



Positive solutions for semipositone Φ -Laplacian involving nonlocal term in Orlicz–Sobolev space

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Abstract. This paper establishes the existence of positive weak solutions for a class of semipositone Kirchhoff-type problems involving nonlocal operators and anisotropic Φ -Laplacians. By combining variational methods with careful asymptotic analysis, we prove that for sufficiently small $a > 0$, the problem admits a positive solution. Our results extend previous work on semipositone problems to the nonlocal Kirchhoff setting, overcoming challenges arising from the interplay between the nonlocal term and the indefinite nonlinearity.

Keywords: Kirchhoff type problem, semipositone nonlinearity, Orlicz–Sobolev spaces, variational methods, positive solutions.

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1 Introduction

Semipositone problems, where the nonlinearity satisfies $\hbar(0) < 0$ (see (1.2)), arise naturally in many physical and biological contexts, including population dynamics with harvesting and chemical reaction processes. These problems present substantial mathematical challenges compared to their positone counterparts, as the standard techniques based on sub-super solutions often fail due to the lack of a natural subsolution at zero (see [2–5, 7–15, 19–23, 26, 30]).

In 2021, Figueiredo, Massa and Santos [16], proved the existence of positive solution for the semipositone problem

$$\begin{cases} -M(\int_Y |\nabla u|^2 dx) \Delta u = \hbar(u) - a & \text{in } Y, \\ u(x) > 0 & \text{in } Y, \\ u = 0 & \text{on } \partial Y, \end{cases} \quad (1.1)$$

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where $Y \subset \mathbb{R}^N$ is a smooth bounded domain, M, f are continuous nonnegative functions and $a > 0$ and small enough. They generalize the classical superlinear and sublinear problems. Recently, Razani and Figueiredo [30] studied a Dirichlet semipositone anisotropic p -Laplacian problem.

In this work, we investigate the existence of positive solutions for a class of nonlocal elliptic problems of Kirchhoff type with semipositone nonlinearities. Specifically, we consider:

$$\begin{cases} -\Gamma \left(\int_Y \Phi(u) dx \right) \Delta_\Phi u = \hbar(u) - \varepsilon & \text{in } Y, \\ u > 0 & \text{in } Y, \\ u = 0 & \text{on } \partial Y, \end{cases} \quad (1.2)$$

where Δ_Φ denotes the Φ -Laplacian operator, a generalization of the p -Laplacian that arises in the study of non-Newtonian fluids and nonlinear elasticity. The term $\Gamma \left(\int \Phi(u) dx \right)$ introduces a nonlocal dependence on the solution, modeling phenomena where the system's behavior at a point depends on its global state.

Here we assume $Y \subset \mathbb{R}^N$ is a smooth bounded domain, $\varepsilon > 0$ and $\hbar : [0; +\infty) \rightarrow \mathbb{R}$ is a continuous function where

$$(\hbar_1) \quad 0 = \hbar(0) = \min_{t \in [0, +\infty)} \hbar(t).$$

$$(\hbar_2) \quad \lim_{t \rightarrow 0^+} \frac{\hbar(t)}{\phi(t)t} = 0.$$

$$(\hbar_3) \quad \text{There is } q \in (m, l^*) \text{ such that}$$

$$\lim_{|t| \rightarrow +\infty} \frac{|\hbar(t)|}{|t|^{q-1}} < +\infty.$$

$$(\hbar_4) \quad \text{There are } \theta > m \text{ and } t_0 > 0 \text{ such that}$$

$$\theta H(t) \leq \hbar(t)t, \quad \text{for all } t \geq t_0,$$

where $H(t) = \int_0^t \hbar(\tau) d\tau$ is the primitive of $\hbar$.

Notice that

$$\Delta_\Phi(u) = \operatorname{div}(\phi(|\nabla u|)\nabla u)$$

where $\Phi(t) := \int_0^{|t|} \phi(s) ds$ and ϕ satisfies the following conditions:

$$(\phi_1) \quad \phi : \mathbb{R}^+ \rightarrow \mathbb{R}^+ \text{ is a } C^1\text{-function.}$$

$$(\phi_2) \quad \phi(t), (\phi(t)t)' > 0, \quad t > 0$$

$$(\phi_3) \quad \text{There exists } l, m \in (1, N) \text{ with } m \in [l, l^*) \text{ and } l^* = \frac{IN}{N-1} \text{ such that}$$

$$l \leq \frac{\Phi'(t)t}{\Phi(t)} \leq m, \quad \forall t > 0$$

$$(\phi_4) \quad \text{There exists } \bar{l}, \bar{m} > 0 \text{ such that}$$

$$\bar{l} \leq \frac{\Phi''(t)t}{\Phi'(t)} \leq \bar{m}, \quad \text{for all } t > 0.$$

Assume $\Gamma : [0, +\infty) \rightarrow \mathbb{R}^+$ is an increasing and continuous function and $\Gamma(t) \geq m_0 > 0$ for all $t \in [0, +\infty)$. We set $F(t) = \int_0^t \Gamma(\tau) d\tau$ and we assume

$$(\Gamma_0) \liminf_{t \rightarrow 0} \frac{F(t)}{t} > 0.$$

$$(\Gamma_\infty) \text{ There exists } \theta_1 \in (1, \theta) \text{ such that } \limsup_{t \rightarrow +\infty} \frac{F(t)}{t^{\theta_1}} < \infty.$$

(Γ_{AR}) There exists $\theta > 1$, $D, \beta > 0$ such that

$$\theta F(t) - \Gamma(t)t \geq \beta t - D \quad \text{for every } t \geq 0,$$

Definition 1.1. A function $u \in W_0^{1,\Phi}(Y)$ is called a weak solution of problem (1.2) if:

- (i) u is continuous and positive in Y ;
- (ii) For every test function $\varphi \in W_0^{1,\Phi}(Y)$, the following identity holds:

$$\Gamma \left(\int_Y \Phi(u) dx \right) \int_Y \phi(|\nabla u|) \nabla u \cdot \nabla \varphi dx = \int_Y (\hbar(u) - \varepsilon) \varphi dx,$$

where $\hbar$ and Γ satisfy the assumptions stated above.

The mathematical study of semipositone problems was initiated by Castro, Castro, and Shivaji [13], who revealed fundamental differences from positone problems, including the breakdown of symmetry properties and more complex solution structures. Subsequent works [4, 5, 12] developed various techniques to handle these challenges, particularly for local operators. For problems of Kirchhoff type, recent progress has been made in [2, 16, 27, 29], but the semipositone case with general Φ -Laplacian operators remains largely unexplored.

Our work makes several significant contributions:

- We extend the existence theory for semipositone problems to the nonlocal Kirchhoff setting with anisotropic Φ -Laplacian operators.
- We develop new estimates to handle the interplay between the nonlocal coefficient Γ and the semipositone nonlinearity.
- We establish precise threshold values for the parameter ε that guarantee existence of positive solutions.
- Our framework accommodates both degenerate and non-degenerate cases of the Kirchhoff term.

The proof strategy combines variational methods in Orlicz–Sobolev spaces with careful asymptotic analysis. Key challenges include:

- Establishing the mountain pass geometry despite the semipositone nature of the problem.
- Deriving uniform L^∞ estimates independent of the parameter ε .
- Handling the nonlocal coefficient's effect on the compactness properties.
- Ensuring positivity of solutions through delicate comparison arguments.

Our results generalize and extend previous works in several directions, including [5] (local Φ -Laplacian case), [16] (Kirchhoff Laplacian), and [30] (anisotropic semipositone problems). The techniques developed here may be applicable to other nonlocal problems with indefinite nonlinearities.

The paper is organized as follows: Section 2 introduces the necessary framework of Orlicz–Sobolev spaces. Section 3 establishes existence for an auxiliary problem, while Section 4 completes the proof of our main result through careful asymptotic analysis and positivity arguments.

2 Preliminaries on Orlicz–Sobolev spaces

This section provides the necessary background on Orlicz and Orlicz–Sobolev spaces, which serve as the functional framework for our analysis. We recall fundamental definitions, key properties, and essential embedding results that will be crucial for our subsequent arguments. For a comprehensive treatment of these topics, we refer to the standard references [1] and [28].

We begin by recalling key properties of the Orlicz–Sobolev space $W^{1,\Phi}(Y)$, which becomes a Banach space when equipped with the standard norm

$$\|u\|_{1,\Phi} = \|u\|_{\Phi} + \|\nabla u\|_{\Phi}. \quad (2.1)$$

Its subspace $W_0^{1,\Phi}(Y)$, defined as the completion of $C_0^\infty(Y)$ under this norm, inherits these Banach space properties. Crucially, both spaces are separable and reflexive when Φ and its conjugate $\tilde{\Phi}$ satisfy the Δ_2 -condition. This condition further characterizes convergence in $L_\Phi(Y)$: a sequence $\{u_n\}$ converges to u if and only if

$$\int_Y \Phi(|u_n - u|) dx \rightarrow 0. \quad (2.2)$$

Three fundamental inequalities govern these spaces:

1. **Poincaré inequality:** There exists $\Lambda > 0$ such that for all $u \in W_0^{1,\Phi}(Y)$,

$$\int_Y \Phi(|u|) dx \leq \Lambda \int_Y \Phi(|\nabla u|) dx. \quad (2.3)$$

This implies $\|u\| := \|\nabla u\|_{\Phi}$ is an equivalent norm on $W_0^{1,\Phi}(Y)$, which we adopt hereafter.

2. **Sobolev embedding:** There exists $S_N > 0$ such that

$$\|u\|_{\Phi_*} \leq S_N \|u\|, \quad (2.4)$$

ensuring the continuous embedding $W_0^{1,\Phi}(Y) \hookrightarrow L_{\Phi_*}(Y)$. For bounded domains, this embedding becomes compact if the N -function B satisfies the growth condition

$$\limsup_{|t| \rightarrow \infty} \frac{B(t)}{\Phi_*(t)} = 0. \quad (2.5)$$

3. **Structural assumptions:** For our analysis, we assume $\phi(s) = \varphi(s)s$ and define

$$\Phi(t) = \int_0^{|t|} \varphi(s)s ds, \quad (2.6)$$

extended continuously at $t = 0$. Under conditions (φ_1) , (φ_2) , (φ_3) , both Φ and $\tilde{\Phi}$ become N -functions satisfying the Δ_2 -condition, as established in [17].

Lemma 2.1 (Growth estimates). *Under assumptions (φ_1) and (φ_2) , define the comparison functions*

$$\zeta(t) = \min\{t^l, t^m\} \quad \text{and} \quad \kappa(t) = \max\{t^l, t^m\}, \quad t \geq 0.$$

Then the following estimates hold:

(i) *Scaling property: For all $\rho, t \geq 0$,*

$$\zeta(\rho)\Phi(t) \leq \Phi(\rho t) \leq \kappa(\rho)\Phi(t).$$

(ii) *Norm control: For any $u \in L_\Phi(Y)$,*

$$\zeta(\|u\|_\Phi) \leq \int_Y \Phi(|u|)dx \leq \kappa(\|u\|_\Phi).$$

Lemma 2.2 (Convergence criterion). *Assume (φ_1) , (φ_2) , (φ_3) hold and $Y \subset \mathbb{R}^N$ is a smooth bounded domain. Let $u \in W_0^{1,\Phi}(Y)$ and $\{u_n\}$ be a bounded sequence in $W_0^{1,\Phi}(Y)$ satisfying*

$$\begin{aligned} \int_Y \left(\Gamma \left(\int_Y \Phi(u_n)dx \right) \phi(|\nabla u_n|) \nabla u_n \right. \\ \left. - \Gamma \left(\int_Y \Phi(u)dx \right) \phi(|\nabla u|) \nabla u \right) \cdot \nabla (u_n - u) dx \rightarrow 0. \end{aligned} \quad (2.7)$$

Then $u_n \rightarrow u$ strongly in $W_0^{1,\Phi}(Y)$.

The proof follows from [5, Lemma 2.2].

Lemma 2.3 (Regularity result). *Under assumptions (φ_1) , (φ_2) , (φ_3) , (φ_4) , let $h \in L^\infty(Y)$ and $u \in W_0^{1,\Phi}(Y)$ be a weak solution of*

$$\begin{cases} -\Gamma \left(\int_Y \Phi(u)dx \right) \operatorname{div}(\phi(|\nabla u|) \nabla u) = h & \text{in } Y, \\ u = 0 & \text{on } \partial Y. \end{cases}$$

Then:

- $u \in C^{1,\alpha}(\overline{Y})$ for some $\alpha > 0$ depending on $\bar{l}$ and $\bar{m}$
- There exists $K = K(\|h\|_\infty, \bar{l}, \bar{m}, Y) > 0$ such that $|u|_{1,\alpha} \leq K$
- $K \rightarrow 0$ as $\|h\|_\infty \rightarrow 0$

The proof follows from the regularity results in [24, 25].

Definition 2.4 (Differential inequality). For $u, v \in W_0^{1,\Phi}(Y)$, we say that

$$-\Gamma \left(\int_Y \Phi(u)dx \right) \Delta_\Phi u \leq -\Gamma \left(\int_Y \Phi(v)dx \right) \Delta_\Phi v \quad \text{in } Y$$

if for all $w \in W_0^{1,\Phi}(Y)$ with $w \geq 0$,

$$\Gamma \left(\int_Y \Phi(u)dx \right) \int_Y \phi(|\nabla u|) \nabla u \cdot \nabla w dx \leq \Gamma \left(\int_Y \Phi(v)dx \right) \int_Y \phi(|\nabla v|) \nabla v \cdot \nabla w dx.$$

Lemma 2.5 (Comparison principle). *Let $u, v \in W_0^{1,\Phi}(Y)$ satisfy*

$$\Gamma \left(\int_Y \Phi(u) dx \right) \int_Y \phi(|\nabla u|) \nabla u \cdot \nabla w dx \leq \Gamma \left(\int_Y \Phi(v) dx \right) \int_Y \phi(|\nabla v|) \nabla v \cdot \nabla w dx$$

for all nonnegative $w \in W_0^{1,\Phi}(Y)$. Then $u \leq v$ in Y .

Proof. Consider $(u - v)^+ \in W_0^{1,\Phi}(Y)$. On the set $Y_1 = \{x \in Y : u(x) - v(x) \geq 0\}$, we have:

$$\int_{Y_1} \langle \Gamma \left(\int_Y \Phi(u) dx \right) \phi(|\nabla u|) \nabla u - \Gamma \left(\int_Y \Phi(v) dx \right) \phi(|\nabla v|) \nabla v, \nabla(v - u) \rangle dx \leq 0.$$

By strict convexity of Φ , this implies $\nabla(u - v) = 0$ a.e. in Y_1 . Since $(u - v)^+ = 0$ on ∂Y , we conclude $u \leq v$ in Y . $\square$

The following maximum principle was established by Guedda et al. [18]:

Theorem 2.6 (Maximum principle). *Let $Y \subset \mathbb{R}^N$ be a smooth bounded domain and $u \in C^1(Y)$ a nonnegative function satisfying*

$$-\Delta_\Phi u \geq 0 \quad \text{in } Y. \quad (2.8)$$

Then

- *If $u \neq 0$, then $u > 0$ in Y*
- *For $u \in C^1(Y \cup \{x_0\})$ with $u(x_0) = 0$ at $x_0 \in \partial Y$, the normal derivative satisfies $\frac{\partial u}{\partial \nu}(y) \geq c > 0$ for all $y \in \partial Y$.*
- *In particular, for $u \in C^1(\bar{Y})$ with $u = 0$ on ∂Y , there exists $c > 0$ such that $\frac{\partial u}{\partial \nu}(y) \geq c$ for all $y \in \partial Y$.*

3 Auxiliary problem and functional framework

We begin by defining the modified nonlinearity $\tilde{h}_\varepsilon : \mathbb{R} \rightarrow \mathbb{R}$:

$$\tilde{h}_\varepsilon(t) = \begin{cases} \tilde{h}(t) - \varepsilon & \text{if } t \geq 0, \\ -\varepsilon(t + 1) & \text{if } t \in [-1, 0], \\ 0 & \text{if } t \leq -1, \end{cases}$$

where $0 < \varepsilon < 1$ and $-\varepsilon = \min_{t \in \mathbb{R}} \tilde{h}_\varepsilon(t)$. This leads us to study the auxiliary problem:

$$\begin{cases} -\Gamma \left(\int_Y \Phi(u) dx \right) \Delta_\Phi u = \tilde{h}_\varepsilon(u) & \text{in } Y, \\ u > 0 & \text{in } Y, \\ u = 0 & \text{on } \partial Y. \end{cases} \quad (3.1)$$

The corresponding energy functional $I_\varepsilon : W_0^{1,\Phi}(Y) \rightarrow \mathbb{R}$ is defined by:

$$I_\varepsilon(u) = F \left(\int_Y \Phi(|\nabla u|) dx \right) - \int_Y H_\varepsilon(u) dx,$$

where H_ε is the primitive of h_ε :

$$H_\varepsilon(t) = \begin{cases} H(t) - \varepsilon t & \text{if } t \geq 0, \\ -\frac{\varepsilon}{2}(1 - (t+1)^2) & \text{if } t \in [-1, 0], \\ \frac{\varepsilon}{2} & \text{if } t \leq -1. \end{cases}$$

The functional I_ε is Fréchet differentiable and I'_ε is given by

$$\langle I'_\varepsilon(u), v \rangle = \Gamma \left(\int_Y \Phi(|\nabla u|) dx \right) \int_Y \phi(|\nabla u|) \nabla u \nabla v dx - \int_Y h_\varepsilon(u) v dx,$$

for all $v \in W_0^{1,\Phi}(Y)$.

Lemma 3.1 (Mountain Pass Geometry). *There exist constants $r, \varepsilon_1 > 0$ such that for any $\rho \in (0, r)$ and $\|u\| = \rho$, we have:*

- $I_\varepsilon(u) \geq \alpha(\rho) > 0$ for all $\varepsilon \in (0, \varepsilon_1)$.
- The constants r and ρ are independent of $\varepsilon \in (0, \varepsilon_1)$.

Proof. Notice that $H_\varepsilon(t) \leq \frac{1}{\Lambda} \Phi(t) + C \frac{|t|^q}{q} + \frac{\varepsilon}{2}$ for all $t \in \mathbb{R}$ and some $C > 0$. Also by (Γ_0) one can have $F(t) \geq C_2 t$. Therefore

$$\begin{aligned} I_\varepsilon(u) &= F \left(\int_Y \Phi(|\nabla u|) dx \right) - \int_Y H_\varepsilon(u) dx \\ &\geq F \left(\int_Y \Phi(|\nabla u|) dx \right) - \frac{C}{q} \int_Y |u|^q dx - \frac{\varepsilon}{2} |Y| \\ &\geq \zeta(\|u\|) - \frac{C}{q} \int_Y |u|^q dx - \frac{\varepsilon}{2} |Y|, \end{aligned}$$

where ζ was given in Lemma 2.1. Thus, there is $C_1 > 0$ verifying

$$I_\varepsilon(u) \geq \zeta(\|u\|) - \frac{C}{q} \int_Y |u|^q dx - \frac{\varepsilon}{2} |Y|.$$

Taking $\|u\| = \rho$ with ρ small enough and using Lemma 2.1, we get

$$I_\varepsilon(u) \geq \rho^m (1 - C_1 \rho^{q-m}) - \frac{\varepsilon}{2} |Y|.$$

Now, we fix $\varepsilon_1 = \varepsilon_1(\rho, q, m) > 0$ and $r > 0$ such that

$$\rho^m (1 - C_1 \rho^{q-m}) - \frac{\varepsilon}{2} |Y| \geq \frac{\rho^m (1 - C_1 \rho^{q-m})}{2} > 0,$$

for all $\varepsilon \in (0, \varepsilon_1)$ and $\rho \in (0, r)$.

From this, $I_\varepsilon(u) \geq \alpha > 0$ if $\|u\| = \rho$ where $\alpha := \frac{\rho^m (1 - C_1 \rho^{q-m})}{2}$. □

Lemma 3.2 (Mountain Pass Condition). *For each $\varepsilon \in (0, \varepsilon_1)$, there exists $v \in W_0^{1,\Phi}(Y)$ with $\|v\| > \rho$ and $I_\varepsilon(v) < 0$.*

Proof. Let $\varphi \in C_0^\infty(Y)$ be a function verifying

$$\varphi > 0, \text{ in } Y \text{ and } \|\varphi\| = 1.$$

Note that for all $t > 1$,

$$\begin{aligned} I_\varepsilon(t\varphi) &= F\left(\int_Y \Phi(|\nabla t\varphi|)dx\right) - \int_Y H_\varepsilon(t\varphi)dx \\ &= F\left(\int_Y \Phi(|\nabla t\varphi|)dx\right) - \int_Y H_\varepsilon(t\varphi)dx + \varepsilon \int_Y t\varphi dx. \end{aligned}$$

By $(\hbar_4)$, there are $A_1, B_1 > 0$ verifying

$$H(t) \geq A_1|t|^\theta - B_1, \text{ for all } t \in \mathbb{R}. \quad (3.2)$$

and By (Γ_∞) there exists $C, D > 0$ such that $F(t) \leq Ct^{\theta_1} + D$ for $t \geq 0$. The last inequality together with Lemma 2.1 leads to

$$\begin{aligned} I_\varepsilon(t\varphi) &\leq C\kappa(t)\kappa(\|\varphi\|) - \varepsilon_1 t^\theta \int_Y \varphi^\theta dx + ta\|\varphi\|_1 + (D + B_1)|Y| \\ &\leq Ct^m - \varepsilon_1 t^\theta \|\varphi\|_\theta^\theta + ta\|\varphi\|_1 + (D + B_1)|Y|. \end{aligned}$$

where κ was given in Lemma 2.1. Since $\theta > m$ and $\varepsilon \in (0, \varepsilon_1)$, we can fix $t_0 > 1$ large enough so that $I_\varepsilon(v) < 0$, where $v = t_0\varphi \in W_0^{1,\Phi}(Y)$. $\square$

3.1 Compactness and existence results

Lemma 3.3 (Palais–Smale Condition). *The functional I_ε satisfies the Palais–Smale condition for all $a > 0$.*

Proof. Let $\{u_n\}$ be a sequence in $W_0^{1,\Phi}(Y)$ such that $\{I_\varepsilon(u_n)\}$ is bounded and $I'_\varepsilon(u_n) \rightarrow 0$. Hence, there exists $n_0 \in \mathbb{N}$ such that $|\langle I'_\varepsilon(u_n), u_n \rangle| \leq \|u_n\|$ for $n > n_0$. Thus,

$$\begin{aligned} -\|u_n\| &\leq \langle I'_\varepsilon(u_n), u_n \rangle \\ &= \Gamma\left(\int_Y \Phi(|\nabla u_n|)dx\right) \int_Y \phi(|\nabla u_n|) \nabla u_n \nabla u_n dx - \int_Y \hbar_\varepsilon(u_n) u_n dx \\ &\leq \theta F\left(\int_Y \Phi(|\nabla u_n|)dx\right) + \beta\left(\int_Y \Phi(|\nabla u_n|)dx\right) - D - \int_Y \hbar_\varepsilon(u_n) u_n dx, \end{aligned}$$

so

$$-\|u_n\| - \theta F\left(\int_Y \Phi(|\nabla u_n|)dx\right) - \beta\left(\int_Y \Phi(|\nabla u_n|)dx\right) + D \leq - \int_Y \hbar_\varepsilon(u_n) u_n dx. \quad (3.3)$$

On the other hand, as there exists $K > 0$ such that $|I_\varepsilon(u_n)| \leq K$ for all $n = 1, 2, \dots$, it follows that

$$F\left(\int_Y \Phi(|\nabla u_n|)dx\right) - \int_Y H_\varepsilon(u_n)dx \leq K, \quad \text{for all } n \in \mathbb{N}. \quad (3.4)$$

This implies that

$$F\left(\int_Y \Phi(|\nabla u|)dx\right) - \frac{1}{\theta} \int_Y \hbar_\varepsilon(u_n) u_n dx + \frac{1}{\theta} M|Y| \leq K, \quad \text{for all } n \in \mathbb{N}. \quad (3.5)$$

This shows

$$F\left(\int_Y \Phi(|\nabla u|)dx\right) - \frac{1+\beta}{\theta} \|u_n\| \leq K - \frac{1}{\theta} (M|Y| + D),$$

for n large enough, which is a contradiction. This shows that $\{u_n\}$ is bounded in $W_0^{1,\Phi}(Y)$. Thus, without loss of generality, we may assume that $u_n \rightharpoonup u$ in $W_0^{1,\Phi}(Y)$ and since that $B(t) = |t|^q$ verifies (2.5), we get $u_n \rightarrow u$ in $L^q(Y)$, $m < q < l^*$. By conditions of $\hbar$,

$$\int_Y \hbar_\varepsilon(u_n)(u_n - u)dx \rightarrow 0.$$

Since

$$\begin{aligned} o_n &= I'_\varepsilon(u_n)(u_n - u) \\ &= \Gamma\left(\int_Y \Phi(|\nabla u_n|)dx\right) \int_Y \phi(|\nabla u_n|) \nabla u_n \nabla(u_n - u)dx - \int_Y \hbar_\varepsilon(u_n)(u_n - u)dx \end{aligned}$$

we have

$$\Gamma\left(\int_Y \Phi(|\nabla u_n|)dx\right) \int_Y \phi(|\nabla u_n|) \nabla u_n \nabla(u_n - u)dx \rightarrow 0. \quad (3.6)$$

The weak convergence $u_n \rightharpoonup u$ in $W_0^{1,\Phi}(Y)$ yields

$$\Gamma\left(\int_Y \Phi(|\nabla u|)dx\right) \int_Y \phi(|\nabla u|) \nabla u \nabla(u_n - u)dx \rightarrow 0. \quad (3.7)$$

This implies that

$$\int_Y \left\{ \Gamma\left(\int_Y \Phi(|\nabla u_n|)dx\right) \phi(|\nabla u_n|) \nabla u_n - \Gamma\left(\int_Y \Phi(|\nabla u|)dx\right) \phi(|\nabla u|) \nabla u \right\} \nabla(u_n - u)dx \rightarrow 0.$$

This shows $u_n \rightarrow u$ in $W_0^{1,\Phi}(Y)$ by Lemma 2.2. $\square$

Lemma 3.4 (Existence of solutions). *For $\varepsilon \in (0, \varepsilon_1)$, problem (1.2) has a solution $u_\varepsilon \in W_0^{1,\Phi}(Y)$ with $I_\varepsilon(u_\varepsilon) \leq C(\varepsilon_1, \theta, m, |Y|)$.*

Proof. The verification of the Mountain Pass Geometry (Lemma 3.1), linking condition (Lemma 3.2), and Palais–Smale condition (Lemma 3.3) allows us to apply the Mountain Pass Theorem [6], yielding for each $a \in (0, \varepsilon_1)$ a critical point $u_\varepsilon \in W_0^{1,\Phi}(Y)$ at level $d_\varepsilon > 0$, where d_ε is the Mountain Pass level associated with I_ε .

Now, letting $\phi \in C_0^\infty(Y)$, $\phi \geq 0$, $t > 0$ and using (3.2), we obtain

$$\begin{aligned} I_\varepsilon(t\phi) &= F\left(\int_Y \Phi(|\nabla t\phi|)dx\right) - \int_Y H_\varepsilon(t\phi)dx \\ &\leq C\kappa(t)\kappa(\|\Phi\|) - \varepsilon_1 \int_Y (t\phi)^\theta dx + B_1|Y| + \int_Y \varepsilon t\phi dx \end{aligned}$$

for all $\varepsilon \in (0, \varepsilon_1)$. Then

$$I_\varepsilon(t\phi) \leq C_1\kappa(t) - C_2t^\theta + C_3t + C_4,$$

where $C_1 = \kappa(\|\phi\|)$, $C_2 = A_1\|\phi\|_\theta^\theta$, $C_3 = \varepsilon_1\|\phi\|_1$ and $C_4 = B_1|Y|$. Setting $h(t) = C_1\kappa(t) - C_2t^\theta + C_3t + C_4$, we find

$$d_\varepsilon \leq \max\{I_\varepsilon(t\phi), t \geq 0\} \leq \max_{t \geq 0} h(t) = C(\varepsilon_1, m, \theta, |Y|) > 0.$$

Thus $I_\varepsilon(u_\varepsilon) \leq C(\varepsilon_1, m, \theta, |Y|)$. $\square$

Lemma 3.5 (Uniform boundedness). *There exists $K = K(\varepsilon_1, q, m, \theta, |Y|) > 0$ such that $\|u_\varepsilon\| \leq K$ for all $\varepsilon \in (0, \varepsilon_1)$.*

Proof. Notice that

$$\begin{aligned}
C(\varepsilon_1, m, \theta, |Y|) &\geq I_\varepsilon(u_\varepsilon) - \frac{1}{\theta} I'_\varepsilon(u_\varepsilon) u_\varepsilon \\
&= F \left(\int_Y \Phi(|\nabla u_\varepsilon|) dx \right) - \int_Y H_\varepsilon(u_\varepsilon) dx \\
&\quad - \frac{1}{\theta} \left(\Gamma \left(\int_Y \Phi(|\nabla u_\varepsilon|) dx \right) \int_Y \phi(|\nabla u_\varepsilon|) |\nabla u_\varepsilon|^2 dx - \int_Y \hbar_\varepsilon(u_\varepsilon) u_\varepsilon dx \right) \\
&= \left(F \left(\int_Y \Phi(|\nabla u_\varepsilon|) dx \right) - \frac{1}{\theta} \Gamma \left(\int_Y \Phi(|\nabla u_\varepsilon|) dx \right) \int_Y \phi(|\nabla u_\varepsilon|) |\nabla u_\varepsilon|^2 dx \right) \\
&\quad - \left(\int_Y H_\varepsilon(u_\varepsilon) dx - \frac{1}{\theta} \int_Y \hbar_\varepsilon(u_\varepsilon) u_\varepsilon dx \right) \\
&\geq \left(F \left(\int_Y \Phi(|\nabla u_\varepsilon|) dx \right) - \frac{m}{\theta} \Gamma \left(\int_Y \Phi(|\nabla u_\varepsilon|) dx \right) \int_Y \Phi(|\nabla u_\varepsilon|) dx \right) \\
&\quad - \left(\int_Y H_\varepsilon(u_\varepsilon) dx - \frac{1}{\theta} \int_Y \hbar_\varepsilon(u_\varepsilon) u_\varepsilon dx \right) \\
&\geq \frac{1}{\theta} \int_Y \Phi(|\nabla u_\varepsilon|) dx - \frac{1}{\theta} D - \frac{M}{\theta} |Y|.
\end{aligned}$$

Now the Δ_2 -condition combines with the last inequality to gives

$$\|u_\varepsilon\| \leq K \quad \text{for all } \varepsilon \in (0, \varepsilon_1),$$

for some $K = K(\varepsilon_1, q, m, \theta, |Y|)$. □

3.2 Regularity and positivity

Lemma 3.6 (L^∞ -regularity). *There exists $\varepsilon_2 \in (0, \varepsilon_1)$ such that:*

- $u_\varepsilon \in L^\infty(Y)$ for all $\varepsilon \in (0, \varepsilon_2)$.
- $\|u_\varepsilon\|_\infty \leq C$ uniformly for $\varepsilon \in (0, \varepsilon_2)$.

Proof. To establish the uniform L^∞ bound, consider any sequence $\varepsilon_j \rightarrow 0^+$ with corresponding solutions $u_j := u_{\varepsilon_j} \in W_0^{1,\Phi}(Y)$. Our proof proceeds in three steps:

1. Compactness: Since $\{u_j\}$ is bounded in $W_0^{1,\Phi}(Y)$ by Lemma 3.5, the subcritical growth of $\hbar$ and the compact embedding $W_0^{1,\Phi}(Y) \hookrightarrow L^q(Y)$ (for $q < l^*$) yield a subsequence (still denoted $\{u_j\}$) converging strongly to some $u \in W_0^{1,\Phi}(Y)$.
2. Regularity transfer: Each u_j satisfies the equation

$$-\Gamma \left(\int_Y \Phi(u_j) \right) \Delta_\Phi u_j = \hbar(u_j) - \varepsilon_j.$$

The right-hand side remains bounded in $L^s(Y)$ for some $s > 1$ due to the growth condition on $\hbar$ and the uniform $W_0^{1,\Phi}$ bounds.

3. Bootstrap: Applying the regularity arguments from [31, Theorem 3.1], we obtain uniform L^∞ estimates:

$$\|u_j\|_{L^\infty(Y)} \leq C(\|\hbar(u_j)\|_{L^s} + \|\varepsilon_j\|_\infty + 1) \leq C',$$

where C' depends only on the structural constants and is independent of j . $\square$

Lemma 3.7 (Lower bound). *There exist $\varepsilon_3 \in (0, \varepsilon_2)$ and $\delta > 0$ such that $\|u_\varepsilon\|_\infty \geq \delta$ for all $\varepsilon \in (0, \varepsilon_3)$.*

Proof. If u_ε is a solution of (1.2), then

$$\Gamma \left(\int_Y \Phi(|\nabla u_\varepsilon|) dx \right) \int_Y \phi(|\nabla u_\varepsilon|) \nabla u_\varepsilon \nabla \phi dx = \int_Y \hbar_\varepsilon(u_\varepsilon) \phi dx$$

for all $\phi \in W_0^{1,\Phi}(Y)$. For $\phi = u_\varepsilon$

$$\int_Y \hbar_\varepsilon(u_\varepsilon) u_\varepsilon dx = \Gamma \left(\int_Y \Phi(|\nabla u_\varepsilon|) dx \right) \int_Y \phi(|\nabla u_\varepsilon|) |\nabla u_\varepsilon|^2 dx \geq m l \int_Y \Phi(|\nabla u_\varepsilon|) dx.$$

By Δ_2 -condition,

$$\int_Y \Phi(|\nabla u_\varepsilon|) dx \rightarrow 0 \quad \text{as } \varepsilon \rightarrow 0 \quad \Longleftrightarrow \quad \|u_\varepsilon\| \rightarrow 0 \quad \text{as } \varepsilon \rightarrow 0.$$

Since $I_\varepsilon(u_\varepsilon) \geq \alpha$ for all $\varepsilon \in (0, \varepsilon_2)$, there exists $\varepsilon_3 \in (0, \varepsilon_2)$ and $\alpha_0 > 0$ such that

$$\int_Y \Phi(|\nabla u_\varepsilon|) dx \geq \alpha_0, \quad \text{for all } \varepsilon \in (0, \varepsilon_3). \quad (3.8)$$

Thus,

$$\int_Y \hbar_\varepsilon(u_\varepsilon) u_\varepsilon dx \geq l m \alpha =: A > 0, \quad \text{for all } \varepsilon \in (0, \varepsilon_3).$$

There exists $C > 0$ such that

$$|\hbar_\varepsilon(t)| \leq C(|t|^q + \phi(|t|)|t|) + \varepsilon, \quad \text{for all } t \in \mathbb{R} \text{ and for all } \varepsilon \in (0, \varepsilon_3).$$

Therefore

$$\begin{aligned} A &\leq \int_Y (C(|u_\varepsilon|^q + \phi(|u_\varepsilon|)|u_\varepsilon|) + \varepsilon |u_\varepsilon|) dx \\ &\leq (C(\|u_\varepsilon\|_\infty^q + \phi(\|u_\varepsilon\|_\infty)|u_\varepsilon|) + \varepsilon \|u_\varepsilon\|_\infty). \end{aligned}$$

Thus $\|u_\varepsilon\|_\infty \geq \delta$ for some $\delta > 0$ for all $\varepsilon \in (0, \varepsilon_3)$, decreasing ε_3 if necessary. $\square$

4 Existence of positive solution

The study of positive solutions for semipositone problems has long been challenging due to the lack of natural subsolutions at zero. When combined with nonlocal Kirchhoff-type operators and anisotropic Φ -Laplacian nonlinearities, these problems present intricate mathematical structures that require novel analytical approaches. In this section, we establish our principal existence result, which not only extends classical semipositone theory but also provides new insights into the interplay between nonlocal effects and anisotropic operators.

Theorem 4.1 (Existence of positive solutions). *Under the structural conditions:*

(S1) *Regularity and growth assumptions on ϕ ((φ_1)–(φ_4)),*

(S2) *Nonlocal term properties ((Γ_0), (Γ_∞), (Γ_{AR})),*

(S3) *Nonlinearity behavior ((h_1)–(h_4)),*

there exists a critical threshold $\varepsilon^ > 0$ such that for all $\varepsilon \in (0, \varepsilon^*)$, the semipositone Kirchhoff-type problem (1.2) admits at least one positive weak solution $u_\varepsilon \in W_0^{1,\Phi}(Y)$. Moreover, the solution satisfies:*

(i) $u_\varepsilon \in C^{1,\alpha}(\bar{Y})$ for some $\alpha > 0$.

(ii) $\inf_K u_\varepsilon > 0$ for any compact $K \subset Y$,

(iii) *Uniform bounds $\delta \leq \|u_\varepsilon\|_\infty \leq C$ independent of ε ,*

Proof. First, we prove that the solution u_ε is positive for $\varepsilon \in (0, \varepsilon_3)$, decreasing ε_3 if necessary. Indeed, let $\{\varepsilon_j\} \in (0, \varepsilon_3)$ be a sequence with $\varepsilon_j \rightarrow 0$ as $j \rightarrow \infty$, and let u_j be a solution of (1.2) with $\varepsilon = \varepsilon_j$. Setting $\tilde{h}_j(u_j) = h_{\varepsilon_j}(u_j)$, we have

$$\begin{cases} -\Gamma(\int_Y \Phi(u_j) dx) \Delta \Phi(u_j) = \tilde{h}_j(u_j) & \text{in } Y, \\ u_j = 0 & \text{on } \partial Y. \end{cases}$$

By Lemma 3.6, there is $C > 0$ such that $\|u_j\|_\infty \leq C$ for all $j \in \mathbb{N}$, then $\|\tilde{h}_j(u_j)\|_\infty \leq C_1$ for all $j \in \mathbb{N}$ and some $C_1 > 0$. Thereby, by Lemma 2.3, $u_j \in C^{1,\beta}(\bar{Y})$ for some $\beta \in (0, 1)$, and

$$|u_j|_{1,\beta} \leq \Gamma(\|\tilde{h}_j(u_j)\|_\infty) \leq M_2, \quad \text{for all } j \in \mathbb{N} \text{ and for some } M_2 > 0.$$

From compactness embedding $C^{1,\beta}(\bar{Y}) \hookrightarrow C^{1,\tau}(\bar{Y})$ for $\tau \in (0, \beta)$, there exists a subsequence for $\{u_j\}$, still denoted by $\{u_j\}$, and $u \in C^{1,\tau}(\bar{Y})$ such that $u_j \rightarrow u$ in $C^{1,\tau}(\bar{Y})$.

Now we estimate

$$-\Delta \Phi u_j = \frac{\tilde{h}_j(u_j)}{\Gamma(\int_Y \Phi(v_j) dx)} \geq \frac{-\varepsilon_j}{h_{\varepsilon_3}}$$

Assume v_j is the solution of

$$\begin{cases} -\Delta \Phi(v_j) = k_j := \frac{-\varepsilon_j}{h_{\varepsilon_3}} & \text{in } Y, \\ v_j = 0 & \text{on } \partial Y, \end{cases}$$

where $k_j = \min\{\tilde{h}_j(t), t \in \mathbb{R}\} = -\varepsilon_j \rightarrow 0^-$ as $j \rightarrow \infty$. Then $u_j, v_j \in W_0^{1,\Phi}(Y)$ and

$$-\Delta \Phi v_j \leq -\Delta \Phi u_j$$

and by comparison principle

$$v_j \leq u_j \quad \text{in } Y, \text{ for all } j \in \mathbb{N}. \quad (4.1)$$

On the other hand, as $k_j \rightarrow 0$, the Lemma 2.3 gives

$$\|v_j\|_\infty \rightarrow 0. \quad (4.2)$$

Then, (4.1) combines with (4.2) to give $u \geq 0$ in Y . Notice that

- $\nabla u_j \rightarrow \nabla u$ uniformly in $\bar{Y}$,
- $\int_Y \Phi(|\nabla u_j|)dx \rightarrow \int_Y \Phi(|\nabla u|)dx$ and then

$$\Gamma \left(\int_Y \Phi(|\nabla u_j|)dx \right) \rightarrow \Gamma \left(\int_Y \Phi(|\nabla u|)dx \right),$$

- $\{\hbar_j(u_j)\}$ is bounded in $L^s(Y)$, $s > 1$,
- $\hbar_j(u_j) \rightharpoonup z$ in $L^s(Y)$,
- $\hbar_j(u_j(x)) \rightarrow \hbar_0(u(x))$ a.e. $x \in Y$,

where $\hbar_0(t) = \hbar(t)$ if $t \geq 0$, and $\hbar_0(t) = 0$ if $t < 0$. Thus $z = \hbar_0(u)$, and for any $\phi \in C_0^\infty(Y)$,

$$\begin{aligned} \Gamma \left(\int_Y \Phi(|\nabla u|)dx \right) \int_Y \phi(|\nabla u|) \nabla u \nabla \phi dx &= \lim_{j \rightarrow +\infty} \Gamma \left(\int_Y \Phi(|\nabla u_j|)dx \right) \int_Y \phi(|\nabla u_j|) \nabla u_j \nabla \phi dx \\ &= \lim_{j \rightarrow +\infty} \int_Y \hbar_j(u_j) \phi dx \\ &= \int_Y z \phi dx, \end{aligned}$$

consequently

$$\begin{cases} -\Delta_\Phi(u) = \frac{z}{\Gamma(\int_Y \Phi(u)dx)} & \text{in } Y, \\ u \geq 0 & \text{in } Y, \\ u = 0 & \text{on } \partial Y. \end{cases}$$

As $\|u_j\|_\infty \geq C_0$ for all $j \in \mathbb{N}$, we derive that $\|u_j\|_\infty \geq C_0$, and so $u \neq 0$. As $z \geq 0$, by Theorem 2.6,

$$u > 0, \quad \text{in } Y \quad \text{and} \quad \frac{\partial u}{\partial \eta} < 0 \quad \text{on } \partial Y$$

and by

$$u_j \rightarrow u \quad \text{in } C^{1,\tau}(\bar{Y})$$

we have $u_j(x) > 0$, $x \in Y$, for j large enough. Decreasing ε_3 if necessary, the above analysis guarantee that u_ε is positive for $\varepsilon \in (0, \varepsilon_3)$. $\square$

Conclusion

In this work, we have established the existence of positive weak solutions for a class of semipositone Kirchhoff-type problems involving nonlocal operators and anisotropic Φ -Laplacians. By employing variational methods in the framework of Orlicz–Sobolev spaces, we successfully overcame the challenges posed by three key features of the problem:

- The semipositone nature of the nonlinearity ($\hbar(0) < 0$).
- The nonlocal Kirchhoff term $\Gamma(\int \Phi(u)dx)$.
- The anisotropic behavior of the Φ -Laplacian operator.

Our main result (Theorem 4.1) demonstrates that for sufficiently small positive values of the parameter ε , the problem admits at least one positive solution. The proof combined mountain pass techniques with careful asymptotic analysis and delicate estimates to handle the interaction between the nonlocal term and the indefinite nonlinearity.

Several interesting directions for future research emerge from this work:

- Investigation of multiplicity results for positive solutions.
- Extension to more general nonlocal operators.
- Study of critical growth cases in this framework.
- Analysis of related parabolic versions of the problem.

The methods developed in this paper may be applicable to other classes of nonlocal problems with indefinite nonlinearities, particularly those arising in mathematical physics and biology where such structural features naturally occur.

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