Electronic Journal of Qualitative Theory of Differential Equations
2026, No. 1, 1–18; https://doi.org/10.14232/ejqtde.2026.1.1 www.math.u-szeged.hu/ejqtde/

Multiple solutions for a Kirchhoff-type double phase problem involving variable exponents and two parameters

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Received 20 June 2025, appeared 1 January 2026
Communicated by Gabriele Bonanno

Abstract. In this paper, we study a class of Kirchhoff-type double phase equations with sublinear terms and two parameters in the framework of variable exponent function spaces. By employing a variant of the critical point theorem due to G. Bonanno, we establish the existence of at least three distinct solutions to the problem. Our result extends and complements recent contributions concerning double phase equations in the superlinear setting.

Keywords: Kirchhoff-type double phase problems, variable exponents, Musielak–Orlicz–Sobolev spaces, critical point theory.

2020 Mathematics Subject Classification: 35J60, 35D30, 35J70, 35A15, 03H10.

1 Introduction

Let $\Omega \subseteq \mathbb{R}^N$ $(N \geq 2)$ represents a bounded domain with Lipschitz boundary $\partial\Omega$. In this paper, we study the following Kirchhoff-type double phase problem

$$\begin{cases} -M\left[\int_\Omega \left(\frac{|\nabla u|^{p(x)}}{p(x)} + \eta(x)\frac{|\nabla u|^{q(x)}}{q(x)}\right)dx\right]\left(\Delta_{p(x)}u + \Delta^{\eta(x)}_{q(x)}u\right) = \lambda f(x,u) + \mu g(x,u), & x \in \Omega, \\ u = 0, \quad x \in \partial\Omega, \end{cases} \tag{1.1}$$

where $0 \leq \eta(\cdot) \in L^\infty(\Omega)$, $\lambda > 0$ and $\mu \geq 0$ are two parameters, functions $p, q \in C(\overline{\Omega})$ obeying the following relationship

$$1 < p(x) < N, \quad p(x) < q(x) < p^*(x) \text{ for all } x \in \overline{\Omega}, \tag{1.2}$$

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with $p^*(x) := \frac{Np(x)}{N-p(x)}$ for each $x \in \overline{\Omega}$. Here, $\Delta_{p(x)}u + \Delta^{w}_{q(x)}u$ denotes the double phase operator given by the formula

$$\Delta_{p(x)}u + \Delta^{\eta(x)}_{q(x)}u := \operatorname{div}\left(|\nabla u|^{p(x)-2}\nabla u + \eta(x)|\nabla u|^{q(x)-2}\nabla u\right).$$

We consider problem (1.1) when the nonlinear terms $f, g : \Omega \times \mathbb{R} \to \mathbb{R}$ are two Carathéodory functions verifying an appropriate growth condition and the Kirchhoff function M satisfies the following condition:

(H_0) $M : [0, \infty) \to \mathbb{R}$ is supposed to be a continuous non-decreasing function satisfying $\exists m_0 > 0$, such that

$$M(t) \geq m_0, \quad \forall t \geq 0.$$

In recent years, considerable attention has been devoted to problems involving the double phase operator, primarily due to its applications in mathematical physics and engineering, particularly in the modeling of strongly anisotropic materials and elasticity theory [10,27]. The primary objective of this paper is to establish the existence of multiple solutions to problem (1.1). We begin by recalling some relevant contributions that have motivated our study. In [25], Liu and Dai considered the following double-phase problem

$$\begin{cases} -\operatorname{div}\left(|\nabla u|^{p-2}\nabla u + \eta(x)|\nabla u|^{q-2}\nabla u\right) = f(x,u), & x \in \Omega, \\ u = 0, & x \in \partial\Omega, \end{cases} \tag{1.3}$$

where Ω is a bounded domain with Lipschitz boundary in $\mathbb{R}^N$, $N \geq 2$ and $1 < p < q < N$, $\frac{q}{p} < 1 + \frac{1}{N}$, $\eta : \overline{\Omega} \to [0, \infty)$ is a Lipschitz continuous function and $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function. Using variational methods, the authors investigated the existence and multiplicity of solutions to problem (1.1) in the case where f is q-superlinear at infinity. Since then, numerous works have explored this topic. We refer to [9,15,17] for studies on the double-phase problem with constant exponents, and to [2,6,19,23] for results in the variable exponent setting.

Note that problem (1.1) involves integrals over the domain Ω, so the first two equations are no longer pointwise identities. Consequently, the problem is often referred to as a nonlocal problem. Such problems arise in the modeling of various physical and biological systems, where the unknown function u represents a quantity that depends on its own average such as population density, see [11]. Furthermore, problem (1.1) can be seen as a stationary version of the Kirchhoff equation

$$\rho\frac{\partial^2 u}{\partial t^2} - \left(\frac{P_0}{h} + \frac{E}{2L}\int_0^L \left|\frac{\partial u}{\partial x}\right|^2 dx\right)\frac{\partial^2 u}{\partial x^2} = 0 \tag{1.4}$$

presented by Kirchhoff in 1883, see [24]. Equation (1.4) extends the classical d'Alembert wave equation by taking into account the effects of changes in the length of the string during vibration. In recent years, problems involving Kirchhoff-type operators have attracted considerable attention and have been studied extensively; see, for example, [5,12–14,18,20].

In addition to Kirchhoff-type problems involving the p-Laplacian and the $p(x)$-Laplacian, several works have focused on Kirchhoff-double phase problems, see [1,4,16,21,28]. In [16], Fiscella et al. studied Kirchhoff-double phase problems with superlinear nonlinearities and established existence and multiplicity results using the mountain pass theorem and the fountain theorem. Some of these results were extended in [1] by Ahmadi et al., where the superlinear terms no longer satisfy the Ambrosetti-–Rabinowitz condition. In [4], Arora et al. addressed Kirchhoff-double phase problems involving singular nonlinearities, employing the fibering method in the framework of the Nehari manifold. Furthermore, Ho et al. [21] investigated a class of elliptic equations driven by the variable exponent double phase operator with a Kirchhoff-type term, under minimal assumptions on the locally defined right-hand side. Using the method of sub- and supersolutions, Zuo et al. [28] established the existence of solutions for Kirchhoff-double phase problems involving concave–convex nonlinearities. In a recent 2025 study, Kefi and Al-Shomrani [22] explored a double-phase elliptic Dirichlet problem involving nonlocal interactions within a bounded domain $\Omega \subset \mathbb{R}^N$ ($N \geq 2$), governed by the operator

$$-\operatorname{div}(|\nabla w|^{p-2}\nabla w + \eta(x)|\nabla w|^{q-2}\nabla w) = \lambda f(x,w)\left(\int_\Omega F(x,w)\,dx\right)^\gamma,$$

where $\gamma > 0$ is a positive constant, $1 \leq \beta(\gamma+1) < p < N$, $p < q < p^* = \frac{Np}{N-p}$, and $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function satisfying the growth condition $m_1|w|^{\alpha-1} \leq f(x,w) \leq m_2|w|^{\beta-1}$ with $0 < \alpha \leq \beta$. Employing variational methods and critical point theorems, they establish the existence of at least one weak solution and, under specific conditions, three distinct weak solutions in the Musielak–Orlicz–Sobolev space.

Inspired by the aforementioned studies, we investigate the existence of at least three solutions to problem (1.1), which involves sublinear nonlinearities f and g, by employing a critical point theorem developed by Bonanno et al. in [8]. To the best of our knowledge, there are only a limited number of results addressing Kirchhoff-type double phase problems with sublinear terms and variable exponents depending on two parameters. It is worth noting that the result established here remains new even in the particular case when $M(t) = 1$ or the functions $p(\cdot)$ and $q(\cdot)$ are constants, we refer to [7,13,14,18,22]. Our contribution can thus be seen as a natural complement to the papers [1,2,16,17,23], where the double phase problem was explored in a superlinear framework, even under the assumption of constant exponents.We believe that the results presented in this paper can be extended to the Kirchhoff-type multi-phase problem with variable exponents, as recently introduced by Vetro in [26].

Furthermore, we point out that the variable exponents $p(\cdot)$ and $q(\cdot)$ are not required to satisfy the condition

$$\frac{q(x)}{p(x)} < 1 + \frac{1}{N} \quad \text{for all } x \in \overline{\Omega} \tag{1.5}$$

which has been a necessary assumption in many previous works, such as [21,23,28], or [9,17, 25] in the constant exponent case. This relaxation is justified by a recent result in [6], where the authors showed that the functional space $W_0^{1,\mathcal{H}}(\Omega)$ can be equipped with the equivalent norm $\|\nabla \cdot\|$ without assuming (1.5).

The remainder of this paper is organized as follows: Section 2 outlines the fundamental properties of the working space and reviews several preliminary lemmas that will be used in subsequent sections. Section 3 states the main theorem, while Section 4 and Section 5 are dedicated to its proof.

2 Musielak–Orlicz–Sobolev spaces and variational principles

To study double phase systems, we need to introduce our working space and recalling some facts about it. Let

$$C_+(\overline{\Omega}) := \left\{ s \in C(\overline{\Omega}),\ s(x) > 1 \text{ for all } x \in \overline{\Omega} \right\}.$$

For any $s \in C_+(\overline{\Omega})$, we put

$$s^+ := \max_{x\in\bar{\Omega}} s(x), \qquad s^- := \min_{x\in\bar{\Omega}} s(x).$$

The variable exponent Lebesgue space is described as

$$L^{s(x)}(\Omega) = \left\{ u \mid u : \Omega \to \mathbb{R} \text{ is measurable}, \int_\Omega |u(x)|^{s(x)}\, dx < \infty \right\}$$

with the norm

$$|u|_s := \inf \left\{ \xi > 0 : \quad \int_\Omega \left| \frac{u(x)}{\xi} \right|^{s(x)} dx \le 1 \right\}.$$

Furthermore, one has

Proposition 2.1 (see [3]). *Let $p, q, \theta \ge 1$ be measurable functions defined on Ω and satisfy the condition*

$$\frac{1}{\theta(x)} = \frac{1}{p(x)} + \frac{1}{q(x)}, \quad \textit{for a.e. } x \in \Omega.$$

If $f \in L^{p(x)}(\Omega)$ and $g \in L^{q(x)}(\Omega)$, then we have $fg \in L^{\theta(x)}(\Omega)$ and the following Hölder inequality holds

$$|fg|_{\theta(x)} \le 2|f|_{p(x)}|g|_{q(x)}.$$

Define the function $\mathcal{H} : \Omega \times [0, \infty) \to [0, \infty)$ by

$$\mathcal{H}(x, t) = t^{p(x)} + \eta(x)t^{q(x)},$$

where $1 < p(x) < N$, $p(x) < q(x) < p^*(x) = \frac{Np(x)}{N-p(x)}$ and $\eta : \Omega \to [0, \infty)$ is measurable function such that $\eta \in L^\infty(\Omega)$.

Consider

$$\rho_{\mathcal{H}}(u) := \int_\Omega \mathcal{H}(x, |u|)dx.$$

The Musielak–Orlicz–Lebesgue space is described as

$$L^{\mathcal{H}}(\Omega) = \{ u \mid u : \Omega \to \mathbb{R} \text{ is measurable}, \rho_{\mathcal{H}}(u) < \infty \},$$

endowed with the norm

$$|u|_{\mathcal{H}} := \inf \left\{ \xi > 0 : \rho_{\mathcal{H}}\left(\frac{u}{\xi}\right) \le 1 \right\}.$$

By [6, Proposition 2.13] we arrive at the following relation between $\|u\|_{\mathcal{H}}$ and $\rho_{\mathcal{H}}$.

Proposition 2.2. *If $u \in L^{\mathcal{H}}(\Omega)$, then we have*

(i) $\min \left\{ |u|_{\mathcal{H}}^{p^-}, |u|_{\mathcal{H}}^{q^+} \right\} \le \rho_{\mathcal{H}}(u) \le \max \left\{ |u|_{\mathcal{H}}^{p^-}, |u|_{\mathcal{H}}^{q^+} \right\}$;

(ii) $|u|_{\mathcal{H}} \to 0 \iff \rho_{\mathcal{H}}(u) \to 0$;

(*iii*) $|u|_{\mathcal{H}} \to \infty \iff \rho_{\mathcal{H}}(u) \to \infty$;

The Musielak–Orlicz–Sobolev space is described as

$$W^{1,\mathcal{H}}(\Omega) := \left\{ u \in L^{\mathcal{H}}(\Omega) : |\nabla u| \in L^{\mathcal{H}}(\Omega) \right\}$$

including the norm

$$\|u\|_{1,\mathcal{H}} := |\nabla u|_{\mathcal{H}} + |u|_{\mathcal{H}}$$

in which $|\nabla u|_{\mathcal{H}} = \Big| |\nabla u| \Big|_{\mathcal{H}}$.

We define our working space $\mathbb{X} := W_0^{1,\mathcal{H}}(\Omega)$ as the closure of $C_0^\infty(\Omega)$ with respect to the norm $\|u\|_{\mathbb{X}} = |\nabla u|_{\mathcal{H}}$. This space is a separable and reflexive Banach space (see [6, Proposition 2.12]). In the spirit of Proposition 2.16 in [6], we obtain the following embedding lemma.

Proposition 2.3. *For any $s \in C_+(\overline{\Omega})$ with $s(x) \le p^*(x)$ for all $x \in \overline{\Omega}$, the embedding $\mathbb{X} \hookrightarrow L^{s(x)}(\Omega)$ is continuous; the embedding is compact if $s(x) < p^*(x)$.*

Invoking the forenamed Sobolev embedding theorem, we denote by c_s the best constant obeying the following relationship

$$|u|_s \le c_s \|u\|_{\mathbb{X}}, \quad \forall u \in \mathbb{X}, \tag{2.1}$$

and for any $s \in C(\overline{\Omega})$, we denote

$$\widetilde{c}_s := \max \left\{ c_s^{s^+}, c_s^{s^-} \right\}. \tag{2.2}$$

Proposition 2.4 (see [6]). *Assume that condition (1.2) holds, and define the operator $\mathcal{A} : \mathbb{X} \to \mathbb{X}^*$ by the formula*

$$\langle \mathcal{A}(u), \varphi \rangle = \int_\Omega \left(|\nabla u|^{p(x)-2} \nabla u + \eta(x) |\nabla u|^{q(x)-2} \nabla u \right) \nabla \varphi \, dx$$

for all $u, \varphi \in \mathbb{X}$, where $\langle \cdot, \cdot \rangle$ denotes the duality pairing between $\mathbb{X}$ and its dual $\mathbb{X}^*$. Then, the operator $\mathcal{A}$ satisfies the following properties:

(*i*) *$\mathcal{A}$ is continuous, bounded, and strictly monotone;*

(*ii*) *$\mathcal{A}$ is of type $(S)_+$, namely, if $u_n \rightharpoonup u$ weakly in $\mathbb{X}$ as $n \to \infty$ and $\limsup_{n\to\infty} \langle \mathcal{A}(u_n), u_n - u \rangle \le 0$, then $u_n \to u$ strongly in $\mathbb{X}$ as $n \to \infty$;*

(*iii*) *$\mathcal{A}$ is coercive and a homeomorphism.*

Eventually, we recall the folowing theorem, obtained in [8] which plays an essential role in the proof of our main result.

Proposition 2.5. *Assume that $\Phi : X \to \mathbb{R}$ is a coercive, continuously Gâteaux differentiable, and sequentially weakly lower semicontinuous functional defined on a real Banach space X, whose Gâteaux derivative admits a continuous inverse on X^*. Let $\Psi : X \to \mathbb{R}$ be a continuously Gâteaux differentiable functional whose Gâteaux derivative is compact. Moreover, assume that $\Phi(0) = \Psi(0) = 0$.*

Assume that there exist a constant $\gamma > 0$ and $u_0 \in X$, with $\Phi(u_0) > \gamma$ such that

(T_1) $\dfrac{\sup_{u\in\Phi^{-1}(-\infty,\gamma]} \Psi(u)}{\gamma} < \dfrac{\Psi(u_0)}{\Phi(u_0)}$;

(T_2) *for each $\lambda \in \Lambda_\gamma := \left(\frac{\Phi(u_0)}{\Psi(u_0)}, \frac{\gamma}{\sup_{u\in\Phi^{-1}(-\infty,\gamma]} \Psi(u)} \right)$, the functional $\Phi - \lambda\Psi$ is coercive on X.*

Then, for every $\lambda \in \Lambda_\gamma$, the functional $\Phi - \lambda\Psi$ admits at least three distinct critical points in the space X.

3 Existence of three weak solutions

In this section, we will present some notations and the main result of the paper.

Definition 3.1. By a weak solution of problem (1.1) we mean $u \in \mathbb{X}$ obeying the following relationship

$$M\left[\int_\Omega \left(\frac{|\nabla u|^{p(x)}}{p(x)} + \eta(x)\frac{|\nabla u|^{q(x)}}{q(x)}\right) dx\right] \int_\Omega \left(|\nabla u|^{p(x)-2}\nabla u + \eta(x)|\nabla u|^{q(x)-2}\nabla u\right)\nabla\varphi\, dx$$
$$-\lambda \int_\Omega f(x,u)\varphi\, dx - \mu \int_\Omega g(x,u)\varphi\, dx = 0,$$

for all $\varphi \in \mathbb{X}$.

For the sake of describing a result on the existence of three nontrivial solutions for (1.1), we introduce the functionals $\Phi, \Psi, J_\lambda : \mathbb{X} \to \mathbb{R}$ by

$$\Phi(u) = \widehat{M}\left[\int_\Omega \left(\frac{|\nabla u|^{p(x)}}{p(x)} + \eta(x)\frac{|\nabla u|^{q(x)}}{q(x)}\right) dx\right],$$
$$\Psi(u) = \int_\Omega \left[F(x,u) + \frac{\mu}{\lambda}G(x,u)\right] dx$$

and

$$J_\lambda(u) = \Phi(u) - \lambda\Psi(u), \quad \forall u \in \mathbb{X},$$

where $\widehat{M}(t) = \int_0^t M(\tau)d\tau$, $F(x,t) = \int_0^t f(x,\tau)d\tau$, $G(x,t) = \int_0^t g(x,\tau)d\tau$ and J_λ represents the so-called energy functional. We know that Φ, Ψ and J_λ are continuously Gâteaux differentiable whose Gâteaux derivative are characterized by

$$\langle \Phi'(u), \varphi\rangle = M\left[\int_\Omega \left(\frac{|\nabla u|^{p(x)}}{p(x)} + \eta(x)\frac{|\nabla u|^{q(x)}}{q(x)}\right) dx\right]$$
$$\times \int_\Omega \left(|\nabla u|^{p(x)-2}\nabla u + \eta(x)|\nabla u|^{q(x)-2}\nabla u\right)\nabla\varphi\, dx,$$
$$\langle \Psi'(u), \varphi\rangle = \int_\Omega \left[f(x,u) + \frac{\mu}{\lambda}g(x,u)\right]\varphi\, dx$$

and

$$\langle J'_\lambda(u), \varphi\rangle = \langle \Phi'(u), \varphi\rangle - \lambda\langle \Psi'(u), \varphi\rangle$$

for every $\varphi \in \mathbb{X}$.

Invoking Definition 3.1, the weak solutions of problem (1.1) correspond exactly to the critical points of the functional J_λ. To present our main result, we first introduce some necessary notations.

To outline the main ideas and tools, we first introduce some notations related to our assumptions. Let $\rho := \sup_{x\in\Omega} \operatorname{dist}(x, \partial\Omega)$. Then, there exists a point $x^0 \in \Omega$ such that the ball $B_\rho(x^0) \subseteq \Omega$, where $B_\rho(x^0)$ denotes the ball centered at x^0 with radius ρ. We indicate with V_ρ the Lebesgue measure of $B_\rho(x^0)$ in $\mathbb{R}^N$ given by

$$V_\rho := \left|B_\rho(x^0)\right| = \frac{\pi^{\frac{N}{2}}}{\Gamma(1+\frac{N}{2})}\rho^N,$$

where Γ denotes the Gamma function.

Besides, for any $\sigma \in \mathbb{R}^+$ and $\alpha \in C_+(\overline{\Omega})$, we put

$$\mathcal{K}_{\sigma,\alpha} := c_1 \max\left\{ \left(\frac{q^+}{m_0 \sigma^{p^- -1}}\right)^{\frac{1}{p^-}}, \left(\frac{q^+}{m_0 \sigma^{q^+ -1}}\right)^{\frac{1}{q^+}} \right\} + \widetilde{c}_\alpha \max\left\{ \left(\frac{q^+}{m_0 \sigma^{\frac{p^- - \alpha^+}{\alpha^+}}}\right)^{\frac{\alpha^+}{p^-}}, \left(\frac{q^+}{m_0 \sigma^{\frac{q^+ - \alpha^-}{\alpha^-}}}\right)^{\frac{\alpha^-}{q^+}} \right\},$$

where c_1 and $\widetilde{c}_\alpha$ are the same values defined by (2.1) and (2.2).

Now, we state our main result.

Theorem 3.2. *Assume that the hypotheses* (1.2) *and* (H_0) *hold and there exist two positive constants* σ_1 *and* σ_2 *verifying*

$$\delta_1 \sigma_1 < \min\left\{ \sigma_2^{p^-}, \sigma_2^{p^+} \right\}, \tag{3.1}$$

with

$$\delta_1 := \frac{p^+ \max\{\rho^{p^-}, \rho^{p^+}\}}{m_0 2^{p^- - N}(2^N - 1) V_\rho},$$

such that

(H_1) *there exist* $A_1 > 0$ *and* $\alpha_1 \in C_+(\overline{\Omega})$ *with* $\alpha_1^+ < p^-$ *such that*

$$|f(x,t)| \le A_1(1 + |t|^{\alpha_1(x)-1})$$

for a.e. $x \in \Omega$ *and for all* $t \in \mathbb{R}$;

(H_2) $F(x,t) \ge 0$ *for a.e.* $x \in \Omega$ *and for all* $t \in [0, \sigma_2]$;

(H_3) $A_1 \mathcal{K}_{\sigma_1,\alpha_1} < \frac{\delta_2 \int_{B_{\rho/2}(x^0)} F(x,\sigma_2) dx}{\max\{\sigma_2^{p^-}, \sigma_2^{q^+}\}}$ *with*

$$\delta_2 := \frac{p^- \min\{\rho^{p^-}, \rho^{q^+}\}}{m_1 2^{q^+ + 1 - N}(2^N - 1) \max\{1, |\eta|_\infty\} V_\rho},$$

and

$$m_1 := M\left(\frac{2^{q^+ + 1 - N}(2^N - 1) \max\{1, |\eta|_\infty\}. \max\{\sigma_2^{p^-}, \sigma_2^{q^+}\} V_\rho}{p^- \min\{\rho^{p^-}, \rho^{q^+}\}} \right).$$

Moreover, for every parameter

$$\lambda \in \Lambda_{\sigma_1,\sigma_2} := \left(\frac{\max\{\sigma_2^{p^-}, \sigma_2^{q^+}\}}{\delta_2 \int_{B_{\frac{\rho}{2}}(x^0)} F(x,\sigma_2)\, dx}, \frac{1}{A_1 \mathcal{K}_{\sigma_1,\alpha_1}} \right)$$

and for all Carathéodory function $g : \Omega \times \mathbb{R} \to \mathbb{R}$ *verifying*

(H_4) *there exist* $A_2 > 0$ *and* $\alpha_2 \in C_+(\overline{\Omega})$ *with* $\alpha_2^+ < p^-$ *such that*

$$|g(x,t)| \le A_2(1 + |t|^{\alpha_2(x)-1})$$

for a.e. $x \in \Omega$ *and for all* $t \in \mathbb{R}$;

(H_5) *$G(x,t) \geq 0$ for a.e. $x \in \Omega$ and for all $t \in \mathbb{R}^+$.*

Then, there is a positive constant $\mu^*_{\lambda,g}$ given by

$$\mu^*_{\lambda,g} := \frac{1 - \lambda A_1 \mathcal{K}_{\sigma_1,\alpha_1}}{A_2 \mathcal{K}_{\sigma_1,\alpha_2}},$$

such that, for each $\mu \in [0, \mu^*_{\lambda,g})$, problem (1.1) possesses at least three distinct weak solutions in the space $\mathbb{X}$.

4 Regularity assumptions of Φ and Ψ

Our goal is to prove Theorem 3.2 by applying Proposition 2.5. To this end, we aim to verify that the functionals Φ and Ψ satisfy all the regularity assumptions required by Proposition 2.5.

Lemma 4.1. *The functional $\Phi' : \mathbb{X} \to \mathbb{X}^\star$ is both coercive and strictly monotone in $\mathbb{X}^*$.*

Proof. Let $u \in \mathbb{X} \setminus \{0\}$ be such that $\|u\|_{\mathbb{X}} > 1$. By the hypothesis (H_0) and Proposition 2.2, we observe that

$$\langle \Phi'(u), u \rangle = m_0 \rho_{\mathcal{H}}(\nabla u) \geq m_0 \min\left\{ \|u\|_{\mathbb{X}}^{p^-}, \|u\|_{\mathbb{X}}^{q^+} \right\} = m_0 \|u\|_{\mathbb{X}}^{p^-},$$

which, together with the fact that $p^- > 1$, implies that

$$\lim_{\|u\|_{\mathbb{X}} \to \infty} \frac{\langle \Phi'(u), u \rangle}{\|u\|_{\mathbb{X}}} = \infty.$$

Hence, the Gâteaux derivative Φ' is coercive.

Now, to prove that Φ' is strictly monotone in $\mathbb{X}^*$, for any $u, v \in \mathbb{X}$ with $u \neq v$, we may assume, without loss of generality, that

$$\begin{aligned} \mathcal{L}(u) &:= \int_\Omega \left(\frac{|\nabla u|^{p(x)}}{p(x)} + \eta(x) \frac{|\nabla u|^{q(x)}}{q(x)} \right) dx \\ &\geq \int_\Omega \left(\frac{|\nabla v|^{p(x)}}{p(x)} + \eta(x) \frac{|\nabla v|^{q(x)}}{q(x)} \right) dx =: \mathcal{L}(v). \end{aligned} \tag{4.1}$$

Note that the functional $\mathcal{L}$ is convex, see for example [6], and its Gâteaux derivative is given by the formula

$$\langle \mathcal{L}'(u), v \rangle = \langle \mathcal{A}(u), v \rangle,$$

where $\mathcal{A}$ denotes the strictly monotone operator introduced in Proposition 2.4. Furthermore, for all $u, v \in \mathbb{X}$, the derivative of Φ satisfies

$$\langle \Phi'(u), v \rangle = M(\mathcal{L}(u)) \langle \mathcal{A}(u), v \rangle,$$

and thus,

$$\begin{aligned} \langle \Phi'(u) - \Phi'(v), u - v \rangle &= \langle M(\mathcal{L}(u))\mathcal{A}(u) - M(\mathcal{L}(v))\mathcal{A}(v), u - v \rangle \\ &= \left(M(\mathcal{L}(u)) - M(\mathcal{L}(v)) \right) \langle \mathcal{A}(u), u - v \rangle \\ &\quad + M(\mathcal{L}(v)) \langle \mathcal{A}(u) - \mathcal{A}(v), u - v \rangle. \end{aligned}$$

By the convexity of $\mathcal{L}$, we have

$$\mathcal{L}(v) \geq \mathcal{L}(u) + \langle \mathcal{A}(u), v - u \rangle,$$

which implies by (4.1) that

$$\langle \mathcal{A}(u), u - v \rangle \geq \mathcal{L}(u) - \mathcal{L}(v) \geq 0.$$

Since M is a non-decreasing function, and under the assumption that condition (H_0) holds, we deduce that $M(\mathcal{L}(u)) \geq M(\mathcal{L}(v) \geq m_0$, it follows that

$$\langle \Phi'(u) - \Phi'(v), u - v \rangle \geq 0.$$

Therefore, the operator Φ' is monotone on $\mathbb{X}^*$. Moreover, the strict monotonicity of $\mathcal{A}$ and the properties of M imply that Φ' is strictly monotone on $\mathbb{X}^*$. □

Lemma 4.2. *The functional Φ' is a mapping of $(S)_+$-type in $\mathbb{X}^*$.*

Proof. Let $(u_n) \subset \mathbb{X}$ be a sequence such that $u_n \rightharpoonup u$ in $\mathbb{X}$ as $n \to \infty$, and it satisfies the following condition

$$\limsup_{n\to\infty} \langle \Phi'(u_n), u_n - u \rangle \leq 0. \tag{4.2}$$

Clearly, $\lim_{n\to\infty} \langle \Phi'(u), u_n - u \rangle = 0$ due to the fact that $u_n \rightharpoonup u$ in $\mathbb{X}$ as $n \to \infty$. By the strict monotonicity of Φ' in $\mathbb{X}^*$, we obtain

$$0 \leq \lim_{n\to\infty} \langle \Phi'(u_n) - \Phi'(u), u_n - u \rangle = \lim_{n\to\infty} \langle \Phi'(u_n), u_n - u \rangle = \lim_{n\to\infty} M(\mathcal{L}(u_n)) \langle \mathcal{A}(u_n), u_n - u \rangle \leq 0.$$

Since $M(t) \geq m_0$, for any $t \geq 0$, ones has $\lim_{n\to\infty} \langle \mathcal{A}(u_n), u_n - u \rangle = 0$. The proof is achieved by (ii) of Proposition 2.4. □

Lemma 4.3. *The operator Φ' is an homeomorphism.*

Proof. The strict monotonicity of Φ' ensures that it is injective. Given that Φ' is also coercive, it follows that Φ' is surjective. Consequently, Φ' possesses an inverse mapping. We now demonstrate that the inverse mapping $(\Phi')^{-1}$ is continuous.

Let $\widetilde{\theta}_n, \widetilde{\theta} \in \mathbb{X}^*$ be such that $\widetilde{\theta}_n \to \widetilde{\theta}$ as $n \to \infty$. Our objective is to prove that

$$\lim_{n\to\infty} (\Phi')^{-1}(\widetilde{\theta}_n) = (\Phi')^{-1}(\widetilde{\theta}).$$

Let us define $w_n = (\Phi')^{-1}(\widetilde{\theta}_n)$, $n = 1, 2, \ldots$ and $w = (\Phi')^{-1}(\widetilde{\theta})$, so that

$$\Phi'(w_n) = \widetilde{\theta}_n \quad \text{and} \quad \Phi'(w) = \widetilde{\theta}.$$

Due to the coercivity of Φ', the sequence (w_n) is bounded. Without loss of generality, assume that $w_n \rightharpoonup w$ as $n \to \infty$, which leads to the following

$$\lim_{n\to\infty} \langle \Phi'(w_n) - \Phi'(w), w_n - w \rangle = \lim_{n\to\infty} \langle \widetilde{\theta}_n - \widetilde{\theta}, w_n - w \rangle = 0.$$

Since Φ' is of $(S)_+$-type, it follows that $w_n \to w$ in $\mathbb{X}$ as $n \to \infty$, which ensures that

$$\lim_{n\to\infty} \Phi'(w_n) = \Phi'(w).$$

Combining this with $\Phi'(w_n) \to \Phi'(w)$, as $n \to \infty$, we deduce that

$$\Phi'(w) = \Phi'(w).$$

Since Φ' is injective, it follows that $w = w$, and hence $w_n \to w$ as $n \to \infty$. Therefore, we have

$$(\Phi')^{-1}(\widetilde{\theta}_n) \to (\Phi')^{-1}(\widetilde{\theta}),$$

proving that $(\Phi')^{-1}$ is continuous. □

Moreover, one has

Lemma 4.4. *The operator $\Psi' : \mathbb{X} \to \mathbb{X}^*$ is compact.*

Proof. In what follows, consider Ψ_1' and Ψ_2' such that

$$\langle \Psi'(u), v\rangle = \langle \Psi_1'(u), v\rangle + \frac{\mu}{\lambda}\langle \Psi_2'(u), v\rangle.$$

Condition (H_1) and the compact embedding $\mathbb{X} \hookrightarrow L^{\alpha_1(x)}(\Omega), 1 < \alpha_1^+ < p^-$ implies the compactness of $\Psi_1'(u)$.

In fact, let $(u_n)_n \subset \mathbb{X}$ be a sequence such that $u_n \rightharpoonup u$ as $n \to \infty$. Noting that the embedding $\mathbb{X} \hookrightarrow L^{\alpha_1(x)}(\Omega)$, $1 < \alpha_1^+ < p^-$ is compact, thus there is a subsequence, still denoted by $(u_n)_n$, such that $u_n \to u$ strongly in $L^{\alpha_1(x)}(\Omega)$ as $n \to \infty$.

We claim that the Nemytskii operator $\mathcal{N}_f(u)(x) := f(x, u(x))$ is continuous since $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function satisfying (H_1), thus, $\mathcal{N}_f(u_n) \to \mathcal{N}_f(u)$ in $L^{\frac{\alpha_1(x)}{\alpha_1(x)-1}}(\Omega)$ as $n \to \infty$. In view of Hölder's inequality mentioned in Proposition 2.1 and the compact embedding $\mathbb{X} \hookrightarrow L^{\alpha_1(x)}(\Omega)$, $1 < \alpha_1^+ < p^-$, for all $v \in \mathbb{X}$, one has

$$\begin{aligned}
|\Psi_1'(u_n)(v) - \Psi_1'(u)(v)| &= \left| \int_\Omega f(x, u_n) v dx - \int_\Omega f(x, u) v\, dx \right| \\
&\leq \int_\Omega |(f(x, u_n) - f(x, u))v|\, dx \\
&\leq 2 \left|\mathcal{N}_f(u_n) - \mathcal{N}_f(u)\right|_{\frac{\alpha_1(x)}{\alpha_1(x)-1}} |v|_{\alpha_1(x)} \\
&\leq 2c_{\alpha_1} \left|\mathcal{N}_f(u_n) - \mathcal{N}_f(u)\right|_{\frac{\alpha_1(x)}{\alpha_1(x)-1}} \|v\|_{\mathbb{X}},
\end{aligned}$$

where c_{α_1} is the embedding constant of the embedding $\mathbb{X} \hookrightarrow L^{\alpha_1(x)}(\Omega)$ as in (2.1). Thus, $\Psi_1'(u_n) \to \Psi_1'(u)$ in $\mathbb{X}^*$ as $n \to \infty$, i.e. Ψ_1' is completely continuous, so we conclude that Ψ_1' is compact. A similar argument can be made to prove that Ψ_2' is compact, thereby completing the proof. □

5 Proof of Theorem 3.2

Let $x^0 \in \Omega$ be such that $B_\rho(x_0) \subseteq \Omega$. Now, for positive constant σ_2 as in the statement of Theorem 3.2, let us consider the function $u_* \in \mathbb{X}$ given by

$$u_*(x) := \begin{cases} 0 & \text{if } x \in \Omega \backslash B_\rho(x^0), \\ \sigma_2 & \text{if } x \in B_{\frac{\rho}{2}}(x^0), \\ \frac{2\sigma_2}{\rho}\left(\rho - |x - x^0|\right) & \text{if } x \in B_\rho(x^0) \backslash B_{\frac{\rho}{2}}(x^0). \end{cases} \tag{5.1}$$

From the definition of the function u_* in (5.1) we have $0 \leq u_*(x) \leq \sigma_2$ for all $x \in \Omega$ and thus, $F(x, u_*(x)) \geq 0$ for all $x \in \Omega$, due to the hypothesis (H_2). Moreover, we then deduce for each $i = 1, 2, \ldots, N$ that $\partial_{x_i} u_* = -\frac{2\sigma_2}{\rho} \frac{(x - x_i^0)}{|x - x^0|}$, so we get

$$|\nabla u_*| = \left(\sum_{i=1}^N |\partial_{x_i} u_*(x)|^2 \right)^{\frac{1}{2}} = \frac{2\sigma_2}{\rho}.$$

Hence,

$$\begin{aligned}
\frac{|\nabla u_*|^{p(x)}}{p(x)} + \eta(x) \frac{|\nabla u_*|^{q(x)}}{q(x)} &= \frac{1}{p(x)} (\frac{2\sigma_2}{\rho})^{p(x)} + \frac{\eta(x)}{q(x)} \left(\frac{2\sigma_2}{\rho} \right)^{q(x)} \\
&\geq \frac{1}{p(x)} \left(\frac{2\sigma_2}{\rho} \right)^{p(x)} \\
&\geq \frac{2^{p^-}}{p^+} \min \left\{ \left(\frac{\sigma_2}{\rho} \right)^{p^-}, \left(\frac{\sigma_2}{\rho} \right)^{p^+} \right\}.
\end{aligned}$$

Taking into account the condition (H_0), we arrive at

$$\begin{aligned}
\Phi(u_*) &= \widehat{M} \left[\int_\Omega \left(\frac{|\nabla u_*|^{p(x)}}{p(x)} + \eta(x) \frac{|\nabla u_*|^{q(x)}}{q(x)} \right) dx \right] \\
&\geq m_0 \int_\Omega \left(\frac{|\nabla u_*|^{p(x)}}{p(x)} + \eta(x) \frac{|\nabla u_*|^{q(x)}}{q(x)} \right) dx \\
&\geq \frac{m_0 2^{p^-}}{p^+} \min \left\{ (\frac{\sigma_2}{\rho})^{p^-}, (\frac{\sigma_2}{\rho})^{p^+} \right\} \int_{B_\rho(x^0) \setminus B_{\frac{\rho}{2}}(x^0)} dx \\
&\geq \frac{m_0 2^{p^-}}{p^+} \frac{\min\{\sigma_2^{p^-}, \sigma_2^{p^+}\}}{\max\{\rho^{p^-}, \rho^{p^+}\}} . (V_\rho - V_{\frac{\rho}{2}}) \\
&= \frac{m_0 2^{p^- - N}(2^N - 1) \min\{\sigma_2^{p^-}, \sigma_2^{p^+}\} . V_\rho}{p^+ \max\{\rho^{p^-}, \rho^{p^+}\}} \\
&= \frac{1}{\delta_1} \min\{\sigma_2^{p^-}, \sigma_2^{p^+}\}. \qquad (5.2)
\end{aligned}$$

From (5.2), it implies by using the condition (3.1) that

$$\Phi(u_*) > \sigma_1.$$

On the other hand, since $p(x) < q(x)$ for all $x \in \overline{\Omega}$, we also have

$$\begin{aligned}
\frac{|\nabla u_*|^{p(x)}}{p(x)} + \eta(x) \frac{|\nabla u_*|^{q(x)}}{q(x)} &= \frac{1}{p(x)} \left(\frac{2\sigma_2}{\rho} \right)^{p(x)} + \frac{\eta(x)}{q(x)} \left(\frac{2\sigma_2}{\rho} \right)^{q(x)} \\
&\leq \frac{2^{q^+}}{p^-} \max\{1, |\eta|_\infty\} \left(\left(\frac{\sigma_2}{\rho} \right)^{p(x)} + \left(\frac{\sigma_2}{\rho} \right)^{q(x)} \right) \\
&\leq \frac{2^{q^+}}{p^-} \max\{1, |\eta|_\infty\} . 2 \max \left\{ \left(\frac{\sigma_2}{\rho} \right)^{p^-}, \left(\frac{\sigma_2}{\rho} \right)^{q^+} \right\} \\
&\leq \frac{2^{q^+ + 1}}{p^-} \max\{1, |\eta|_\infty\} \frac{\max\{\sigma_2^{p^-}, \sigma_2^{q^+}\}}{\min\{\rho^{p^-}, \rho^{q^+}\}}.
\end{aligned}$$

Due to the fact that M is a continuous nondecreasing function on $[0,\infty)$, it implies that

$$\widehat{M}(t)=\int_0^t M(\tau)\,d\tau \le M(t)\int_0^t d\tau = M(t)t, \quad \forall t\ge 0,$$

which helps us to get

$$\begin{aligned}
\Phi(u_*) &= \widehat{M}\left[\int_\Omega\left(\frac{|\nabla u_*|^{p(x)}}{p(x)}+\eta(x)\frac{|\nabla u_*|^{q(x)}}{q(x)}\right)dx\right]\\
&\le M\left[\int_\Omega\left(\frac{|\nabla u_*|^{p(x)}}{p(x)}+\eta(x)\frac{|\nabla u_*|^{q(x)}}{q(x)}\right)dx\right]\int_\Omega\left(\frac{|\nabla u_*|^{p(x)}}{p(x)}+\eta(x)\frac{|\nabla u_*|^{q(x)}}{q(x)}\right)dx\\
&\le M\left[\int_{B_\rho(x^0)\setminus B_{\frac{\rho}{2}}(x^0)}\left(\frac{2^{q^+ +1}}{p^-}\max\{1,|\eta|_\infty\}\frac{\max\{\sigma_2^{p^-},\sigma_2^{q^+}\}}{\min\{\rho^{p^-},\rho^{q^+}\}}\right)dx\right]\times\\
&\qquad\times\int_{B_\rho(x^0)\setminus B_{\frac{\rho}{2}}(x^0)}\left(\frac{2^{q^+ +1}}{p^-}\max\{1,|\eta|_\infty\}\frac{\max\{\sigma_2^{p^-},\sigma_2^{q^+}\}}{\min\{\rho^{p^-},\rho^{q^+}\}}\right)dx\\
&\le \frac{m_1 2^{q^+ +1-N}(2^N-1)\max\{1,|\eta|_\infty\}.\max\{\sigma_2^{p^-},\sigma_2^{q^+}\}V_\rho}{p^-\min\{\rho^{p^-},\rho^{q^+}\}}\\
&= \frac{1}{\delta_2}\max\{\sigma_2^{p^-},\sigma_2^{q^+}\}. \qquad (5.3)
\end{aligned}$$

For every $u\in\mathbb{X}$ with $\Phi(u)\le\sigma_1$, due to Proposition 2.2, we infer that

$$q^+\sigma_1\ge q^+\Phi(u)\ge m_0\rho_{\mathcal{H}}(\nabla u)\ge m_0\min\left\{\|u\|_{\mathbb{X}}^{p^-},\|u\|_{\mathbb{X}}^{q^+}\right\},$$

which implies that

$$\begin{aligned}
\Phi^{-1}(-\infty,\sigma_1] &= \{u\in\mathbb{X}:\ \Phi(u)\le\sigma_1\}\\
&\subseteq\left\{u\in\mathbb{X}:\ \|u\|_{\mathbb{X}}\le\max\left\{\left(\frac{q^+\sigma_1}{m_0}\right)^{\frac{1}{p^-}},\left(\frac{q^+\sigma_1}{m_0}\right)^{\frac{1}{q^+}}\right\}\right\}.
\end{aligned}$$

Using the conditions (H_1), (H_4) and the fact that $\alpha_i\in C_+(\overline{\Omega})$, $i=1,2$, we obtain

$$|F(x,t)|\le A_1|t|+\frac{A_1}{\alpha_1(x)}|t|^{\alpha_1(x)}\le A_1(|t|+|t|^{\alpha_1(x)})$$

and

$$|G(x,t)|\le A_2|t|+\frac{A_2}{\alpha_2(x)}|t|^{\alpha_2(x)}\le A_2(|t|+|t|^{\alpha_2(x)})$$

for all $(x,t)\in\overline{\Omega}\times\mathbb{R}$.

Hence, by Proposition 2.2 and Proposition 2.3 and the definition of the functional Ψ, for all $u \in \Phi^{-1}(-\infty, \sigma_1]$, we arrive at

$$
\begin{aligned}
\Psi(u) &\le A_1 \int_\Omega (|u| + |u|^{\alpha_1(x)})\, dx + \frac{\mu}{\lambda} \int_\Omega A_2(|u| + |u|^{\alpha_2(x)})\, dx \\
&\le A_1 c_1 \|u\|_{\mathbb{X}} + A_1 \max\left\{ |u|^{\alpha_1^-}_{\alpha(x)}, |u|^{\alpha_1^+}_{\alpha_1(x)} \right\} + \frac{\mu}{\lambda}\left[A_2 c_1 \|u\|_{\mathbb{X}} + A_2 \max\left\{ |u|^{\alpha_2^-}_{\alpha_2(x)}, |u|^{\alpha_2^+}_{\alpha_2(x)} \right\} \right] \\
&\le A_1 c_1 \|u\|_{\mathbb{X}} + A_1 \max\left\{ c_{\alpha_1}^{\alpha_1^-}, c_{\alpha_1}^{\alpha_1^+} \right\} \max\left\{ \|u\|_{\mathbb{X}}^{\alpha_1^-}, \|u\|_{\mathbb{X}}^{\alpha_1^+} \right\} \\
&\quad + \frac{\mu}{\lambda}\left[A_2 c_1 \|u\|_{\mathbb{X}} + A_2 \max\left\{ c_{\alpha_2}^{\alpha_2^-}, c_{\alpha_2}^{\alpha_2^+} \right\} \max\left\{ \|u\|_{\mathbb{X}}^{\alpha_2^-}, \|u\|_{\mathbb{X}}^{\alpha_2^+} \right\} \right] \\
&\le A_1 c_1 \max\left\{ \left(\frac{q^+\sigma_1}{m_0}\right)^{\frac{1}{p^-}}, \left(\frac{q^+\sigma_1}{m_0}\right)^{\frac{1}{q^+}} \right\} + A_1 \widetilde{c}_{\alpha_1} \max\left\{ \left(\frac{q^+\sigma_1}{m_0}\right)^{\frac{\alpha_1^+}{p^-}}, \left(\frac{q^+\sigma_1}{m_0}\right)^{\frac{\alpha_1^-}{q^+}} \right\} \\
&\quad + \frac{\mu}{\lambda}\left[A_2 c_1 \max\left\{ \left(\frac{q^+\sigma_1}{m_0}\right)^{\frac{1}{p^-}}, \left(\frac{q^+\sigma_1}{m_0}\right)^{\frac{1}{q^+}} \right\} + A_2 \widetilde{c}_{\alpha_2} \max\left\{ \left(\frac{q^+\sigma_1}{m_0}\right)^{\frac{\alpha_2^+}{p^-}}, \left(\frac{q^+\sigma_1}{m_0}\right)^{\frac{\alpha_2^-}{q^+}} \right\} \right].
\end{aligned}
$$

Hence,

$$
\begin{aligned}
\frac{\sup_{u \in \Phi^{-1}(-\infty,\sigma_1]} \Psi(u)}{\sigma_1} &\le A_1 c_1 \max\left\{ \left(\frac{q^+}{\sigma_1^{p^- - 1} m_0}\right)^{\frac{1}{p^-}}, \left(\frac{q^+}{\sigma_1^{q^+ - 1} m_0}\right)^{\frac{1}{q^+}} \right\} \\
&\quad + A_1 \widetilde{c}_{\alpha_1} \max\left\{ \left(\frac{q^+}{m_0 \sigma_1^{\frac{p^- - \alpha_1^+}{\alpha_1^+}}}\right)^{\frac{\alpha_1^+}{p^-}}, \left(\frac{q^+}{m_0 \sigma_1^{\frac{q^+ - \alpha_1^-}{\alpha_1^-}}}\right)^{\frac{\alpha_1^-}{q^+}} \right\} \\
&\quad + \frac{\mu}{\lambda}\left[A_2 c_1 \max\left\{ \left(\frac{q^+}{\sigma_1^{p^- - 1} m_0}\right)^{\frac{1}{p^-}}, \left(\frac{q^+}{\sigma_1^{q^+ - 1} m_0}\right)^{\frac{1}{q^+}} \right\} \right. \\
&\quad \left. + A_2 \widetilde{c}_{\alpha_2} \max\left\{ \left(\frac{q^+}{m_0 \sigma_1^{\frac{p^- - \alpha_2^+}{\alpha_2^+}}}\right)^{\frac{\alpha_2^+}{p^-}}, \left(\frac{q^+}{m_0 \sigma_1^{\frac{q^+ - \alpha_2^-}{\alpha_2^-}}}\right)^{\frac{\alpha_2^-}{q^+}} \right\} \right] \\
&= A_1 \mathcal{K}_{\sigma_1,\alpha_1} + \frac{\mu}{\lambda} A_2 \mathcal{K}_{\sigma_1,\alpha_2}. \qquad (5.4)
\end{aligned}
$$

On the other hand, the conditions (H_2) and (H_5) yields

$$
\begin{aligned}
\Psi(u_*) &= \int_{B_{\frac{\rho}{2}}(x^0)} F(x,\sigma_2)dx + \int_{B_\rho(x^0)\setminus B_{\frac{\rho}{2}}(x^0)} F\left(x, \frac{2\sigma_2}{\rho}(\rho - |x - x^0|)\right) dx + \frac{\mu}{\lambda}\int_\Omega G(x,u(x))\, dx \\
&\ge \int_{B_{\frac{\rho}{2}}(x^0)} F(x,\sigma_2)dx. \qquad (5.5)
\end{aligned}
$$

Invoking (5.3) and (5.5), we get

$$
\frac{\Psi(u_*)}{\Phi(u_*)} \ge \delta_2 \frac{\int_{B_{\frac{\rho}{2}}(x^0)} F(x,\sigma_2)dx}{\max\{\sigma_2^{p^-}, \sigma_2^{q^+}\}}. \qquad (5.6)
$$

Because $\mu < \mu^*_{\lambda,g}$, we infer that

$$\mu < \frac{1 - \lambda A_1 \mathcal{K}_{\sigma_1,\alpha_1}}{A_2 \mathcal{K}_{\sigma_1,\alpha_2}},$$

that is,

$$\frac{1}{\lambda} > A_1 \mathcal{K}_{\sigma_1,\alpha_1} + \frac{\mu}{\lambda} A_2 \mathcal{K}_{\sigma_1,\alpha_2},$$

so by (5.4), we arrive at

$$\frac{\sup_{u\in\Phi^{-1}(-\infty,\sigma_1]} \Psi(u)}{\sigma_1} < \frac{1}{\lambda}. \tag{5.7}$$

Furthermore, owing to the choice of the parameter λ, that is,

$$\frac{1}{\lambda} < \delta_2 \frac{\int_{B_{\frac{\varrho}{2}}(x^0)} F(x,\sigma_2)\, dx}{\max\{\sigma_2^{p^-}, \sigma_2^{q^+}\}},$$

which implies by using (5.6) that

$$\frac{\Psi(u_*)}{\Phi(u_*)} > \frac{1}{\lambda}. \tag{5.8}$$

Thanks to (5.7)–(5.8), it follows that the condition (T_1) of Proposition 2.5 is fulfilled with $u_0 = u_*$ and $\gamma = \sigma_1$.

Now, thanks to the conditions (H_1), (H_4), Proposition 2.2 and Proposition 2.3, for $u \in \mathbb{X}$ with $\|u\|_{\mathbb{X}} > 1$, we arrive at

$$\begin{aligned}
\Phi(u) - \lambda\Psi(u) &\geq \frac{m_0}{q^+} \min\left\{\|u\|_{\mathbb{X}}^{p^-}, \|u\|_{\mathbb{X}}^{q^+}\right\} - \lambda \int_\Omega \left[F(x,u) + \frac{\mu}{\lambda} G(x,u)\right] dx \\
&\geq \frac{m_0}{q^+}\|u\|_{\mathbb{X}}^{p^-} - \lambda A_1 \int_\Omega |u|\, dx - \lambda \frac{A_1}{\alpha_1^-} \int_\Omega |u|^{\alpha_1(x)}\, dx - \mu A_2 \int_\Omega |u|\, dx \\
&\quad - \mu \frac{A_2}{\alpha_2^-} \int_\Omega |u|^{\alpha_2(x)}\, dx \\
&\geq \frac{m_0}{q^+}\|u\|_{\mathbb{X}}^{p^-} - \lambda A_1 c_1 \|u\|_{\mathbb{X}} - \lambda \frac{A_1}{\alpha_1^-} \widetilde{c}_{\alpha_1} \max\left\{\|u\|_{\mathbb{X}}^{\alpha_1^-}, \|u\|_{\mathbb{X}}^{\alpha_1^+}\right\} \\
&\quad - \mu A_2 c_1 \|u\|_{\mathbb{X}} - \mu \frac{A_2}{\alpha_2^-} \widetilde{c}_{\alpha_2} \max\left\{\|u\|_{\mathbb{X}}^{\alpha_2^-}, \|u\|_{\mathbb{X}}^{\alpha_2^+}\right\} \\
&= \frac{m_0}{q^+}\|u\|_{\mathbb{X}}^{p^-} - (\lambda A_1 + \mu A_2) c_1 \|u\|_{\mathbb{X}} - \lambda \frac{A_1}{\alpha_1^-} \widetilde{c}_{\alpha_1} \|u\|_{\mathbb{X}}^{\alpha_1^+} - \mu \frac{A_2}{\alpha_2^-} \widetilde{c}_{\alpha_2} \|u\|_{\mathbb{X}}^{\alpha_2^+}.
\end{aligned}$$

Since $1 < \alpha_i^+ < p^-$ for $i = 1, 2$, the coercivity of $\Phi - \lambda\Psi$ is ensured, and the condition (T_2) in Proposition 2.5 is satisfied. Therefore, the application of Proposition 2.5 yields the desired conclusion.

In the sequel, let us consider the following Kirchhoff-type double phase problem with constant exponents and one parameter

$$\begin{cases} -M\left[\int_\Omega \left(\frac{|\nabla u|^p}{p} + \eta(\cdot)\frac{|\nabla u|^q}{q}\right) dx\right] \left(\Delta_p u + \eta \Delta_q^{\eta(\cdot)} u\right) = \lambda f(x,u), & x \in \Omega, \\ u = 0, \quad x \in \partial\Omega, \end{cases} \tag{5.9}$$

where, $\Omega \subseteq \mathbb{R}^N$ $(N \geq 2)$ be a bounded domain with Lipschitz boundary $\partial\Omega$, $0 \leq \eta(\cdot) \in L^{\infty}(\Omega)$, $\lambda > 0$ is a parameter, and p, q are constants satisfying

$$1 < p < N, \quad p < q < \frac{Np}{N-p}, \tag{5.10}$$

Assume that $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function satisfying appropriate growth conditions, and the Kirchhoff function $M(t) = 1 + t$ with $t \geq 0$. For any $\sigma \in \mathbb{R}^+$ and $\alpha > 1$, let us define the number $\mathcal{K}_{\sigma,\alpha}$ as

$$\mathcal{K}_{\sigma,\alpha} := c_1 \max\left\{ \left(\frac{q}{\sigma^{p-1}}\right)^{\frac{1}{p}}, \left(\frac{q}{\sigma^{q-1}}\right)^{\frac{1}{q}} \right\} + c_\alpha^\alpha \max\left\{ \left(\frac{q}{\sigma^{\frac{p-\alpha}{\alpha}}}\right)^{\frac{\alpha}{p}}, \left(\frac{q}{\sigma^{\frac{q-\alpha}{\alpha}}}\right)^{\frac{\alpha}{q}} \right\},$$

we then deduce the following multiplicity result for problem (1.1).

Corollary 5.1. *Assume that the hypothesis* (5.10) *holds and there exist two positive constants* σ_1 *and* σ_2 *verifying*

$$\delta_1 \sigma_1 < \sigma_2^p,$$

with

$$\delta_1 := \frac{p\rho^p}{2^{p-N}(2^N - 1)V_\rho},$$

such that

(H_1) *there exist* $A_1 > 0$ *and* α_1 *with* $1 < \alpha_1 < p$ *such that*

$$|f(x,t)| \leq A_1(1 + |t|^{\alpha_1 - 1})$$

for a.e. $x \in \Omega$ *and for all* $t \in \mathbb{R}$*;*

(H_2) $F(x,t) \geq 0$ *for a.e.* $x \in \Omega$ *and for all* $t \in [0, \sigma_2]$*;*

(H_3) $A_1 \mathcal{K}_{\sigma_1,\alpha_1} < \frac{\delta_2 \int_{B_{\rho/2}(x^0)} F(x,\sigma_2)dx}{\max\{\sigma_2^p, \sigma_2^q\}}$ *with*

$$\delta_2 := \frac{p \min\{\rho^p, \rho^q\}}{m_1 2^{q+1-N}(2^N - 1) \max\{1, |\eta|_\infty\} V_\rho},$$

$$m_1 := 1 + \frac{2^{q+1-N}(2^N - 1) \max\{1, |\eta|_\infty\}. \max\{\sigma_2^p, \sigma_2^q\} V_\rho}{p \min\{\rho^p, \rho^q\}}.$$

Then, for every parameter

$$\lambda \in \Lambda_{\sigma_1,\sigma_2} := \left(\frac{\max\{\sigma_2^p, \sigma_2^q\}}{\delta_2 \int_{B_{\frac{\rho}{2}}(x^0)} F(x,\sigma_2)\,dx}, \frac{1}{A_1 \mathcal{K}_{\sigma_1,\alpha_1}} \right)$$

problem (5.9) *possesses at least three distinct weak solutions in the space* $\mathbb{X}$.

Acknowledgements

The authors would like to thank the referees for their suggestions and helpful comments which improved the presentation of the original manuscript. The authors extend their appreciation to the Deanship of Scientific Research at Northern Border University, Arar, KSA for funding this research work through the project number NBU-FFR-2025-1706-13.

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