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Double phase problems with supercritical and sublinear growth

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Abstract. In this work, our objective is to study the operators involving the nonstandard growth via variational methods. We first study a double phase problem having supercritical growth via minimization technique on convex sets. Alongside, the case of competing (p, q) Laplacian problems with sublinear growth also has been studied by adopting the variational principle developed for double phase problems.

Keywords: supercritical, sublinear, double phase problems, variational principle, (p, q)-Laplacian, competing Laplacian.

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1 Introduction

In this paper, our objective is to study the operators with unbalanced growth of (p, q) type. We first study the problems involving double phase operator and supercritical growth in the sense of Sobolev embedding. Precisely, we establish an existence result for the following problem

$$\begin{cases} -\Delta_p u - \operatorname{div}(a(x)|\nabla u|^{q-2}\nabla u) = |u|^{r-2}u + \mu|u|^{s-2}u & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{P_μ}$$

where $\Omega \subset \mathbb{R}^N$ is a bounded domain with Lipschitz boundary, $N \geq 2, 1 < s < p < q < r, N > p$ and $0 \leq a(x) \in L^\infty(\Omega)$. Note that there is no control on r from above and hence r can take values greater than $p^* = \frac{Np}{N-p}$ which is a critical exponent for the compact embeddings of Musielak–Orlicz spaces (see Proposition 2.15 in [5]). The operator involved in the problem (P_μ) is known as double phase operator. The behaviour of this operator switches between two different elliptic situations depending on the values of the weight function $a : \bar{\Omega} \to [0, \infty)$. In other words, the operator is called as double phase operator as on the set $\{x \in \Omega : a(x) = 0\}$ it will be controlled by the gradient of order p and in $\{x \in \Omega : a(x) \neq 0\}$ it is the gradient of order q.

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Firstly, Zhikov [21] introduced such classes of operators in order to describe models of strongly anisotropic materials, see also [8,22,23] and the monograph [24]. The energy functional corresponding to this operator is

$$u \mapsto \int_{\Omega} (|\nabla u|^p + a(x)|\nabla u|^q)\, dx,$$

where the integrand $\mathcal{H}(x,\xi) := |\xi|^p + a(x)|\xi|^q$ for all $(x,\xi) \in \Omega \times \mathbb{R}^N$ has unbounded growth, i.e.

$$|\xi|^p \leq H(x,\xi) \leq b(1+|\xi|^q) \quad \text{for a.e. } x \in \Omega \text{ and for all } \xi \in \mathbb{R}^N,$$

with $b > 0$. This functional is also related to non-Newtonian fluids, see [11,14,19]. Due to unbalanced growth of the integrand, the appropriate space to deal of this class of problems is generalized Orlicz and generalized Orlicz–Sobolev spaces, see [5,11,18].

The problem (P_μ) has concave-convex non-linearity. In this direction, a very first work has been done by Ambrosetti, Brézis and Cerami in [1] where authors have proved the existence of at least two positive solutions with the non-linearity $u^p + \lambda u^q$ satisfying $0 < q < 1 < p$. The result in [1] received a lot of attention because there was no control on p from the above. There are numerous results, when $p \leq 2^*$, the existence of infinitely many solutions was established for an appropriate choice of λ. We refer interested readers to see [2,4,6,10] also for concave-convex problems with supercritical growth in bounded domains for the existence of infinitely many solutions. In [3] authors discussed the mixed local-nonlocal operator with concave-convex nonlinearities of sublinear and critical growth and are interested in proving one or more positive weak solutions. To show an existence result for the problem (P_μ), we used a variational principle discussed in [15].

Secondly, following the variational principle developed for (P_μ) we show the existence of a solution for a sublinear problem having a competing Laplacian operators. We consider the following problem with homogeneous Dirichlet boundary and sublinear growth condition

$$\begin{cases} -\Delta u + \Delta_q u = \lambda |u|^{s-2}u & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{P_λ}$$

where $\Omega \subset \mathbb{R}^N$ is a bounded domain with Lipschitz boundary, $N \geq 3$ and $1 < s < 2 < q$. Note that the left hand side of the (P_λ) is in the divergence form $-\Delta u + \Delta_q u = -\text{div}(a(|\nabla u|))$ with $a(t) = t^2 - t^q$ for all $t > 0$. The ellipticity condition for an operator in divergence form $-\text{div}(a(|\nabla u|))$ is to have $a(t) > 0$ for all $t > 0$ which is not satisfied in the case of $a(t) = t^2 - t^q$. Thus the ellipticity is lost for the operator in problem (P_λ). For this reason, we call the operator $-\Delta u + \Delta_q u$ as competing $(2,q)$ Laplacian operator. There is not a very rich literature available for this class of problems. The competing $(2,q)$ Laplacian operator with concave-convex type nonlinearity involving supercritical growth is discussed in [16]. Interested readers can find the works of [12,17] dealing with general competing (p,q) Laplacian operator with $1 < p < q$.

1.1 Main results

In this paper, we are dealing with operators having nonstandard growth with two types of nonlinearities: supercritical and sublinear. We prove the following result related to the problem (P_μ).

Theorem 1.1. *Assume that $1 < s < p < q < r$. Then there exists $\mu^* > 0$ such that for each $\mu \in (0,\mu^*)$, problem (P_μ) has at least one non-negative solution with negative energy.*

Finally, the main result concerning problem (P_λ) is the following.

Theorem 1.2. *There exists a* $\lambda^* > 0$ *such that the problem* (P_λ) *has a non-negative solution for* $0 < \lambda < \lambda^*$.

The outline of the paper is as follows. In Section 2, we give a formal introduction of the appropriate spaces to study the problem (P_μ), which are known as Musielak–Orlicz spaces along with an abstract variational principle on closed convex sets and apriori estimate for the weak solutions of the problem (P_μ). In Section 3, we establish Theorem 1.1 by proving some technical lemma. We conclude the paper by studying a competing Laplacian problem in Section 4.

2 Musielak–Orlicz spaces and variational principle

In this section, we recall the main properties of the Musielak–Orlicz spaces along with their suitable embeddings. We also establish the variational setup for the problem (P_μ). To this end, we suppose that $\Omega \subset \mathbb{R}^N$, $N \geq 2$, is a bounded domain with Lipschitz boundary $\partial\Omega$. For any $s \in [1, \infty)$, we denote by $L^s(\Omega)$ and $W_0^{1,s}(\Omega)$ the usual Lebesgue and Sobolev spaces with the norm $\|\cdot\|_s$ and $\|\cdot\|_{1,s}$, respectively. Moreover, the Sobolev space $W_0^{1,s}(\Omega)$ is equipped with an equivalent norm as $\|\nabla \cdot\|_s$.

Recall that the function $\mathcal{H}\colon \Omega \times [0, \infty) \to [0, \infty)$, involving concave-convex terms, is defined as

$$\mathcal{H}(x, t) = t^p + a(x)t^q.$$

Define the space $L^{\mathcal{H}}(\Omega)$ by

$$L^{\mathcal{H}}(\Omega) = \{u\colon \Omega \to \mathbb{R} \text{ measurable function} : \varrho_{\mathcal{H}}(u) < \infty\}$$

and supplied with the Luxemburg norm

$$\|u\|_{\mathcal{H}} = \inf\left\{\tau > 0 : \varrho_{\mathcal{H}}\left(\frac{u}{\tau}\right) \leq 1\right\},$$

where the function $\varrho_{\mathcal{H}}$ is given by

$$\varrho_{\mathcal{H}}(u) = \int_\Omega \mathcal{H}(x, |u|)\, dx = \int_\Omega \left(|u|^p + a(x)|u|^q\right) dx.$$

Furthermore, the space $W^{1,\mathcal{H}}(\Omega)$ is defined as

$$W^{1,\mathcal{H}}(\Omega) = \left\{u \in L^{\mathcal{H}}(\Omega) : |\nabla u| \in L^{\mathcal{H}}(\Omega)\right\}$$

equipped with the norm

$$\|u\|_{1,\mathcal{H}} = \|\nabla u\|_{\mathcal{H}} + \|u\|_{\mathcal{H}},$$

where $\|\nabla u\|_{\mathcal{H}} = \|\, |\nabla u|\, \|_{\mathcal{H}}$. As usual, we denote by $W_0^{1,\mathcal{H}}(\Omega)$ the completion of $C_c^\infty(\Omega)$ in $W^{1,\mathcal{H}}(\Omega)$. From [11], as $1 < p < q$, the norms $\|u\|_{1,\mathcal{H}}$ and $\|\nabla u\|_{\mathcal{H}}$ are equivalent norms on $W_0^{1,\mathcal{H}}(\Omega)$. It is worth recalling (see [5], [11]) that the spaces $L^{\mathcal{H}}(\Omega)$, $W^{1,\mathcal{H}}(\Omega)$ and $W_0^{1,\mathcal{H}}(\Omega)$ are reflexive Banach spaces. We refer the interested readers to see [5],[11] and references therein for the results related with embeddings of Musielak–Orlicz space. Let us recall the following notions of differentiability on Banach spaces.

Definition 2.1 (Gâteaux and Fréchet differentiability)**.** Let X and Y be Banach spaces and let $F : X \to Y$. We say that F is **Gâteaux differentiable** at $x \in X$ if there exists a linear map $DF(x) : X \to Y$ such that

$$\lim_{t\to 0} \frac{F(x+th) - F(x)}{t} = DF(x)[h], \quad \forall\, h \in X.$$

The functional F is **Fréchet differentiable** at x if there exists a bounded linear operator $DF(x)$ satisfying

$$\lim_{\|h\|_X \to 0} \frac{\|F(x+h) - F(x) - DF(x)[h]\|_Y}{\|h\|_X} = 0.$$

Remark 2.2. We say F is a class of $C^1(X, \mathbb{R})$ if and only if F is Fréchet differentiable on X and $DF : X \to X^*$ is continuous.

Now consider the Banach space $\mathcal{W} = W_0^{1,\mathcal{H}}(\Omega) \cap L^r(\Omega)$ equipped with the following norm

$$\|u\|_{\mathcal{W}} = \|u\|_{W_0^{1,\mathcal{H}}} + \|u\|_{L^r(\Omega)}.$$

Next we define the weak solution for the problem (P_μ).

Definition 2.3 (Weak solution)**.** We say $u \in \mathcal{W}$ is a weak solution to problem (P_μ), if

$$\int_\Omega \left(|\nabla u|^{p-2}\nabla u \nabla v + a(x)|\nabla u|^{q-2}\nabla u \nabla v\right)\, dx = \int_\Omega \left(|u|^{r-2}uv + \mu|u|^{s-2}uv\right)\, dx$$

for all $v \in \mathcal{W}$.

Let $J : \mathcal{W} \to \mathbb{R}$ be the Euler–Lagrange functional corresponding to (P_μ),

$$J(u) = \frac{1}{p}\int_\Omega |\nabla u|^p\, dx + \frac{1}{q}\int_\Omega a(x)|\nabla u|^q\, dx - \frac{1}{r}\int_\Omega |u|^r\, dx - \frac{\mu}{s}\int_\Omega |u|^s\, dx.$$

Define $\Phi : \mathcal{W} \to \mathbb{R}$ by

$$\Phi(u) = \frac{1}{r}\int_\Omega |u|^r\, dx + \frac{\mu}{s}\int_\Omega |u|^s\, dx,$$

where $\Phi \in C^1(\mathcal{W}, \mathbb{R})$ and

$$\langle D\Phi(u), v\rangle = \int_\Omega |u|^{r-2}uv\, dx + \mu\int_\Omega |u|^{s-2}uv\, dx, \tag{2.1}$$

where $\langle \cdot, \cdot \rangle$ is duality pairing between $\mathcal{W}$ and its dual. Define the function $\Psi : \mathcal{W} \to \mathbb{R}$ by

$$\Psi(u) = \frac{1}{p}\int_\Omega |\nabla u|^p\, dx + \frac{1}{q}\int_\Omega a(x)|\nabla u|^q\, dx,$$

where Ψ is Gâteaux differentiable. The restriction on Ψ to a weakly closed and convex subset K of $\mathcal{W}$ is denoted by Ψ_K and defined by

$$\Psi_K(u) = \begin{cases} \Psi(u) & \text{if } u \in K, \\ +\infty & \text{if } u \notin K. \end{cases} \tag{2.2}$$

Finally, let us introduce the functional $J_K : \mathcal{W} \to (-\infty, +\infty]$ defined by

$$J_K(u) = \Psi_K(u) - \Phi(u).$$

As in [20], we have the following definition of critical points of J_K.

Definition 2.4. A point $u \in \mathcal{W}$ is said to be a critical point of J_K if $J_K(u) \in \mathbb{R}$ and it satisfies the following inequality,

$$\langle D\Phi(u), u - v\rangle + \Psi_K(v) - \Psi_K(u) \geq 0 \quad \forall v \in \mathcal{W}.$$

Using convexity of Ψ_K, it is easy to show the following result (see [20]).

Proposition 2.5. *If J_K is a proper, convex and lower semi continuous functional, then each local minimum is necessarily a critical point of J_K.*

Inspired by [15] and [16], we have the following result.

Theorem 2.6. *Let K be a convex and weakly closed subset of $\mathcal{W}$. If the following two assertions hold*

(i) The functional $J_K : \mathcal{W} \to \mathbb{R} \cup \{+\infty\}$ defined by $J_K(u) = \Psi_K(u) - \Phi(u)$ has a critical point $\bar{u} \in \mathcal{W}$ as in Definition 2.4 and

(ii) there exists $\bar{v} \in K$ such that

$$-\Delta_p \bar{v} - \operatorname{div}(a(x)|\nabla \bar{v}|^{q-2}\nabla \bar{v}) = D\Phi(\bar{u}),$$

in the weak sense and in the sense of (2.1), then $\bar{u} \in K$ is a solution of the equation

$$-\Delta_p u - \operatorname{div}(a(x)|\nabla u|^{q-2}\nabla u) = u|u|^{r-2} + \mu u|u|^{s-2}.$$

Proof. Since $\bar{u}$ is a critical point of $J_K(u)$, it follows from Definition 2.4 that

$$\Psi_K(v) - \Psi_K(\bar{u}) \geq \langle D\Phi(\bar{u}), v - \bar{u}\rangle \quad \forall v \in \mathcal{W},$$

where $\langle D\Phi(\bar{u}), v - \bar{u}\rangle = \int_\Omega \left(|\bar{u}|^{r-2}\bar{u} + \mu|\bar{u}|^{s-2}\bar{u}\right)(v - \bar{u})\, dx$ which leads to

$$\begin{aligned} &\frac{1}{p}\int_\Omega |\nabla v|^p\, dx + \frac{1}{q}\int_\Omega a(x)|\nabla v|^q\, dx - \frac{1}{p}\int_\Omega |\nabla \bar{u}|^p\, dx \\ &\quad - \frac{1}{q}\int_\Omega a(x)|\nabla \bar{u}|^q\, dx \geq \langle D\Phi(\bar{u}), v - \bar{u}\rangle. \end{aligned} \tag{2.3}$$

Now by second assumption of Theorem 2.6, there exists $\bar{v} \in K$ such that

$$\int_\Omega \left(|\nabla \bar{v}|^{p-2}\nabla \bar{v} + a(x)|\nabla \bar{v}|^{q-2}\nabla \bar{v}\right)\nabla h\, dx = \int_\Omega \left(|\bar{u}|^{r-2}\bar{u} + \mu|\bar{u}|^{s-2}\bar{u}\right) h\, dx \qquad \forall h \in \mathcal{W}.$$

Put $h = \bar{v} - \bar{u}$,

$$\int_\Omega \left(|\nabla \bar{v}|^{p-2}\nabla \bar{v} + a(x)|\nabla \bar{v}|^{q-2}\nabla \bar{v}\right)(\nabla \bar{v} - \nabla \bar{u})\, dx = \int_\Omega \left(|\bar{u}|^{r-2}\bar{u} + \mu|\bar{u}|^{s-2}\bar{u}\right)(\bar{v} - \bar{u})\, dx. \tag{2.4}$$

Put $v = \bar{v}$ in (2.3) and combine with (2.4) to obtain

$$\begin{aligned} &\frac{1}{p}\int_\Omega \left(|\nabla \bar{v}|^p - |\nabla \bar{u}|^p\right) dx + \frac{1}{q}\int_\Omega a(x)\left(|\nabla \bar{v}|^q - |\nabla \bar{u}|^q\right) dx \\ &\quad \geq \int_\Omega \left(|\nabla \bar{v}|^{p-2}\nabla \bar{v} + a(x)|\nabla \bar{v}|^{q-2}\nabla \bar{v}\right)(\nabla \bar{v} - \nabla \bar{u})\, dx \end{aligned} \tag{2.5}$$

On the other hand, by the convexity of Ψ, we get

$$\begin{aligned} &\frac{1}{p}\int_\Omega \left(|\nabla \bar{u}|^p - |\nabla \bar{v}|^p\right) dx + \frac{1}{q}\int_\Omega a(x)\left(|\nabla \bar{u}|^q - |\nabla \bar{v}|^q\right) dx \\ &\quad \geq \int_\Omega \left(|\nabla \bar{v}|^{p-2}\nabla \bar{v} + a(x)|\nabla \bar{v}|^{q-2}\nabla \bar{v}\right)(\nabla \bar{u} - \nabla \bar{v})\, dx \end{aligned} \tag{2.6}$$

which by (2.5) and (2.6) results into

$$\begin{aligned}\frac{1}{p}\int_\Omega (|\nabla\bar{v}|^p - |\nabla\bar{u}|^p)\,dx + \frac{1}{q}\int_\Omega a(x)\,(|\nabla\bar{v}|^q - |\nabla\bar{u}|^q)\,dx \\ = \int_\Omega \left(|\nabla\bar{v}|^{p-2}\nabla\bar{v} + a(x)|\nabla\bar{v}|^{q-2}\nabla\bar{v}\right)(\nabla\bar{v} - \nabla\bar{u})\,dx.\end{aligned} \tag{2.7}$$

Note that $|t_1|^\kappa \le |t_2|^\kappa + \kappa\langle |t_1|^{\kappa-2}t_1, t_1 - t_2\rangle$ for $\kappa > 1$ with strict inequality whenever $t_1 \neq t_2$. Suppose $t_1 \neq t_2$ then apply this inequality for $\kappa = p$ and after substituting $t_1 = \nabla\bar{v}$ and $t_2 = \nabla\bar{u}$, we get

$$\frac{1}{p}|\nabla\bar{v}|^p - \frac{1}{p}|\nabla\bar{u}|^p < \langle |\nabla\bar{v}|^{p-2}\nabla\bar{v}, \nabla\bar{v} - \nabla\bar{u}\rangle. \tag{2.8}$$

Similarly, for $\kappa = q$,

$$\frac{1}{q}|\nabla\bar{v}|^q - \frac{1}{q}|\nabla\bar{u}|^q < \langle |\nabla\bar{v}|^{q-2}\nabla\bar{v}, \nabla\bar{v} - \nabla\bar{u}\rangle. \tag{2.9}$$

Since $a(x) \ge 0$, we multiply (2.9) by $a(x)$ and add it to (2.8). This gives,

$$\frac{1}{p}|\nabla\bar{v}|^p - \frac{1}{p}|\nabla\bar{u}|^p + a(x)\left(\frac{1}{q}|\nabla\bar{v}|^q - \frac{1}{q}|\nabla\bar{u}|^q\right) < \langle |\nabla\bar{v}|^{p-2}\nabla\bar{v} + |\nabla\bar{v}|^{q-2}\nabla\bar{v}, \nabla\bar{v} - \nabla\bar{u}\rangle.$$

Thus,

$$\begin{aligned}\frac{1}{p}\int_\Omega (|\nabla\bar{v}|^p - |\nabla\bar{u}|^p)\,dx + \frac{1}{q}\int_\Omega a(x)\,(|\nabla\bar{v}|^q - |\nabla\bar{u}|^q)\,dx \\ \le \int_\Omega \left(|\nabla\bar{v}|^{p-2}\nabla\bar{v} + a(x)|\nabla\bar{v}|^{q-2}\nabla\bar{v}\right)(\nabla\bar{v} - \nabla\bar{u})\,dx.\end{aligned} \tag{2.10}$$

Moreover, (2.10) implies that equality occurs only if $\nabla\bar{v} = \nabla\bar{u}$ a.e. in Ω. Since $\bar{v}, \bar{u} \in W_0^{1,\mathcal{H}}$, it follows that $\bar{v} = \bar{u}$. □

In order to find the weakly closed convex subset K, we need apriori estimates for the weak solutions of the double phase operator. In [8], [13] and Proposition 3.2 of [19], authors have applied the Moser's iteration for the apriori estimate of the weak solutions of double phase operator. Following [9], we present a different proof of the estimate.

Theorem 2.7. *Let $f \in L^\infty(\Omega)$. If $u \in W_0^{1,\mathcal{H}}(\Omega)$ is the weak solution of the problem*

$$\begin{cases} -\Delta_p u - \operatorname{div}(a(x)|\nabla u|^{q-2}\nabla u) = f(x) & \text{in } \Omega, \\ u = 0 & \text{in } \partial\Omega, \end{cases}$$

where $1 < p < q < N$ and $0 \le a(x) \in L^1(\Omega)$, then

$$C_0\|u\|_{L^\infty} \le \|f\|_{L^\infty}^{\frac{1}{p-1}}.$$

Proof. Let $t > 0$ be a fixed real number and define

$$\theta_h(s) = \begin{cases} 0 & 0 < s \le t, \\ \frac{1}{h}(s-t) & t < s < t+h, \\ 1 & t+h \le s, \end{cases}$$

and $\theta_h(-s) = -\theta_h(s)$. Let $\mu(t) = |\{x \in \Omega : |u(x)| > t\}| \neq 0$ and for every $h > 0$, we define $E_h = \{x \in \Omega : t < |u(x)| \leq t + h\}$. We observe that

$$\int_\Omega |\nabla u|^{p-2}\nabla u \nabla\theta_h(u)\,dx \leq \int_\Omega (|\nabla u|^{p-2}\nabla u + a(x)|\nabla u|^{q-2}\nabla u)\nabla\theta_h(u)\,dx \tag{2.11}$$

and

$$\int_\Omega |\nabla u|^{p-2}\nabla u \nabla\theta_h(u)\,dx = \frac{1}{h}\int_{E_h} |\nabla u|^p\,dx = h^{p-1}\int_{E_h} |\nabla\theta_h(u)|^p\,dx. \tag{2.12}$$

Now using Holder's inequality, we have

$$\left(\int_\Omega |\nabla\theta_h(u)|\,dx\right)^p \leq \left(\int_{E_h} |\nabla\theta_h(u)|^p\,dx\right)|E_h|^{p-1}. \tag{2.13}$$

By (2.11), (2.12) and (2.13)

$$\left(\int_\Omega |\nabla\theta_h(u)|\,dx\right)^p \leq \frac{|E_h|^{p-1}}{h^{p-1}}\int_\Omega (|\nabla u|^{p-2}\nabla u + a(x)|\nabla u|^{q-2}\nabla u)\nabla\theta_h(u)\,dx.$$

Since $u \in W_0^{1,\mathcal{H}}(\Omega) \subseteq W_0^{1,1}(\Omega)$, by Sobolev embedding $W^{1,1}(\Omega) \hookrightarrow L^{\frac{N}{N-1}}(\Omega)$, we get

$$\|\theta_h(u)\|^p_{\frac{N}{N-1}} \leq \frac{C(1,N)^p|E_h|^{p-1}}{h^{p-1}}\int_\Omega (|\nabla u|^{p-2}\nabla u + a(x)|\nabla u|^{q-2}\nabla u)\nabla\theta_h(u)\,dx. \tag{2.14}$$

On the other hand,

$$\int_\Omega f(x)\theta_h(u)\,dx \leq \|f\|_{L^\infty}\int_\Omega |\theta_h(u)|\,dx \leq \|f\|_{L^\infty}\mu(t). \tag{2.15}$$

Now combine (2.14) and (2.15),

$$\|\theta_h(u)\|^p_{\frac{N}{N-1}} \leq \frac{C(1,N)^p|E_h|^{p-1}}{h^{p-1}}\|f\|_{L^\infty}\mu(t).$$

Since $\mu(t)$ is a decreasing function, it is differentiable almost everywhere. Hence, by taking the limit as $h \to 0$,

$$\begin{aligned}\mu(t)^{\frac{(N-1)p}{N}} &\leq C(1,N)^p(-\mu'(t))^{p-1}\mu(t)\|f\|_{L^\infty}\\ 1 &\leq C(1,N)^{\frac{p}{p-1}}\mu(t)^{\frac{1}{p-1}(1-\frac{(N-1)p}{N})}(-\mu'(t))\|f\|_{L^\infty}^{\frac{1}{p-1}}.\end{aligned}$$

On integrating both sides from 0 to t, we have

$$t\left(\frac{\frac{p}{(p-1)N}}{C(1,N)^{\frac{p}{p-1}}\|f\|_{L^\infty}^{\frac{1}{p-1}}}\right) - |\Omega|^{\frac{p}{(p-1)N}} \leq -\mu(t)^{\frac{p}{(p-1)N}} \leq 0.$$

This is the case when $\mu(t) \neq 0$. If, for some $t \geq \frac{C(1,N)^{\frac{p}{p-1}}N(p-1)}{p}|\Omega|^{\frac{p}{(p-1)N}}\|f\|_{L^\infty}^{\frac{1}{p-1}}$, then $\mu(t) = 0$. Thus, by definition of $\mu(t)$, we conclude that

$$\|u\|_{L^\infty} \leq C_0\|f\|_{L^\infty}^{\frac{1}{p-1}},$$

where $C_0 = \frac{C(1,N)^{\frac{p}{p-1}}N(p-1)}{p}|\Omega|^{\frac{p}{(p-1)N}}$. □

Now we can define the required closed convex subset $K \subset W_0^{1,\mathcal{H}}$ as,

$$K(R) = \{u \in \mathcal{W} \cap L^\infty(\Omega) : \|u\|_{L^\infty} \leq R\}$$

for some $R > 0$ to be determined later.

Lemma 2.8. *Let $R > 0$. The set*

$$K(R) = \{u \in \mathcal{W} \cap L^\infty(\Omega) : \|u\|_{L^\infty} \leq R\}$$

is weakly closed subset of $\mathcal{W}$.

Proof. In order to show that its weakly closed, we suppose $\{u_j\}_{j=1}^\infty$ be a sequence in K such that

$$u_j \rightharpoonup u \quad \text{weakly in } \mathcal{W} \cap L^\infty(\Omega).$$

Then in particular, $u_j \rightharpoonup u$ weakly in $W_0^{1,\mathcal{H}}$ also. By using compact embedding $W_0^{1,\mathcal{H}} \hookrightarrow\hookrightarrow L^\tau(\Omega)$ for all $\tau \in [p, p^*)$, we get $u_j \to u$ in $L^\tau(\Omega)$. Then, there exists a subsequence of u_j , call it u_j again, such that

$$u_j \to u \quad \text{a.e. in } \Omega.$$

However, $\{u_j\}_{j=1}^\infty \subset K$, so $\|u_j\|_{L^\infty} \leq R$. Further taking $j \to \infty$, we have $\|u\|_{L^\infty} \leq R$. □

3 Proof of Theorem 1.1

In this section, we study the double phase operator with supercritical growth (P_μ). To prove Theorem 1.1, we present the following lemma addressing condition (ii) of Theorem 2.6.

Lemma 3.1. *Let $1 < s < p < q < r$. Then there exists $\mu^* > 0$ and $R > 0$ with $\bar{u} \in K(R)$ we have*

$$-\Delta_p v - \operatorname{div}(a(x)|\nabla v|^{q-2}\nabla v) = \bar{u}|\bar{u}|^{r-2} + \mu\bar{u}|\bar{u}|^{s-2}$$

for some $v \in K(R)$ and for each $\mu \in (0, \mu^*)$.

Proof. Let $g(x) = \bar{u}|\bar{u}|^{r-2} + \mu\bar{u}|\bar{u}|^{s-2}$. Since $\bar{u} \in K(R)$ we have $g \in L^\infty(\Omega)$. Now define

$$Q(u) = \frac{1}{p}\int_\Omega |\nabla u|^p \, dx + \frac{1}{q}\int_\Omega a(x)|\nabla u|^q \, dx - \int_\Omega g(x)u \, dx.$$

The functional Q is well defined on $W_0^{1,\mathcal{H}}(\Omega)$ due to the continuous embedding of $W_0^{1,\mathcal{H}}(\Omega)$ into $L^1(\Omega)$ and admits its infimum at some $v \in W_0^{1,\mathcal{H}}(\Omega)$ which satisfies

$$-\Delta_p v - \operatorname{div}(a(x)|\nabla v|^{q-2}\nabla v) = g(x) = D\Phi(\bar{u})$$

in the weak sense. Since right hand side is in $L^\infty(\Omega)$, by Theorem 2.7, we have

$$\|v\|_{L^\infty(\Omega)} \leq C_0\|g\|_{L^\infty(\Omega)}^{\frac{1}{p-1}} \leq C_0\left(R^{r-1} + \mu R^{s-1}\right)^{\frac{1}{p-1}}.$$

Since $p - 1 < r - 1$, choose R small enough such that $R^{r-1} \leq \frac{1}{2}(\frac{R}{C_0})^{p-1}$. For this chosen R, we can choose μ^* such that $\mu^* = \frac{R^{p-s}}{2C_0^{p-1}}$ which gives $\mu^* R^{s-1} = \frac{R^{p-1}}{2C_0^{p-1}}$. Thus we have $R^{r-1} + \mu R^{s-1} < (C_0 R)^{p-1}$ for each $\mu \in (0, \mu^*)$. This completes the proof. □

3.1 Proof of Theorem 1.1

Now we sketch the proof of Theorem 1.1 in the following two steps. First assume that R and μ^* be as in Lemma 3.1 with $\mu \in (0, \mu^*)$. Set $K = K(R)$.

Step 1: In this step, we show that there exists $\bar{u} \in K$ such that $J_K(\bar{u}) = \inf_{u \in \mathcal{W}} J_K(u)$ then by Proposition 2.5, $\bar{u}$ is a critical point of J_K. Denote $\eta := \inf_{u \in \mathcal{W}} J_K(u)$. Using definition of $\Psi_K(u)$ for every $u \notin K$ we have $J_K(u) = +\infty$ and therefore $\eta = \inf_K J_K(u)$. For every $u \in K$, we have

$$\Phi(u) = \frac{1}{r}\int_\Omega |u|^r\,dx + \frac{\mu}{s}\int_\Omega |u|^s\,dx \leq \frac{R^{r-1}}{r}\int_\Omega |u|\,dx + \frac{\mu R^{s-1}}{s}\int_\Omega |u|\,dx.$$

By the compact embedding of $W_0^{1,\mathcal{H}}(\Omega)$ into $L^1(\Omega)$ (see Proposition 2.15 and Proposition 2.18 in [5]), we obtain

$$\Phi(u) \leq C_1 \|u\|_{W_0^{1,\mathcal{H}}(\Omega)}$$

where $C_1 = C\left(\frac{R^{r-1}}{r} + \frac{\mu R^{s-1}}{s}\right)$ and $C > 0$ is the constant for the embedding $W_0^{1,\mathcal{H}}(\Omega)$ into $L^1(\Omega)$. Thus for $u \in K$, we have

$$J_K(u) = \Psi_K(u) - \Phi(u) \geq \frac{1}{p}\|u\|_{W_0^{1,\mathcal{H}}(\Omega)} - C_1\|u\|_{W_0^{1,\mathcal{H}}(\Omega)} \tag{3.1}$$

from which we obtain that $\eta > -\infty$. Now suppose that $\{u_n\}$ is sequence in K such that $J_K(u_n) \to \eta$. It follows from (3.1) and the definition of set K that the sequence $\{u_n\}$ is bounded in $W_0^{1,\mathcal{H}}(\Omega) \cap L^\infty(\Omega)$. Passing to a subsequence if necessary there is $\bar{u} \in W_0^{1,\mathcal{H}}(\Omega)$ such that $u_n \rightharpoonup \bar{u}$ in $W_0^{1,\mathcal{H}}(\Omega)$, $u_n \to \bar{u}$ a.e. $x \in \Omega$ which implies that $\bar{u} \in L^\infty(\Omega)$ with $\|\bar{u}\|_{L^\infty(\Omega)} \leq R$. As a consequence of $\bar{u} \in K$, we now show that $\Phi(u_n) \to \Phi(\bar{u})$ in $\mathbb{R}$. We have

$$\frac{1}{r}|u_n|^r + \frac{\mu}{s}|u_n|^s \leq \frac{R^{r-1}}{r}|u_n| + \frac{\mu R^{s-1}}{s}|u_n|$$

by strong convergence in $L^1(\Omega)$ because of the compact embedding $W_0^{1,\mathcal{H}}(\Omega) \hookrightarrow L^1(\Omega)$ and the dominated convergence theorem, we obtain that

$$\Phi(u_n) \to \Phi(\bar{u}).$$

Since Ψ_K is lower semi continuous and $J_K = \Psi_K - \Phi$, it follows that $J_K(\bar{u}) \leq \liminf_{n\to\infty} J_K(u_n)$. So, $J_K(\bar{u}) = \eta = \inf_{u \in \mathcal{W}} J_K(u)$. As $J_K(\bar{u}) = J_K(|\bar{u}|)$, we can assume $\bar{u} \geq 0$.

Step 2: In this step, we show that there exists $v \in K$ such that

$$-\Delta_p v - \operatorname{div}(a(x)|\nabla v|^{q-2}\nabla v) = \bar{u}|\bar{u}|^{r-2} + \mu\bar{u}|\bar{u}|^{s-2}.$$

Using Lemma 3.1 together with the fact that $\bar{u} \in K$, we obtain that $v \in K$. It now follows from Theorem 2.6 together with Step 1 and Step 2 that $\bar{u}$ is a non-negative solution of (P_μ). To complete the proof, we show that $\bar{u}$ is non-trivial. In fact, for $e \in K$ and $t \in [0,1]$, we have that $te \in K$ and therefore, $J_K(te) < 0$ for sufficiently small t as $1 < s < p < q < r$. Thus, we can conclude that $J_K(\bar{u}) < 0$. Consequently, $\bar{u}$ is a non-trivial and non-negative solution of (P_μ).

4 Competing Laplacian problem

In this section, we study the competing Laplacian problem (P_λ). To prove Theorem 1.2, we adopt the variational principle developed in Theorem 2.6.

Let $\mathcal{V} = W_0^{1,2}(\Omega) \cap W^{1,q}(\Omega)$ and define the Euler–Lagrange functional $\mathcal{I} : \mathcal{V} \to \mathbb{R}$ associated to problem (P_λ), as

$$\mathcal{I}(u) = \frac{1}{2}\int_\Omega |\nabla u|^2\, dx - \frac{1}{q}\int_\Omega |\nabla u|^q\, dx - \frac{\lambda}{s}\int_\Omega |u|^s\, dx.$$

Next we define $\phi, \varphi : \mathcal{V} \to \mathbb{R}$, as

$$\phi(u) = \frac{1}{q}\int_\Omega |\nabla u|^q\, dx + \frac{\lambda}{s}\int_\Omega |u|^s\, dx \ \text{ and } \varphi(u) = \frac{1}{2}\int_\Omega |\nabla u|^2\, dx,$$

where $\phi \in C^1(\mathcal{V}, \mathbb{R})$ and φ is the proper convex and lower semi-continuous functional. In addition, the restriction of φ on a subset K of $\mathcal{V}$ is denoted as $\varphi_K(u)$ and defined similar to (2.2). Since $\phi \in C^1(\mathcal{V}, \mathbb{R})$, then we have

$$\langle D\phi(u), v\rangle = \int_\Omega |\nabla u|^{q-2}\nabla u \nabla v\, dx + \lambda\int_\Omega |u|^{s-2}uv\, dx. \tag{4.1}$$

Let $K := K(r)$ be the convex and weakly closed subset of $\mathcal{V}$ defined by

$$K(r) = \left\{ u \in \mathcal{V} : \|u\|_{W^{2,t}(\Omega)} \leq r \right\},$$

where $t > N$ and for some $r > 0$ to be determined later. Define $\mathcal{I}_K(u) : \mathcal{V} \to (-\infty, +\infty]$ as $\mathcal{I}_K(u) = \varphi_K(u) - \phi(u)$. In addition, $\mathcal{I}_K$ is weakly lower semi-continuous and the definition of critical points of $\mathcal{I}_K$ is followed by Definition 2.4. The following lemma is crucial in adopting the approach based on the variational principle in Section 2.

Lemma 4.1. *Let $\Omega \subset \mathbb{R}^N$ be a bounded domain with Lipschitz boundary and assume that $1 < s < 2 < q$ with $t > N$. Then*

$$\|D\phi(u)\|_{L^t(\Omega)} \leq C_1 r^{q-1} + \lambda C_2 r^{s-1}, \quad \forall\, u \in K(r)$$

where C_1 and C_2 are suitable constants for the embedding $W^{2,t}(\Omega) \hookrightarrow W^{1,\infty}(\Omega)$.

Proof. By the definition of $D\phi(u)$, we get

$$\|D\phi(u)\|_{L^t(\Omega)} \leq \| - \Delta_q u\|_{L^t(\Omega)} + \lambda \| |u|^{s-2}u\|_{L^t(\Omega)}. \tag{4.2}$$

By the definition of q-Laplacian, we have

$$\Delta_q u = \operatorname{div}(\nabla u|\nabla u|^{q-2}) = \Delta u|\nabla u|^{q-2} + (q-2)|\nabla u|^{q-4}\sum_{i=1}^{N}\sum_{j=1}^{N} u_{x_i}u_{x_j}u_{x_ix_j}.$$

Taking the $\|\cdot\|_{L^t(\Omega)}$ norm on both sides and by the triangle inequality, we obtain

$$\begin{aligned}
\|\Delta_q u\|_{L^t(\Omega)} &\le \|\Delta u|\nabla u|^{q-2}\|_{L^t(\Omega)} + (q-2)\left\| |\nabla u|^{q-4}\sum_{i,j=1}^{n} u_{x_i}u_{x_j}u_{x_ix_j}\right\|_{L^t(\Omega)} \\
&\le \left(\int_\Omega |\Delta u|\nabla u||^t\,dx\right)^{\frac{1}{t}} + (q-2)\left(\int_\Omega \left||\nabla u|^{q-4}\sum_{i,j=1}^{n} u_{x_i}u_{x_j}u_{x_ix_j}\right|^t dx\right)^{\frac{1}{t}} \\
&\le \|u\|^{q-2}_{W^{1,\infty}(\Omega)}\left(\int_\Omega |\Delta u|^t\,dx\right)^{\frac{1}{t}} + \|u\|^{q-2}_{W^{1,\infty}(\Omega)}\left(\int_\Omega \left|\sum_{i,j=1}^{n} u_{x_ix_j}\right|^t dx\right)^{\frac{1}{t}} \\
&\le \|u\|^{q-2}_{W^{1,\infty}(\Omega)}\|u\|_{W^{2,t}(\Omega)} + (q-2)\|u\|^{q-2}_{W^{1,\infty}(\Omega)}\|u\|_{W^{2,t}(\Omega)}.
\end{aligned}$$

Now, using the Sobolev embedding $W^{2,t}(\Omega) \hookrightarrow W^{1,\infty}(\Omega)$, we have

$$\|\Delta_q u\|_{L^t(\Omega)} \le C'\|u\|^{q-1}_{W^{2,t}(\Omega)} + (q-2)C'\|u\|^{q-1}_{W^{2,t}(\Omega)} \le C_1\|u\|^{q-1}_{W^{2,t}(\Omega)}. \tag{4.3}$$

Moreover,

$$\||u|^{s-2}u\|_{L^t(\Omega)} \le C_2\|u\|^{s-1}_{W^{2,t}(\Omega)}. \tag{4.4}$$

Since $u \in K(r)$, using (4.2), (4.3) and (4.4), we get

$$\|D\phi(u)\|_{L^t(\Omega)} \le C_1 r^{q-1} + \lambda C_2 r^{s-1}.$$

This completes the proof. □

Lemma 4.2. *Let $1 < s < 2 < q$. Then there exists $\lambda^* > 0$ and $r > 0$ such that for each $\bar{u} \in K(r)$ the following holds*

$$-\Delta v = D\phi(\bar{u}) = -\Delta_q \bar{u} + \lambda \bar{u}|\bar{u}|^{s-2} \tag{4.5}$$

in the weak sense for some $v \in K(r)$ and for each $\lambda \in (0, \lambda^*)$.

Proof. Consider the Euler–Lagrange functional, given by

$$\bar{I}(v) = \frac{1}{2}\int_\Omega |\nabla v|^2\,dx - \int_\Omega |\nabla \bar{u}|^{q-2}\nabla\bar{u}\nabla v\,dx - \lambda\int_\Omega |\bar{u}|^{s-2}\bar{u}v\,dx \quad \forall v \in W^{1,2}_0(\Omega).$$

Since $\bar{I}$ is coercive and convex, $\bar{I}$ admits its infimum at some point $v \in W^{1,2}_0(\Omega)$. So, v satisfy (4.5) in weak sense, that is,

$$\int_\Omega \nabla v\nabla\psi\,dx = \int_\Omega |\nabla\bar{u}|^{q-2}\nabla\bar{u}\nabla\psi\,dx + \lambda\int_\Omega |\bar{u}|^{s-2}\bar{u}\psi\,dx \qquad \forall\psi \in W^{1,2}_0(\Omega).$$

Since $D\phi(\bar{u}) \in L^t(\Omega)$, by elliptic regularity theory (see Chapter 6 in [7]), we have that $v \in W^{2,t}(\Omega)$. Since $\|\Delta\cdot\|_{L^t(\Omega)}$ is an equivalent norm on $W^{1,2}_0(\Omega)\cap W^{2,t}(\Omega)$, we have

$$\|v\|_{W^{2,t}(\Omega)} = \|\Delta v\|_{L^t(\Omega)} = \|D\phi(\bar{u})\|_{L^t(\Omega)}.$$

By Lemma 4.1, we obtain $\|v\|_{W^{2,t}(\Omega)} \le C_1 r^{q-1} + \lambda C_2 r^{s-1}$. Since $1 < q-1$, we choose r small enough such that $C_1 r^{q-1} \le \frac{r}{2}$. For this chosen r, we can choose $\lambda^* = \frac{1}{2C_2}r^{2-s}$ which gives $\lambda^* C_2 r^{s-1} = \frac{r}{2}$. Thus, we have $C_1 r^{q-1} + \lambda C_2 r^{s-1} < r$ for each $\lambda \in (0, \lambda^*)$. □

4.1 Proof of Theorem 1.2

Assume that $r > 0$ and λ^* be as in Lemma 4.2 with $\lambda \in (0, \lambda^*)$. Set $K = K(r)$. We have to prove that there exists $\bar{u} \in K$ such that $I_K(\bar{u}) = \inf_{u \in K} \mathcal{I}_K(u)$. Let $m = \inf_{u \in K} \mathcal{I}_K(u)$. By the Sobolev embeddings, $W^{2,t}(\Omega) \hookrightarrow L^s(\Omega)$ and $W^{2,t}(\Omega) \hookrightarrow W^{1,q}(\Omega)$, we get

$$\phi(u) = \frac{1}{q}\|u\|^q_{W^{1,q}(\Omega)} + \frac{\lambda}{s}\|u\|^s_{L^s(\Omega)} \leq C_1\|u\|^q_{W^{2,t}(\Omega)} + C_2\|u\|^s_{W^{2,t}(\Omega)}.$$

Thus for $u \in K$, we have

$$\mathcal{I}_K(u) = \varphi_K(u) - \phi(u) \geq -(C_1 r^q + C_2 r^s) \tag{4.6}$$

from which we obtain that $m > -\infty$. Now suppose that $\{u_n\}$ is a sequence in K such that $\mathcal{I}_K(u_n) \to m$ it follows from (4.6) and definition of K that sequence $\{u_n\}$ is bounded in $W^{1,2}_0(\Omega) \cap W^{2,t}(\Omega)$. Moreover, as $\mathcal{I}_K(u_n) = \mathcal{I}_K(|u_n|)$, we can assume that $u_n \geq 0$. Thus, up to a subsequence, $u_n \rightharpoonup \bar{u}$ for some $0 \leq \bar{u} \in W^{2,t}(\Omega)$. Since K is weakly closed, $\bar{u} \in K$. Moreover, $\mathcal{I}_K$ is lower semi continuous. Consequently, $m \leq \mathcal{I}_K(\bar{u}) \leq \liminf_{n\to\infty} \mathcal{I}_K(u_n) = m$. This gives that $\bar{u}$ is minimizer and by Proposition 2.5 we conclude that $\bar{u}$ is a non-negative critical point of $\mathcal{I}_K(u)$. Finally, to show that there exists $\bar{v} \in K$ such that $-\Delta\bar{v} = -\Delta_q\bar{u} + \lambda\bar{u}|\bar{u}|^{s-2}$, we use Lemma 4.2, with the fact that $\bar{u} \in K$. Therefore, using Theorem 2.6, we conclude that $\bar{u}$ is a non-negative solution of problem (P_λ). Take $e \in K$. For $t \in [0,1]$, we have $te \in K$ and therefore $\mathcal{I}_K(te) < 0$ for sufficiently small t as $1 < s < 2 < q$. Thus, $\mathcal{I}_K(\bar{u}) < 0$. Hence, $\bar{u}$ is a non-trivial and non-negative solution of problem (P_λ).

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References

[1] A. Ambrosetti, H. Brézis, G. Cerami, Combined effects of concave and convex nonlinearities in some elliptic problems, *J. Funct. Anal.* **122**(1994), No. 2, 519–543. https://doi.org/10.1006/jfan.1994.1078; Zbl 0805.35028

[2] D. Amundsen, A. Moameni, R. Y. Temgoua, Mixed local and nonlocal supercritical Dirichlet problems, *Commun. Pure Appl. Anal.*, **22**(2023), No. 10, 3139–3164. https://doi.org/10.3934/cpaa.2023104; Zbl 1527.35461

[3] S. Biagi, E. Vecchi, On the existence of a second positive solution to mixed local-nonlocal concave-convex critical problems, *Nonlinear Anal.* **256**(2025), Art. ID 113795. https://doi.org/10.1016/j.na.2025.113795; Zbl 1562.35281

[4] X. Cabré, P. Majer, Truncation of nonlinearities in some supercritical elliptic problems, *C. R. Acad. Sci. Paris Sér. I Math.* **322**(1996), No. 12, 1157–1162. Zbl 0849.35029

[5] F. Colasuonno, M. Squassina, Eigenvalues for double phase variational integrals, *Ann. Mat. Pura Appl. (4)* **195**(2016), No. 6, 1917–1959. https://doi.org/10.1007/s10231-015-0542-7; Zbl 1364.35226

[6] J. M. do Ó, P. K. Mishra, A. Moameni, Super critical problems with concave and convex nonlinearities in $\mathbb{R}^N$, *Commun. Contemp. Math.* **23**(2021), No. 6. https://doi.org/10.1142/S0219199720500522; Zbl 1472.35192

[7] L. C. Evans, *Partial differential equations*, Second edition, Grad. Stud. Math., Vol. 19, American Mathematical Society, Providence, RI, 2010. https://doi.org/10.1090/gsm/019; Zbl 1194.35001

[8] L. Gasiński, P. Winkert, Constant sign solutions for double phase problems with superlinear nonlinearity, *Nonlinear Anal.* **195**(2020), Art. ID 111739. https://doi.org/10.1016/j.na.2019.111739; Zbl 1437.35233

[9] N. Grenon, L^r estimates for degenerate elliptic problems, *Potential Anal.*, **16**(2002), No. 4, 387–392. https://doi.org/10.1023/A:1014895230754; Zbl 1014.35035

[10] N. Kouhestani, A. Moameni, Multiplicity results for elliptic problems with supercritical concave and convex nonlinearites, *Calc. Var. Partial Differential Equations* **57**(2018), No. 2, 263–279. https://doi.org/10.1016/j.na.2018.12.006; Zbl 1392.35096

[11] W. Liu, G. Dai, Existence and multiplicity results for double phase problem, *J. Differential Equations* **265**(2018), No. 9, 4311–4334. https://doi.org/10.1016/j.jde.2018.06.006; Zbl 1401.35103

[12] Z. Liu, R. Livrea, D. Motreanu, S. Zeng, Variational differential inclusions without ellipticity condition, *Electron. J. Qual. Theory Differ. Equ.* **2020**, No. 43, 1–17. https://doi.org/10.14232/ejqtde.2020.1.43; Zbl 1463.49021

[13] G. Marino, P. Winkert, Moser iteration applied to elliptic equations with critical growth on the boundary, *Nonlinear Anal.* **180**(2019), 154–169. https://doi.org/10.1016/j.na.2018.10.002; Zbl 1426.35122

[14] G. Mingione, V. D. Rădulescu, Recent developments in problems with nonstandard growth and nonuniform ellipticity, *J. Math. Anal. Appl.* **501**(2021), No. 1, 125–197. https://doi.org/10.1016/j.jmaa.2021.125197; Zbl 1467.49003

[15] A. Moameni, Critical point theory on convex subsets with applications in differential equations and analysis, *J. Math. Pures Appl.* **141**(2020), 266–315. https://doi.org/10.1016/j.matpur.2020.05.005; Zbl 1446.58008

[16] A. Moameni, K. L. Wong, Existence of solutions for supercritical $(p,2)$-Laplace equations, *Mediterr. J. Math.* **20**(2023), No. 3. https://doi.org/10.1007/s00009-023-02336-y; Zbl 1512.35341

[17] D. Motreanu, Quasilinear Dirichlet problems with competing operators and convection, *Open Math.* **18**(2020), No. 1, 1510–1517. https://doi.org/10.1515/math-2020-0112; Zbl 1481.35243

[18] J. Musielak, *Orlicz spaces and modular spaces*, Lecture Notes in Mathematics, Vol. 1034, Springer-Verlag, Berlin, 1983. https://doi.org/10.1007/BFb0072210; Zbl 0557.46020

[19] N. S. Papageorgiou, Double phase problems: a survey of some recent results, *Opuscula Math.* **42**(2022), No. 2, 257–278. https://doi.org/10.7494/OpMath.2022.42.2.257; Zbl 1485.35158

[20] A. Szulkin, Minimax principles for lower semicontinuous functions and applications to nonlinear boundary value problems, *Ann. Inst. H. Poincaré Anal. Non Linéaire* **3**(1986), No. 2, 77–109. https://doi.org/10.1016/S0294-1449(16)30389-4; Zbl 0612.58011

[21] V. V. Zhikov, Averaging of functionals of the calculus of variations and elasticity theory, *Izv. Akad. Nauk SSSR, Ser. Mat.* **50**(1986), No. 4, 675–710. https://doi.org/10.1070/IM1987v029n01ABEH000958; Zbl 0599.49031

[22] V. V. Zhikov, On Lavrentiev's phenomenon, *Russ. J. Math. Phys.* **3**(1995), No. 2, 249–269. Zbl 0910.49020

[23] V. V. Zhikov, On some variational problems, *Russ. J. Math. Phys.* **5**(1997), No. 1, 105–116. Zbl 0917.49006

[24] V. V. Zhikov, S. M. Kozlov, O. A. Oleińik, *Homogenization of differential operators and integral functionals*, First edition, Springer, Berlin, 1994. https://doi.org/10.1007/978-3-642-84659-5; Zbl 0838.35001