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Existence and uniqueness of crossing periodic solutions of a discontinuous mixed-type Duffing equation

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Abstract. In this paper, we investigate the existence and uniqueness of crossing periodic solutions for a class of mixed-type second order Duffing equations with discontinuity and asymmetry. By the Poincaré–Bohl theorem, we obtain several existence criteria of regular periodic solutions which are crossing. We also present a uniqueness result.

Keywords: discontinuity, Duffing equation, Poincaré–Bohl theorem, crossing periodic solution, existence and uniqueness.

2020 Mathematics Subject Classification: 34A36, 34A34, 34A12, 34C25.

1 Introduction

Second order Duffing equation

$$x'' + g(x) = p(t), \tag{1.1}$$

is one of the typical models both in ODE and forced vibrations, where g is an elastic force and p is an external excitation. There have been many interesting results reported concerning the existence of periodic solutions. The methods involve the Poincaré–Birkhoff twist theorem, the Poincaré–Bohl theorem, averaging theory, variational method and the Leray–Schauder continuation method of topological degree etc., see for example [3–6, 9, 22–26] and the references therein.

In the paper, we are concerned with Eq. (1.1) with discontinuous vector field, in which g presents a discontinuity at $\alpha \in \mathbb{R}$ and it satisfies asymmetric conditions, and then we called it as discontinuous mixed-type Duffing equations. During the last decades, the amount of publications on the discontinuous differential equations is vast. We refer readers to consult classical monographs [8, 19] for the general theory, and literatures [7, 11, 12, 17, 18] for some related research. In [8], the author extend discontinuous differential equations to differential inclusions, he revealed many aspects of discontinuous differential equations and addressed existence and uniqueness of solutions. More results on the differential inclusion can be found

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in [1]. Inspired by these, we consider the existence and uniqueness of periodic solutions for the following second order Duffing equations allowing for discontinuities at 0

$$x'' + g_{\pm}(x) = p(t), \qquad x \neq 0, \tag{1.2}$$

where $p(t) = p(t + 2\pi)$, $g_{\pm}(x)$ are of class C^1 and satisfy the asymmetric growth condition

$$0 \leq \lim_{x\to-\infty} \frac{g_-(x)}{x} < +\infty, \qquad \lim_{x\to+\infty} \frac{g_+(x)}{x} = +\infty. \tag{1.3}$$

Many existing literatures, whether continuous or not, focused on periodic solution problem of Duffing equations with symmetric growth condition. But there is no study about the discontinuous Duffing equation with the asymmetric condition. According to the relationship of linear growth satisfied by the elastic force g, the Duffing equations can be divided into three categories:

(S_p) Superlinear growth: $\lim_{|x|\to\infty} \frac{g(x)}{x} = +\infty$;

(S_m) Semilinear growth: $0 < \liminf_{|x|\to\infty} \frac{g(x)}{x} \leq \limsup_{|x|\to\infty} \frac{g(x)}{x} < \infty$;

(S_b) Sublinear growth: $\lim_{|x|\to\infty} \frac{g(x)}{x} = 0$.

On the other hand, when g satisfies different linear growth conditions for $x \in \mathbb{R}$, it means that the processing method is also different from the one of symmetric Duffing equations. Recently, there are a few papers on the periodic solution problem of discontinuous symmetric Duffing equations (with impulsive perturbations or discontinuous vector field), in which the elastic force g satisfies one of the above linear growth conditions, see [2,10,12–16,18,20,21] for example and the references therein. However, in this paper we consider the existence criteria of crossing periodic solutions of (1.2) when $g_{\pm}(x)$ satisfy different linear growth conditions (1.3) for $x \in \mathbb{R}$. And a uniqueness theorem is also presented.

To obtain the desirable results, we first assume that $g_{\pm}$ satisfy the following conditions.

$(H1)$ Let $g_+ \in C^1([0, +\infty), \mathbb{R})$, $g_- \in C^1((-\infty, 0], \mathbb{R})$, and the behavior of $g_{\pm}$ at the discontinuity point $x = 0$ satisfies that the left and right limits

$$g_- = \lim_{x\to 0^-} g_-(x), \qquad g_+ = \lim_{x\to 0^+} g_+(x)$$

exist and are finite, and satisfy $g_- < 0 < g_+$.

$(H2)$ There are two real numbers d_1 and d_2 with $d_1 < 0 < d_2$ such that

$$x \leq d_1 \Rightarrow g_-(x) < 0, \qquad x \geq d_2 \Rightarrow g_+(x) > 0.$$

$(H3)$ There are positive constants l_-, l_+, L_- and A_1 such that the asymptotic growth of $g_{\pm}$ is controlled by

$$l_- \leq \frac{g_-(x)}{x} \leq L_-, \quad x \leq -A_1; \qquad \frac{g_+(x)}{x} \geq l_+, \quad x \geq A_1. \tag{1.4}$$

Moreover, there exists an integer $m > 0$ that satisfies

$$\frac{2}{m+1} < \frac{1}{\sqrt{L_-}} \leq \frac{1}{\sqrt{l_-}} + \frac{1}{\sqrt{l_+}} < \frac{2}{m}. \tag{1.5}$$

Then by applying the Poincaré–Bohl theorem to the discontinuous Duffing equation (1.2) with the asymmetric growth condition, we obtain several existence criteria of crossing periodic solutions as follows.

Theorem 1.1. *Assume that* $(H1)$*–*$(H3)$ *hold. Then* (1.2) *has at least one* 2π *crossing periodic solution.*

Note that (1.4) means that g_- satisfies semilinear growth for $x \le -A_1$, while g_+ can satisfy both semilinear growth and superlinear growth when $x \ge A_1$. Further we have that

$$\lim_{x\to-\infty} g_-(x) = -\infty, \qquad \lim_{x\to+\infty} g_+(x) = +\infty. \tag{1.6}$$

Theorem 1.2. *In Theorem 1.1, if* $(H3)$ *is replaced as* (1.6) *and*

$(H3)'$ *There exist* $L_- \in (0, \frac{1}{4})$ *and* $A_1 > 0$ *such that*

$$\frac{g_-(x)}{x} \le L_-, \qquad x \le -A_1.$$

Then (1.2) *has at least one* 2π *crossing periodic solution.*

Theorem 1.3. *In Theorem 1.1, if* $(H3)$ *is replaced as*

$(H3)^*$ *There are positive constants* l_-, L_-, A *and an integer* $m > 0$ *such that* $\lim_{x\to+\infty} \frac{g_+(x)}{x} = +\infty$, *and*

$$\frac{1}{4}m^2 < l_- \le \frac{g_-(x)}{x} \le L_- < \frac{1}{4}(m+1)^2, \qquad x \le -A. \tag{1.7}$$

Then (1.2) *has at least one* 2π *crossing periodic solution.*

Moreover, for the completeness of the paper we further provide a uniqueness criterion.

Theorem 1.4. *Assume that* $g_\pm$ *are Lipschitz continuous and satisfy* $0 < \frac{g_\pm(x)-g_\pm(y)}{x-y} < 1$ *for any* $x, y \in \mathbb{R} \setminus \{0\}$. *Then* (1.2) *has at most one* 2π *crossing periodic solution.*

This paper is organized as follows. In Section 2, we present some useful preliminaries. In Sections 3–4, we give the behavior of solutions for the autonomous and non-autonomous systems, respectively. Then by applying the Poincaré–Bohl theorem, we obtain several existence criteria of 2π crossing periodic solutions. And we also prove the uniqueness. Moreover, to demonstrate the effectiveness of the obtained results, we further provide an example in Section 5. Concluding remarks are outlined in Section 6.

2 Preliminaries

Consider the second order Duffing equation

$$x'' + g(x) = p(t), \tag{2.1}$$

where $p : \mathbb{R} \to \mathbb{R}$ is continuous and 2π-periodic, and denote by

$$E = \max_{t\in[0,2\pi]} \{|p(t)|\}, \qquad \overline{E} = \max_{t\in[0,2\pi]} \{p(t)\}, \qquad \underline{E} = \min_{t\in[0,2\pi]} \{p(t)\}. \tag{2.2}$$

While $g : \mathbb{R} \to \mathbb{R}$ is continuous except for a discontinuity point at $x = 0$. Assume that $\Sigma_0 = \{(x, y) \in \mathbb{R}^2 : x = 0\}$ divide $g(x)$ of the form

$$g(x) = \begin{cases} g_+(x), & x > 0, \\ g_-(x), & x < 0. \end{cases} \tag{2.3}$$

Here Σ_0 is called as a discontinuity line on the plane, and we denote by

$$\Sigma_0^+ = \{(x, y) : x = 0, y > 0\}, \qquad \Sigma_0^- = \{(x, y) : x = 0, y < 0\},$$
$$\Sigma_+ = \{(x, y) \in \mathbb{R}^2 : x > 0\}, \qquad \Sigma_- = \{(x, y) \in \mathbb{R}^2 : x < 0\}.$$

Then $\Sigma_0 = \Sigma_0^+ \cup \{O\} \cup \Sigma_0^-$, $\mathbb{R}^2 = \Sigma_+ \cup \Sigma_0 \cup \Sigma_-$, and the unit normal vector perpendicular to Σ_0 is taken to be $\mathbf{n}^T = (1, 0)$.

We recall an existence result of periodic solutions from the Poincaré–Bohl theorem.

Theorem 2.1 ([26, Theorem 1.2 of Chapter VI])**.** *Suppose that $\mathcal{F} : \mathbb{D} \to \mathbb{R}^2$ is a continuous mapping, where $\mathbb{D}(\subset \mathbb{R}^2)$ is a closed bounded region including the origin as an interior point, the boundary $\partial\mathbb{D}$ is a piecewise smooth simple closed curve. For any $p \in \partial\mathbb{D}$ then the image $q = \mathcal{F}(p)$ satisfies $\overrightarrow{Oq} \neq \lambda\overrightarrow{Op}$, where $\lambda \geq 1$ is a constant. Then $\mathcal{F}$ has at least one fixed point in $\mathbb{D}$.*

It is sufficient to prove that any orbit which starts from $p = (x(0), x'(0)) \in \partial\mathbb{D}$ and moves to $q = (x(2\pi), x'(2\pi)) \in \mathbb{R}^2 \setminus \mathbb{D}$, then the points p and q are not on the same ray starting from the origin.

In what follows, we consider the equivalent system of (2.1) and (2.3)

$$\begin{cases} x' = y, \\ y' = -g_+(x) + p(t), \quad x > 0; \end{cases} \qquad \begin{cases} x' = y, \\ y' = -g_-(x) + p(t), \quad x < 0. \end{cases} \tag{2.4}$$

Let $\mathbf{V}_\pm(t; x, y) = (y, -g_\pm(x) + p(t))^T$ denote the vector field of (2.4). The second component of $\mathbf{V}_\pm$ is discontinuous on Σ_0, so by using Filippov theory to define orbits of (2.4) when they intersect with Σ_0 such that the orbits can be concatenated in a natural way. Concretely, the set-valued extension as follows

$$z'(t) \in \mathbf{V}(t; x, y) = \begin{cases} \mathbf{V}_+(t; x, y), & x > 0, \\ \overline{co}\{\mathbf{V}_+(t; x, y), \mathbf{V}_-(t; x, y)\}, & x = 0, \\ \mathbf{V}_-(t; x, y), & x < 0, \end{cases} \tag{2.5}$$

where $z(t) \doteq (x(t), y(t))^T$, $\mathbf{V}(t; x, y) \doteq \mathbf{V}(t; x(t), y(t))$, and $\overline{co}(A)$ denotes the smallest closed convex set containing A. The convex set with $\mathbf{V}_-$ and $\mathbf{V}_+$ is of the form

$$\overline{co}\{\mathbf{V}_+, \mathbf{V}_-\} = \{(1 - q)\mathbf{V}_- + q\mathbf{V}_+, \ \forall q \in [0, 1]\},$$

here q is a parameter which defines the convex combination. The extension of (2.4) into the convex differential inclusion (2.5) is known as the Filippov's convex method. The existence of solutions of (2.5) is guaranteed with an upper semi-continuity of set-valued functions, and the existence of solutions of an initial value problem (IVP for short)

$$z'(t) \in \mathbf{V}(t; x, y), \qquad z(0) = (x_0, y_0)^T$$

combines with Filippov's convex method define the solutions of (2.4). For more details, we refer readers to consult [1,8] and the references therein.

In this paper, we deal with the periodic solutions that appear when the discontinuity line Σ_0 is considered, and the solution $t \to (x(t), x'(t))$ vanishes twice per period. These solutions are built piecewise, from restrictions to some subintervals of solutions of $x'' + g_+(x) = p(t)$ (*resp.* $x'' + g_-(x) = p(t)$) vanishing at the bounds of the corresponding subinterval. Now we give the following definitions concerning what kind of solutions we are looking for.

Definition 2.2 ([12])**.** Let I be an interval of $\mathbb{R}$ and $x : I \to \mathbb{R}$ be any solution of (2.4). We say that x is singular if there exists $t^* \in I$ such that $x(t^*) = x'(t^*) = 0$. A non-singular solution is said to be regular.

Definition 2.3. We say that $x : \mathbb{R} \to \mathbb{R}$ is a regular solution of (2.4) if it is continuous to t, and there exists a strictly increasing sequence $(t_j)_{j\in\mathbb{Z}}$, with no accumulation points, such that $x(t_j) = 0, x'(t_j) \neq 0$ and the restriction of x on each interval $[t_{j-1}, t_j]$ is twice continuously differentiable. Moreover, the limits $\lim_{t\to t_j^-} x''(t)$ and $\lim_{t\to t_j^+} x''(t)$ exist and are finite.

By $(H1)$, we assume the existence of solutions to the IVP of (2.4) with $z(0) = (x_0, y_0)^T$. And from the above statements, any solution of IVP with the initial condition $(x_0, y_0) \notin \Sigma_0$ is locally unique, because $\mathbf{V}_+(t; x, y)$ and $\mathbf{V}_-(t; x, y)$ are of class C^1. While for the uniqueness problem of IVP with the initial condition on Σ_0, there are three basic ways in which the vector field around Σ_0 can behave as follows.

(1) Transversal intersection.

Any solution of IVP with the initial condition $(x_0, y_0) \in \Sigma_+/$or $(x_0, y_0) \in \Sigma_-$, exists and is unique. A necessary condition for the transversal intersection at Σ_0 is

$$[\mathbf{n}^T\mathbf{V}_-(t; x, y)] \cdot [\mathbf{n}^T\mathbf{V}_+(t; x, y)] > 0, \qquad (x, y) \in \Sigma_0,$$

where $\mathbf{n}^T\mathbf{V}_-$ and $\mathbf{n}^T\mathbf{V}_+$ are projections of $\mathbf{V}_-$ and $\mathbf{V}_+$ on the normal vector to Σ_0.

(2) Attraction sliding mode (i.e. Σ_0 attracts the solution). Here the solution will hit Σ_0 but cannot leave it and then move along Σ_0.

Any solution of IVP exists and is unique in the forward time. During the sliding mode, the solution will continue along Σ_0 with the vector field $\widetilde{\mathbf{V}}$ being given by

$$\widetilde{\mathbf{V}} = \alpha\mathbf{V}_+ + (1-\alpha)\mathbf{V}_-, \qquad \alpha = \frac{n^T\mathbf{V}_-}{n^T(\mathbf{V}_- - \mathbf{V}_+)},$$

where the parameter α is chosen such that the vector field $\widetilde{\mathbf{V}}$ lies along Σ_0. An attraction sliding mode at Σ_0 occurs if

$$\mathbf{n}^T\mathbf{V}_-(t; x, y) > 0, \quad \mathbf{n}^T\mathbf{V}_+(t; x, y) < 0, \qquad (x, y) \in \Sigma_0.$$

(3) Repulsion sliding mode (i.e. the vector field is repulsing from Σ_0). Here the solution is diverging from Σ_0.

The IVP with the initial condition on Σ_0 has three possible solutions. In fact, a solution which starts close to Σ_0 will move away from it, but a solution emanating on Σ_0 can stay on Σ_0, obeying Filippov's solution, or leave Σ_0 by entering either Σ_+ or Σ_-. Hence the solution exists but is not unique in forward time.

Remark 2.4. Note that the solution of (2.4) passing through the origin is not unique. For example, we consider

$$g_+(0) - p(t) < 0, \qquad g_-(0) - p(t) > 0.$$

A qualitative sketch of the dynamics is as shown in Figure 2.1. There are two orbits passing through the origin: one is contained in Σ_+ involving the system with g_+, the other is contained in Σ_- involving the system with g_-, but it is not clear that which one should be the unique solution.

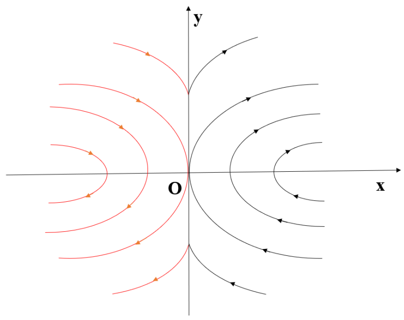

Figure 2.1: Schematic diagram

Definition 2.5. For any given $(0, y) \in \Sigma_0$, if

$$[\mathbf{n}^T \mathbf{V}_-(t; 0^-, y)] \cdot [\mathbf{n}^T \mathbf{V}_+(t; 0^+, y)] \leq 0,$$

$(0, y)$ is called to be *a sliding point*. A set of sliding points is called as a sliding set. We say $(0, y)$ a *crossing point*, if

$$[\mathbf{n}^T \mathbf{V}_-(t; 0^-, y)] \cdot [\mathbf{n}^T \mathbf{V}_+(t; 0^+, y)] > 0.$$

Then the origin is the unique possible sliding point on Σ_0, and it is an isolated and singular sliding point.

Definition 2.6. *A crossing periodic orbit* is defined as a closed orbit with no points in common with the sliding set.

In conclusion, any regular orbit of (2.4) not passing through the origin on the phase plane exists and is unique. In the next text, we focus on the crossing solutions (or orbits) of (2.4) exception for the origin. To be precise, we discuss large-amplitude *crossing periodic orbits* of (2.4). They intersect with Σ_0 transversally, *i.e.* the transversal intersection case. For more information about the existence and uniqueness of solutions of discontinuous differential equations, please see [8, 11, 17] for example and the references therein.

3 Autonomous system

The autonomous equation

$$x'' + g_\pm(x) = 0, \qquad x \neq 0, \tag{3.1}$$

which is equivalent to the system

$$\begin{cases} x' = y, \\ y' = -g_+(x), \quad x > 0; \end{cases} \qquad \begin{cases} x' = y, \\ y' = -g_-(x), \quad x < 0. \end{cases} \tag{3.2}$$

Let $G(x) = \int_0^x g_\pm(u)du$ with $G(0) = 0$, it follows from $(H3)$ that

$$\lim_{x\to-\infty} G(x) = \lim_{x\to+\infty} G(x) = +\infty. \tag{3.3}$$

Then the function G has a minimum on $\mathbb{R}$, by adding a suitable constant we can assume without loss of generality that $G(x) \geq 0$ for every $x \in \mathbb{R}$.

For the system (3.2) we have a Hamiltonian function $H : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ defined as

$$H(x, y) = \frac{1}{2}y^2 + G(x).$$

Note that H is not of class C^1 in $\mathbb{R}^2$, since G is continuous but not differentiable at $x = 0$. The orbits of the crossing solutions are symmetric with respect to the horizontal axis, and they lie on the level sets of the Hamiltonian function, i.e.,

$$L^c = \{(x, y) \in \mathbb{R}^2 : H(x, y) = c\}.$$

The level sets L^c are compact, and consequently the crossing solutions of (3.1) are globally defined.

We are now concentrate on the dynamics both near the origin $O(0, 0)$.

Lemma 3.1. *The origin is a local center for* (3.2)*; if $c > 0$ is small sufficiently, the level set L^c is a closed curve, star-shaped with respect to the origin, corresponding to the crossing periodic solution of* (3.2)*. The same is true for $c > 0$ being large enough.*

Proof. By $(H1)$, there exists a $\delta > 0$ such that for every $x \in [-\delta, 0) \cup (0, \delta]$ we have that

$$xg_\pm(x) > x^2.$$

Let $z(t) = (x(t), y(t))$ be a crossing solution of (3.2) with $x(t) \in [-\delta, 0) \cup (0, \delta]$ for some $t \in \mathbb{R}$. Making the following polar coordinates transformation

$$x(t) = r(t)\cos\theta(t), \qquad y(t) = r(t)\sin\theta(t), \tag{3.4}$$

where $r : [0, 2\pi] \to (0, +\infty)$ and $\theta : [0, 2\pi] \to \mathbb{R}$ are continuous functions, we have that

$$-\theta'(t) = \frac{x'(t)y(t) - x(t)y'(t)}{x^2(t) + y^2(t)} = \frac{x(t)g_\pm(x(t)) + y^2(t)}{x^2(t) + y^2(t)} \geq 1. \tag{3.5}$$

We observe that for $x \in [-\delta, 0) \cup (0, \delta]$,

$$\nabla H(x, y) \cdot (x, y)^T = xg_\pm(x) + y^2 > 0.$$

Hence near the origin, $H(x, y)$ is strictly increasing along the rays emanating from the origin, at least in a small neighborhood of the origin. Then a solution starting with initial point $(x(0), y(0)) \neq (0, 0)$ sufficiently near the origin, on a given ray, will remain in the strip $\{(x, y) : |x| < \delta\}$ for every $t \in [0, 2\pi]$, and during this time it will perform an entire rotation, returning to the initial ray, by (3.5). Since $H(x, y)$ is constant along the orbit, it must indeed return to the same initial point. This proves that the small-amplitude crossing solutions of (3.2) are periodic and from (3.5) they have star-shaped orbits with respect to the origin.

We next consider a large-amplitude crossing solutions of (3.2). From $(H2)$ and set $M = \max\{|xg_\pm(x)| : x \in [d_1, d_2]\}$. Let $(x(t), y(t))$ be a crossing solution of (3.2) with $(x(t), y(t)) \neq [d_1, d_2] \times [-\sqrt{M}, \sqrt{M}]$ and $x(t) \neq 0$ for some $t \in \mathbb{R}$. Then it follows from (3.4) that

$$-\theta'(t) \begin{cases} \geq \frac{-M+y^2(t)}{x^2(t)+y^2(t)} > 0, & x(t) \in [d_1, d_2], \\ > \frac{y^2(t)}{x^2(t)+y^2(t)} \geq 0, & x(t) \notin [d_1, d_2]. \end{cases}$$

The same argument shows that far from the origin then $H(x, y)$ is strictly increasing along the rays emanating from the origin. Let

$$\overline{S} = \max\{H(x, y) : (x, y) \in [d_1, d_2] \times [-\sqrt{M}, \sqrt{M}]\},$$

and set $K = \sqrt{2\overline{S}}$. By $(H3)$ there exist $\widehat{d}_1, \widehat{d}_2$ with $\widehat{d}_1 \leq d_1 < 0 < d_2 \leq \widehat{d}_2$ such that

$$x \notin [\widehat{d}_1, \widehat{d}_2] \Rightarrow G(x) > \overline{S}.$$

We claim that if (x, y) is a crossing solution of (3.2) with $(x(0), y(0)) \notin [\widehat{d}_1, \widehat{d}_2] \times [-K, K]$, then $(x(t), y(t)) \notin [d_1, d_2] \times [-\sqrt{M}, \sqrt{M}]$ for every $t \in \mathbb{R}$. Once the claim is proved, the above argument used for the small-amplitude crossing solutions can be applied, and consequently the large-amplitude crossing solutions of (3.2) are periodic and have star-shaped orbits with respect to the origin.

We let (x, y) be a crossing solution of (3.2) such that $(x(0), y(0)) \notin [\widehat{d}_1, \widehat{d}_2] \times [-K, K]$. There are two cases. Either $x(0) \notin [\widehat{d}_1, \widehat{d}_2]$, in which case $G(x(0)) > \overline{S}$, hence

$$H(x(t), y(t)) = H(x(0), y(0)) \geq G(x(0)) > \overline{S}, \qquad t \in \mathbb{R};$$

or $x(0) \in [\widehat{d}_1, \widehat{d}_2]$ and $y(0) \notin [-K, K]$, in which case $\frac{1}{2}y^2(0) > \overline{S}$, hence

$$H(x(t), y(t)) = H(x(0), y(0)) \geq \frac{1}{2}y^2(0) > \overline{S}, \qquad t \in \mathbb{R}.$$

By the definition $\overline{S}$, in both cases then $(x(t), y(t)) \notin [d_1, d_2] \times [-\sqrt{M}, \sqrt{M}]$ for every $t \in \mathbb{R}$. □

For every positive numbers $c_1 < c_2$, define the set

$$A(c_1, c_2) = \left\{(x, y) \in \mathbb{R}^2 : c_1 \leq \frac{1}{2}y^2 + G(x) \leq c_2\right\}.$$

Then it follows from Lemma 3.1 that when $c_1 > 0$ is small enough and $c_2 > c_1$ is large enough, the set $A(c_1, c_2)$ is an annulus with strictly star-shaped boundary curves. Moreover, all crossing solutions of (3.2) with the associated initial value

$$(x(0), y(0)) = (x_0, y_0) \in \mathbb{R} \times \mathbb{R}$$

are regular, with the only exception for $(x_0, y_0) = (0, 0)$, in which case the solution is not defined. Further we agree that the origin will be considered as a singular point defined as in Definition 2.2.

Remark 3.2. If we consider c_1 and c_2 satisfying the above properties as fixed in Lemma 3.1. It is easy to see that we can choose a constant $\rho > 1$ such that for every $(x, y) \in \mathbb{R}^2$

$$\frac{1}{2}y^2 + G(x) \geq c_1 \Rightarrow x^2 + y^2 > \rho^{-1},$$
$$\frac{1}{2}y^2 + G(x) \leq c_2 \Rightarrow x^2 + y^2 < \rho.$$

Lemma 3.1 implies that all regular solutions (or regular orbits) of (3.1), exception for the origin, cross Σ_0 transversally in a clockwise fashion. For example see Figure 3.1: crossing periodic orbits of $x'' + x + \operatorname{sgn}(x) = 0$. It shows that the crossing periodic orbits are nonsmooth but continuous at the intersection points of the orbits and Σ_0.

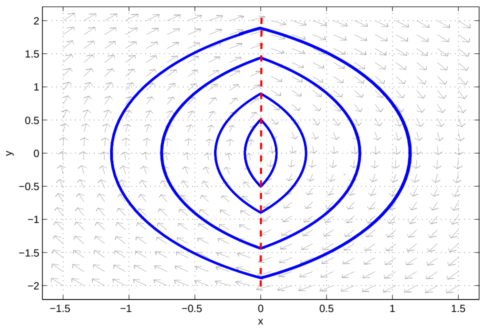

Figure 3.1: Crossing periodic orbits of $x'' + x + \operatorname{sgn}(x) = 0$. Red dashed line denotes Σ_0.

4 Perturbed system

We rewrite the equivalent system associated to (2.1) and (2.3) as follows

$$\begin{cases} x' = y, \\ y' = -g_+(x) + p(t), \quad x > 0; \end{cases} \qquad \begin{cases} x' = y, \\ y' = -g_-(x) + p(t), \quad x < 0. \end{cases} \tag{4.1}$$

Any function $z(t) = (x(t), y(t)) : [0, 2\pi] \to \mathbb{R} \times \mathbb{R}$ such that $(x(t), y(t)) \neq (0,0)$ for every $t \in [0, 2\pi]$. By the polar coordinates transformation (3.4), the resulting equation of (4.1) for $r(t) > 0$ and $\theta(t) \neq \frac{\pi}{2} + k\pi, k \in \mathbb{Z}$ is of the form

$$\begin{cases} \dfrac{dr(t)}{dt} = r\cos\theta\sin\theta + [p(t) - g_\pm(r\cos\theta)]\sin\theta, \\ \dfrac{d\theta(t)}{dt} = -\sin^2\theta + \dfrac{1}{r}[p(t) - g_\pm(r\cos\theta)]\cos\theta. \end{cases} \tag{4.2}$$

Correspondingly, we let $r(t) = r(t; r_0, \theta_0), \theta(t) = \theta(t; r_0, \theta_0)$ be the solution of (4.2), which satisfies $(r(0), \theta(0)) \doteq (r_0, \theta_0) \neq (0,0)$, and $x_0 = r_0\cos\theta_0,\ y_0 = r_0\sin\theta_0$.

Denote by $\mathbb{D}(R_0) \doteq \{(x,y) \in \mathbb{R}^2 : x^2 + y^2 < R_0^2\}$ and $\mathbb{D}[R_0] \doteq \{(x,y) \in \mathbb{R}^2 : x^2 + y^2 \leq R_0^2\}$ are, respectively, the open and closed discs in the plane (*i.e.* two-dimensional balls), and $\mathbb{S}_{R_0} = \{(x,y) \in \mathbb{R}^2 : x^2 + y^2 = R_0^2\}$, with center the origin $O(0,0)$ and radius $R_0 > 0$.

Note that any solution function $z(t) = (x(t), y(t))$ of (4.1) is supposed to be defined on a maximal interval of existence. By (1.6) and the boundedness of $p(t)$, it is easy to show that global existence occurs that $z(t)$ is defined for all $t \in \mathbb{R}$.

Lemma 4.1. *The solution $z(t) = (x(t), y(t))$ of (4.1) is defined on the whole t-axis.*

Proof. Define a function $W(x,y) : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ as

$$W(x,y) = \frac{1}{2}y^2 + G(x) - G_{\min} + 1,$$

where $G_{\min} = \min\{G(x) : x \in \mathbb{R}\}$. It follows that $W(x,y) > \frac{1}{2}y^2 + \frac{1}{2} > 0$ for every $(x,y) \in \mathbb{R}^2$, and

$$\lim_{x^2+y^2 \to +\infty} W(x,y) = +\infty. \tag{4.3}$$

We set $W(t) \doteq W(z(t)) = W(x(t),y(t))$ with $W(t_0) = W_0 = W(z_0) = W(x_0,y_0)$. When $x(t) \neq 0$ for every $t \in \mathbb{R}$, consider the derivative of W along the solution $z(t) = (x(t),y(t))$ of (4.1) one has that

$$\left|\frac{dW(t)}{dt}\right| = |y(t)p(t)| \leq E|y(t)| \leq EW(t).$$

Using the elementary differential inequalities we have that

$$W(t) \leq W_0 e^{E|t-t_0|},$$

for t in a maximal interval containing t_0. Further we have proved that $W(z(t))$ is bounded for t in any bounded interval. And then there is no blow-up at finite time and the global existence is proved. □

Lemma 4.2. *There exists $R_0 > 0$ such that when $r > R_0$, we have that $\frac{d\theta(t)}{dt} < 0$ for all $t \in [0, 2\pi]$.*

Proof. By (1.6), there exists $N > 0$ sufficiently large such that

$$g_+(x) > \overline{E}, \quad x \geq N, \qquad g_-(x) < \underline{E}, \quad x \leq -N,$$

where $\overline{E}$ and $\underline{E}$ come from (2.2). Let $R_0 > N$. When $r > R_0$ and $|r\cos\theta| \geq N$ one has that

$$\frac{d\theta(t)}{dt} = -\sin^2\theta - \frac{[g_\pm(r\cos\theta) - p(t)]\cos^2\theta}{r\cos\theta} < 0.$$

While for $r > R_0$ and $|r\cos\theta| < N$ with $r\cos\theta \neq 0$, one has that

$$\frac{d\theta(t)}{dt} < -\sin^2\theta + \frac{|p(t) - g_\pm(r\cos\theta)|}{r} < -\frac{R_0^2 - N^2 - R_0\delta}{R_0^2},$$

where $\delta = E + \max_{|x|\leq N}|g_\pm(x)|$ with E being from (2.2). Further, $\frac{d\theta(t)}{dt} < 0$ for $t \in [0, 2\pi]$ as long as $R_0 > \frac{1}{2}(\delta + \sqrt{4N^2 + \delta^2})$. □

Lemma 4.2 implies that there exists $R_0 > 0$ sufficiently large such that the regular orbits of (4.1) rotate clockwise outside of $\mathbb{S}_{R_0}$ as t $(\in [0, 2\pi])$ increases.

Let $|w| = |(a,b)| \doteq \sqrt{a^2+b^2}$, we denote the usual Euclidean norm of the point $w = (a,b) \in \mathbb{R}^2$. Next we prove that any solution $z(t) = (x(t),y(t))$ of (4.1) possesses an elastic property as follows.

Lemma 4.3. *For each $\alpha > 0$, there is $\beta = \beta(\alpha) > 0$ such that $|z(0)| \geq \beta$ implies that*

$$|z(t)| > \alpha, \qquad t \in [0, 2\pi].$$

Proof. Let $W(x,y) = \frac{1}{2}y^2 + G(x)$, where $G(x)$ is shown as in (3.3). By (1.6), there exists $X > 0$ such that $G(x) > 0$ for $|x| > X$. Further, $W(x,y) > 0$ for $|x| > X$ and (4.3) holds. We set $G = \min_{x\in\mathbb{R}}\{G(x)\} \le 0$. Consider any solution $z(t;z_0) \doteq (x(t;x_0,y_0), y(t;x_0,y_0))$ of (4.1) passing through $z_0 = (x_0,y_0)$, and we denote by $W(t) \doteq W(x(t;x_0,y_0), y(t;x_0,y_0))$ satisfying $W(0) \doteq W_0 = W(x_0,y_0)$. When $x(t) \neq 0$, along the solution of (4.1) one has that

$$\frac{dW(t)}{dt} = y(t)p(t) \ge -E\sqrt{2(W(t)-G)},$$

where E is from (2.2). Further

$$\sqrt{2(W(t)-G)} \ge \sqrt{2(W_0-G)} - Et, \qquad t \in [0,2\pi].$$

Let $\gamma(\alpha) = \max\{W(x,y) : x^2+y^2 \le \alpha^2\}$. By (4.3), there exists $\beta = \beta(\alpha)(>\alpha)$ sufficiently large such that for $x^2+y^2 \ge \beta^2$ then

$$2(W(x,y)-G) > (\sqrt{2(\gamma(\alpha)-G)} + 2E\pi)^2.$$

Hence $x_0^2+y_0^2 \ge \beta^2$, we have that

$$\begin{aligned}\sqrt{2(W(x(t;x_0,y_0),y(t;x_0,y_0))-G)} &\ge \sqrt{2(W_0-G)} - Et\\ &> \sqrt{2(\gamma(\alpha)-G)} + 2E\pi - Et \ge \sqrt{2(\gamma(\alpha)-G)},\ t\in[0,2\pi].\end{aligned}$$

This implies that $x^2(t;x_0,y_0) + y^2(t;x_0,y_0) > \alpha^2$ for $t \in [0,2\pi]$. $\square$

Lemmas 4.2–4.3 imply that there exists $\alpha > 0$ sufficiently large such that the regular orbits of (4.1) rotate clockwise in an annulus as t $(\in [0,2\pi])$ increases.

Consider any solution $(x(t),y(t))$ of (4.1) with the initial point $(x(0),y(0)) = (x_0,y_0) \in \Sigma_-$. In the polar coordinates, the solution is denoted by $(r(t),\theta(t))$ satisfying $(r_0,\theta_0) \doteq (r(0),\theta(0))$. By Lemma 4.2, the corresponding regular orbit rotates in a clockwise fashion in Σ_-. Assume that it reaches Σ_0^+ at $t = t_0(>0)$, that is $(x(t_0),y(t_0)) = (0,y_{t_0}) \in \Sigma_0^+$. Next we present a result, and the discussion for $(x_0,y_0) \in \Sigma_+$ is the same and so omitted here.

Lemma 4.4. *There is $\epsilon_0 > 0$ such that the solution of (4.1) with $(x(t_0+\epsilon_0), y(t_0+\epsilon_0)) \in \Sigma_+$ satisfies*

$$\lim_{\epsilon_0\to 0^+}(x(t_0+\epsilon_0), y(t_0+\epsilon_0)) = (0,y_{t_0}).$$

Proof. By (4.2), the solution $(x(t_0+\epsilon_0), y(t_0+\epsilon_0)) \in \Sigma_+$ satisfies

$$\int_{\frac{\pi}{2}}^{\theta(t_0+\epsilon_0)} \frac{d\theta}{-\sin^2\theta + \frac{1}{r}[p(t)-g_+(r\cos\theta)]\cos\theta} = \int_{t_0}^{t_0+\epsilon_0} dt = \epsilon_0, \tag{4.4}$$

$$\int_{y_{t_0}}^{r(t_0+\epsilon_0)} \frac{dr}{r\cos\theta\sin\theta + [p(t)-g_+(r\cos\theta)]\sin\theta} = \int_{t_0}^{t_0+\epsilon_0} dt = \epsilon_0. \tag{4.5}$$

We rewrite (4.4) as the form

$$\int_{\frac{\pi}{2}}^{\theta(t_0+\epsilon_0)} \frac{d\theta}{-\sin^2\theta + \frac{1}{r(\theta)}[p(t)-g_+(r(\theta)\cos\theta)]\cos\theta} = \epsilon_0.$$

Further, when ϵ_0 is small enough, (4.4) becomes

$$\int_{\frac{\pi}{2}}^{\theta(t_0+\epsilon_0)} \frac{d\theta}{-\sin^2\theta} = \epsilon_0. \tag{4.6}$$

It is easy to find $\theta^*(t_0+\epsilon_0)$ such that (4.6) holds. And on (r,θ) coordinate plane, $\theta^*(r,t_0+\epsilon_0)$ is a continuous curve close to $\theta = \frac{\pi}{2}$ for a given ϵ_0.

Similarly, we can find the corresponding $r^*(\theta, t_0+\epsilon_0)$ such that (4.5) holds. And on (r,θ) coordinate plane, $r^*(\theta)$ is a continuous curve near $r = y_{t_0}$ for a given ϵ_0.

Put $\theta^*(r,t_0+\epsilon_0)$ and $r^*(\theta,t_0+\epsilon_0)$ into the (r,θ) coordinate plane for the same given ϵ_0. Then there is at least one intersection point $(r^*(t_0+\epsilon_0), \theta^*(t_0+\epsilon_0))$. And the solution of (4.4)–(4.5) with this intersection point as an initial value is the result. □

The above lemmas prompt us to provide the following definition, which can be used to define the Poincaré mapping of (4.1) on the plane.

Definition 4.5. For each $(0,y) \in \Sigma_0^+$, let $(0,-p_R(y)) \in \Sigma_0^-$ be the first intersection point of the regular orbit starting from $(0,y)$ and Σ_0. Then the planar mapping $P_R : (0,y) \to (0,-p_R(y))$ is called as a right Poincaré mapping. Similarly, for each $(0,-y) \in \Sigma_0^-$ let $(0,p_L(y)) \in \Sigma_0^+$ be the first intersection point of the regular orbit starting from $(0,-y)$ and Σ_0, and we call $P_L : (0,-y) \to (0,p_L(y))$ as a left Poincaré mapping.

By the continuous dependence of solutions on the initial values, P_L and P_R are continuous. Consequently, the component functions p_L and p_R are also continuous. For a regular orbit of (4.1) starting from a given point $(0,y) \in \Sigma_0^+$, by Definition 4.5 it intersects with Σ_0^- at $(0,-p_R(y))$ and Σ_0^+ then at $(0,p_L(p_R(y)))$. Clearly the regular orbit is periodic if and only if $y = p_L(p_R(y))$.

Further, the Poincaré mapping $\mathcal{P}$ of (4.1) is well defined as follows

$$\begin{aligned} \mathcal{P}(\doteq P_L \circ P_R) &: \mathbb{R}^2 \setminus \{O\} \to \mathbb{R}^2 \setminus \{O\}, \\ (x_0,y_0) &\to (x(2\pi;x_0,y_0), y(2\pi;x_0,y_0)), \\ (r_0,\theta_0) &\to (r(2\pi;r_0,\theta_0), \theta(2\pi;r_0,\theta_0)), \end{aligned}$$

where $(x_0,y_0) \neq (0,0)$ and $(r_0,\theta_0) \neq (0,0)$. Moreover, note that $\mathcal{P}$ is continuous, and fixed points of $\mathcal{P}$ correspond to the initial points of the regular periodic solutions.

Based on the above several lemmas, we now give the proofs of Theorems 1.1–1.4.

Proof of Theorem 1.1. By $(H3)$, there exists $A \geq A_1$ such that

$$g_+(x) > \overline{E}, \quad x \geq A, \qquad g_-(x) < \underline{E}, \quad x \leq -A.$$

By Lemmas 4.2–4.3, there exists $R_0 > 0$ large enough such that regular orbits of (4.1) rotate clockwise outside $\mathbb{S}_{R_0}(\doteq \{(x,y) : x^2+y^2 = R_0^2\})$. We consider any regular orbit $L_{M_0}(\subset \{(x,y) : x^2+y^2 > R_0^2\})$ which starts from $M_0(r_0,\theta_0)$ at $t=0$, where $r_0 = r(0)$ and $\theta_0 = \theta(0)$. Assume that $R_0 > A$, $M_0 \in \{(x,y) : x > A\}$, and L_{M_0} successively intersects with $\{x = A\}, \Sigma_0^-, \{x=-A\}, \{x=-A\}, \Sigma_0^+, \{x=A\}$ and $\{\theta = \theta_0\}$ at points $M_1, M_2, M_3, M_4, M_5, M_6$ and M_7 (see Figure 4.1), and the corresponding moments and arguments are denoted by $\theta_i = \theta(t_i)$ for $i = 1,2,3,4,5,6,7$. We next estimate the time of L_{M_0} rotating clockwise a circle around $\mathbb{S}_{R_0}$ when $R_0 > 0$ is sufficiently large.

According to Lemmas 4.2–4.3, we choose $\varepsilon > 0$ small sufficiently and $R_0 = \frac{E}{\varepsilon} > A$. Let

$$\psi_\pm = \sin^2\theta + l_\pm \cos^2\theta - \varepsilon,$$
$$\varphi_- = \sin^2\theta + L_- \cos^2\theta + \varepsilon.$$

Then $\psi_\pm > 0$ and $\varphi_- > 0$ when ε is sufficiently small. By the second equality of (4.2),

$$-\varphi_- < \theta' < -\psi_-, \qquad x \le -A,$$
$$\theta' < -\psi_+, \qquad x \ge A,$$

where $\theta' = \frac{d\theta(t)}{dt}$. When $x \ge A$, one has that

$$t_1 = \int_{\theta_0}^{\theta_1} \frac{d\theta}{\theta'} < \int_{\theta_1}^{\theta_0} \frac{d\theta}{\psi_+},$$
$$t_7 - t_6 = \int_{\theta_6}^{\theta_7} \frac{d\theta}{\theta'} < \int_{\theta_7}^{\theta_6} \frac{d\theta}{\psi_+} = \int_{\theta_7+2\pi}^{\theta_6+2\pi} \frac{d\theta}{\psi_+}.$$

Since $\theta_7 + 2\pi = \theta_0$, it follows that

$$t_7 - t_6 + t_1 < \int_{\theta_1}^{\theta_6+2\pi} \frac{d\theta}{\psi_+} < \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{d\theta}{\psi_+} = \frac{\pi}{\sqrt{(l_+ - \varepsilon)(1-\varepsilon)}}.$$

Similarly, for $x \le -A$ then

$$t_4 - t_3 < \int_{-\frac{3\pi}{2}}^{-\frac{\pi}{2}} \frac{d\theta}{\psi_-} = \frac{\pi}{\sqrt{(l_- - \varepsilon)(1-\varepsilon)}}.$$

While for $|x| \le A$, it follows that

$$|\theta'| \ge \sin^2\theta - \frac{|g_\pm(x)| + |p(t)|}{r}$$
$$\ge \frac{R_0^2 - A^2}{R_0^2} - \frac{\delta}{R_0} = 1 - \frac{\delta\varepsilon}{E} - \frac{A^2\varepsilon}{E^2},$$

where $\delta = E + \max_{|x|\le A} |g_\pm(x)|$. Note that the time T that L_{M_0} rotates a circle around $\mathbb{S}_{R_0}$ satisfies

$$T < \frac{\pi}{\sqrt{(l_- - \varepsilon)(1-\varepsilon)}} + \frac{\pi}{\sqrt{(l_+ - \varepsilon)(1-\varepsilon)}} + \left(\int_{\theta_1}^{\theta_2} + \int_{\theta_2}^{\theta_3} + \int_{\theta_4}^{\theta_5} + \int_{\theta_5}^{\theta_6}\right) \frac{d\theta}{\theta'}.$$

It is obvious that $1 - \frac{\delta\varepsilon}{E} - \frac{A^2\varepsilon}{E^2} \to 1$ as $\varepsilon \to 0$. So when $0 < \varepsilon$ is sufficiently small,

$$\left|\left(\int_{\theta_1}^{\theta_2} + \int_{\theta_2}^{\theta_3} + \int_{\theta_4}^{\theta_5} + \int_{\theta_5}^{\theta_6}\right) \frac{d\theta}{\theta'}\right| \le \left(\int_{\theta_1}^{\theta_2} + \int_{\theta_2}^{\theta_3} + \int_{\theta_4}^{\theta_5} + \int_{\theta_5}^{\theta_6}\right) \frac{d\theta}{|\theta'|}$$
$$\le \left(\int_{\theta_1}^{\theta_2} + \int_{\theta_2}^{\theta_3} + \int_{\theta_4}^{\theta_5} + \int_{\theta_5}^{\theta_6}\right) \frac{d\theta}{1 - \frac{\delta\varepsilon}{E} - \frac{A^2\varepsilon}{E^2}} \le \frac{4\sigma}{1 - \frac{\delta\varepsilon}{E} - \frac{A^2\varepsilon}{E^2}},$$

where σ is an angle shown as in Figure 4.1. Since $R_0 = \frac{E}{\varepsilon} \to +\infty$ as $\varepsilon \to 0$, and then $\sigma \to 0$. Hence when R_0 is large sufficiently (*i.e.* $\varepsilon \to 0$), it follows from (1.5) that

$$T < \frac{2\pi}{m}.$$

On the other hand, when $x \le -A$ it follows from $\theta' \ge -\varphi_-$ that

$$T > t_4 - t_3 > \int_{\theta_3}^{\theta_4} \frac{d\theta}{\theta'} > \int_{\theta_4}^{\theta_3} \frac{d\theta}{\varphi_-} = \int_{-\frac{3\pi}{2}}^{-\frac{\pi}{2}} \frac{d\theta}{\varphi_-} - \left(\int_{-\frac{3\pi}{2}}^{\theta_4} + \int_{\theta_3}^{-\frac{\pi}{2}} \right) \frac{d\theta}{\varphi_-}$$
$$= \frac{\pi}{\sqrt{(L_- + \varepsilon)(1+\varepsilon)}} - \left(\int_{-\frac{3\pi}{2}}^{\theta_4} + \int_{\theta_3}^{-\frac{\pi}{2}} \right) \frac{d\theta}{\varphi_-},$$

and

$$\left| \left(\int_{-\frac{3\pi}{2}}^{\theta_4} + \int_{\theta_3}^{-\frac{\pi}{2}} \right) \frac{d\theta}{\varphi_-} \right| \le \frac{2\sigma}{\min\{1, L_-\}} \to 0 \quad \text{as } \varepsilon \to 0.$$

In conclusion, when $R_0 > 0$ is large sufficiently we have that

$$\frac{2\pi}{m+1} < T < \frac{2\pi}{m}.$$

This implies that the number of rotation of any regular orbit of (4.1) in $[0, 2\pi]$ is greater than m but less than $m+1$. Hence by Theorem 2.1, (4.1) has at least one 2π crossing periodic solution. The proof is complete. □

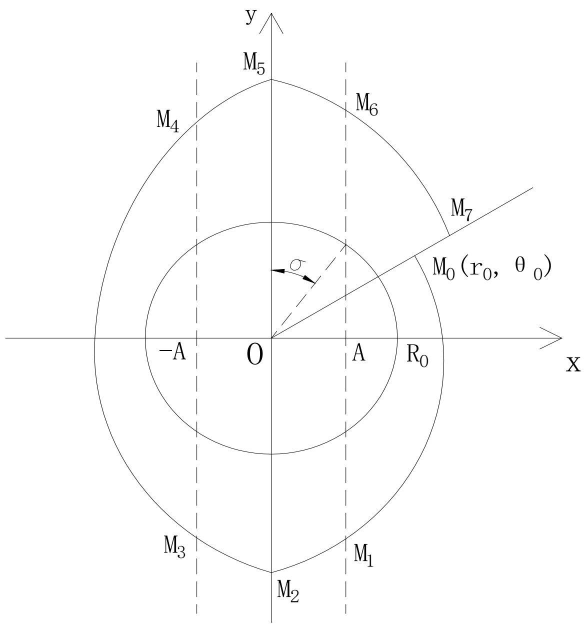

Figure 4.1: Schematic diagram

Proof of Theorem 1.2. By $(H3)'$, there is $A(> A_1)$ large enough such that

$$\theta' \doteq \frac{d\theta}{dt} > -(\sin^2\theta + L_- \cos^2\theta + \varepsilon), \qquad x \le -A.$$

Further, one has that

$$T > \int_{\theta_3}^{\theta_4} \frac{d\theta}{\theta'} > \int_{\theta_4}^{\theta_3} \frac{d\theta}{\sin^2\theta + L_-\cos^2\theta + \varepsilon}$$
$$= \int_{-\frac{3\pi}{2}}^{-\frac{\pi}{2}} \frac{d\theta}{\sin^2\theta + L_-\cos^2\theta + \varepsilon} - \left(\int_{-\frac{3\pi}{2}}^{\theta_4} + \int_{\theta_3}^{-\frac{\pi}{2}} \right) \frac{d\theta}{\sin^2\theta + L_-\cos^2\theta + \varepsilon},$$

and

$$\left|\left(\int_{-\frac{3\pi}{2}}^{\theta_4}+\int_{\theta_3}^{-\frac{\pi}{2}}\right)\frac{d\theta}{\sin^2\theta+L_-\cos^2\theta+\varepsilon}\right|\le\frac{2\sigma}{L_-},$$

where σ is the same as in the proof of Theorem 1.1. Further, by $0<L_-<\frac{1}{4}$, when $R_0>0$ is sufficiently large one has that

$$T>\frac{\pi}{\sqrt{(L_-+\varepsilon)(1+\varepsilon)}}>2\pi.$$

This implies that the number of rotation for any regular orbit rotating clockwise outside $\mathbb{S}_{R_0}$ in $[0,2\pi]$ is less than one. Hence by Theorem 2.1, the conclusion holds. □

Proof of Theorem 1.3. By $\lim\limits_{x\to+\infty}\dfrac{g_+(x)}{x}=+\infty$, there are $A_1(>A)$ and $l_+>0$ large enough such that

$$\frac{g_+(x)}{x}\ge l_+,\qquad x\ge A_1.$$

When $m\ge 1$, by (1.7) we can find certain arbitrary large l_+ such that

$$\frac{1}{\sqrt{l_-}}+\frac{1}{\sqrt{l_+}}<\frac{2}{m}.$$

Hence all conditions in Theorem 1.1 hold, and when $m=0$ it is Theorem 1.2. □

Proof of Theorem 1.4. Let $(x_1(t),y_1(t))$ and $(x_2(t),y_2(t))$ be any two 2π crossing periodic solutions, and denote by

$$u(t)=x_1(t)-x_2(t),\qquad v(t)=y_1(t)-y_2(t).$$

When $x_i(t)\ne 0$ for $i=1,2$, one has that

$$\begin{aligned}\frac{du(t)}{dt}&=v(t),\\ \frac{dv(t)}{dt}&=-[g_\pm(x_1(t))-g_\pm(x_2(t))],\end{aligned}\tag{4.7}$$

where $g_\pm(x_1(t))-g_\pm(x_2(t))$ are 2π-periodic. Making a polar coordinates transformation

$$u(t)=r(t)\cos\theta(t),\qquad v(t)=r(t)\sin\theta(t),$$

then (4.7) is transformed into the form

$$\begin{aligned}\frac{dr(t)}{dt}&=r\sin\theta\cos\theta-[g_\pm(x_1(t))-g_\pm(x_2(t))]\sin\theta,\\ \frac{d\theta(t)}{dt}&=-\sin^2\theta-\frac{1}{r}[g_\pm(x_1(t))-g_\pm(x_2(t))]\cos\theta.\end{aligned}\tag{4.8}$$

Let $(r(t;\theta_0,r_0),\theta(t;\theta_0,r_0))$ be any regular solution of (4.8) with the initial value condition $(\theta_0,r_0)\ne(0,0)$. Then $r(t)>0$ for $t\in[0,2\pi]$. Note that

$$0<\frac{1}{r}[g_\pm(x_1(t))-g_\pm(x_2(t))]\cos\theta=\frac{g_\pm(x_1(t))-g_\pm(x_2(t))}{x_1(t)-x_2(t)}\cos^2\theta<\cos^2\theta.$$

So we have that

$$-2\pi<\theta(2\pi)-\theta(0)=\int_{\theta(0)}^{\theta(2\pi)}\theta'dt<0.$$

Further (4.7) has no nontrivial 2π crossing periodic solution, and then $u(t)\equiv 0$. □

5 An example

Let $g_+(x) = e^x + c_1$ and $g_-(x) = ax - c_2$ in (4.1), where $c_1 \geq 0, c_2 > 0, a \in (\frac{1}{3}, \frac{1}{2})$. Choosing $l_1 = \frac{1}{3}, L_- = \frac{1}{2}$ and $A_1 > 0$ sufficiently large. It is easy to verify $(H1)$–$(H3)$ as follows.

$(H1)$ $g_+ \in C^1([0, +\infty), \mathbb{R})$, $g_- \in C^1((-\infty, 0], \mathbb{R})$, and

$$\lim_{x \to 0^-} (ax - c_2) = -c_2 < 0, \qquad \lim_{x \to 0^+} (e^x + c_1) = 1 + c_1 > 0.$$

$(H2)$ $g_-(x) < 0$ for $x < 0$ and $g_+(x) > 0$ for $x > 0$, and $\lim_{x \to -\infty}(ax - c_2) = -\infty$, $\lim_{x \to +\infty}(e^x + c_1) = +\infty$.

$(H3)$ When $A_1 > 0$ is large sufficiently, there holds

$$\frac{1}{3} < \lim_{x \to -\infty} \frac{g_-(x)}{x} < \frac{1}{2}, \qquad \lim_{x \to +\infty} \frac{g_+(x)}{x} = +\infty.$$

Moreover, we can choose l_+ large arbitrary such that when $m = 1$ we have that

$$1 < \frac{1}{\sqrt{L_-}} \leq \frac{1}{\sqrt{l_-}} + \frac{1}{\sqrt{l_+}} < 2.$$

Hence by using Theorem 1.1, it follows that (4.1) has at least one 2π crossing periodic solution.

6 Concluding remarks

In this paper, we have studied the existence and uniqueness of crossing periodic solutions for a mixed-type second order discontinuous Duffing equation (1.2), where $g_\pm$ satisfy some asymmetric growth conditions for $x \in \mathbb{R}$. However in the existing literatures [2, 10, 12, 13], the authors considered the periodic solution problem of discontinuous symmetric Duffing equation. In [2, 12], the authors considered $g_\pm$ being some specific functions and obtained the existence of periodic solutions. In [2], $g_\pm = x(1 - \frac{1}{\sqrt{a^2+x^2}})$ with $a = 0$ being a discontinuous case; in [12] $g_\pm = \eta \operatorname{sgn}(y)$ with $\eta \in \mathbb{R}, \eta > 0$. While in [10, 13], by the Poincaré–Birkhoff theorem, the authors studied the multiplicity of periodic solutions with large-amplitude. So to some sense, our results are improvement and generalization of the ones in the literatures [2, 10, 12, 13].

We first presented the definitions of regular solutions and crossing periodic solutions (or orbits) of (1.2), and state the existence and uniqueness of any regular solution to initial value problem of (1.2) exception for the origin. Next, we analyzed the behavior of regular solutions (or orbits) for the autonomous and non-autonomous systems of (1.2), respectively. After some technical lemmas, by giving the definition of left and right Poincaré mappings, we constructed the Poincaré mapping of (1.2) on the whole plane. Then by using the Poincaré–Bohl theorem (*i.e*, Theorem 2.1), we found a 2π crossing periodic solution. And several corollaries are presented by giving different asymmetric linear growth conditions to $g_\pm$ as $|x| \to +\infty$. Next, to demonstrate the effectiveness of the obtained results, we further provided an example. Finally, a uniqueness criterion of the crossing periodic solution was also given when $g_\pm$ satisfy a kind of Lipschitz condition on the pieces of the nonlinearity.

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