_{\mathbb{R}^N}\frac{|u_n(x)-u_n(y)|^p}{|x-y|^{N+ps}}\,dy\right)dx,$$

$$\nu_\infty=\lim_{R\to\infty}\limsup_{n\to\infty}\int_{\{x\in\mathbb{R}^N:|x|>R\}}\int_{\mathbb{R}^N}\frac{|u_n(y)|^{p_\mu^*}}{|x-y|^\mu}\,dy\,|u_n(x)|^{p_\mu^*}\,dx.$$

According to Theorem 4.4, we know that ξ_∞ and ν_∞ are well defined and we have

$$\limsup_{n\to\infty}\int_{\mathbb{R}^N}\left(|\nabla u_n|^p+\int_{\mathbb{R}^N}\frac{|u_n(x)-u_n(y)|^p}{|x-y|^{N+ps}}\,dy\right)dx=\xi(\mathbb{R}^N)+\xi_\infty, \tag{5.21}$$

$$\limsup_{n\to\infty}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|u_n(y)|^{p_\mu^*}}{|x-y|^\mu}\,dy\,|u_n(x)|^{p_\mu^*}\,dx=\nu(\mathbb{R}^N)+\nu_\infty. \tag{5.22}$$

To rule out the possibility of concentration for mass at infinity, we take a suitable cutoff function $\chi_R\in C^\infty(\mathbb{R}^N)$ satisfying $\chi_R\in[0,1]$, $\chi_R(x)=0$ for $|x|<R$ and $\chi_R(x)=1$ for $|x|>2R$, $|\nabla\chi_R|<\frac{2}{R}$. Following a discussion analogous to that of Theorem 4.4, we derive

$$\xi_\infty=\lim_{R\to\infty}\limsup_{n\to\infty}\int_{\mathbb{R}^N}|\nabla u_n|^p\chi_R(x)\,dx+\int_{\mathbb{R}^N}\frac{|u_n(x)-u_n(y)|^p}{|x-y|^{N+ps}}\chi_R(x)\,dy\,dx, \tag{5.23}$$

and

$$\nu_\infty=\lim_{R\to\infty}\limsup_{n\to\infty}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|u_n(y)\chi_R(y)|^{p_\mu^*}}{|x-y|^\mu}\,dy\,|u_n(x)\chi_R(x)|^{p_\mu^*}\,dx. \tag{5.24}$$

Moreover, we have

$$S_{H,M}\nu_\infty^{\frac{p}{2p_\mu^*}}<\xi_\infty. \tag{5.25}$$

Given that $\|u_n\|^p$ and $\|u_n\|_{HL}^{2p_\mu^*}$ are bounded, up to subsequence, we can assume that $\|u_n\|^p$ and $\|u_n\|_{HL}^{2p_\mu^*}$ are both convergent. Then by (5.21) and (5.22) we obtain

$$\lim_{n\to\infty} \|u_n\|^p = \int_{\mathbb{R}^N} d\xi + \xi_\infty, \tag{5.26}$$

$$\iint_{\mathbb{R}^{2N}} \frac{|u_n(y)|^{p_\mu^*}}{|x-y|^\mu}\, dy |u_n(x)|^{p_\mu^*}\, dx = \int_{\mathbb{R}^N} d\nu + \nu_\infty. \tag{5.27}$$

The fact $\langle I'_\lambda(u_n), \chi_R u_n\rangle \to 0$ as $n \to \infty$ entails that

$$\begin{aligned}
&\left(a + b\|u_n\|^{(\theta-1)p}\right)\left(\int_{\mathbb{R}^N} |\nabla u_n|^{p-2}\nabla u_n \nabla(\chi_R u_n)\, dx\right.\\
&\quad \left.+ \iint_{\mathbb{R}^{2N}} \frac{|u_n(x)-u_n(y)|^{p-2}(u_n(x)-u_n(y))(\chi_R(x)u_n(x)-\chi_R(y)u_n(y))}{|x-y|^{N+ps}}\, dx\, dy\right)\\
&= \iint_{\mathbb{R}^{2N}} \frac{|u_n(y)|^{p_\mu^*}|u_n(x)|^{p_\mu^*-2}u_n(x)u_n(x)\chi_R(x)}{|x-y|^\mu}\, dx\, dy + \lambda\int_{\mathbb{R}^N} f(x)\chi_R u_n\, dx + o(1).
\end{aligned} \tag{5.28}$$

By (4.8), (4.9) and the Hölder inequality, we get

$$\lim_{R\to\infty}\lim_{n\to\infty} \int_{\mathbb{R}^N} |\nabla u_n|^{p-2}\nabla u_n \nabla\chi_R u_n\, dx = 0, \tag{5.29}$$

$$\lim_{R\to\infty}\limsup_{n\to\infty} \iint_{\mathbb{R}^{2N}} \frac{|u_n(x)-u_n(y)|^{p-2}(u_n(x)-u_n(y))u_n(y)(\chi_R(x)-\chi_R(y))}{|x-y|^{N+ps}}\, dx\, dy = 0. \tag{5.30}$$

Hence, it follows from $\theta > 1$, (5.23), (5.26) and (5.28)–(5.30) that

$$\begin{aligned}
&\lim_{R\to\infty}\limsup_{n\to\infty} \left(a + b\|u_n\|^{(\theta-1)p}\right)\left(\int_{\mathbb{R}^N} |\nabla u_n|^p \chi_R\, dx + \iint_{\mathbb{R}^{2N}} \frac{|u_n(x)-u_n(y)|^p\chi_R(x)}{|x-y|^{N+ps}}\, dy\, dx\right)\\
&= \left[a + \left(\int_{\mathbb{R}^N} d\xi + \xi_\infty\right)^{\theta-1}\right]\\
&\quad\times \lim_{R\to\infty}\limsup_{n\to\infty}\left(\int_{\{x\in\mathbb{R}^N:|x|>R\}} |\nabla u_n|^p\, dx + \int_{\{x\in\mathbb{R}^N:|x|>R\}}\int_{\mathbb{R}^N} \frac{|u_n(x)-u_n(y)|^p}{|x-y|^{N+ps}}\, dy\, dx\right)\\
&\geq \left(a + b\xi_\infty^{\theta-1}\right)\xi_\infty = a\xi_\infty + b\xi_\infty^\theta.
\end{aligned} \tag{5.31}$$

Besides, it is straightforward to observe that

$$\lim_{R\to\infty}\limsup_{n\to\infty} \int_{\mathbb{R}^N} f(x)\chi_R u_n\, dx = \lim_{R\to\infty}\int_{\mathbb{R}^N} f(x)\chi_R u\, dx = 0. \tag{5.32}$$

Therefore, we combine (5.28)-(5.32) with (5.24) and obtain that

$$a\xi_\infty + b\xi_\infty^\theta \leq \nu_\infty.$$

In conjunction with (5.25), this leads to

$$aS_{H,M}\nu_\infty^{\frac{p}{2p_\mu^*}} + bS_{H,M}^\theta \nu_\infty^{\frac{\theta p}{2p_\mu^*}} \leq \nu_\infty,$$

indicating that $\nu_\infty = 0$ or

$$\nu_\infty \geq \begin{cases} \left(\dfrac{aS_{H,M}}{1-bS_{H,M}^{\theta}}\right)^{\frac{\theta}{\theta-1}} & \text{if } \theta = \frac{2N-\mu}{N-p},\ a>0,\ 0<b<S_{H,M}^{-\theta}, \\ \left(\dfrac{bS_{H,M}^{\theta}+\sqrt{b^2S_{H,M}^{2\theta}+4aS_{H,M}}}{2}\right)^{\frac{2\theta-1}{\theta-1}} & \text{if } \theta = \frac{3N-\mu-p}{2(N-p)},\ a>0,\ b>0. \end{cases} \tag{5.33}$$

Suppose that (5.33) holds, then we have

$$\begin{aligned} c &= \lim_{n\to\infty}\left(I_\lambda(u_n) - \frac{1}{\theta p}\langle I_\lambda'(u_n),u_n\rangle\right) \\ &= \lim_{n\to\infty}\left(\left(\frac{a}{p}-\frac{a}{\theta p}\right)\|u_n\|^p + \left(\frac{1}{\theta p}-\frac{1}{2p_\mu^*}\right)\iint_{\mathbb{R}^{2N}}\frac{|u_n(y)|^{p_\mu^*}|u_n(x)|^{p_\mu^*}}{|x-y|^\mu}\,dy\,dx \right.\\ &\quad \left. -\left(1-\frac{1}{\theta p}\right)\lambda\int_{\mathbb{R}^N} fu_n\,dx\right) \\ &= \left(\frac{a}{p}-\frac{a}{\theta p}\right)\int_{\mathbb{R}^N} d\xi + \left(\frac{a}{p}-\frac{a}{\theta p}\right)\xi_\infty + \left(\frac{1}{\theta p}-\frac{1}{2p_\mu^*}\right)\int_{\mathbb{R}^N} d\nu + \left(\frac{1}{\theta p}-\frac{1}{2p_\mu^*}\right)\nu_\infty \\ &\quad -\left(1-\frac{1}{\theta p}\right)\lambda\int_{\mathbb{R}^N} fu\,dx \\ &\geq \left(\frac{a}{p}-\frac{a}{\theta p}\right)\|u\|^p + \left(\frac{a}{p}-\frac{a}{\theta p}\right)\xi_\infty + \left(\frac{1}{\theta p}-\frac{1}{2p_\mu^*}\right)\iint_{\mathbb{R}^{2N}}\frac{|u(y)|^{p_\mu^*}|u(x)|^{p_\mu^*}}{|x-y|^\mu}\,dy\,dx \\ &\quad + \left(\frac{1}{\theta p}-\frac{1}{2p_\mu^*}\right)\nu_\infty - \left(1-\frac{1}{\theta p}\right)\lambda\int_{\mathbb{R}^N} fu\,dx. \end{aligned}$$

Ultimately, from equations (5.20) and (5.33), we infer that $c \geq \Lambda - \Theta(\lambda)$, leading to a contradiction. Hence, $\nu_\infty = 0$. In view of $J = \emptyset$ and (5.27), we conclude that

$$\lim_{n\to\infty}\iint_{\mathbb{R}^{2N}}\frac{|u_n(y)|^{p_\mu^*}}{|x-y|^\mu}\,dy\,|u_n(x)|^{p_\mu^*}\,dx = \iint_{\mathbb{R}^{2N}}\frac{|u(y)|^{p_\mu^*}}{|x-y|^\mu}\,dy\,|u(x)|^{p_\mu^*}\,dx. \tag{5.34}$$

Step 3. We are now in a position to prove that $u_n \to u$ in $X^{1,p}(\mathbb{R}^N)$. Suppose $d := \inf_{n\geq 1}\|u_n\| > 0$. For any $w,v \in X^{1,p}(\mathbb{R}^N)$, we define

$$\begin{aligned} \langle L(w),v\rangle = &\int_{\mathbb{R}^N}|\nabla w_n|^{p-2}\nabla w_n\nabla v\,dx \\ &+ \iint_{\mathbb{R}^{2N}}\frac{|w_n(x)-w_n(y)|^{p-2}(w_n(x)-w_n(y))(v_n(x)-v_n(y))}{|x-y|^{N+ps}}\,dx\,dy. \end{aligned}$$

Clearly, $L(w)$ is a continuous linear functional in $X^{1,p}(\mathbb{R}^N)$. Then by the boundedness of $\{u_n\}_n$ and $u_n \rightharpoonup u$ in $X^{1,p}(\mathbb{R}^N)$, we deduce

$$\lim_{n\to\infty}\left(a+b\|u_n\|^{(\theta-1)p}\right)\langle L(u),u_n-u\rangle = 0,$$

and

$$\lim_{n\to\infty}\left(a+b\|u\|^{(\theta-1)p}\right)\langle L(u),u_n-u\rangle = 0.$$

Therefore, as $n \to \infty$, there holds

$$\begin{aligned}&\lim_{n\to\infty}\left(a+b\|u_n\|^{(\theta-1)p}\right)\langle L(u_n),u_n-u\rangle-\left(a+b\|u\|^{(\theta-1)p}\right)\langle L(u),u_n-u\rangle\\&\quad=\lim_{n\to\infty}\left(a+b\|u_n\|^{(\theta-1)p}\right)\left(\langle L(u_n),u_n-u\rangle-\langle L(u),u_n-u\rangle\right).\end{aligned}$$

It follows from $\langle I'_\lambda(u_n)-I'_\lambda(u),u_n-u\rangle \to 0$ that

$$\begin{aligned}&\left(a+b\|u_n\|^{(\theta-1)p}\right)\langle L(u_n),u_n-u\rangle-\left(a+b\|u\|^{(\theta-1)p}\right)\langle L(u),u_n-u\rangle\\&=\iint_{\mathbb{R}^{2N}}\left(\frac{|u_n(y)|^{p_\mu^*}}{|x-y|^\mu}\,dy\,|u_n(x)|^{p_\mu^*-2}u_n(x)-\frac{|u(y)|^{p_\mu^*}}{|x-y|^\mu}\,dy\,|u(x)|^{p_\mu^*-2}u(x)\right)(u_n(x)-u(x))\,dx\\&\quad+o(1).\end{aligned}$$

Hence, we conclude from (5.34) that

$$\lim_{n\to\infty}\left(a+b\|u_n\|^{(\theta-1)p}\right)\left(\langle L(u_n),u_n-u\rangle-\langle L(u),u_n-u\rangle\right)=0.$$

Since $d := \inf_{n\geq 1}\|u_n\| > 0$ and $b > 0$, we have

$$\lim_{n\to\infty}\left(\langle L(u_n),u_n-u\rangle-\langle L(u),u_n-u\rangle\right)=0.$$

Now recall the famous Simon inequalities:

$$|\alpha-\beta|^p\leq\begin{cases}C'_p\left(|\alpha|^{p-2}\alpha-|\beta|^{p-2}\beta\right)\cdot(\alpha-\beta) & \text{if } p\geq 2,\\ C''_p\left[\left(|\alpha|^{p-2}\alpha-|\beta|^{p-2}\beta\right)\cdot(\alpha-\beta)\right]^{\frac{p}{2}}\left(|\alpha|^p+|\beta|^p\right)^{\frac{2-p}{2}} & \text{if } 1<p<2,\end{cases}\tag{5.35}$$

for all $\alpha,\beta\in\mathbb{R}^N$, where C'_p and C''_p are positive constants depending only on p.

If $p \geq 2$, from equation (5.35), we derive that

$$\begin{aligned}&\int_{\mathbb{R}^N}|\nabla(u_n-u)|^p\,dx+\iint_{\mathbb{R}^{2N}}\frac{|u_n(x)-u_n(y)-u(x)+u(y)|^p}{|x-y|^{N+ps}}\,dx\,dy\\&\quad\leq C'_p\left(\langle L(u_n),u_n-u\rangle-\langle L(u),u_n-u\rangle\right)\to 0\end{aligned}$$

as $n \to \infty$. Hence $u_n \to u$ in $X^{1,p}(\mathbb{R}^N)$.

For the case $1 < p < 2$, let $A = u_n(x) - u_n(y),\ B = u(x) - u(y)$. Using (5.35) and concavity we have

$$\begin{aligned}&\int_{\mathbb{R}^N}|\nabla(u_n-u)|^p\,dx+\iint_{\mathbb{R}^{2N}}\frac{|u_n(x)-u_n(y)-u(x)+u(y)|^p}{|x-y|^{N+ps}}\,dx\,dy\\&\quad\leq C''_p\int_{\mathbb{R}^N}\left[\left(|\nabla u_n|^{p-2}\nabla u_n-|\nabla u|^{p-2}\nabla u\right)\cdot(\nabla u_n-\nabla u)\right]^{\frac{p}{2}}\left(|\nabla u_n|^p+|\nabla u|^p\right)^{\frac{2-p}{2}}dx\\&\qquad+C''_p\iint_{\mathbb{R}^{2N}}\frac{\left[\left(|A|^{p-2}A-|B|^{p-2}B\right)\cdot(A-B)\right]^{\frac{p}{2}}\left(|A|^p+|B|^p\right)^{\frac{2-p}{2}}}{|x-y|^{N+ps}}\,dx\,dy\\&\quad\leq C\left(\langle L(u_n),u_n-u\rangle-\langle L(u),u_n-u\rangle\right)^{\frac{p}{2}}\to 0\end{aligned}$$

as $n \to \infty$. Hence $u_n \to u$ in $X^{1,p}(\mathbb{R}^N)$. In conclusion, we get $u_n \to u$ strongly in $X^{1,p}(\mathbb{R}^N)$ as $n \to \infty$.

Finally, we consider $\inf_n \|u_n\|_{X^{1,p}(\mathbb{R}^N)} = 0$. Then either 0 is an accumulation point of the sequence $\{u_n\}_n$ and so there exists a subsequence of $\{u_n\}_n$ strongly converging to $u = 0$, or 0 is an isolated point of the sequence $\{u_n\}_n$ and so there exists a subsequence, still denoted by $\{u_n\}_n$, such that $\inf_n \|u_n\| > 0$. In the first case we are done, while in the latter case we can process as above. □

Set $B_\rho(0) := \{u \in X^{1,p}(\mathbb{R}^N) : \|u\| < \rho\}$.

Theorem 5.6. *There exists $\lambda^* > 0$ such that for all $\lambda \in (0, \lambda^*)$, problem (1.1) admits a nonnegative weak solution u_1 in $X^{1,p}(\mathbb{R}^N)$ with $I_\lambda(u_1) < 0$.*

Proof. By Lemma 5.3, there exist $\lambda_0 > 0$ and $\gamma > 0$ such that

$$I_\lambda(u) \geq \gamma > 0 \quad \text{for } \lambda \in (0, \lambda_0) \quad \text{and} \quad u \in X^{1,p}(\mathbb{R}^N) \quad \text{with } \|u\| = \rho.$$

In the complete metric space $\overline{B}_\rho(0) := \{u \in X^{1,p}(\mathbb{R}^N) : \|u\| < \rho\}$, we apply Ekeland's variational principle [21] to prove that there exists a $(PS)_{c_0}$ sequence $\{u_n\}_n \subset B_\rho(0)$ with

$$c_0 = \inf\{I_\lambda(u) : u \in \overline{B}_\rho(0)\}.$$

Next, we aim to show that $-\infty < c_0 < 0$. Choose $\varphi_0 \in X^{1,p}(\mathbb{R}^N)$ such that $\int_{\mathbb{R}^N} f\varphi_0\, dx > 0$ and fix $\lambda \in (0, \lambda^*)$. Then for any $t > 0$, we have

$$I_\lambda(t\varphi_0) = \frac{at^p}{p}\|\varphi_0\|^p + \frac{bt^{\theta p}}{\theta p}\|\varphi_0\|^{\theta p} - \frac{t^{2p_\mu^*}}{2p_\mu^*}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|\varphi_0{}^+(y)|^{p_\mu^*}}{|x-y|^\mu}\,dy\,|\varphi_0{}^+(x)|^{p_\mu^*}\,dx - \lambda t\int_{\mathbb{R}^N} f(x)\varphi_0\,dx.$$

Then we can deduce that there exists a sufficiently small $t > 0$ such that $\|t\varphi_0\| \leq \rho$ and $I_\lambda(t\varphi_0) < 0$. Hence, according to the definition of c_0, it follows that $c_0 < I_\lambda(t\varphi_0) < 0$ and it's evident that $c_0 > -\infty$.

Selecting $\lambda^* \in (0, \lambda_0]$ such that $0 < \Lambda - \Theta(\lambda)$ for all $\lambda \in (0, \lambda^*)$. From Lemma 5.5 and $c_0 < 0$, we can obtain that there exist a subsequence of $\{u_n\}_n$ and $u_1 \in X^{1,p}(\mathbb{R}^N)$ such that $u_n \to u_1$ strongly in $X^{1,p}(\mathbb{R}^N)$. Therefore, $I'_\lambda(u_1) = 0$ and $I_\lambda(u_1) = c_0 < 0$. Hence we complete the proof of Theorem 1.2. □

Drawing on the ideas introduced by Chabrowski [16], Alves [2] and Goncalves and Alves [28], we are going to explore that equation (1.1) has anther solution. Firstly we know that I_λ satisfies the mountain pass geometry by Lemmas 5.3 and 5.4. Hence applying the mountain pass theorem [3], there exists a $(PS)_c$ sequence $\{v_n\}_n$ with

$$c = \inf_{\gamma\in\Gamma}\max_{t\in[0,1]} I_\lambda(\gamma(t)),$$

$$\Gamma = \{\gamma \in C([0,1], X^{1,p}(\mathbb{R}^N)) : \gamma(0) = 0, \gamma(1) = e\}.$$

Lemma 5.7. *Suppose $a > 0$, $b > 0$, and either $\theta = \frac{3N-\mu-p}{2(N-p)}$ or $a > 0$, $0 < b < S_{H,M}^{-\theta}$, $\theta = \frac{2N-\mu}{N-p}$. Under the assumption (f), there exists $\lambda^{**} \in (0, \lambda^*)$, such that for any $\lambda \in (0, \lambda^{**})$, one has*

$$c \leq \sup_{t\geq 0} I_\lambda(tU) < \Lambda - \Theta(\lambda).$$

Proof. Employing the function U in Lemma 5.4, we consider the funtions

$$g(t) := I_\lambda(tU) = \frac{a\|U\|^p}{p}t^p + \frac{b\|U\|^{\theta p}}{\theta p}t^{\theta p} - \frac{t^{2p_\mu^*}}{2p_\mu^*} - \lambda t\int_{\mathbb{R}^N} fU\,dx$$

and

$$\tilde{g}(t) := \frac{a\|U\|^p}{p}t^p + \frac{b\|U\|^{\theta p}}{\theta p}t^{\theta p} - \frac{t^{2p_\mu^*}}{2p_\mu^*}$$

for all $t \geq 0$. There exists $t_1 > 0$ such that $\tilde{g}'(t_1) = 0$ and $\max_{t\geq 0}\tilde{g}(t) = \tilde{g}(t_1) = \Lambda + h(\varepsilon) > 0$, where t_1 satisfies

$$t_1 = \begin{cases} \left(\dfrac{a\|U\|^p}{1-b\|U\|^{\theta p}}\right)^{\frac{1}{\theta p-p}} & \text{if } \theta = \frac{2N-\mu}{N-p},\ a>0,\ 0<b<S_{H,M}^{-\theta}, \\ \left(\dfrac{b\|U\|^{\theta p} + \sqrt{b^2\|U\|^{2\theta p} + 4a\|U\|^p}}{2}\right)^{\frac{1}{\theta p-p}} & \text{if } \theta = \frac{3N-\mu-p}{2(N-p)},\ a>0,\ b>0. \end{cases}$$

When $c = 0$, the proof is similar to Theorem 1.2. When $c > 0$, we can choose $\lambda_1 \in (0, \lambda^*]$ such that $\Lambda - \Theta(\lambda_1) > 0$. It is easy to observe that

$$\lim_{t\to 0^+}\left(\frac{a\|U\|^p}{p}t^p + \frac{b\|U\|^{\theta p}}{\theta p}t^{\theta p}\right) = 0.$$

So there exists $t_2 \in (0, t_1)$ such that for any $\lambda \in (0, \lambda_1)$, we have

$$\max_{0\leq t\leq t_2} g(t) \leq \max_{0\leq t\leq t_2}\left(\frac{a\|U\|^p}{p}t^p + \frac{b\|U\|^{\theta p}}{\theta p}t^{\theta p}\right) \leq \Lambda - \Theta(\lambda_1) < \Lambda - \Theta(\lambda).$$

Choosing $\lambda^{**} \in (0, \lambda_1]$ such that for any $\lambda \in (0, \lambda^{**})$, there holds

$$\lambda t_2 \int_{\mathbb{R}^N} fU^+\,dx > \frac{p-1}{p}\left[\left(a - \frac{a}{\theta}\right)S\right]^{-\frac{1}{p-1}}\lambda^{\frac{p}{p-1}}|f|_{(p^*)'}^{\frac{p}{p-1}} + h(\varepsilon).$$

Then, for any $\lambda \in (0, \lambda^{**})$, one can obtain

$$\begin{aligned} \sup_{t\geq t_2} g(t) &\leq \sup_{t\geq t_2}\tilde{g}(t) - \lambda t_2\int_{\mathbb{R}^N} fU^+\,dx \\ &< \tilde{g}(t_1) - \frac{p-1}{p}\left[\left(a - \frac{a}{\theta}\right)S\right]^{-\frac{1}{p-1}}\lambda^{\frac{p}{p-1}}|f|_{(p^*)'}^{\frac{p}{p-1}} - h(\varepsilon) \\ &= \Lambda - \Theta(\lambda). \end{aligned}$$

Therefore, for any $\lambda \in (0, \lambda^{**})$ we have

$$c \leq \sup_{t\geq 0} I_\lambda(tU) = \sup_{t\geq 0} g(t) < \Lambda - \Theta(\lambda).$$

Hence, the proof is complete. □

Theorem 5.8. *There exists $\lambda^{**} \in (0, \lambda^*]$ such that for all $\lambda \in (0, \lambda^{**})$, problem (1.1) has another nonnegative weak solution u_2 in $X^{1,p}(\mathbb{R}^N)$ with $I_\lambda(u_2) > 0$.*

Proof. Using Lemmas 5.5 and 5.7, there exist a subsequence of $\{v_n\}_n$, still denoted by $\{v_n\}_n$, and a function $u_2 \in X^{1,p}(\mathbb{R}^N)$ such that $v_n \to u_2$ strongly in $X^{1,p}(\mathbb{R}^N)$. Therefore,

$$I'_\lambda(u_2) = 0 \quad \text{and} \quad I_\lambda(u_2) = c > 0 = I_\lambda(0),$$

indicating that u_2 is a nontrivial and nonnegative solution of (1.1). □

Proof of Theorem 1.2. Combining the proof of Theorem 5.6 with Theorem 5.8, we can obtain Theorem 1.2. □

Statements and declarations

The authors declare that they have no conflict of interest.

Data availability

Date sharing is not applicable to this article as no new data were created analyzed in this study.

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