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Local and global solutions for abstract integro-differential equations with state-dependent delay and applications to a system with coupled memory and diffusion

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Abstract. In this work we study the local and global existence, uniqueness and well-posedness of solutions for abstract integro-differential equations with state-dependent delay. We establish these results without requiring the conventional assumption of local Lipschitz continuity in the forcing terms. Furthermore, our goal is to apply the abstract theory to the study of a problem involving a memory and diffusion-coupled integro-differential system with state-dependent delay.

Keywords: integro-differential equations, state-dependent delay, local and global existence of solutions, uniqueness of solutions, local well-posedness, diffusion-coupled system.

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1 Introduction

In this work, we are first interested in the investigation of the existence and uniqueness, regularity, well-posedness, and global existence of solutions, problems that are crucial and challenging when dealing with differential equations involving state-dependent delays. In this article, we study the integro-differential abstract equations modeled by

$$\frac{d}{dt}\left(u(t)+\int_0^t P(t-s)u(s)ds\right)$$
$$= Au(t)+\int_0^t Q(t-s)u(s)ds + G(t,u(t-\sigma(t,u_t))), t\in[0,a], \tag{1.1}$$
$$u_0=\varphi\in\mathcal{B}=C([-p,0];Z), \tag{1.2}$$

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where A and $Q(t)$ for $t \geq 0$ are closed linear operators defined on a common domain $D(A)$ that is dense in Z, while $(P(t))_{t\geq 0}$ is a family of bounded linear operators on Z. The variable u_t represents the history of $u(\cdot)$ at time t ($u_t \in \mathcal{B}$ and $u_t(\theta) = u(t+\theta)$ for $\theta \in [-p,0]$, $t > 0$) and $G(\cdot)$ and $\sigma(\cdot)$ are continuous functions.

Secondly, we apply abstract theory to study the existence and uniqueness of mild and classical local and global solutions for a diffusion-coupled integro-differential system with state-dependent delay described by

$$\begin{aligned}
&\frac{\partial}{\partial t}\left[u(t,x) + \int_0^t (t-s)^\alpha e^{-\omega(t-s)} v(s,x)ds\right] \\
&\qquad = \Delta u(t,x) + \int_0^t B(t-s)\Delta v(s,x)ds + \zeta(t)\int_\Omega b(u(\varrho(t,u_t),y))f(x-y)dy, \\
&\frac{\partial}{\partial t}\left[v(t,x) + \int_0^t (t-s)^\alpha e^{-\gamma(t-s)} u(s,x)ds\right] \\
&\qquad = \Delta v(t,x) + \int_0^t C(t-s)\Delta u(s,x)ds + g(v(\varrho(t,u_t),x)), \\
&u(t,\cdot)|_{\partial\Omega} = 0, \;\; v(t,\cdot)|_{\partial\Omega} = 0 \\
&\quad u(\theta,y) = \varphi(\theta,y), \;\; v(\theta,y) = \phi(\theta,y), \qquad \theta \in [-p,0],\, y \in \Omega,
\end{aligned}$$

for $t \in [0,a]$ and $x \in \Omega$, where $f \in C(\widetilde{\Omega};\mathbb{R})$, $\widetilde{\Omega} = \{x-y : x,y \in \Omega\}$, $g \in C_{\text{Lip}}(\mathbb{R}^n;\mathbb{R}^n)$, $b \in C_{\text{Lip}}(\mathbb{R}^n;\mathbb{R}^n)$, $\varrho \in C_{\text{Lip}}([0,a]\times\mathcal{B};\mathbb{R}), d-p \leq \varrho(t,\psi) \leq t$ for all $(t,\psi) \in [0,a]\times\mathcal{B}$ and $\zeta \in C_{\text{Lip}}(\mathbb{R};\mathbb{R})$.

It is well-known that integro-differential equations in abstract spaces play a significant role in the literature due to their ability to model a wide range of important equations arising in physical and biological systems. For example, in the classical theory developed by Gurtin and Pipkin [14], Miller [27], and Nunziato [29], integro-differential equations are employed to describe heat conduction in materials with fading memory. They are also used to model the non-equilibrium thermal response of glass-forming substances with dynamic (time-dependent) heat capacity under fast thermal perturbations (see [28] for more details).

In [2], the author considers a coupled linear integro-differential wave system with memory. In this work, the existence and uniqueness of global solutions are studied, and the uniform stabilization of the total energy is proved as time goes to infinity. Another example of the importance of integro-differential equations is the McKean–Vlasov equation, which has attracted considerable attention due to its broad applicability. It models not only continuous phenomena in areas such as statistical physics, stochastic control, mean field games, financial mathematics, Bayesian inference, and neural networks, but also discontinuous phenomena in mean-field dynamics—such as the motion of particles in polymer dynamics, the onset of financial crises, and discontinuities in control systems [23].

An important framework for analyzing partial integro-differential equations in abstract spaces is the theory of resolvent operators. This theory is widely used to establish the existence, uniqueness, and qualitative properties of solutions to such equations. The resolvent operator approach has been developed in several works. Firstly, by Grimmer et al. [11–13]. Secondly, Da Prato et al. [3,4] and Lunardi [24,25] studied systems with $P \equiv 0$. Recently, Dos Santos et al. [7,8,16] addressed the general case where both $P \neq 0$ and $Q \neq 0$.

An important topic in the theory of differential equations, abstract differential equations with state-dependent delay are a field of great significance due to their applications (see [1,15]) and because their qualitative theory is rather different from the usual theory on functional differential equations with constant time-dependent delay. The literature on state-dependent delay problems is extensive. For ordinary differential equations (ODE) in finite-dimensional spaces, notable works include [1, 9, 10, 15] and their associated references. For ODEs and partial differential equations (PDEs) with state-dependent delay, one can refer to the papers [17–19, 19–22, 26]. Many works in the literature prove the existence of solutions of integro-differential equations with state-dependent delay [5, 7, 30, 32]. But these papers establish only local solutions by fixed point results using compactness; they do not address uniqueness, the regularity of solutions, or the existence and uniqueness of maximal solutions for integro-differential equations. The central difference emerges from the fact that functions of the form $u \mapsto u(\cdot - \sigma(\cdot, u_{(\cdot)}))$ and $u \mapsto u_{\mu(\cdot, u_{(\cdot)})}$ are not Lipschitz in spaces of continuous functions, and this has implications concerning the uniqueness of solutions and the well-posedness of certain differential equations with state-dependent delay.

Inspired by the ideas from [17–19], we establish the local and global existence, uniqueness and local well-posedness of the problem (1.1)–(1.2) in spaces of Lipschitz functions. In this context, we emphasize the importance of inequalities like the following:

$$\begin{aligned}\|u(\sigma(\cdot, u(\cdot))) \;-\; v(\sigma(\cdot, v(\cdot)))\|_{C([0,b];Z)} \\ \leq (1 + [v]_{C_{\mathrm{Lip}}([-p,b];Z)}[\sigma]_{C_{\mathrm{Lip}}([0,b]\times X;\mathbb{R})})\|u - v\|_{C([0,b];Z)}.\end{aligned}$$

where

$$[v]_{C_{\mathrm{Lip}}([-p,b];Z)} := \sup_{t,s\in[-p,b], t\neq s} \frac{\|v(t) - v(s)\|_Z}{|t - s|}.$$

Moreover, such analysis becomes even more challenging in the context of integro-differential equations, since the associated resolvent operator does not satisfy the semigroup property. This difficulty motivates the present work, whose main contributions can be summarized as follows:

First, we establish existence and uniqueness results—both local and global—as well as well-posedness for the integro-differential problem described above in spaces of Lipschitz functions. This contribution is significant because the absence of the semigroup property in the associated resolvent operator makes the construction of maximal solutions considerably more intricate. Our results therefore extend the theory beyond the scope of classical semigroup-based approaches.

Second, we introduce and rigorously analyze a new class of coupled integro-differential systems that incorporate fading memory terms and diffusion operators with state-dependent delays. To the best of our knowledge, this paper provides the first systematic treatment of such systems, thereby opening new avenues of research in the field.

Third, we address additional aspects that, to date, have not appeared in the literature, reinforcing the novelty and breadth of our contributions.

Finally, the paper is organized into five sections. Section 2 introduces the notation and preliminary results used throughout the study. Section 3 develops the existence and uniqueness theory for problem (1.1)–(1.2), both locally and globally. Section 4 establishes the local well-posedness of (1.1)–(1.2) in the spaces $\mathcal{D}_a$ and $C([0, a]; Z)$. Section 5 concludes with the analysis of a non-trivial coupled diffusive integro-differential system with state-dependent delay, further illustrating the scope of our framework.

2 Preliminaries

Let $(Z, \|\cdot\|_Z)$ and $(W, \|\cdot\|_W)$ be Banach spaces. In this work $\mathcal{B} = C([-p, 0]; Z)$ endowed with the usual norm $\|\cdot\|_{\mathcal{B}}$, $C_{\mathrm{Lip}}([b, c]; Z)$ is the space of the Lipschitz functions from $[b, c]$ into Z endowed with the norm

$$\|\cdot\|_{C_{\mathrm{Lip}}([b,c];Z)} = \|\cdot\|_{C([b,c];Z)} + [\cdot]_{C_{\mathrm{Lip}}([b,c];Z)},$$

where $[\xi]_{C_{\text{Lip}}([b,c];Z)} = \sup_{t,s\in[b,c],t\neq s} \frac{\|\xi(t)-\xi(s)\|_Z}{|t-s|}$ and $C^\gamma([b,c];Z)$, with $\gamma \in (0,1)$, is the space of the γ-Hölder functions from $[b,c]$ into Z, endowed with the norm

$$\|\cdot\|_{C^\gamma([b,c];Z)} = \|\cdot\|_{C([b,c];Z)} + [\cdot]_{C^\gamma([b,c];Z)},$$

where $[\xi]_{C^\gamma([b,c];Z)} = \sup_{t,s\in[b,c],t\neq s} \frac{\|\xi(t)-\xi(s)\|_Z}{|t-s|^\gamma}$.

Furthermore, $\mathcal{L}(Z,W)$ is the space of bounded linear operators from Z into W endowed with the operator norm $\|\cdot\|_{\mathcal{L}(Z,W)}$ and we write $\mathcal{L}(Z)$ and $\|\cdot\|$ if $Z = W$. Finally, for $r > 0$ and $z \in Z$, $B_r(z,Z) = \{x \in Z; \|x - z\|_Z \leq r\}$.

As cited in the introduction, our results on existence and uniqueness of solutions are proved without assuming that $G(\cdot)$ and $\sigma(\cdot)$ are locally Lipschitz, they belong to a more general class of functions. The next definition comes from [18].

Definition 2.1. Let Y_i, $i = 1,2$, be Banach spaces and $p \geq 1$. A function $G \in C([c,d] \times Y_1; Y_2)$ is a L^p-Lipschitz function, if there exists an integrable function $[G]_{(\cdot,\cdot)} : [c,d] \times [c,d] \to \mathbb{R}^+$ and a non-decreasing function $\mathcal{W}_G : \mathbb{R}^+ \to \mathbb{R}^+$ such that $[G]_{(t,\cdot)} \in L^p([c,t];\mathbb{R}^+)$ and $[G]_{(\cdot,0)} \in L^p([c,t];\mathbb{R}^+)$ for all $t \in (c,d]$, and

$$\|G(t,x) - G(s,y)\|_{Y_2} \leq \mathcal{W}_G(\max\{\|x\|_{Y_1}, \|y\|_{Y_1}\})[G]_{(t,s)}(|t-s| + \|x-y\|_{Y_1}),$$

for all $x,y \in Y_1$ and $c \leq s \leq t \leq d$. We use the notation $L^p_{\text{Lip}}([c,d] \times Y_1; Y_2)$ for the set formed by this type of functions.

The next result can be found in [18, Lemma 1] and there for $p,b > 0$, $u \in C([-p,b];Z)$ and $\sigma \in C([0,a] \times \mathcal{B}; [0,p])$, $u^\sigma : [0,b] \to Z$ is the function given by $u^\sigma(t) = u(t - \sigma(t,u_t))$.

Lemma 2.2 ([18, Lemma 1]). *Assume $0 < b \leq a$, $u,v \in C([-p,b];Z)$, $\sigma \in L^r_{\text{Lip}}([0,a] \times \mathcal{B}; [0,p])$ and let $\rho = \max\{\|u\|_{C([-p,b];Z)}, \|v\|_{C([-p,b];Z)}\}$. If $u_{|_{[-p,0]}} \in C_{\text{Lip}}([-p,0];Z)$ and $u_{|_{[0,b]}} \in C_{\text{Lip}}([0,b];Z)$, then $u \in C_{\text{Lip}}([-p,b];Z)$, $u_{(\cdot)} \in C_{\text{Lip}}([0,b];\mathcal{B})$ and*

$$\max\{[u]_{C_{\text{Lip}}([-p,b];Z)}, [u_{(\cdot)}]_{C_{\text{Lip}}([0,b];\mathcal{B})}\} \leq [u_0]_{C_{\text{Lip}}([-p,0];Z)} + [u_{|_{[0,b]}}]_{C_{\text{Lip}}([0,b];\mathcal{B})},$$
$$\|u^\sigma(s+h) - u^\sigma(s)\| \leq [u]_{C_{\text{Lip}}([-p,b];Z)}(1 + [\sigma]_{(s+h,s)}\mathcal{W}_\sigma(\rho)(1 + [u]_{C_{\text{Lip}}([-p,b];Z)}))h,$$
$$\|u^\sigma(s) - v^\sigma(s)\| \leq (1 + [u]_{C_{\text{Lip}}([-p,b];Z)}[\sigma]_{(s,s)}\mathcal{W}_\sigma(\rho))\|u - v\|_{C([-p,b];Z)},$$

for all $h,s \in [0,b]$ with $s + h \in [0,b]$.

In addition, if $w \in C([-p,b];Z)$, $\alpha \in (0,1)$, $w_{|_{[0,b]}} \in C^\alpha([0,b];Z)$ and $w_0 \in C^\alpha([-p,0];Z)$, then $w_{(\cdot)} \in C^\alpha([0,b];\mathcal{B})$, $w^\sigma \in C^{\alpha^2}([0,b];Z)$ and

$$[w_{(\cdot)}]_{C^\alpha([0,b];\mathcal{B})} \leq \varpi(w,\alpha) := [w]_{C^\alpha([0,b];X)} + [w_0]_{C^\alpha([-p,0];Z)},$$
$$[w^\sigma]_{C^{\alpha^2}([0,b];Z)} \leq [w]_{C^\alpha([-p,b];X)}(b^{1-\alpha} + [\sigma]_{C_{\text{Lip}}([0,a]\times\mathcal{B};[0,p])}(b^{1-\alpha} + [w]_{C^\alpha([-p,b];Z)}))^\alpha,$$
$$[w^\sigma]_{C^{\alpha^2}([0,b];Z)} \leq [w]_{C^\alpha([-p,b];Z)}(b^{\alpha-\alpha^2} + [\sigma]^\alpha_{C_{\text{Lip}}([0,a]\times\mathcal{B};[0,p])}(b^{\alpha-\alpha^2} + [w]^\alpha_{C^\alpha([-p,b];Z)})).$$

To achieve our results, we mention the main points about the theory of the resolvent operator $(\mathcal{V}(t))_{t\geq 0}$ on Z to the integro-differential abstract Cauchy problem.

$$\frac{d}{dt}\left[u(t) + \int_0^t P(t-s)u(s)\,ds\right] = Au(t) + \int_0^t Q(t-s)u(s)\,ds, \tag{2.1}$$
$$u(0) = z \in Z. \tag{2.2}$$

where $A : D(A) \subseteq Z \to Z$, is a closed linear operator and the notation $[D(A)]$ represents the domain of A endowed with the graph norm, $\|z\|_1 = \|z\|_Z + \|Az\|_W$, $z \in D(A)$, and $Q(t)$ for $t \geq 0$ are closed linear operators defined on a common domain $D(A)$ that is dense in Z, while $P(t)$ for $t \geq 0$ is a family of bounded linear operators on Z.

Next we present the definition of resolvent operator family for the problem (2.1)–(2.2).

Definition 2.3. A one parameter family of bounded linear operators $(\mathcal{V}(t))_{t\geq 0}$ on Z is called a resolvent operator of (2.1)–(2.2) if the following conditions are verified.

(a) The function $\mathcal{V}(\cdot) : [0,\infty) \to \mathcal{L}(Z)$ is strongly continuous, exponentially bounded and $\mathcal{V}(0)z = z$ for all $z \in Z$.

(b) For $z \in D(A)$, $\mathcal{V}(\cdot)z \in C([0,\infty); [D(A)]) \bigcap C^1((0,\infty); Z)$, and

$$\frac{d}{dt}\left[\mathcal{V}(t)z + \int_0^t P(t-s)\mathcal{V}(s)zds\right] = A\mathcal{V}(t)z + \int_0^t Q(t-s)\mathcal{V}(s)zds, \tag{2.3}$$

$$\frac{d}{dt}\left[\mathcal{V}(t)z + \int_0^t \mathcal{V}(t-s)P(s)zds\right] = \mathcal{V}(t)Az + \int_0^t \mathcal{V}(t-s)Q(s)zds, \tag{2.4}$$

for every $t \geq 0$.

The existence of a resolvent operator for problem (2.1)–(2.2) was studied in [7,8]. In these works we considered the following conditions:

(C1) The operator $A : D(A) \subseteq Z \to Z$ is the infinitesimal generator of an analytic semigroup $(T(t))_{t\geq 0}$ on Z, and there are constants $M_0 > 0$ and $\vartheta \in (\pi/2, \pi)$ such that $\rho(A) \supseteq \Lambda_\vartheta = \{\lambda \in \mathbb{C} \setminus \{0\} : |\arg(\lambda)| < \vartheta\}$ and $\|R(\lambda, A)\| \leq M_0|\lambda|^{-1}$ for all $\lambda \in \Lambda_\vartheta$.

(C2) The function $P : [0,\infty) \to \mathcal{L}(Z)$ is strongly continuous and $\widehat{P}(\lambda)z$ is absolutely convergent for $z \in Z$ and $Re(\lambda) > 0$. There exists $\alpha > 0$ and an analytical extension of $\widehat{P}(\lambda)$ (still denoted by $\widehat{P}(\lambda)$) to Λ_ϑ such that $\|\widehat{P}(\lambda)\| \leq N_0|\lambda|^{-\alpha}$ for every $\lambda \in \Lambda_\vartheta$, and $\|\widehat{P}(\lambda)z\| \leq N_1|\lambda|^{-1}\|z\|_1$ for every $\lambda \in \Lambda_\vartheta$ and $z \in D(A)$.

(C3) For all $t \geq 0$, $Q(t) : D(Q(t)) \subseteq Z \to Z$ is a closed linear operator, $D(A) \subseteq D(Q(t))$ and $Q(\cdot)z$ is strongly measurable on $(0,\infty)$ for each $z \in D(A)$. There exists $q(\cdot) \in L^1_{\text{loc}}(\mathbb{R}^+)$ such that $\widehat{q}(\lambda)$ exists for $Re(\lambda) > 0$ and $\|Q(t)z\| \leq q(t)\|z\|_1$ for all $t > 0$ and $z \in D(A)$. Moreover, the operator valued function $\widehat{Q} : \Lambda_{\pi/2} \to \mathcal{L}([D(A)]; Z)$ has an analytical extension (still denoted by $\widehat{Q}$) to Λ_ϑ such that $\|\widehat{Q}(\lambda)z\| \leq \|\widehat{Q}(\lambda)\|\,\|z\|_1$ for all $z \in D(A)$, and $\|\widehat{Q}(\lambda)\| \to 0$ as $|\lambda| \to \infty$.

(C4) There exists a subspace $D \subseteq D(A)$ dense in $[D(A)]$ and positive constants C_i, $i = 1, 2$, such that $A(D) \subseteq D(A)$, $\widehat{Q}(\lambda)(D) \subseteq D(A)$, $\widehat{P}(\lambda)(D) \subseteq D(A)$, $\|A\widehat{Q}(\lambda)z\| \leq C_1\|z\|$ and $\|\widehat{P}(\lambda)z\|_1 \leq C_2|\lambda|^{-\alpha}\|z\|_1$ for every $z \in D$ and all $\lambda \in \Lambda_\vartheta$.

The following result has been established in [8, Theorem 2.1].

Theorem 2.4 ([8, Theorem 2.1])**.** *Assume that conditions* **(C1)–(C4)** *are fulfilled. Then there exists a unique resolvent operator for problem* (2.1)–(2.2)*.*

For the development of this work we will assume that the conditions **(C1)–(C4)** are satisfied.

Remark 2.5. From Theorem 2.4 it follows that there exists a constant $M > 0$ such that the resolvent satisfies $\|\mathcal{V}(t)\| \leq M$, for all $t \in [0, a]$ and there exists a constant $C > 0$ such that $\|A\mathcal{V}(t)\| \leq \frac{C}{t}$ for all $t \in (0, a]$ and $z \in Z$.

Let E be the space consisting of $z \in Z$ such that

$$\mathcal{V}(\cdot)z \in C([0,\infty),[D(A)]) \cap C^1([0,\infty);Z).$$

It is clear that $E \subseteq D(A)$ and $\frac{d}{dt}\mathcal{V}(t)z|_{t=0} = Az - P(0)z$ for $z \in E$. For more details see [8, Remark 2.2]

The following definitions and results are related to the non-homogeneous problem

$$\frac{d}{dt}\left[u(t) + \int_0^t P(t-s)u(s)ds\right] = Au(t) + \int_0^t Q(t-s)u(s)\,ds + G(t), \qquad t \in I = [0,b], \quad (2.5)$$

with initial condition (2.2), where $G : [0,b] \to Z$ is a continuous function.

Definition 2.6. A function $u \in C([0,b];Z)$, $0 < b \le a$ is called a mild solution of (2.5)–(2.2) on $[0,b]$ if $u(0) = z$ and

$$u(t) = \mathcal{V}(t)z + \int_0^t \mathcal{V}(t-s)G(s)ds, \qquad \forall\, t \in [0,b]. \quad (2.6)$$

Definition 2.7. A function $u : [0,b] \to Z$, $0 < b \le a$, is called a classical solution of (2.5)–(2.2) on $[0,b]$ if $u \in C([0,b],[D(A)]) \cap C^1((0,b];Z)$, the condition (2.2) holds and the equation (2.5) is verified on $[0,b]$. Moreover, if $u \in C([0,b],[D(A)]) \cap C^1([0,b];Z)$, the function u is said to be a strict solution of (2.5)–(2.2) on $[0,b]$.

Remark 2.8. From the above definition, it is clear that $\mathcal{V}(\cdot)z$ is a classical solution of problem (2.5)–(2.2) on $(0,\infty)$ for $z \in D(A)$ and $G(t) \equiv 0$.

In the work [8] the following results are established for the problem (2.5)–(2.2)

Theorem 2.9 ([8, Theorem 2.4])**.** *Let $z \in D(A)$. Assume that $G \in C([0,b];Z)$ and $u(\cdot)$ is a classical solution of problem (2.5)–(2.2) on $(0,b]$. Then*

$$u(t) = \mathcal{V}(t)z + \int_0^t \mathcal{V}(t-s)G(s)\,ds, \qquad t \in [0,b]. \quad (2.7)$$

Theorem 2.10 ([8, Lemma 3.1.1])**.** *If $R(\lambda_0, A)$ is compact for some $\lambda_0 \in \rho(A)$, then $\mathcal{V}(t)$ is compact for all $t > 0$.*

Theorem 2.11 ([8, Theorem 2.6])**.** *Let $z \in D(A)$ and let $G \in W^{1,1}([0,a];Z)$. Then the mild solution $u(\cdot)$ of problem (2.5)–(2.2) is a classical solution on $[0,a]$. Further, if $z \in E$, then $u(\cdot)$ is a strict solution on $[0,a]$.*

Corollary 2.12 ([8, Corollary 2.2])**.** *Let $z \in D(A)$ and $G \in C([0,a];Z)$. Let $u(\cdot)$ be the mild solution of problem (2.5)–(2.2). If $u \in C([0,a],[D(A)])$, then $u(\cdot)$ is a classical solution on $(0,a]$.*

3 Existence and uniqueness of solution

In this section we begin by studying the existence and uniqueness of mild solution for the integro-differential problem (1.1)–(1.2). First, we establish some concepts and assumptions that will be used in this work. For this remember that for $p, b > 0$, $u \in C([-p,b];Z)$ and $\sigma \in C([0,a] \times \mathcal{B};[0,p])$, $u^\sigma : [0,b] \to Z$ denotes the function given by $u^\sigma(t) = u(t - \sigma(t,u_t))$.

Definition 3.1. A function $u \in C([-p,b];Z)$, $0 < b \le a$ is called a mild solution of (1.1)–(1.2) on $[-p,b]$ if $u_0 = \varphi$ on $[-p,0]$ and

$$u(t) = \mathcal{V}(t)\varphi(0) + \int_0^t \mathcal{V}(t-s)G\left(s,u^\sigma(s)\right)ds, \qquad \forall\, t \in [0,b]. \quad (3.1)$$

Definition 3.2. A function $u \in C([-p,b];Z)$, $0 < b \le a$, is said to be a strict solution (resp. a classical solution) of (1.1)–(1.2) on $[-p,b]$ if $u_{|_{[0,b]}} \in C^1([0,b];Z) \cap C([0,b];[D(A)])$ (resp. $u_{|_{[0,b]}} \in C^1((0,b];Z) \cap C((0,b];[D(A)]))$, $u_0 = \varphi$ and $u(\cdot)$ satisfies (1.1) on $[0,b]$ (resp. satisfies (1.1) on $(0,b]$).

To establish the main results of this section, we introduce the following condition.

Condition $\boldsymbol{H^{q,r}_{G,\sigma,a}}$: $q, r \ge 1$, $\Theta(q,r) = \frac{1}{q} + \frac{1}{r} \le 1$, $G \in L^q_{\mathrm{Lip}}([0,a] \times Z; Z)$ and $\sigma \in L^r_{\mathrm{Lip}}([0,a] \times \mathcal{B}; [0,p])$.

Notation 3.3. If condition $\boldsymbol{H^{q,r}_{G,\sigma,a}}$ holds, we use the notations

$$\Sigma_1(b) := \sup_{h,c \in [0,b], h+c \in [0,b]} \|[G]_{(\cdot+h,\cdot)}(1 + [\sigma]_{(\cdot+h,\cdot)})\|_{L^1([0,c])},$$

$$\Sigma_2(b) := \sup_{c \in [0,b]} \|[G]_{(\cdot,0)}(1 + [\sigma]_{(\cdot,0)})\|_{L^1([0,c])}.$$

In addition, if $\Theta(q,r) < 1$, $\Lambda(q,r)$ is the number defined by the relation $\Theta(q,r) + \frac{1}{\Lambda(q,r)} = 1$.

Now we can present the first theorem of this work.

Theorem 3.4. *Suppose $\varphi \in C_{\mathrm{Lip}}([-p,0];Z)$, $\mathcal{V}(\cdot)\varphi(0) \in C_{\mathrm{Lip}}([0,a];Z)$. If condition $\boldsymbol{H^{q,r}_{G,\sigma,a}}$ is satisfied and $\Sigma_1(b) + \Sigma_2(b) \to 0$ as $b \to 0$, then there exists a unique mild solution $u \in C_{\mathrm{Lip}}([-p,b];Z)$ of the problem* (1.1)–(1.2) *on $[-p,b]$ for some $0 < b \le a$.*

Proof. Consider $R_\varphi > [\mathcal{V}(\cdot)\varphi(0)]_{C_{\mathrm{Lip}}([0,a];Z)} + \partial|\varphi\|_{C_{\mathrm{Lip}}([-p,0];Z)} + M\|G(0,\varphi(-\sigma(0,\varphi))\|$. From the hypothesis on $\Sigma_i(\cdot)$, $i = 1,2$, we can choose $0 < b \le a$ such that

$$[\mathcal{V}(\cdot)\varphi(0)]_{C_{\mathrm{Lip}}([0,a];Z)} + \|\varphi\|_{C_{\mathrm{Lip}}([-p,0];Z)} + M\|G(0,\varphi(-\sigma(0,\varphi))\| \\ + (1+R_\varphi)^2 \mathcal{W}_G(\rho)(1 + \mathcal{W}_\sigma(\rho))M\left(\Sigma_1(b) + \Sigma_2(b)\right) < R_\varphi, \tag{3.2}$$

$$(1+R_\varphi)\mathcal{W}_G(\rho)M(1+\mathcal{W}_\sigma(\rho))\Sigma_1(b) < 1 \tag{3.3}$$

with $\rho := R_\varphi(b+p) + \|\varphi\|_{\mathcal{B}}$.

Define the space

$$\mathcal{Z}(b,R_\varphi) = \{u \in C([-p,b];Z) : u_0 = \varphi,\ [u]_{C_{\mathrm{Lip}}([-p,b];Z)} \le R_\varphi\},$$

endowed with the metric $d(u,v) = \|u - v\|_{C([0,b];Z)}$ and let $\Gamma : \mathcal{Z}(b,R_\varphi) \to C([-p,b];Z)$ be the map defined by $\Gamma u(t) = \varphi(t)$ for $t \in [-p,0]$ and

$$\Gamma u(t) = \mathcal{V}(t)\varphi(0) + \int_0^t \mathcal{V}(t-s)G\left(s, u^\sigma(s)\right) ds, \quad \text{for } t \in [0,b].$$

We state that $\Gamma(\cdot)$ is a contraction. Indeed, for $u \in \mathcal{Z}(b,R_\varphi)$ and $s \in [0,b]$, we note that

$$\begin{aligned} \max\{\|u(s)\|, \|u^\sigma(s)\|, \|u_s\|_{\mathcal{B}}\} &\le \sup_{\tau \in [-p,b]} \|u(\tau) - \varphi(0)\| + \|\varphi(0)\| \\ &\le [u]_{C_{\mathrm{Lip}}([-p,b];X)}(b+p) + \|\varphi\|_{\mathcal{B}} \\ &\le R_\varphi(b+p) + \|\varphi\|_{\mathcal{B}} \le \rho. \end{aligned} \tag{3.4}$$

Using the former estimate and Lemma 2.2, for $s, h \in [0, b]$ with $s + h \in [0, b]$, we have that

$$\begin{aligned}
&\|G(s+h, u^\sigma(s+h)) - G(s, u^\sigma(s))\| \\
&\quad \leq [G]_{(s+h,s)} \mathcal{W}_G(\rho)(h + [u]_{C_{\mathrm{Lip}}([-p,b];Z)}(h + \mathcal{W}_\sigma(\rho)[\sigma]_{(s+h,s)}(h + [u]_{C_{\mathrm{Lip}}([-p,b];Z)} h)) \\
&\quad \leq [G]_{(s+h,s)} \mathcal{W}_G(\rho)(1 + R_\varphi(1 + \mathcal{W}_\sigma(\rho)[\sigma]_{(s+h,s)}(1 + R_\varphi)))h \\
&\quad \leq (1 + R_\varphi)^2 \mathcal{W}_G(\rho)(1 + \mathcal{W}_\sigma(\rho))[G]_{(s+h,s)}(1 + [\sigma]_{(s+h,s)})h.
\end{aligned} \tag{3.5}$$

For $0 \leq s \leq h \leq b$ and arguing as above, we also have that

$$\|G(s, u^\sigma(s)) - G(0, \varphi(-\sigma(0, \varphi)))\| \leq (1 + R_\varphi)^2 \mathcal{W}_G(\rho)(1 + \mathcal{W}_\sigma(\rho))[G]_{(s,0)}(1 + [\sigma]_{(s,0)})h. \tag{3.6}$$

Using the previous inequalities, for $h, t \in [0, b]$ with $t + h \in [0, b]$, we obtain

$$\begin{aligned}
&\|\Gamma u(t+h) - \Gamma u(t)\| \\
&\quad \leq [\mathcal{V}(\cdot)\varphi(0)]_{C_{\mathrm{Lip}}([0,b];Z)} h + \int_0^h \|\mathcal{V}(t+h-s))G(0, \varphi(-\sigma(0,\varphi)))\| ds \\
&\qquad + \int_0^h \|\mathcal{V}(t+h-s))\| \|G(s, u^\sigma(s)) - G(0, \varphi(-\sigma(0, \varphi)))\| ds \\
&\qquad + \int_0^t \|\mathcal{V}(t-s)\| \|G(s+h, u^\sigma(s+h)) - G(s, u^\sigma(s))\| ds \\
&\quad \leq [\mathcal{V}(\cdot)\varphi(0)]_{C_{\mathrm{Lip}}([0,b];Z)} h + M\|G(0, \varphi(-\sigma(0, \varphi))\| h \\
&\qquad + (1 + R_\varphi)^2 \mathcal{W}_G(\rho)(1 + \mathcal{W}_\sigma(\rho)) M \int_0^h [G]_{(s,0)}(1 + [\sigma]_{(s,0)}) s ds \\
&\qquad + (1 + R_\varphi)^2 \mathcal{W}_G(\rho)(1 + \mathcal{W}_\sigma(\rho)) M \int_0^t [G]_{(s+h,s)}(1 + [\sigma]_{(s+h,s)}) h ds \\
&\quad \leq [\mathcal{V}(\cdot)\varphi(0)]_{C_{\mathrm{Lip}}([0,b];Z)} h + M\|G(0, \varphi(-\sigma(0, \varphi))\| h \\
&\qquad + (1 + R_\varphi)^2 \mathcal{W}_G(\rho)(1 + \mathcal{W}_\sigma(\rho)) M(\Sigma_1(b) + \Sigma_2(b))h.
\end{aligned} \tag{3.7}$$

Then $[\Gamma u]_{C_{\mathrm{Lip}}([0,b];Z)} \leq R_\varphi$ and $[\Gamma u]_{C_{\mathrm{Lip}}([-p,b];Z)} \leq R_\varphi$ since $\|\varphi\|_{C_{\mathrm{Lip}}([-p,0];Z)} < R_\varphi$. This proves that $\Gamma(\cdot)$ is a $\mathcal{Z}(b, R_\varphi)$-valued function.

Also, from Lemma 2.2, for $u, v \in \mathcal{Z}(b, R_\varphi)$ and $t \in [0, b]$ it follows that

$$\begin{aligned}
&\|\Gamma u(t) - \Gamma v(t)\| \\
&\quad \leq \int_0^t \|\mathcal{V}(t-s)\| \|G(s, u^\sigma(s)) - G(s, v^\sigma(s))\| ds \\
&\quad \leq \mathcal{W}_G(\rho) \int_0^t M[G]_{(s,s)} \|u^\sigma(s) - v^\sigma(s)\| ds \\
&\quad \leq \mathcal{W}_G(\rho) \int_0^t M[G]_{(s,s)}(1 + [u]_{C_{\mathrm{Lip}}([-p,b];Z)} \mathcal{W}_\sigma(\rho)[\sigma]_{(s,s)}) \|u - v\|_{C([0,s];Z)} ds \\
&\quad \leq \mathcal{W}_G(\rho) \int_0^t M[G]_{(s,s)}(1 + R_\varphi \mathcal{W}_\sigma(\rho)[\sigma]_{(s,s)}) \|u - v\|_{C([0,s];Z)} ds \\
&\quad \leq (1 + R_\varphi) \mathcal{W}_G(\rho)(1 + \mathcal{W}_\sigma(\rho)) M \int_0^t [G]_{(s,s)}(1 + [\sigma]_{(s,s)}) ds \|u - v\|_{C([0,b];Z)} \\
&\quad \leq (1 + R_\varphi) \mathcal{W}_G(\rho)(1 + \mathcal{W}_\sigma(\rho)) \Sigma_1(b) M \|u - v\|_{C([0,b];Z)},
\end{aligned} \tag{3.8}$$

and, from (3.3), we conclude that $\Gamma(\cdot)$ is a contraction on $\mathcal{Z}(b, R_\varphi)$ and then there exists a unique mild solution $u \in C_{\mathrm{Lip}}([-p, b]; Z)$ of (1.1)–(1.2) on $[-p, b]$. □

For the next result we will use the Radon–Nikodym property. It is well-known that every reflexive space has the Radon–Nikodym property (abbreviated, RNP) and that there are non reflexive spaces that have the RNP. We refer to [6] several characterizations of the RNP. For this reason, in the sequel, we consider spaces that have the RNP.

Proposition 3.5. *If conditions in Theorem 3.4 are valid, Z has the RNP, the functions $[G]_{(\cdot,\cdot)}$ and $[\sigma]_{(\cdot,\cdot)}$ are continuous, and $u \in C_{\mathrm{Lip}}([-p,b];Z)$ is the unique associated mild solution of the problem (1.1)–(1.2), then $u(\cdot)$ is a classical solution if $\varphi(0) \in D(A)$ and a strict solution if $\varphi(0) \in E$.*

Proof. Note that

$$\begin{aligned}&\|G(t,u^{\sigma}(t)) - G(s,u^{\sigma}(s))\| \\ &\quad \leq [G]_{(t,s)}\mathcal{W}_G(\rho)(1+[u]_{C_{\mathrm{Lip}}([-p,b];Z)}(1+\mathcal{W}_\sigma(\rho)[\sigma]_{(t,s)}(1+[u]_{C_{\mathrm{Lip}}([-p,b];Z)}))(t-s) \\ &\quad \leq \mathcal{K}|t-s|,\end{aligned}$$

where $\rho = \|u\|_{C([-p,b];Z)}$ and

$$\mathcal{K} = \sup_{(t,s)\in[0,a]\times[0,a]} ([G]_{(t,s)}\mathcal{W}_G(\rho)(1+[u]_{C_{\mathrm{Lip}}([-p,b];Z)}(1+\mathcal{W}_\sigma(\rho)[\sigma]_{(t,s)}(1+[u]_{C_{\mathrm{Lip}}([-p,b];Z)}))).$$

Consequently, the function $t \mapsto G(t) = G(t,u^{\sigma}(t))$ is Lipschitz continuous and since the space Z has satisfies the RNP, the function $G(\cdot) \in W^{1,1}([0,b];Z)$. The statement is now consequence of Theorem 2.11. □

In the next corollary, $\Theta(q,r)$ and $\Lambda(q,r)$ are the numbers defined in Notation 3.3.

Corollary 3.6. *Suppose that $\varphi \in C_{\mathrm{Lip}}([-p,0];Z)$, $\mathcal{V}(\cdot)\varphi(0) \in C_{\mathrm{Lip}}([0,a];Z)$ and assume that $\Theta(q,r) < 1$. If there exists $0 < c \leq a$ such that*

$$\Sigma_3(c) = \sup_{b\in[0,c],d,h\in[0,b],d+h\leq b} \|[G]_{(\cdot+h,\cdot)}\|_{L^q([0,d])}\|(1+[\sigma]_{(\cdot+h,\cdot)})\|_{L^r([0,d])} < \infty.$$

Then there exists a unique mild solution $u \in C_{\mathrm{Lip}}([-p,b];Z)$ of the problem (1.1)–(1.2) on $[-p,b]$ for some $0 < b \leq a$. In particular, the assertion is valid if $G(\cdot)$ and $\sigma(\cdot)$ are Lipschitz.

Proof. From Theorem 3.4, it is sufficient to show that $\Sigma_1(b) + \Sigma_2(b) \to 0$ as $b \to 0$. For $b \in [0,c]$, $t,h \in [0,b)$ with $t+h \leq b$, it is easy to see that

$$\int_0^t [G]_{(s+h,s)}(1+[\sigma]_{(s+h,s)})ds \leq \Sigma_3(c),$$

$$\int_0^t [G]_{(s,0)}(1+[\sigma]_{(s,0)})ds \leq \|[G]_{(\cdot,0)}\|_{L^q([0,b])}\|(1+[\sigma]_{(\cdot,0)})\|_{L^r([0,b])},$$

which implies that $\Sigma_1(b) + \Sigma_2(b) \to 0$ as $b \to 0$. The last claim is direct. □

Now we will discuss solutions defined on a pre-defined interval $[-p,a]$ and on the whole semi-axis $[-p,\infty)$.

Remark 3.7. If condition $\boldsymbol{H^{q,r}_{G,\sigma,a}}$ is satisfied, for $x > 0$ and $b \in [0,a]$, we assume that

$$\Sigma_{1,b}(x) := \sup_{h,c\in[0,b],h+c\in[0,b]} \|\mathcal{W}_{F,\sigma}(\rho(x,\cdot))[G]_{(\cdot+h,\cdot)}(1+[\sigma]_{(\cdot+h,\cdot)})\|_{L^1([0,c])},$$

$$\Sigma_{2,b}(x) := \sup_{c\in[0,b]} \|\mathcal{W}_{F,\sigma}(\rho(x,\cdot))[G]_{(\cdot,0)}(1+[\sigma]_{(\cdot,0)})\|_{L^1([0,c])},$$

where $\mathcal{W}_{F,\sigma}(\rho(x,s)) = \mathcal{W}_G(\rho(x,s))(1+\mathcal{W}_\sigma(\rho(x,s)))$ and $\rho(x,s) = x(s+p) + \|\varphi\|_{\mathcal{B}}$, are finite. Additionally, if the condition $\boldsymbol{H^{q,r}_{G,\sigma,a}}$ is satisfied for all $a > 0$, we use the notations $\Sigma_{1,\infty}(x) := \sup_{b\geq 0}\Sigma_{1,b}(x)$ and $\Sigma_{2,\infty}(x) := \sup_{b\geq 0}\Sigma_{2,b}(x)$.

From the proof of Theorem 3.4, we deduce the next result.

Proposition 3.8. *Suppose that condition $\boldsymbol{H}^{q,r}_{G,\sigma,a}$ is satisfied, $\varphi \in C_{\mathrm{Lip}}([-p,0];Z)$, $\mathcal{V}(\cdot)\varphi(0) \in C_{\mathrm{Lip}}([0,a];Z)$ and let $\mathcal{P}_a : \mathbb{R} \mapsto \mathbb{R}$ be the function given by*

$$\mathcal{P}_a(x) := [\mathcal{V}(\cdot)\varphi(0)]_{C_{\mathrm{Lip}}([0,a];Z)} + \|\varphi\|_{C_{\mathrm{Lip}}([-p,0];Z)} + M\|G(0,\varphi(-\sigma(0,\varphi))\| \\ + (1+x)^2 M(\Sigma_{1,a}(x)+\Sigma_{2,a}(x)) + (1+x)M\Sigma_{1,a}(x)x - x.$$

If $\mathcal{P}_a(R_\varphi) < 0$ for some $R_\varphi > 0$, then there exists a unique mild solution $u \in C_{\mathrm{Lip}}([-p,a];Z)$ of problem (1.1)–(1.2) on $[-p,a]$.

Proof. The demonstration of this proposition follows from the proof of Theorem 3.4 and we will include some details to complete the proof. We remark, from the definition of $\mathcal{P}_a(\cdot)$ and our hypothesis about $\mathcal{P}_a(R_\varphi)$, that

$$[\mathcal{V}(\cdot)\varphi(0)]_{C_{\mathrm{Lip}}([0,a];X)} + \|\varphi\|_{C_{\mathrm{Lip}}([-p,0];X)} + M\|G(0,\varphi(-\sigma(0,\varphi))\| \\ + M(1+R_\varphi)^2\left(\Sigma_{1,a}(R_\varphi)+\Sigma_{2,a}(R_\varphi)\right) + M(1+R_\varphi)\Sigma_{1,a}(R_\varphi)R_\varphi < R_\varphi. \quad (3.9)$$

Consider $\mathcal{Z}(a,R_\varphi)$ and $\Gamma(\cdot)$ as in the proof Theorem 3.4, but using a instead of b and let $u \in \mathcal{Z}(a,R_\varphi)$.

Proceeding as in the proof of Theorem 3.4 and from the definition of $\rho(R_\varphi,s)$, for $s \in [0,a]$ we obtain

$$\max\{\|u(s)\|, \|u^\sigma(s)\|, \|u_s\|_{\mathcal{B}}\} \le R_\varphi(a+p) + \|\varphi\|_{\mathcal{B}} \le \rho(R_\varphi,s).$$

From the inequality above and the proof of Theorem 3.4, for $s \in [0,a]$ and $\tau, h \in [0,a]$ with $\tau + h \in [0,a]$, it follows that

$$\|G(\tau+h,u^\sigma(\tau+h)) - G(\tau,u^\sigma(\tau))\| \\ \le (1+R_\varphi)^2\mathcal{W}_G(\rho(R_\varphi,\tau)))(1+\mathcal{W}_\sigma(\rho(R_\varphi,\tau)))[G]_{(\tau+h,\tau)}(1+[\sigma]_{(\tau+h,\tau)})h,$$

$$\|G(s,u^\sigma(s)) - G(0,\varphi(-\sigma(0,\varphi)))\| \\ \le (1+R_\varphi)^2\mathcal{W}_G(\rho(R_\varphi,s))(1+\mathcal{W}_\sigma(\rho(R_\varphi,s)))[G]_{(s,0)}(1+[\sigma]_{(s,0)})h.$$

From the previous inequalities and arguing as in the estimates (3.7) and (3.8) we have that

$$\|\Gamma u(t+h) - \Gamma u(t)\| \le [\mathcal{V}(\cdot)\varphi(0)]_{C_{\mathrm{Lip}}([0,a];X)}h + M\|G(0,\varphi(-\sigma(0,\varphi))\|h \\ + (1+R_\varphi)^2 M(\Sigma_{1,a}(R_\varphi)+\Sigma_{2,a,}(R_\varphi))h,$$
$$\|\Gamma u(t) - \Gamma v(t)\| \le (1+R_\varphi)M\Sigma_{1,a}(R_\varphi)\|u-v\|_{C([0,a];X)},$$

for all $t \in [0,a]$. Then, $\Gamma(\cdot)$ is a contraction on $\mathcal{Z}(a,R_\varphi)$ and we conclude that there exists a unique mild solution $u \in C_{\mathrm{Lip}}([-p,a];X)$ of (1.1)–(1.2) on $[-p,a]$. □

Proceeding as in the proof of the Proposition 3.8, we can demonstrate the next result.

Corollary 3.9. *Let $\varphi \in C_{\mathrm{Lip}}([-p,0];Z)$ and $\mathcal{V}(\cdot)\varphi(0) \in C_{\mathrm{Lip}}([0,a];Z)$.*

(a) *Assume that $G \in C_{\mathrm{Lip}}([0,a] \times B_r(0,X);Z)$ and that $\sigma \in C_{\mathrm{Lip}}([0,a] \times B_r(0,\mathcal{B});[0,p])$ for all $r > 0$ and let $\mathcal{P}_a : \mathbb{R} \mapsto \mathbb{R}$ be the function given by*

$$\mathcal{P}_a(x) = [\mathcal{V}(\cdot)\varphi(0)]_{C_{\mathrm{Lip}}([0,a];Z)} + \|\varphi\|_{C_{\mathrm{Lip}}([-p,0];Z)} + M\|G(0,\varphi(-\sigma(0,\varphi))\| \\ + [(1+x)^2 + (1+x)][G]_{C_{\mathrm{Lip},\rho(x)}}(1+[\sigma]_{C_{\mathrm{Lip},\rho(x)}})Ma - x,$$

where $\rho(x) = x(a+2p) + \|\varphi\|_{\mathcal{B}}$ and $[G]_{C_{\mathrm{Lip},\rho(x)}}, [\sigma]_{C_{\mathrm{Lip},\rho(x)}}$ are the Lipschitz constants of $G(\cdot)$ and $\sigma(\cdot)$ on $[0,a] \times B_{\rho(x)}(0,X)$ and $[0,a] \times B_{\rho(x)}(0,\mathcal{B})$. If $\mathcal{P}_a(y) < 0$ for some $y > 0$, then there exists a unique mild solution $u \in C_{\mathrm{Lip}}([-p,a];Z)$ of (1.1)–(1.2) on $[-p,a]$.

(b) *If $G(\cdot)$ and $\sigma(\cdot)$ are Lipschitz and $\mathcal{P}_a(y) < 0$ for some $y > 0$, where*

$$\begin{aligned}\mathcal{P}_a(x) &= [\mathcal{V}(\cdot)\varphi(0)]_{C_{\mathrm{Lip}}([0,a];Z)} + \|\varphi\|_{C_{\mathrm{Lip}}([-p,0];Z)} + M\|G(0,\varphi(-\sigma(0,\varphi))\| \\ &\quad + (1+x)^2[G]_{C_{\mathrm{Lip}}}(1+[\sigma]_{C_{\mathrm{Lip}}})Ma + (1+x)[G]_{C_{\mathrm{Lip},\rho(x)}}(1+[\sigma]_{C_{\mathrm{Lip},\rho(x)}})Ma - x \\ &= \theta + (1+x)^2\mu_a + (1+x)\mu_a - x, \end{aligned} \tag{3.10}$$

then there exists a unique mild solution $u \in C_{\mathrm{Lip}}([-p,a];Z)$ of (1.1)–(1.2) on $[-p,a]$. In particular, the affirmation holds if $1 > \frac{4\mu_a(\theta+2\mu_a)}{(3\mu_a-1)^2}$.

With respect to the existence and uniqueness of the solutions defined in $[-p,\infty)$, we have the next corollary.

Corollary 3.10. *Assume that $\mathcal{V}(\cdot)\varphi(0) \in C_{\mathrm{Lip}}([0,\infty);Z)$, $\varphi \in C_{\mathrm{Lip}}([-p,0];Z)$, that the condition $\boldsymbol{H}^{q,r}_{G,\sigma,a}$ is satisfied for all $a > 0$ and that $\Sigma_{i,\infty}(x) < \infty$ for $i = 1,2$ and every $x > 0$. Let $\mathcal{P}_\infty : [0,\infty) \mapsto [0,\infty)$ be the function given by*

$$\begin{aligned}\mathcal{P}_\infty(x) &= [\mathcal{V}(\cdot)\varphi(0)]_{C_{\mathrm{Lip}}([0,\infty;Z)} + \|\varphi\|_{C_{\mathrm{Lip}}([-p,0];Z)} + M\|G(0,\varphi(-\sigma(0,\varphi))\| \\ &\quad + (1+x)^2M(\Sigma_{1,\infty}(x)+\Sigma_{2,\infty}(x)) + (1+x)M\Sigma_{1,\infty}(x)x - x. \end{aligned} \tag{3.11}$$

If $\mathcal{P}_\infty(y) < 0$ for some $y > 0$, then there exists a unique locally Lipschitz mild solution $u \in C([-p,\infty);Z)$ of the problem (1.1)–(1.2) on $[-p,\infty)$.

Proof. Given $a > 0$, let $\mathcal{P}_a(\cdot)$ be the function in Proposition 3.8. As $0 < \mathcal{P}_a(x) \leq \mathcal{P}_\infty(x)$ for all $x \geq 0$, it follows that $0 < \mathcal{P}_a(y) < 0$ for all $a > 0$.

From Proposition 3.8, there exists a unique mild solution $u^a \in C_{\mathrm{Lip}}([-p,a];Z)$ of (1.1)–(1.2) on $[-p,a]$. Since the solution is unique and setting $u : [0,\infty) \mapsto X$ as $u(t) = u^a(t)$ for $t \in [0,a]$, we get a unique locally Lipschitz mild solution of (1.1)–(1.2) defined on the semi-axis $[-p,\infty)$. □

Remark 3.11. The first example of Section 5 contains comments with respect to the existence of the numbers $\Sigma_{i,\infty}(x)$ and an application of Corollary 3.10.

In the next result we investigate the existence and uniqueness of a maximal locally Lipschitz mild solution for (1.1)–(1.2). For this purpose, we say that $\eta : [c,d] \to \mathbb{R}$ is a Grönwall's type function, if the Grönwall's inequality is valid for inequalities of the form $\gamma(t) \leq l_1 + l_2\int_c^t \eta(s)\gamma(s)ds$ with $\gamma(\cdot)$ continuous, that is, this inequality implies that $\gamma(t) \leq l_1 e^{l_2\int_0^t \eta(s)ds}$ for all $t \in [c,d]$.

Proposition 3.12. *Suppose that the hypotheses of Theorem 3.4 hold, $q > 1$, $[G]_{(s,s)}$ is a Grönwall-type function, $\mathcal{W}_\infty := \sup_{s\geq 0}\mathcal{W}_G(s)(1+\mathcal{W}_\sigma(s)) < \infty$, and there is $\alpha \in (0,\frac{1}{q'})$, such that*

$$\sup_{t\in[0,a]}\left\|\frac{[G]_{(t,\cdot)}}{(t-\cdot)^{1-\alpha}}[\sigma]^\alpha_{(t,\cdot)}\right\|_{L^1([0,t])} < \infty$$

and that

$$\Sigma_3(c,\nu) := \sup_{h\in[0,\nu],d\in[c,c+\nu]}\|[G]_{(\cdot+h,\cdot)}(1+[\sigma]_{(\cdot+h,\cdot)})\|_{L^1([c-\nu,d])} \to 0, \tag{3.12}$$

as $\nu \downarrow 0$, for all $c \in [0,a]$. Then there exists a unique maximal mild solution $u \in C_{\mathrm{Lip}}([-p,a];Z)$ of the problem (1.1)–(1.2) on $[-p,a]$.

Proof. Let $u \in C_{\text{Lip}}([-p,b];Z)$ be the mild solution obtained in Theorem 3.4. From the demonstration of Theorem 3.4, we can show that there exists $\delta > 0$ and a unique mild solution $v \in C_{\text{Lip}}([-p,b+\delta];Z)$ of (1.1)–(1.2) such that $v = u$ on $[-p,b]$.

Indeed, let R_φ, b and ρ presented in the proof of Theorem 3.4 and assume $a > b$. Consider $R_1 > [\mathcal{V}(\cdot)\varphi(0)]_{C_{\text{Lip}}([b,a];Z)} + 2R_\varphi$. Using (3.12), we can choose $0 < \delta < b$ small such that

$$2R_\varphi + [\mathcal{V}(\cdot)\varphi(0)]_{C_{\text{Lip}}([b,a];X)} + (1+R_1)^2 \mathcal{W}_G(\rho_1)(1+\mathcal{W}_\sigma(\rho_1))M\Sigma_3(b,\delta) < R_1,$$

where $\rho_1 := R_1(2b+p) + \|\varphi\|_{\mathcal{B}}$. Now consider the space

$$\mathfrak{Y} = \{w \in C([-p,b+\delta];Z)\,;\, w_{|_{[-p,b]}} = u_{|_{[-p,b]}},\ [w]_{C_{\text{Lip}([-p,b+\delta];Z)}} \leq R_1\}, \tag{3.13}$$

endowed with the metric $d(u,v) = \|u-v\|_{C([0,b+\delta];Z)}$ and define the map $\Gamma : \mathfrak{Y} \to C([-p,b+\delta];Z)$ as $(\Gamma w)_{|_{[-p,b]}} = u_{|_{[-p,b]}}$ and

$$\Gamma w(t) = \mathcal{V}(t)\varphi(0) + \int_0^b \mathcal{V}(t-s)G\left(s,u^\sigma(s)\right)ds + \int_b^t \mathcal{V}(t-s)G\left(s,w^\sigma(s)\right)ds, \tag{3.14}$$

for $t \in [b,b+\delta]$.

Let $w \in \mathfrak{Y}$. Then, $[\Gamma w]_{C_{\text{Lip}}([-p,b];Z)} \leq R_\varphi$ since $\Gamma w(\cdot) = u(\cdot)$ on $[-p,b]$. We remark that the inequalities (3.4), (3.5) and (3.6) are verified. Moreover, using (3.4), for $s \in [b,b+\delta]$ we obtain

$$\begin{aligned}
&\max\{\|w(s)\|, \|w^\sigma(s)\|, \|w_s\|_{\mathcal{B}}\}\\
&\quad \leq \max\{\|w(\cdot)\|_{C([b,b+\delta];Z)}, \|w\|_{C([b-p,b+\delta];Z)}, \|w\|_{C([b-p,b+\delta];Z)}\}\\
&\quad \leq \max\{\|w(\cdot)\|_{C([b,b+\delta];Z)}, \|u\|_{C([b-p,b];Z)}\}\\
&\quad \leq \max\{\|w-u(b)\|_{C([b,b+\delta];X)} + \|u(b)\|, \|u\|_{C([b-p,b];Z)}\}\\
&\quad \leq [w]_{C_{\text{Lip}}([b,b+\delta];Z)}\delta + R_\varphi(b+p) + \|\varphi\|_{\mathcal{B}}\\
&\quad \leq R_1\delta + R_\varphi(b+p) + \|\varphi\|_{\mathcal{B}}\\
&\quad \leq R_1 b + R_1(b+p) + \|\varphi\|_{\mathcal{B}} \leq \rho_1.
\end{aligned}$$

In addition, since $R_\varphi \leq R_1$ and $\rho \leq \rho_1$ and , for $s,h \in [0,b+\delta]$ with $s+h \in [b,b+\delta]$ and arguing as in the estimate (3.5), it follows that

$$\begin{aligned}
&\|G(s+h,w^\sigma(s+h)) - G(s,w^\sigma(s))\|\\
&\quad \leq \mathcal{W}_G(\rho_1)(1+\mathcal{W}_\sigma(\rho_1))(1+R_1)^2[G]_{(s+h,s)}(1+[\sigma]_{(s+h,s)})h.
\end{aligned}$$

Let $t \in [b,b+\delta)$ and $h > 0$ with $t+h \in [b,b+\delta]$. From the above inequalities, the inequalities (3.4)–(3.6), the fact that $0 < h < \delta < b$, and proceeding as in the estimate (3.7), we get

$$\begin{aligned}
&\|\Gamma w(t+h) - \Gamma w(t)\|\\
&\quad \leq [\mathcal{V}(\cdot)\varphi(0)]_{C_{\text{Lip}}([b,b+\delta];Z)}h + \int_0^h \|\mathcal{V}(t+h-s))G(0,\varphi(-\sigma(0,\varphi)))\|ds\\
&\qquad + \int_0^h \|\mathcal{V}(t+h-s))\|\|G(s,u^\sigma(s)) - G(0,\varphi(-\sigma(0,\varphi)))\|ds\\
&\qquad + \int_0^t \|\mathcal{V}(t-s)\|\|G(s+h,w^\sigma(s+h)) - G(s,w^\sigma(s))\|ds
\end{aligned}$$

$$
\begin{aligned}
&\le [\mathcal{V}(\cdot)\varphi(0)]_{C_{\mathrm{Lip}}([b,b+\delta];Z)} h + M\|G(0,\varphi(-\sigma(0,\varphi))\|h \\
&\quad + \mathcal{W}_G(\rho)(1+\mathcal{W}_\sigma(\rho))(1+R_\varphi)^2 \int_0^h M[G]_{(s,0)}(1+[\sigma]_{(s,0)})hds \\
&\quad + \int_0^{b-h} \|\mathcal{V}(t-s)(G(s+h,u^\sigma(s+h)) - G(s,u^\sigma(s)))\|ds \\
&\quad + \int_{b-h}^{t} \|\mathcal{V}(t-s)(G(s+h,w^\sigma(s+h)) - G(s,w^\sigma(s)))\|ds \\
&\le [\mathcal{V}(\cdot)\varphi(0)]_{C_{\mathrm{Lip}}([b,b+\delta];Z)} h + M\|G(0,\varphi(-\sigma(0,\varphi))\|h \\
&\quad + \mathcal{W}_G(\rho)(1+\mathcal{W}_\sigma(\rho))(1+R_\varphi)^2 \int_0^h M[G]_{(s,0)}(1+[\sigma]_{(s,0)})hds \\
&\quad + \mathcal{W}_G(\rho)(1+\mathcal{W}_\sigma(\rho))(1+R_\varphi)^2 \int_0^{b-h} M[G]_{(s+h,s)}(1+[\sigma]_{(s+h,s)})hds \\
&\quad + \mathcal{W}_G(\rho_1)(1+\mathcal{W}_\sigma(\rho_1))(1+R_1)^2 \int_{b-h}^{t} M[G]_{(s+h,s)}(1+[\sigma]_{(s+h,s)})hds \\
&\le [\mathcal{V}(\cdot)\varphi(0)]_{C_{\mathrm{Lip}}([b,b+\delta];Z)} h + I_1 + I_2 + I_3 \\
&\quad + \mathcal{W}_G(\rho_1)(1+\mathcal{W}_\sigma(\rho_1))(1+R_1)^2 \int_{b-h}^{t} M[G]_{(s+h,s)}(1+[\sigma]_{(s+h,s)})hds.
\end{aligned}
$$

Observing that in the estimate (3.7) the term $I_1 + I_2 + I_3$ also appears, we deduce that

$$
\begin{aligned}
&\|\Gamma w(t+h) - \Gamma w(t)\| \\
&\quad \le [\mathcal{V}(\cdot)\varphi(0)]_{C_{\mathrm{Lip}}([b,b+\delta];Z)} h + R_\varphi h \\
&\qquad + \mathcal{W}_G(\rho_1)(1+\mathcal{W}_\sigma(\rho_1))(1+R_1)^2 M \int_{b-h}^{t} [G]_{(s+h,s)}(1+[\sigma]_{(s+h,s)})hds \\
&\quad \le R_\varphi h + [\mathcal{V}(\cdot)\varphi(0)]_{C_{\mathrm{Lip}}([b,a];Z)} h + \mathcal{W}_G(\rho_1)(1+\mathcal{W}_\sigma(\rho_1))(1+R_1)^2 M \Sigma_3(b,\delta) h \le R_1 h,
\end{aligned}
$$

which allows us to conclude that $[w_{|_{[b,b+\delta]}}]_{C_{\mathrm{Lip}}([b,b+\delta];Z)} \le R_1$ and $[w]_{C_{\mathrm{Lip}}([-p,b+\delta];Z)} \le R_1$ because $[w_{|_{[-p,b]}}]_{C_{\mathrm{Lip}}([-p,b];Z)} = [u_{|_{[-p,b]}}]_{C_{\mathrm{Lip}}([-p,b];Z)} \le R_\varphi \le R_1$. Then, $\Gamma(\cdot)$ is a $\mathfrak{Y}$-valued function.

Moreover, arguing as in (3.8), for $t \in [b, b+\delta]$ and $w, v \in \mathfrak{Y}$ we have that

$$
\begin{aligned}
&\|\Gamma w(t) - \Gamma v(t)\| \\
&\quad \le \int_b^t M\mathcal{W}_G(\rho_1)[G]_{(s,s)} \|w^\sigma(s) - v^\sigma(s)\| ds \\
&\quad \le \int_b^t M\mathcal{W}_G(\rho_1)[G]_{(s,s)}(1 + [w]_{C_{\mathrm{Lip}}([-p,b+\delta];Z)} \mathcal{W}_\sigma(\rho_1)[\sigma]_{(s,s)}) \|w - v\|_{C([0,s];Z)} ds \\
&\quad \le M(1+R_1)\mathcal{W}_G(\rho_1)(1+\mathcal{W}_\sigma(\rho_1)) \int_b^t [G]_{(s,s)}(1+[\sigma]_{(s,s)}) ds \|w - v\|_{C([b,b+\delta];Z)} \\
&\quad \le M(1+R_1)\mathcal{W}_G(\rho_1)(1+\mathcal{W}_\sigma(\rho_1)) \Sigma_3(b,\delta) \|w - v\|_{C([b,b+\delta];Z)},
\end{aligned}
$$

and then $\Gamma(\cdot)$ is a contraction and there exists unique mild solution $w \in C_{\mathrm{Lip}}([-p, b+\delta]; Z)$ of (1.1)–(1.2) on $[-p, b+\delta]$ such that $w = u$ on $[-p, b]$.

Proceeding this way, we obtain a maximal "locally Lipschitz" mild solution $u \in C(I_{\max}; Z)$ of (1.1)–(1.2).

Now let $b_\varphi = \sup I_{\max}$. We state that $I_{\max} = [-p, b_\varphi]$ and that $u \in C_{\mathrm{Lip}}([0, b_\varphi]; Z)$. Indeed, note that $\|u^\sigma(t)\| \le \|\varphi\|_{\mathcal{B}} + \|u\|_{C([0,t];Z)}$, for all $t \in [0, b_\varphi)$ and then

$$\begin{aligned}\|u(t)\| &\le \|\mathcal{V}(\cdot)\|\|\varphi(0)\| + \int_0^t \|\mathcal{V}(t-s)G(s,\varphi(-\sigma(0,\varphi)))\|ds \\ &\quad + \int_0^t \|\mathcal{V}(t-s)\|\|G(s,u^\sigma(s)) - G(s,\varphi(-\sigma(0,\varphi)))\|ds \\ &\le M(\|\varphi\|_{\mathcal{B}} + \int_0^t \|G(s,\varphi(-\sigma(0,\varphi)))\|ds) \\ &\quad + M\int_0^t \mathcal{W}_\infty[G]_{(s,s)}(\|u\|_{C([0,s];Z)} + 2\|\varphi\|_{\mathcal{B}})ds,\end{aligned}$$

that is, $u(\cdot)$ is bounded on $[0, b_\varphi)$ and $G(\cdot, u^\sigma(\cdot)) \in L^q([0, b_\varphi); Z) \cap C([0, b_\varphi); Z)$ because $[G]_{(\cdot,\cdot)}$ is a Grönwall-type function.

Moreover, we can adapt the argument in [3, Proposition 6.3] to the resolvent family and show that $u \in C^\alpha([b, b_\varphi); Z)$ and $\lim_{t\to b_\varphi} u(t)$ exists. Let $v : [-p, b_\varphi] \to Z$ be defined by $v(t) = u(t)$ for $t < b_\varphi$ and $v(b_\varphi) = \lim_{t\to b_\varphi} u(t)$. Obviously, $v \in C([0, b_\varphi]; Z)$, $v(\cdot)$ is a mild solution of (1.1)–(1.2) on $[-p, b_\varphi]$ and $v \in C^\alpha([0, b_\varphi]; Z)$. From the above, for $t \in (0, b_\varphi]$ and $\rho = [v]_{C^\alpha([-p,b_\varphi);Z)} + \|v\|_{C([-p,b_\varphi];Z)}$, we have that

$$\begin{aligned}&\int_0^t \|A\mathcal{V}(t-s)(G(s,v^\sigma(s)) - G(t,v^\sigma(t)))\|ds \\ &\le \int_0^t C\mathcal{W}_\infty \frac{[G]_{(t,s)}}{(t-s)}((t-s) + \rho(a^{\alpha-\alpha^2} + \mathcal{W}_\sigma(\rho)^\alpha[\sigma]^\alpha_{(t,s)}(a^{\alpha-\alpha^2}) + \rho^\alpha)(t-s)^{\alpha^2})ds \\ &\le C\mathcal{W}_\infty \sup_{t\in[0,a]} \left\| [G]_{(t,\cdot)} + \frac{[G]_{(t,\cdot)}}{(t-\cdot)^{1-\alpha^2}}\rho(a^{\alpha-\alpha^2} + \mathcal{W}_\sigma(\rho)^\alpha[\sigma]^\alpha_{(t,\cdot)}(a^{\alpha-\alpha^2}) + \rho^\alpha)\right\|_{L^1([0,t])}. \quad (3.15)\end{aligned}$$

Denote by $\mathcal{L}_\infty$ the right hand side of (3.15) and note that

$$\begin{aligned}v(t) &= \mathcal{V}(t)\varphi(0) + \int_0^t \mathcal{V}(t-s)G(t,v^\sigma(t))ds + \int_0^t \mathcal{V}(t-s)(G(s,v^\sigma(s)) - G(t,v^\sigma(t)))ds \\ &= \mathcal{V}(t)\varphi(0) + \int_0^t \mathcal{V}(s)G(t,v^\sigma(t))ds + \int_0^t \mathcal{V}(t-s)(G(s,v^\sigma(s)) - G(t,v^\sigma(t)))ds. \quad (3.16)\end{aligned}$$

Defining $V(t) = \int_0^t \mathcal{V}(s)G(t,v^\sigma(t))ds$ and using the same argument of the [8, Lemma 2.10] we can show the function $AV(t) : [0, b_\varphi] \to Z$ is continuous. Now we infer

$$\begin{aligned}\sup_{t\in[b,b_\varphi]} \|Av(t)\| &\le \sup_{t\in[b,b_\varphi]} \left(\|A\mathcal{V}(t)\varphi(0)\| + \left\|A\int_0^t \mathcal{R}(s)G(t,v^\sigma(t))ds\right\| + \mathcal{L}_\infty\right) \\ &\le \frac{C_1}{b}\|\varphi(0)\| + \|AV(\cdot)\|_{C([0,b_\varphi];Z)} + \mathcal{L}_\infty. \quad (3.17)\end{aligned}$$

We get $v(t) \in D(A)$ for all $t \in (0, b_\varphi]$, and from Corollary 2.12 we conclude that $v(\cdot)$ is a classical solution of (1.1)–(1.2) on $[-p, b_\varphi]$, which implies that $\sup_{t\in[b,b_\varphi]} \|v'(t)\| < \infty$, $v \in C_{\text{Lip}}([-p, b_\varphi]; Z)$, $u(\cdot) = v(\cdot)$ on $[0, b_\varphi]$ and $I_{\max} = [-p, b_\varphi]$.

Assume $b_\varphi < a$. This fact allows us to conclude that there exists $\delta' > 0$ and a unique Lipschitz mild solution $w \in C_{\text{Lip}}([-p, b_\varphi + \delta']; Z)$ of (1.1)–(1.2) on $[-p, b_\varphi + \delta']$ such that $w = u$ on $[-p, b_\varphi]$, which is contrary to the maximality of $u(\cdot)$. Thus, $b_\varphi = a$, $u \in C_{\text{Lip}}([-p, a]; Z)$ and $u(\cdot)$ is the unique mild solution of (1.1)–(1.2) on $[-p, a]$. □

For the next result we use Schauder's fixed point Theorem to investigate the existence of a mild solution.

Proposition 3.13. *Assume that condition $\boldsymbol{H}^{q,r}_{G,\sigma,a}$ holds, $\varphi \in \mathcal{B}$ and $\mathcal{V}(t)$ is compact for all $t > 0$. Then there exists a mild solution $u \in C([-p,b];Z)$ of the problem (1.1)–(1.2) on $[-p,b]$ for some $0 < b \leq a$.*

Proof. Let $r > 0$, $\rho := r + \|\varphi\|_{\mathcal{B}}$ and $\Lambda(r) := \mathcal{W}_G(\rho)\rho$. Since $s \to [G]_{(s,s)}$ is integrable on $[0,t]$ for all $t \in [0,a]$ then

$$\Sigma_4(0,c) := \sup_{t\in[0,c]} \int_0^t [G]_{(s,s)} ds \to 0,$$

as $c \to 0$. Moreover, as $\lim_{t\downarrow 0} \mathcal{V}(t)\varphi(0) = \varphi(0)$, we can choose $b > 0$ such that

$$\|\mathcal{V}(\cdot)\varphi(0) - \varphi(0)\|_{C([0,b];Z)} + M\left(\Sigma_4(0,b)\mathcal{W}_G(\rho)\rho + \|G(\cdot,0)\|_{C([0,a];Z)}b\right) < r. \tag{3.18}$$

Consider the space

$$\mathfrak{Y}(b) = \{u \in C([-p,b];Z) : u_0 = \varphi, \|u(\cdot) - \varphi(0)\|_{C([0,b];Z)} \leq r\},$$

endowed with the metric $d(u,v) = \|u - v\|_{C([0,b];Z)}$ and define $\Gamma : \mathfrak{Y}(b) \to C([-p,b];Z)$ as in the proof of Theorem 3.4.

If $u \in \mathfrak{Y}(b)$, from the definition of $\mathfrak{Y}(b)$, for $s \in [0,b]$ we note that

$$\begin{aligned}\max\{\|u(s)\|, \|u^\sigma(s)\|, \|u_s\|_{\mathcal{B}}\} &\leq \sup_{\tau\in[0,b]} \|u(\tau) - \varphi(0)\| + \|\varphi\|_{\mathcal{B}} \\ &\leq r + \|\varphi\|_{\mathcal{B}} = \rho,\end{aligned}$$

which shows that

$$\begin{aligned}\|G(s,u^\sigma(s))\| &\leq \|G(s,u^\sigma(s)) - G(s,0)\| + \|G(\cdot,0)\|_{C([0,b];Z)} \\ &\leq [G]_{(s,s)}\mathcal{W}_G(\rho)\|u^\sigma(s)\| + \|G(\cdot,0)\|_{C([0,b];Z)} \\ &\leq [G]_{(s,s)}\mathcal{W}_G(\rho)\rho + \|G(\cdot,0)\|_{C([0,b];Z)}.\end{aligned}$$

Using the previous inequalities, it is easy to see that

$$\begin{aligned}\|\Gamma u(t) - \varphi(0)\| &\leq \|\mathcal{V}(t)\varphi(0) - \varphi(0)\| + M\int_0^t [G]_{(s,s)}\mathcal{W}_G(\rho)\rho ds \\ &\quad + M\|G(\cdot,0)\|_{C([0,b];Z)}b \\ &\leq \|\mathcal{V}(\cdot)\varphi(0) - \varphi(0)\|_{C([0,b];Z)} \\ &\quad + M(\Sigma_4(0,b)\mathcal{W}_G(\rho)\rho + \|G(\cdot,0)\|_{C([0,a];Z)}b) \leq r,\end{aligned}$$

which implies that $\Gamma(\cdot)$ is a $\mathfrak{Y}(b)$-valued function.

Furthermore, for $t \in (0,b]$ and $h > 0$ with $t + h < b$, we obtain

$$\begin{aligned}&\|\Gamma u(t+h) - \Gamma u(t)\| \\ &\quad\leq \|\mathcal{V}(t+h)\varphi(0) - \mathcal{V}(t)\varphi(0)\| \\ &\qquad + \int_0^t \|\mathcal{V}(t+h-s) - \mathcal{V}(t-s)\|([G]_{(s,s)}\mathcal{W}_G(\rho)\rho + \|G(\cdot,0)\|_{C([0,b];Z)})ds \\ &\qquad + \int_t^{t+h} M([G]_{(s,s)}\mathcal{W}_G(\rho)\rho + \|G(\cdot,0)\|_{C([0,b];Z)})ds \\ &\quad\leq \|\mathcal{V}(t+h)\varphi(0) - \mathcal{V}(t)\varphi(0)\| \\ &\qquad + \int_0^t \|\mathcal{V}(t+h-s) - \mathcal{V}(t-s)\|([G]_{(s,s)}\mathcal{W}_G(\rho)\rho + \|G(\cdot,0)\|_{C([0,b];Z)})ds \\ &\qquad + M(\|[G]_{(\cdot,\cdot)}\|_{L^1([0,a])}\mathcal{W}_G(\rho)\rho + \|G(\cdot,0)\|_{C([0,b];Z)})h.\end{aligned}$$

We can write the last inequality in the form

$$\|\Gamma u(t+h) - \Gamma u(t)\| \leq \epsilon^1_\varphi(t, t+h) + \int_0^t \epsilon^2_\varphi(t, t+h, s) G(s) ds,$$

where $G(s) = [G]_{(s,s)} \mathcal{W}_G(\rho)\rho + \|G(\cdot, 0)\|_{C([0,b];Z)}$.

Note that $\epsilon^1_\varphi(t, t+h) \to 0$ as $h \to 0$ uniformly for $(t, u) \in [0, b] \times \mathfrak{Y}(b)$. Moreover, since $\epsilon^2_\varphi(t, t+h, s)G(s) \to 0$ for all $s \in [0, t]$ as $h \downarrow 0$, $\epsilon^2_\varphi(t, t+h, s)G(s) \leq MG(s)$ and $MG(\cdot)$ is integrable on $[0, t]$, we conclude that the second term at the right hand side converge to zero as $h \to 0$, uniformly for $u \in \mathfrak{Y}(b)$.

Then, the set $\{\Gamma u : u \in \mathfrak{Y}(b)\}$ is right equicontinuous at t. Similarly, we show the left equicontinuity at $t \in (0, b]$ and the right equicontinuity at $t = 0$.

Now, we need to prove that $\{\Gamma u(t) : u \in \mathfrak{Y}(b)\}$ is relatively compact in Z for all $t \in [0, b]$. Let $t \in (0, b]$ and $\varepsilon > 0$. By noting that

$$\int_{t-\delta}^t \|\mathcal{V}(t-s)\| \|G(s, u^\sigma(s))\| ds \leq M\delta^{\frac{1}{q}} \|G(\cdot)\|_{L^q([0,a])},$$

for $u \in \mathfrak{Y}(b)$, we can select $0 < \delta < t$ such that

$$\int_{t-\delta}^t \|\mathcal{V}(t-s)\| \|G(s, u^\sigma(s))\| ds \leq \varepsilon, \qquad \forall u \in \mathfrak{Y}(b).$$

From the continuity of $\mathcal{V}(\cdot)$ on $[\delta, t]$, we can choose $N \in \mathbb{N}$ and numbers $0 < t_1 < t_2 < \ldots < t_N = t - \delta < t_{N+1} = t$ such that $\|\mathcal{V}(t-s) - \mathcal{V}(t-t_i)\| \leq \frac{\varepsilon}{1+R}$ for all $s \in [t_i, t_{i+1}]$ and $i = 1, \ldots, N-1$.

Under the previous conditions and remarking that

$$\int_\mu^{t_{i+1}} \|G(s, u^\sigma(s))\| ds \leq \int_0^t \|G(s)\| ds \leq \Theta := b^{\frac{1}{q}} \|G(\cdot)\|_{L^q([0,a])},$$

for all $i = 1, \ldots, N-1$ and every $\mu < t$, for $u \in \mathfrak{Y}(b)$ we obtain

$$\begin{aligned} \Gamma u(t) &= \mathcal{V}(t)\varphi(0) + \int_{t_N}^t \mathcal{V}(t-s) G(s, u^\sigma(s)) ds + \sum_{i=1}^{N-1} \mathcal{V}(t-t_i) \int_{t_i}^{t_{i+1}} G(s, u^\sigma(s)) ds \\ &\quad + \sum_{i=1}^{N-1} \int_{t_i}^{t_{i+1}} (\mathcal{V}(t-s) - \mathcal{V}(t-t_i)) G(s, u^\sigma(s)) ds \\ &\in \mathcal{V}(t)\varphi(0) + B_\varepsilon(0; Z) + \sum_{i=1}^{N-1} \mathcal{V}(t-t_i) B_\Theta(0; Z) + \varepsilon B_\Theta(0; Z), \end{aligned}$$

and hence, $\{\Gamma u(t) : u \in \mathfrak{Y}(b)\} \subset D_\varepsilon + K_\varepsilon$, where the diameter of D_ε go to 0 as $\varepsilon \to 0$ and K_ε is relatively compact in Z. Then, $\{\Gamma u(t) : u \in \mathfrak{Y}(b)\}$ is relatively compact in Z.

By hypothesis $\boldsymbol{H}^{q,r}_{G,\sigma,a}$ it follows that $\Gamma(\cdot)$ is continuous and from the previous commentaries we can conclude that $\Gamma(\cdot)$ is completely continuous. Therefore, from Schauder's Fixed Point Theorem, there exists a mild solution $u \in C([-p, b]; Z)$ of (1.1)–(1.2) on $[-p, b]$. □

4 Well-posedness

To begin our study on the local well-posedness of the problem (1.1)–(1.2), we first introduce the next concept.

Definition 4.1. Let $(\mathcal{D}, \|\cdot\|_{\mathcal{D}}) \hookrightarrow (\mathcal{B}, \|\cdot\|_{\mathcal{B}})$ be a Banach space and $a > 0$. We say that the problem (1.1)–(1.2) is locally well-posed related to the spaces $(\mathcal{D}, \|\cdot\|_{\mathcal{D}})$ and $C([0,a]; Z)$, if for all $\varphi \in \mathcal{D}$ and $r > 0$, there exists $c_\varphi > 0$ such that, for all $\psi \in B_r(\varphi, \mathcal{D})$ there exists a unique mild solution $u(\cdot, \psi) \in C([-p, c_\varphi]; Z)$ of the problem (1.1)–(1.2) on $[-p, c_\varphi]$ and

$$\|u(\cdot, \varphi) - u(\cdot, \psi)\|_{C([0,c_\varphi];Z)} \to 0 \quad \text{as } \|\varphi - \psi\|_{\mathcal{D}} \to 0. \tag{4.1}$$

Combining the proofs of Theorem 3.4 and Corollary 3.6, we have the next result, with

$$\mathcal{D} = \mathcal{D}_a = \{\psi \in C_{\text{Lip}}([-p,0]; Z) : \mathcal{V}(\cdot)\psi(0) \in C_{\text{Lip}}([0,a]; Z),\ \psi(0) \in E\},$$

endowed with the metric $d(\psi_1, \psi_2) = \|\psi_1 - \psi_2\|_{C_{\text{Lip}}([-p,0];Z)} + \|\mathcal{V}(\cdot)(\psi_1(0) - \psi_2(0))\|_{C_{\text{Lip}}([0,a];Z)}$.

Proposition 4.2. *Suppose that condition $\boldsymbol{H^{q,r}_{G,\sigma,a}}$ is satisfied and $\Theta(q, r) < 1$. Then the problem (1.1)–1.2 is locally well-posed related to the spaces $(\mathcal{D}_a, \|\cdot\|_{\mathcal{D}_a})$ and $C([0,a]; Z)$.*

Proof. Let $\varphi \in \mathcal{D}_a$, $r > 0$ and $R^*_\varphi > 0$ be such that

$$R^*_\varphi > \Lambda := \sup_{\psi \in B_r(\varphi;\mathcal{D}_a)} \Big([\mathcal{V}(\cdot)\psi(0)]_{C_{\text{Lip}}([0,a];Z)} + \|\psi\|_{C_{\text{Lip}}([-p,0];Z)} + M\|G(0, \psi(-\sigma(0,\psi)))\| \Big).$$

Since $\Sigma_i(c) \to 0$ as $c \to 0$ for $i = 1, 2$, we can choose $0 < c_\varphi \le a$ such that

$$\Lambda + (1 + R^*_\varphi)^2 \mathcal{W}_G(\rho^*)(1 + \mathcal{W}_\sigma(\rho^*)) \left(\Sigma_1(c_\varphi) + \Sigma_2(c_\varphi)\right) < R^*_\varphi,$$

where $\rho^* := R^*_\varphi(a + 2p) + \|\varphi\|_{\mathcal{B}} + r$.

Consider $\psi \in B_r(\varphi, \mathcal{D}_a)$. Proceeding as in the proof of Theorem 3.4, with $\psi(\cdot)$ instead of $\varphi(\cdot)$, we can show that there exists a unique mild solution $u(\cdot, \psi) \in C_{\text{Lip}}([-p, c_\varphi]; Z)$ of the problem (1.1) with initial condition $\psi(\cdot)$. In addition, it follows that $[u(\cdot, \psi)]_{C_{\text{Lip}}([-p,c_\varphi];Z)} \le R^*_\varphi$ and $\|u(\cdot, \psi)\|_{C([-p,c_\varphi];Z)} \le \rho^*$.

We define $\mathcal{K}(\cdot, R^*_\varphi) = [G]_{(\cdot,\cdot)}(1 + [\sigma]_{(\cdot,\cdot)})$. Let $\psi \in B_r(\varphi; \mathcal{D}_a)$, $u(\cdot) = u(\cdot, \varphi)$ and $v(\cdot) = u(\cdot, \psi)$. From the above remarks and the last inequality in Lemma 2.2, for $t \in [0, c_\varphi]$ we get

$$\begin{aligned}
&\|u(t) - v(t)\| \\
&\quad \le \|\mathcal{V}(t)\varphi(0) - \mathcal{V}(t)\psi(0)\| + \int_0^t \|\mathcal{V}(t-s)\| \|G(s, u^\sigma(s)) - G(s, v^\sigma(s))\| ds \\
&\quad \le d(\varphi, \psi) + \mathcal{W}_G(\rho^*) \int_0^t M[G]_{(s,s)} \|u^\sigma(s) - v^\sigma(s)\| ds \\
&\quad \le d(\varphi, \psi) + \mathcal{W}_G(\rho^*) \int_0^t M[G]_{(s,s)} (1 + R^*_\varphi \mathcal{W}_\sigma(\rho^*)[\sigma]_{(s,s)}) \|u - v\|_{C([0,s];Z)} ds \\
&\quad \le d(\varphi, \psi) + (1 + R^*_\varphi)\mathcal{W}_G(\rho^*)(1 + \mathcal{W}_\sigma(\rho^*))M \int_0^t \mathcal{K}(s, R^*_\varphi) \|u - v\|_{C([0,s];Z)} ds,
\end{aligned}$$

and hence,

$$\|u - v\|_{C([0,t];Z)} \le d(\varphi, \psi) e^{(1+R^*_\varphi)\mathcal{W}_G(\rho^*)(1+\mathcal{W}_\sigma(\rho^*))M\|\mathcal{K}(\cdot,R^*_\varphi)\|_{L^1([0,c_\varphi])}}.$$

Therefore, there exists $\Lambda > 0$ such that $\|u(\cdot, \varphi) - v(\cdot, \psi)\|_{C([0,c_\varphi];Z)} \le \Lambda d(\varphi, \psi)$, for all $\psi \in B_r(\varphi; \mathcal{D})$, which completes the proof. □

Remark 4.3. Every result proved in this work can be easily adapted for the integro-differential equations

$$\frac{du(t)}{dt} = Au(t) + \int_0^t Q(t-s)u(s)ds + G(t, u(t - \sigma(t, u_t))), t \in [0, a], \tag{4.2}$$

$$u_0 = \varphi \in \mathcal{B} = C([-p, 0]; Z), \tag{4.3}$$

where A and $Q(t)$ for $t \geq 0$ are closed linear operators defined on a common domain $D(A)$ that is dense in Z where the functions $G(\cdot)$ and $\sigma(\cdot)$ are continuous. Simply use the resolvent theory for integro-differential equations developed by Grimmer et al. [11–13].

5 Memory and diffusion-coupled system

To finish this paper, we study now the existence of solutions for a problem concerning the coupled memory and diffusive integro-differential system with state-dependent delay. Specifically, we consider the integro-differential system

$$\begin{aligned} &\frac{\partial}{\partial t}\left[u(t,x)+\int_0^t (t-s)^{\alpha}e^{-\omega(t-s)}v(s,x)ds\right] \\ &\qquad = \Delta u(t,x)+\int_0^t B(t-s)\Delta v(s,x)ds+\zeta(t)\int_{\Omega} b(u(\varrho(t,u_t),y))f(x-y)dy, &(5.1)\\ &\frac{\partial}{\partial t}\left[v(t,x)+\int_0^t (t-s)^{\alpha}e^{-\gamma(t-s)}u(s,x)ds\right] \\ &\qquad = \Delta v(t,x)+\int_0^t C(t-s)\Delta u(s,x)ds+g(v(\varrho(t,u_t),x)), &(5.2)\\ &u(t,\cdot)|_{\partial\Omega}=0, \;\; v(t,\cdot)|_{\partial\Omega}=0, &(5.3)\\ &u(\theta,y)=\varphi(\theta,y), \;\; v(\theta,y)=\phi(\theta,y), \qquad \theta\in[-p,0],\, y\in\Omega, &(5.4)\end{aligned}$$

for $t \in [0,a]$ and $x \in \Omega$, where $f \in C(\widetilde{\Omega};\mathbb{R})$, $\widetilde{\Omega} = \{x-y : x,y \in \Omega\}$, $g \in C_{\mathrm{Lip}}(\mathbb{R}^n;\mathbb{R}^n)$, $b \in C_{\mathrm{Lip}}(\mathbb{R}^n;\mathbb{R}^n)$, $\varrho \in C_{\mathrm{Lip}}([0,a]\times\mathcal{B};\mathbb{R}), d-p \leq \varrho(t,\psi) \leq t$ for all $(t,\psi) \in [0,a]\times\mathcal{B}$ and $\zeta \in C_{\mathrm{Lip}}(\mathbb{R};\mathbb{R})$.

To show the existence of solutions for the problem (5.1)–(5.4) we consider the operators $A, Q(t) : D(A) \subseteq X \to X$, $t \geq 0$, where $X = L^2(\Omega;\mathbb{R})$ given by $Ax = \Delta x$, for $x \in D(A) = H^2(\Omega)\cap H^1_0(\Omega)$ and $P_\omega(t)x = t^\alpha e^{-\omega t}x$ and $P_\gamma(t)x = t^\alpha e^{-\gamma t}x$ for $x \in X$. The operator A is the infinitesimal generator of an analytic semigroup and for all $\vartheta \in (\pi/2,\pi)$ there exists constant $M_\vartheta > 0$ such that $\|R(\lambda,A)\| \leq M_\vartheta|\lambda|^{-1}$ for all $\lambda \in \Lambda_\vartheta$. We assume that $B(\cdot)$ and $C(\cdot)$ in $L^1_{\mathrm{loc}}(\mathbb{R}^+)$, furthermore $B(\cdot)$ and $C(\cdot)$ has Laplace transformed absolutely convergent for $Re(\lambda) > 0$ which analytical extension to Λ_ϑ and $|\widehat{B}(\lambda)|+|\widehat{C}(\lambda)| = O(\frac{1}{|\lambda|})$, as $|\lambda| \to \infty$. To study this problem, we define the functions $\sigma : [0,\infty)\times\mathcal{B} \to \mathbb{R}$ by $\sigma(t,\psi) = t-\varrho(t,\psi)$, $F_1(t,u)(x) = \zeta(t)\int_\Omega b(u(y))f(x-y)dy$ and $F_2(u)(x) = g(u(x))$.

Let $\mathcal{U} = \left(\begin{smallmatrix} u \\ v\end{smallmatrix}\right)$ in this problem the space Z is defined by $Z = L^2(\Omega;\mathbb{R})\times L^2(\Omega;\mathbb{R})$ under the norm $\|\left(\begin{smallmatrix} u \\ v\end{smallmatrix}\right)\| = \left(\int_\Omega |u|^2+|v|^2dx\right)^{\frac{1}{2}}$, it is easy to see that Z is a Hilbert space. We define the $D(\mathcal{A})$ by $D(\mathcal{A}) = \left\{\left(\begin{smallmatrix} u \\ v\end{smallmatrix}\right) \in X : u,v \in H^2(\Omega)\cap H^1_0(\Omega)\right\}$, and $\mathcal{G} : [0,a]\times Z \to Z$

$$\begin{aligned}&\frac{\partial}{\partial t}\left[\begin{pmatrix} u(t,x)\\ v(t,x)\end{pmatrix}+\int_0^t \begin{pmatrix} 0 & (t-s)^\alpha e^{-\omega(t-s)}\\ (t-s)^\alpha e^{-\gamma(t-s)} & 0\end{pmatrix}\begin{pmatrix} u(s,x)\\ v(s,x)\end{pmatrix}ds\right]\\ &\quad = \begin{pmatrix}\Delta & 0\\ 0 & \Delta\end{pmatrix}\begin{pmatrix} u(t,x)\\ v(t,x)\end{pmatrix}+\int_0^t\begin{pmatrix} 0 & B(t-s)\Delta\\ C(t-s)\Delta & 0\end{pmatrix}\begin{pmatrix} u(s,x)\\ v(s,x)\end{pmatrix}ds\\ &\qquad +\begin{pmatrix} F_1(t,u^\sigma(t,x))\\ F_2(t,v^\sigma(t,x))\end{pmatrix}.\end{aligned}$$

Therefore, we can represent the system (5.1)–(5.4) in the abstract form

$$\frac{\partial}{\partial t}\left(\mathcal{U}(t)+\int_0^t \mathcal{P}(t-s)\mathcal{U}(s)ds\right)=\mathcal{A}\mathcal{U}(t)+\int_0^t \mathcal{Q}(t-s)\mathcal{U}(s)ds+\mathcal{G}(t,\mathcal{U}^{\sigma}(t)), \tag{5.5}$$

$$\mathcal{U}_0=\begin{pmatrix}\varphi\\ \phi\end{pmatrix}\in\mathcal{B}\times\mathcal{B}, \tag{5.6}$$

where

$$\mathcal{A}=\begin{pmatrix}\Delta & 0\\ 0 & \Delta\end{pmatrix},\ \mathcal{P}(t)=\begin{pmatrix}0 & P_{\omega}(t)\\ P_{\gamma}(t) & 0\end{pmatrix},\ \mathcal{Q}(t)=\begin{pmatrix}0 & A(t)\Delta\\ B(t)\Delta & 0\end{pmatrix},\ \text{and}$$

$$\mathcal{G}(t,\mathcal{U}^{\sigma}(t))=\begin{pmatrix}F_1(t,u^{\sigma}(t,x))\\ F_2(t,v^{\sigma}(t,x))\end{pmatrix}.$$

Now we show that $\mathcal{A}$ is a sectorial operator, this fact follows that

$$(\lambda-\mathcal{A})\begin{pmatrix}u_1\\ u_2\end{pmatrix}=\begin{pmatrix}f_1,\\ f_2.\end{pmatrix},$$

this is equivalent to

$$\begin{cases}(\lambda-A)u_1=f_1,\\ (\lambda-A)u_2=f_2.\end{cases}$$

Assume that $|\arg(\lambda)|<\vartheta$. Then $\lambda\in\rho(\mathcal{A})$, and the resolvent is written as

$$\begin{cases}u_1=(\lambda-A)^{-1}f_1,\\ u_2=(\lambda-A)^{-1}f_2.\end{cases}$$

we get

$$\begin{pmatrix}u_1\\ u_2\end{pmatrix}=\begin{pmatrix}(\lambda-A)^{-1} & 0\\ 0 & (\lambda-A)^{-1}\end{pmatrix}\begin{pmatrix}f_1\\ f_2\end{pmatrix}.$$

Now we can show

$$\begin{aligned}\|(\lambda I-\mathcal{A})^{-1}\| &\le \|(\lambda-A)^{-1}\|+\|(\lambda-A)^{-1}\|\\ &\le 2M_{\vartheta}|\lambda|^{-1}.\end{aligned}$$

The previous argument shows that $\mathcal{A}$ is a sectorial operator.

From the definitions of $\mathcal{Q}(t)$ we have for $\mathcal{U}\in D(\mathcal{A})$

$$\begin{aligned}\|\mathcal{Q}(t)\mathcal{U}\|^2 &= \left\|\begin{pmatrix}0 & A(t)\Delta x\\ B(t)\Delta x & 0\end{pmatrix}\begin{pmatrix}u\\ v\end{pmatrix}\right\|^2\\ &= \int_{\Omega}\|B(t)\Delta v\|^2+\|A(t)\Delta u\|^2dx\\ &\le (A(t)+B(t))^2\left(\int_{\Omega}\|\Delta v\|^2dx+\int_{\Omega}\|\Delta u\|^2dx\right)\\ &\le (A(t)+B(t))^2(\|\mathcal{A}\mathcal{U}\|+\|\mathcal{U}\|)^2.\end{aligned}$$

Therefore

$$\|\mathcal{Q}(t)U\|\le k(t)(\|\mathcal{A}\mathcal{U}\|+\|\mathcal{U}\|), \tag{5.7}$$

where $k(t) = A(t) + B(t)$. From (5.7) we obtain the Laplace transform of $(\mathcal{Q}(t))_{t\geq 0}$ is absolutely convergent for $Re(\lambda) > 0$, admits an analytical extension to Λ_ϑ and

$$\|\widehat{\mathcal{Q}}(\lambda)\|_{\mathcal{L}([D(\mathcal{A}),X])} = O\left(\frac{1}{|\lambda|}\right), \quad \text{as } |\lambda| \to \infty.$$

By the definitions of $\mathcal{P}(t)$ we get for $\mathcal{U} \in Z$

$$\begin{aligned}
\|\mathcal{P}(t)\mathcal{U}\|^2 &= \left\|\begin{pmatrix} 0 & P_\omega(t) \\ P_\gamma(t) & 0 \end{pmatrix}\begin{pmatrix} u \\ v \end{pmatrix}\right\|^2 \\
&= \int_\Omega \|P_\gamma(t)u\|^2 + \|P_\omega(t)v\|^2 dx \\
&\leq (P_\omega(t) + P_\gamma(t))^2 \left(\int_\Omega \|u\|^2 dx + \int_\Omega \|v\|^2 dx\right) \\
&\leq (P_\omega(t) + P_\gamma(t))^2 \|\mathcal{U}\|^2.
\end{aligned}$$

It is clear that the function $\mathcal{P} : [0,\infty) \to \mathcal{L}(Z)$ is strongly continuous and $\widehat{\mathcal{P}}(\lambda)z$ is absolutely convergent for $z \in Z$ and $Re(\lambda) > 0$. There exists an analytical extension of $\widehat{\mathcal{P}}(\lambda)$ (still denoted by $\widehat{\mathcal{P}}(\lambda)$) to Λ_ϑ such that $\|\widehat{\mathcal{P}}(\lambda)\| \leq N_0|\lambda|^{-\alpha-1}$ for every $\lambda \in \Lambda_\vartheta$, and $\|\widehat{\mathcal{P}}(\lambda)z\| \leq N_1|\lambda|^{-1}\|z\|_1$ for every $\lambda \in \Lambda_\vartheta$ and $z \in D(\mathcal{A})$. As well as $D = C_0^\infty(\Omega) \times C_0^\infty(\Omega)$, where $C_0^\infty(\Omega)$ is the space of infinitely differentiable functions that vanish at $\partial\Omega$. It is easy to see that conditions **(C1)–(C4)** are satisfied with the previous conditions.

By the [8, Theorem 2.] we obtain the next result.

Proposition 5.1. *Assume that the above conditions are fulfilled. Then, there exists an analytical resolvent operator family $(\mathcal{V}(t))_{t\geq 0}$ associated to (5.5)–(5.6).*

Next, we prove that $\mathcal{G}(\cdot)$ is a L^q- Lipschitz function. Assume $(\int_\Omega \int_\Omega f(x-y)^2 dydx) < \infty$. It is easy to see that

$$\begin{aligned}
&\|F_1(t,u) - F_1(s,v)\| \\
&\quad \leq (\|b\|_{C_{\text{Lip}}} \int_\Omega \int_\Omega f(x-y)^2 dydx) m(\Omega))(\|u\|)([\zeta]_{(t,s)} + \zeta(s))(|t-s| + \|u-v\|), \\
&\qquad \|F_2(t,u) - F_2(s,v)\| \leq \|g\|_{C_{\text{Lip}}}(|t-s| + \|u-v\|),
\end{aligned}$$

where $m(\Omega)$ represent the measure of Ω. Thus, $\mathcal{G}(\cdot)$ is a L^q-Lipschitz function, the condition $\boldsymbol{H^{q,r}_{G,\sigma,a}}$ is satisfied with $W_\sigma(r) = 1$, $[\sigma]_{C_{\text{Lip}}} = [\varrho]_{C_{\text{Lip}}} + 1$ and

$$\begin{aligned}
&\mathcal{W}_{\mathcal{G}}(r) = r + 1, \\
&[\mathcal{G}]_{(t,s)} = (\|b\|_{C_{\text{Lip}}} \int_\Omega \int_\Omega f(x-y)^2 dydx) m(\Omega) + \|g\|_{C_{\text{Lip}}})([\zeta]_{(t,s)} + \zeta(s)).
\end{aligned}$$

We have the next result.

Proposition 5.2. *Assume $q > 1$ and $\mathcal{V}(\cdot)\varphi(0) \in C_{\text{Lip}}([0,a];Z)$.*

(a) If $\sup_{b\in[0,c],d,h\in[0,b],d+h\leq b} \|[\zeta]_{(\cdot+h,\cdot)}\|_{L^q([0,d])} < \infty$, for some $0 < c \leq a$ or $\zeta(\cdot)$ is Lipschitz, then there exists a unique mild solution $u \in C_{\text{Lip}}([-p,b];Z)$ of the problem (5.1)–(5.4) on $[-p,b]$ for some $0 < b \leq a$.

(b) If $\sup_{b\in[0,c],d,h\in[0,b],d+h\leq b} \|[\zeta]_{(\cdot+h,\cdot)}\|_{L^q([0,d])} < \infty$, for some $0 < c \leq a$ or $\zeta(\cdot)$ is Lipschitz, then if $\varphi(0,\cdot)$ and $\phi(0,\cdot) \in H^2(\Omega) \cap H^1_0(\Omega)$, then there exists a unique classical solution $u(\cdot)$ of the problem (5.1)–(5.4) on $[-p,b]$ for some $0 < b \leq a$.

(c) *If* $\sup_{b\in[0,c],d,h\in[0,b],d+h\le b}\|[\zeta]_{(\cdot+h,\cdot)}\|_{L^q([0,d])}<\infty$, *for some* $0<c\le a$ *or* $\zeta(\cdot)$ *is Lipschitz, moreover, if* $\zeta(\cdot)$ *is Lipschitz and* $1>\frac{4\beta(\theta+2\beta)}{(3\beta-1)^2}$. *where* $\theta:=[\mathcal{V}(\cdot)\varphi(0)]_{C_{\mathrm{Lip}}([0,a];Z)}+\|\varphi\|_{C_{\mathrm{Lip}}([-p,0];Z)}+M\|G(0,\varphi(-\sigma(0,\varphi))\|$ *and* $\beta:=(\|b\|_{C_{\mathrm{Lip}}}\int_\Omega\int_\Omega f(x-y)^2dydx)m(\Omega)+\|g\|_{C_{\mathrm{Lip}}})Ma$, *then there exists a unique mild solution* $u\in C_{\mathrm{Lip}}([-p,a];X)$ *of* (5.1)–(5.4) *on* $[-p,a]$.

Proof. The first assumption (a) follows from Corollary 3.6.

It is well-known that every reflexive Banach space has the Radon–Nikodym property (RNP). Since Hilbert spaces are reflexive, they automatically enjoy the RNP. By $\varphi(0,\cdot),\phi(0,\cdot)\in H^2(\Omega)\cap H^1_0(\Omega)$, $[\mathcal{G}]_{(t,s)}$ and $\zeta(\cdot)$ are continuous, the second assumption (b) follows from Proposition 3.5.

The assumption (c), the existence of a Lipschitz mild solution defined on $[-p,a]$ follows from Proposition 3.8 noting that the function $P_a:\mathbb{R}\mapsto\mathbb{R}$ given by

$$\begin{aligned}\mathcal{P}_a(x)&=[\mathcal{V}(\cdot)\varphi(0)]_{C_{\mathrm{Lip}}([0,a];Z)}+\|\varphi\|_{C_{\mathrm{Lip}}([-p,0];Z)}+M\|G(0,\varphi(-\sigma(0,\varphi))\|\\&\quad+(1+x)^2(\|b\|_{C_{\mathrm{Lip}}}\int_\Omega\int_\Omega f(x-y)^2dydx)m(\Omega)+\|g\|_{C_{\mathrm{Lip}}})Ma\\&\quad+(1+x)(\|b\|_{C_{\mathrm{Lip}}}\int_\Omega\int_\Omega f(x-y)^2dydx)m(\Omega)+\|g\|_{C_{\mathrm{Lip}}})Ma\\&=\theta+(1+x)^2\beta+(1+x)\beta-x,\end{aligned}$$

has two positive root if $1>\frac{4\beta(\theta+2\beta)}{(3\beta-1)^2}$, the results follow by Corollary 2.12 □

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7 Conflict of interest

The authors declare that they have no conflicts of interest.

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