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Existence and nonexistence of positive solutions of nonlinear elliptic equations involving singular term and convection term

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Abstract. This paper establishes the existence and nonexistence of positive solutions for a class of Dirichlet problems involving singular nonlinearities and gradient-dependent terms. The nonlinearities considered are subcritical in the Sobolev sense. Our approach is based on an approximation scheme.

Keywords: quasilinear elliptic equation, positive solution, singular nonlinearity, dependence on the gradient, convection term.

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1 Introduction

Let $\Omega \subset \mathbb{R}^N$ be a bounded domain with a C^2-boundary, $N \geq 2$, and $0 < \alpha < 1$. In this paper, we investigate the existence of at least one positive solution to the following singular elliptic equation with Dirichlet boundary conditions, driven by the p-Laplacian operator $-\Delta_p u = -\mathrm{div}\left(|\nabla u|^{p-2}\nabla u\right)$, with $1 < p < \infty$:

$$\begin{cases} -\Delta_p u = \lambda \left(\dfrac{1}{u^\alpha} + |\nabla u|^{r_2} \right) + f(x,u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where $0 < \lambda < \lambda^*$ is a parameter, r_2 is a constant such that $0 < r_2 < p-1$, and f is a continuous nonlinearity satisfying

$$0 \leq f(x,t) \leq a_2 |t|^{r_1}, \tag{1.2}$$

with constants $a_2 > 0$ and $r_1 \in (0, p-1) \cup (p-1, p^* - 1)$.

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We denote by p^* the critical Sobolev exponent, that is, $p^* = \frac{Np}{N-p}$ if $1 < p < N$, and $p^* \in (p, \infty)$ if $p = N$.

In recent years, singular elliptic equations have posed considerable analytical challenges. A vast body of literature has been devoted to such problems, particularly from a theoretical standpoint. For example, the following problem has been extensively investigated:

$$\begin{cases} -\Delta_p u = \eta(x) u^{-\alpha} & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.3}$$

where $\eta(x) \geq 0$ in Ω and $0 < \alpha < 1$. Canino, Sciunzi, and Trombetta [6] established the existence and uniqueness of solutions to (1.3) for $p \neq 2$ under various configurations. For the particular case $p = 2$, see, for instance, [14, 20, 27].

Giacomoni, Schindler, and Takáč [13] studied the existence and multiplicity of weak solutions for the following p-Laplacian equation:

$$\begin{cases} -\Delta_p u = \lambda u^{-\delta} + u^q & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

where $1 < p < \infty$, $p - 1 < q \leq p^* - 1$, $\lambda > 0$, and $0 < \delta < 1$.

Further results concerning singular problems without convection terms can be found in [3, 5, 12, 25] and the references therein.

Elliptic problems involving convection terms have also attracted substantial attention, as they introduce additional analytical difficulties. For instance, Francesca and Puglisi [11], employing the subsolution–supersolution method, truncation techniques, nonlinear regularity theory, the Leray–Schauder alternative principle, and tools from set-valued analysis, proved the existence of a positive solution for the problem,

$$\begin{cases} -\Delta u = f(x, u(x), \nabla u(x)) + g(x, u(x)) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

where $f : \Omega \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ satisfies an appropriate growth condition, and the semilinear function $g : \Omega \times (0, \infty) \to \mathbb{R}$ is singular at $s = 0$, that is,

$$\lim_{s \to 0^+} g(x, s) = +\infty.$$

It is worth noting that condition (1.2) in the present paper is considerably less restrictive than the assumptions imposed on f in [11] (see also [23] and the references therein).

Faria, Miyagaki, and Motreanu [10] established the existence of a positive solution for the following quasilinear elliptic problem involving the (p, q)-Laplacian and a convection term, by means of the Galerkin method and a Schauder basis:

$$\begin{cases} -\Delta_p u - \mu \Delta_q u = f(x, u, \nabla u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

Additional works concerning problems with convection terms can be found in [4, 26, 31] and the references therein.

Beyond the study of variational and nonvariational elliptic problems, the analytical framework of this paper is also connected to broader developments in partial differential equations with irregular coefficients and singular terms. Ragusa [30] proved existence and regularity results for elliptic boundary value problems under vanishing mean oscillation (VMO) assumptions on the coefficients, highlighting how solvability can be preserved even in the absence of smoothness. Polidoro and Ragusa [28] extended these ideas to degenerate ultraparabolic equations with singular lower-order terms, establishing a Harnack inequality and proving the existence of a Green's function for hypoelliptic operators. Kozhanov and Shipina [18] examined loaded elliptic equations and linear inverse problems, where source terms must be simultaneously recovered with the solution itself. Guariglia [15] developed a fractional-calculus framework for the Lerch zeta function, contributing to the analytic study of operators with nonlocal and fractional structures. The work of Guariglia and Silvestrov [16] successfully bridges harmonic analysis with fractional differential operators in complex domains. They achieved this by introducing a new fractional-wavelet analysis tailored for the spaces of positive definite distributions ($\mathcal{D}'(\mathbb{C})$). These contributions collectively advance the analytic study of nonlocal, fractional, and singular operators relevant to modern elliptic theory.

Unlike the cited works, which only address singular nonlinearities without gradient dependence or equations with regular lower-order terms, our study is unique because it simultaneously incorporates a singular term of the form $u^{-\alpha}$ and a convection term involving $|\nabla u|^{r_2}$. This interaction destroys the problem's variational structure and introduces substantial analytical challenges, in obtaining uniform estimates and ensuring the positivity of weak solutions. To address these difficulties, we develop an approximation scheme based on Galerkin approximations using a Schauder basis and non-variational *a priori* bounds, and pass to the limit to obtain existence results. This strategy not only circumvents the lack of a variational framework but also extends the existing theory to a broader class of nonlinear singular elliptic problems. A key contribution of this study is the proof of existence for an equation with a singular nonlinearity where the exponent r_1 of $|u|$ satisfies $r_1 < p^* - 1$.

We now state the main results of the paper.

Theorem 1.1. *Assume that f is a continuous function satisfying* (1.2). *Then there exists a constant $\lambda^* > 0$ such that, for every $\lambda \in (0, \lambda^*)$, problem* (1.1) *admits at least one positive solution $u_\lambda \in W_0^{1,p}(\Omega)$.*

Proposition 1.2. *Assume that $f(t) = |t|^{r_1}$, where $r_1 \in (p-1, p^*-1)$, and define*

$$\lambda^* = \sup\{\lambda > 0 : (1.1) \textit{ admits a solution } u_\lambda \textit{ in } W_0^{1,p}(\Omega)\}.$$

Then $\lambda^* < \infty$.

The paper is organized as follows. In Section 2, we introduce an auxiliary problem and apply the Galerkin method to prove the existence of λ^* such that the approximate problem admits at least one positive solution for each $\lambda \in (0, \lambda^*)$ (see Lemma 2.2). In Section 3, we prove Theorem 1.1. Using the *a priori* estimate obtained in Lemma 2.2, we demonstrate that the sequence of solutions to problem (2.1) converges, in an appropriate sense, to the solution of problem (1.1). To establish the positivity of the solution, we employ a comparison principle together with the solution to problem (3.2).

2 Approximation problem

For any $\varepsilon > 0$, we consider the auxiliary problem

$$\begin{cases} -\Delta_p u = \lambda \left(\dfrac{1}{(|u| + \varepsilon)^\alpha} + |\nabla u|^{r_2} \right) + f(x,u) & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{2.1}$$

We begin by proving the existence of a positive solution to problem (2.1) using the Galerkin method. This result will later be used in the proof of Theorem 1.1.

Let $\mathcal{B} = \{e_1, \ldots, e_m, \ldots\}$ be a Schauder basis of $W_0^{1,p}(\Omega)$. For each $m \in \mathbb{N}$, define

$$V_m := [e_1, \ldots, e_m]$$

as the m-dimensional subspace of $W_0^{1,p}(\Omega)$ generated by $\{e_1, \ldots, e_m\}$, equipped with the norm of $W_0^{1,p}(\Omega)$. Let $\xi = (\xi_1, \ldots, \xi_m) \in \mathbb{R}^m$ and observe that

$$|\xi|_m := \left\| \sum_{j=1}^m \xi_j e_j \right\|_{W_0^{1,p}(\Omega)}$$

defines a norm; see [2] for details. We denote by $\mathbb{R}^m$ the m-dimensional space $\mathbb{R}^m$ endowed with this norm. As a consequence of the isometric linear transformation

$$u = \sum_{j=1}^m \xi_j e_j \in W_0^{1,p}(\Omega) \mapsto \xi = (\xi_1, \ldots, \xi_m) \in \mathbb{R}^m,$$

the spaces V_m and $(\mathbb{R}^m, |\cdot|_m)$ can be identified.

The following result is a simple consequence of Brouwer's fixed point theorem; see [1] for its proof. We denote by $|x| = \sqrt{\langle x, x \rangle}$ the usual Euclidean norm in $\mathbb{R}^m$ induced by the inner product $\langle \cdot, \cdot \rangle$.

Lemma 2.1. *Let $F : (\mathbb{R}^m, |\cdot|_m) \to (\mathbb{R}^m, |\cdot|_m)$ be a continuous function such that $\langle F(\xi), \xi \rangle \geq 0$ for every $\xi \in \mathbb{R}^m$ with $|\xi| = R$, for some $R > 0$. Then there exists z_0 in the closed ball $\overline{B}_R^m(0)$ such that $F(z_0) = 0$.*

We now establish the existence of a solution to the auxiliary problem (2.1).

Lemma 2.2. *For any $\varepsilon > 0$, there exists $\lambda^* > 0$ such that (2.1) admits a weak positive solution $u_\varepsilon \in W_0^{1,p}(\Omega) \cap C^{1,\sigma}(\overline{\Omega})$ for some $0 < \sigma < 1$, for every $\lambda \in (0, \lambda^*)$.*

Proof. Define the function

$$F : \mathbb{R}^m \to \mathbb{R}^m, \qquad F(\xi) = (F_1(\xi), F_2(\xi), \ldots, F_m(\xi)),$$

where, for $j = 1, 2, \ldots, m$,

$$F_j(\xi) = \int_\Omega |\nabla u|^{p-2} \nabla u \nabla e_j - \lambda \left(\int_\Omega \frac{e_j}{(\varepsilon + |u|)^\alpha} + \int_\Omega |\nabla u|^{r_2} e_j \right) - \int_\Omega f(x,u) e_j.$$

It is straightforward to verify that F is continuous.

Since the embedding $W_0^{1,p}(\Omega) \hookrightarrow L^q(\Omega)$ is continuous for $1 \le q \le p^*$, it follows from hypothesis (1.2) and Hölder's inequality that

$$\langle F(\xi), \xi \rangle \ge \|u\|^p_{W_0^{1,p}(\Omega)} - \lambda \left(c_1 \|u\|^{1-\alpha}_{W_0^{1,p}(\Omega)} + c_2 \|u\|^{r_2+1}_{W_0^{1,p}(\Omega)} \right) - c_3 \|u\|^{r_1+1}_{W_0^{1,p}(\Omega)},$$

where c_1, c_2, and c_3 are positive constants independent of m and u.

If $r_1 \in (0, p-1)$, since $r_2 \in (0, p-1)$ and $1 - \alpha < p$, we have that $\langle F(\xi), \xi \rangle$ is coercive; that is, for each $\lambda > 0$, there exists $R = R(\lambda) > 0$ sufficiently large such that $|\xi|_m = \|u\|_{W_0^{1,p}(\Omega)} = R$ implies $\langle F(\xi), \xi \rangle \ge 0$.

If $r_1 \in (p-1, p^*-1)$, assume that $\|u\|_{W_0^{1,p}(\Omega)} = R$ for some $R > 0$ and choose R such that

$$R^p - c_3 R^{r_1+1} \ge \frac{R^p}{2},$$

that is,

$$R \le \left(\frac{1}{2c_3} \right)^{\frac{1}{r_1+1-p}}.$$

Therefore,

$$\langle F(\xi), \xi \rangle \ge \frac{R^p}{2} - \lambda (c_1 R^{1-\alpha} + c_2 R^{r_2+1}) \ge 0 \quad \Longleftrightarrow \quad \lambda \le \frac{R^p}{2(c_1 R^{1-\alpha} + c_2 R^{r_2+1})}.$$

Now, taking $\lambda^* := \frac{R^p}{2(c_1 R^{1-\alpha} + c_2 R^{r_2+1})}$, it follows that $\langle F(\xi), \xi \rangle \ge 0$ for every $\lambda \in (0, \lambda^*)$.

By Lemma 2.1, for every $m \in \mathbb{N}$ there exists $y \in \mathbb{R}^m$, with $|y|_m \le R$, such that $F(y) = 0$. Therefore, there exists $u_m \in V_m$ satisfying

$$\|u_m\|_{W_0^{1,p}(\Omega)} \le R, \quad \text{for every } m \in \mathbb{N},$$

such that, for any $v \in V_m$,

$$\int_\Omega |\nabla u_m|^{p-2} \nabla u_m \nabla v = \lambda \left(\int_\Omega \frac{v}{(\varepsilon + |u_m|)^\alpha} + \int_\Omega |\nabla u_m|^{r_2} v \right) + \int_\Omega f(x, u_m) v. \tag{2.2}$$

Since R does not depend on m, the sequence (u_m) is bounded in $W_0^{1,p}(\Omega)$, and for some subsequence, there exists $u \in W_0^{1,p}(\Omega)$ such that

$$\begin{aligned} u_m &\rightharpoonup u \quad && \text{weakly in } W_0^{1,p}(\Omega), \\ u_m &\to u && \text{strongly in } L^q(\Omega),\ 1 \le q < p^*, \\ u_m &\to u && \text{a.e. in } \Omega, \end{aligned} \tag{2.3}$$

since the embedding $W_0^{1,p}(\Omega) \hookrightarrow L^q(\Omega)$ is compact. Moreover,

$$\|u\|_{W_0^{1,p}(\Omega)} \le \liminf_{m\to\infty} \|u_m\|_{W_0^{1,p}(\Omega)} \le R. \tag{2.4}$$

We claim that

$$u_m \to u \quad \text{in } W_0^{1,p}(\Omega). \tag{2.5}$$

Since $\mathcal{B}$ is a Schauder basis of $W_0^{1,p}(\Omega)$, for every $u \in W_0^{1,p}(\Omega)$ there exists a unique sequence $(\alpha_n)_{n\ge1}$ in $\mathbb{R}$ such that $u = \sum_{j=1}^\infty \alpha_j e_j$. Hence,

$$\phi_m := \sum_{j=1}^m \alpha_j e_j \to u \quad \text{in } W_0^{1,p}(\Omega) \text{ as } m \to \infty. \tag{2.6}$$

It follows from (1.2) that

$$\int_\Omega |f(x,u_m)|^{\frac{r_1+1}{r_1}} \le a_2^{\frac{r_1+1}{r_1}} \int_\Omega |u_m|^{r_1+1} \le C, \quad \forall m \in \mathbb{N}, \tag{2.7}$$

which shows that $(f(x,u_m))_{m\in\mathbb{N}}$ is bounded in $L^{\frac{r_1+1}{r_1}}(\Omega)$.

Taking the test function $(u_m - \phi_m) \in V_m$ in (2.2), and using (1.2), Hölder's inequality, the boundedness of $\|u_m\|_{W_0^{1,p}(\Omega)}$, (2.3), and (2.6), we obtain

$$\begin{aligned}
&\int_\Omega |\nabla u_m|^{p-2}\nabla u_m \nabla(u_m-\phi_m)\\
&\quad = \lambda\left(\int_\Omega \frac{(u_m-\phi_m)}{(\varepsilon+|u_m|)^\alpha} + \int_\Omega |\nabla u_m|^{r_2}(u_m-\phi_m)\right) + \int_\Omega f(x,u_m)(u_m-\phi_m)\\
&\quad \le \frac{\lambda}{\varepsilon^\alpha}\|u_m-\phi_m\|_{L^1(\Omega)} + \lambda\tilde{c}_1\|u_m\|^{r_2}_{W_0^{1,p}(\Omega)}\|u_m-\phi_m\|_{L^{r_2+1}(\Omega)}\\
&\qquad + \|f(\cdot,u_m)\|_{L^{\frac{r_1+1}{r_1}}(\Omega)}\|u_m-\phi_m\|_{L^{r_1+1}(\Omega)} \to 0 \quad \text{as } m\to\infty,
\end{aligned}$$

because $r_1+1 < p^*$. Thus, we obtain

$$\lim_{m\to\infty}\int_\Omega |\nabla u_m|^{p-2}\nabla u_m\nabla(u_m-u) = 0.$$

Now, it suffices to apply the (S_+) property of $-\Delta_p$ (see [24, Proposition 3.5]) to deduce (2.5).

Let $k\in\mathbb{N}$. Then, for every $m \ge k$ and every $v_k \in V_k$, we have

$$\int_\Omega |\nabla u_m|^{p-2}\nabla u_m\nabla v_k = \lambda\left(\int_\Omega \frac{v_k}{(\varepsilon+|u_m|)^\alpha} + \int_\Omega |\nabla u_m|^{r_2}v_k\right) + \int_\Omega f(x,u_m)v_k.$$

Note that for $x,y\in\mathbb{R}^N$ there exists a positive constant C_p, depending only on p, such that

$$\left|\,|x|^{p-2}x - |y|^{p-2}y\,\right| \le C_p\left(|x|+|y|\right)^{p-2}|x-y|, \tag{2.8}$$

see [9, Lemma 2.1].

Thus, Hölder's inequality and (2.8) yield

$$\begin{aligned}
&\left|\int_\Omega (|\nabla u_m|^{p-2}\nabla u_m - |\nabla u|^{p-2}\nabla u)\nabla v_k\right|\\
&\quad \le \int_\Omega \left||\nabla u_m|^{p-2}\nabla u_m - |\nabla u|^{p-2}\nabla u\right||\nabla v_k|\\
&\quad \le C_p\int_\Omega (|\nabla u_m|+|\nabla u|)^{p-2}|\nabla(u_m-u)||\nabla v_k|\\
&\quad \le C_p\left(\|\nabla u_m\|_{L^p(\Omega)} + \|\nabla u\|_{L^p(\Omega)}\right)^{p-2}\|\nabla(u_m-u)\|_{L^p(\Omega)}\|\nabla v_k\|_{L^p(\Omega)}\\
&\quad \to 0 \quad \text{as } m\to\infty.
\end{aligned}$$

Therefore,

$$\int_\Omega |\nabla u_m|^{p-2}\nabla u_m\nabla v_k \to \int_\Omega |\nabla u|^{p-2}\nabla u\nabla v_k. \tag{2.9}$$

Analogously, we have

$$\int_\Omega |\nabla u_m|^{r_2} v_k \to \int_\Omega |\nabla u|^{r_2} v_k. \tag{2.10}$$

Since (2.7) implies that $(f(x,u_m))_{m\in\mathbb{N}}$ is bounded in $L^{\frac{r_1+1}{r_1}}(\Omega)$, and since $u_m \to u$ a.e. in Ω, we obtain $f(x,u_m) \to f(x,u)$ a.e. in Ω. Therefore, by [17, Theorem 13.44], we have

$$f(x,u_m) \rightharpoonup f(x,u) \text{ in } L^{\frac{r_1+1}{r_1}}(\Omega).$$

Hence, for all $v_k \in V_k \subset W_0^{1,p}(\Omega)$, it follows that

$$\int_\Omega f(x,u_m) v_k \to \int_\Omega f(x,u) v_k. \tag{2.11}$$

Thus, (2.5), (2.9), (2.10), and (2.11) imply that for every $v_k \in V_k$ it holds that

$$\int_\Omega |\nabla u|^{p-2} \nabla u \nabla v_k = \lambda \left(\int_\Omega \frac{v_k}{(\varepsilon + |u|)^\alpha} + \int_\Omega |\nabla u|^{r_2} v_k \right) + \int_\Omega f(x,u) v_k.$$

By the density of $[V_k]_{k\in\mathbb{N}}$ in $W_0^{1,p}(\Omega)$, we conclude that for all $v \in W_0^{1,p}(\Omega)$,

$$\int_\Omega |\nabla u|^{p-2} \nabla u \nabla v = \lambda \left(\int_\Omega \frac{v}{(\varepsilon + |u|)^\alpha} + \int_\Omega |\nabla u|^{r_2} v \right) + \int_\Omega f(x,u) v.$$

Furthermore, since $u^- \in W_0^{1,p}(\Omega)$, we conclude that

$$\int_\Omega |\nabla u|^{p-2} \nabla u \nabla u^- = \lambda \left(\int_\Omega \frac{u^-}{(\varepsilon + |u|)^\alpha} + \int_\Omega |\nabla u|^{r_2} u^- \right) + \int_\Omega f(x,u) u^-,$$

which implies that

$$-\|u^-\|_{W_0^{1,p}(\Omega)} = \lambda \left(\int_{\Omega/\{u(x)>0\}} \frac{u^-}{(\varepsilon + |u|)^\alpha} + \int_{\Omega/\{u(x)>0\}} |\nabla u|^{r_2} u^- \right) \geq 0,$$

and we obtain that $u^- \equiv 0$ a.e. in Ω. Therefore, for every $v \in W_0^{1,p}(\Omega)$, it holds that

$$\int_\Omega |\nabla u|^{p-2} \nabla u \nabla v = \lambda \left(\int_\Omega \frac{v}{(\varepsilon + u)^\alpha} + \int_\Omega |\nabla u|^{r_2} v \right) + \int_\Omega f(x,u) v.$$

The equation in (2.1) ensures that $u \not\equiv 0$. By [19, Theorem 7.1], we infer that $u \in L^\infty(\Omega)$. Moreover, the regularity result up to the boundary in [21, Theorem 1] guarantees that $u \in C^{1,\sigma}(\overline{\Omega})$ for some $\sigma \in (0,1)$. By applying the strong maximum principle [29, Theorem 5.4.1], we conclude that $u > 0$ in Ω, thereby proving that u is a positive solution of problem (2.1). □

3 The main results

In order to prove Theorem 1.1, we recall the following results.

Lemma 3.1. *Let f be a non-negative continuous function such that $f(u)$ is nonincreasing for $u \in (0,\infty)$, where $1 < p$. Assume that $u, v \in W^{1,p}(\Omega)$ are weak solutions such that $u \geq v$ in $\partial\Omega$ and*

$$\begin{cases} -\Delta_p u \geq f(u), & u > 0 \text{ in } \Omega, \\ -\Delta_p v \leq f(v), & v > 0 \text{ in } \Omega. \end{cases}$$

Then $u \geq v$ in Ω.

The proof of Lemma 3.1 can be found in [8, Proposition 2.3].

Let $d_\Omega : \Omega \to \mathbb{R}_+$ be the distance function to the boundary defined by

$$d_\Omega(x) = \inf_{y\in\partial\Omega} |x-y|.$$

The next result can be found in [20, Lemma 1].

Lemma 3.2. *Suppose that $\Omega \subset \mathbb{R}^N$ is a domain of class C^k for some $k \geq 1$. Then*

$$\int_\Omega d_\Omega(x)^{-\sigma} < \infty \quad \text{for all} \quad 0 < \sigma < 1.$$

We denote by λ_1 and ϕ_1 the first eigenpair of the p-Laplacian, that is,

$$\begin{cases} -\Delta_p \phi_1 = \lambda_1 \phi_1^{p-1} & \text{in } \Omega \\ \phi_1 = 0 & \text{on } \partial\Omega, \end{cases} \tag{3.1}$$

with $\phi_1 > 0$ satisfying $\|\phi_1\|_\infty = 1$. Note that the strong maximum and boundary point principles (see [32, Theorem 5]) guarantee that $\phi_1 > 0$ in Ω and $\frac{\partial \phi_1}{\partial \nu} < 0$ on $\partial\Omega$, respectively. Hence, since $\phi_1 \in C^1(\bar{\Omega})$, there are constants ℓ and $L, 0 < \ell < L$, such that

$$\ell d_\Omega(x) \leq \phi_1(x) \leq L d_\Omega(x)$$

for all $x \in \Omega$.

Let us consider $u_0 \in W_0^{1,p}(\Omega) \cap C^{1,\beta}(\bar{\Omega})$ (where $0 < \beta < 1$), the only solution to the p-Laplacian equation

$$\begin{cases} -\Delta_p u_0 = u_0^{-\alpha} & \text{in } \Omega \\ u_0 > 0 & \text{in } \Omega \\ u_0 = 0 & \text{on } \partial\Omega, \end{cases} \tag{3.2}$$

where $0 < \alpha < 1$. A more complete study of problem (3.2) can be found in [6, Theorem 1.3].

The solution of problem (3.2) combined with Lemma 3.1 will guarantee the positivity of the solution to problem (1.1).

3.1 Proof of Theorem 1.1.

Lemma 2.2 guarantees that for each $\lambda \in (0, \lambda^*)$ and $n \in \mathbb{N}$, we obtain a solution $u_n \in C^{1,\sigma}(\bar{\Omega})$ of the p-Laplacian equation

$$\begin{cases} -\Delta_p u_n = \lambda \left(\dfrac{1}{(u_n + \frac{1}{n})^\alpha} + |\nabla u_n|^{r_2} \right) + f(x, u_n) & \text{in } \Omega, \\ u_n > 0 & \text{in } \Omega, \\ u_n = 0 & \text{on } \partial\Omega. \end{cases} \tag{3.3}$$

That is, for all $v \in W_0^{1,p}(\Omega)$ we have

$$\int_\Omega |\nabla u_n|^{p-2} \nabla u_n \nabla v = \lambda \left(\int_\Omega \frac{v}{(\frac{1}{n} + u_n)^\alpha} + \int_\Omega |\nabla u_n|^{r_2} v \right) + \int_\Omega f(x, u_n) v. \tag{3.4}$$

In particular, equality (3.4) holds for all $v \in C_0^1(\Omega)$.

It follows from (2.4) that (u_n) is a bounded sequence. Thus, by the arguments used to obtain (2.5), we can extract a subsequence, still denoted by (u_n), such that

$$u_n \to u \quad \text{in } W_0^{1,p}(\Omega). \tag{3.5}$$

We have that u_n satisfies

$$\begin{aligned} -\Delta_p\left(u_n + \frac{1}{n}\right) &= -\Delta_p u_n \geq \frac{\lambda}{\left(u_n + \frac{1}{n}\right)^{\alpha}}, \\ \left(u_n + \frac{1}{n}\right)\Big|_{\partial\Omega} &> 0. \end{aligned}$$

Let $w = \lambda^{\frac{1}{p-1-\alpha}} u_0$, where u_0 is the unique solution of (3.2). Then w satisfies the p-Laplacian equation

$$\begin{cases} -\Delta_p w = \dfrac{\lambda}{w^{\alpha}} & \text{in } \Omega, \\ w > 0 & \text{in } \Omega, \\ w = 0 & \text{on } \partial\Omega. \end{cases} \tag{3.6}$$

It follows that $u_n + \frac{1}{n}$ is a supersolution of (3.6). By Lemma 3.1, we obtain

$$u_n + \frac{1}{n} \geq w.$$

Let ϕ_1 be the solution of (3.1). For each $\beta \in \left(0, \lambda_1^{-\frac{1}{p-1-\alpha}}\right)$, we have that $\beta\phi_1$ satisfies

$$\begin{cases} -\Delta_p(\beta\phi_1) \leq \dfrac{1}{(\beta\phi_1)^{\alpha}} & \text{in } \Omega, \\ \beta\phi_1 > 0 & \text{in } \Omega, \\ \beta\phi_1 = 0 & \text{on } \partial\Omega. \end{cases}$$

In particular, $\beta\phi_1$ is a subsolution of (3.2). By Lemma 3.1, we have $u_0 \geq \beta\phi_1$. Since there exists a constant $\ell > 0$ such that $\ell\, d_\Omega \leq \phi_1$, we obtain

$$u_0 \geq k\, d_\Omega, \tag{3.7}$$

where $k = \ell\beta$. Therefore, from (3.7) we obtain

$$w = \lambda^{\frac{1}{p-1-\alpha}} u_0 \geq k_1\, d_\Omega, \tag{3.8}$$

where $k_1 = k\lambda^{\frac{1}{p-1-\alpha}}$. Since (3.8) implies that

$$u_n + \frac{1}{n} \geq w \geq k_1 d_\Omega,$$

it follows that

$$\frac{|v|}{\left(u_n + \frac{1}{n}\right)^{\alpha}} \leq \frac{|v|}{(k_1 d_\Omega)^{\alpha}}, \quad \forall\, v \in C_0^1(\Omega). \tag{3.9}$$

Thus, by Lemma 3.2, we conclude that $\frac{v}{(k_1 d_\Omega)^\alpha}$ is integrable.

Since $u_n + \frac{1}{n} \to u$ a.e. in Ω and

$$\frac{v}{(\frac{1}{n} + u_n)^\alpha} \to \frac{v}{u^\alpha} \quad \text{a.e. in } \Omega \text{ as } n \to +\infty,$$

the Lebesgue dominated convergence theorem yields

$$\int_\Omega \frac{v}{(\frac{1}{n} + u_n)^\alpha} \to \int_\Omega \frac{v}{u^\alpha} \quad \text{as } n \to +\infty. \tag{3.10}$$

From (3.5), (3.10), and the fact that u_n satisfies (3.4), together with the density of $C_0^1(\Omega)$ in $W_0^{1,p}(\Omega)$, letting $n \to +\infty$ we conclude that

$$\int_\Omega |\nabla u|^{p-2} \nabla u \nabla v = \lambda \left(\int_\Omega \frac{v}{u^\alpha} + \int_\Omega |\nabla u|^{r_2} v \right) + \int_\Omega f(x,u)v, \quad \forall v \in W_0^{1,p}(\Omega).$$

Finally, to conclude the proof, we observe that (3.9) and $u_n + \frac{1}{n} \to u$ imply

$$u \geq k_1 d_\Omega > 0 \quad \text{a.e. in } \Omega.$$

□

3.2 Proof of Proposition 1.2

In this proof, we adopt some ideas from [7]. For the sake of contradiction, suppose that $\lambda^* = \infty$. Thus, there exists a sequence $\lambda_n \to \infty$ and corresponding solutions $u_{\lambda_n} > 0$ in Ω, given by Theorem 1.1. Define

$$P_1(t) = \lambda t^{-\alpha} + t^{r_1}.$$

We claim that there exists a constant $C_\lambda > 0$ such that

$$P_1(t) \geq C_\lambda t^{p-1} \quad \text{for every } t > 0.$$

Indeed, define the function $Q(t) = P_1(t)\, t^{-(p-1)}$. Since $p - 1 < r_1$, we have $Q(t) \to \infty$ as $t \to 0^+$ and as $t \to \infty$. The minimum value is $Q(t_1) = C_\lambda$, where $t_1 > 0$ is the unique solution of

$$Q'(t) = 0,$$

that is,

$$t_1 = \left[\frac{\lambda(p-1-\alpha)}{r_1 - (p-1)} \right]^{\frac{1}{r_1+\alpha}}.$$

Consequently, by the argument above, $t_1 > 0$ is the unique minimizer of Q, with $Q(t_1) = C_\lambda$.

Let $\sigma_1 > 0$ and $\varphi_1 > 0$ denote, respectively, the principal eigenvalue and the corresponding eigenfunction of the eigenvalue problem

$$\begin{cases} -\Delta_p \varphi_1 = \sigma_1 |\varphi_1|^{p-2} \varphi_1 & \text{in } \Omega, \\ \varphi_1 = 0 & \text{on } \partial\Omega. \end{cases}$$

Since C_λ increases as λ_n increases (because $r_1 > p - 1$), for $\delta > 0$ small enough, there exists λ_0 in this sequence such that the corresponding constant satisfies $C_{\lambda_0} \geq \sigma_1 + \delta$. Hence, the solution u_{λ_0} of (1.1) associated with λ_0 satisfies $u_{\lambda_0} > 0$ in Ω and

$$\begin{cases} -\Delta_p u_{\lambda_0} \geq C_{\lambda_0} |u_{\lambda_0}|^{p-2} u_{\lambda_0} \geq (\sigma_1 + \delta) |u_{\lambda_0}|^{p-2} u_{\lambda_0} & \text{in } \Omega, \\ u_{\lambda_0} \geq 0 & \text{on } \partial\Omega. \end{cases}$$

Otherwise, taking $\varepsilon > 0$ small enough, we obtain $\varepsilon\varphi_1 < u_{\lambda_0}$ in Ω and

$$\begin{cases} -\Delta_p(\varepsilon\varphi_1) = \sigma_1|\varepsilon\varphi_1|^{p-2}(\varepsilon\varphi_1) \leq (\sigma_1+\delta)|\varepsilon\varphi_1|^{p-2}(\varepsilon\varphi_1) & \text{in } \Omega, \\ \varphi_1 = 0 & \text{on } \partial\Omega. \end{cases}$$

By the subsolution and supersolution method (see [7, Theorem 2.1]), there exists a solution $\varepsilon\varphi_1 \leq \omega \leq u_{\lambda_0}$ in Ω of

$$\begin{cases} -\Delta_p\omega = (\sigma_1+\delta)|\omega|^{p-2}\omega & \text{in } \Omega, \\ \omega = 0 & \text{on } \partial\Omega. \end{cases}$$

Hence, we reach a contradiction to the fact that σ_1 is isolated (see [22]). Therefore, $\lambda^* < \infty$.

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