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Differential inclusion systems with fractional competing operator and multi-valued fractional convection

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Abstract. The existence of solutions to a family of inclusion systems with fractional, possibly competing, elliptic operators, fractional convection, and homogeneous Dirichlet boundary conditions is established. The proof uses Galerkin's method, a surjectivity result for multifunctions in finite dimensional spaces, and approximation techniques.

Keywords: differential inclusion system, fractional competing operator, fractional convection, Dirichlet boundary condition, Galerkin's method.

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1 Introduction

Let Ω be a bounded domain in $\mathbb{R}^N$, $N \geq 3$, with a smooth boundary $\partial\Omega$, let $\mu_1, \mu_2 \in \mathbb{R}$, and let $F_1, F_2 : \Omega \times \mathbb{R}^2 \times \mathbb{R}^{2N} \to 2^{\mathbb{R}}$ be two compact convex-valued multifunctions. Consider the differential inclusion system

$$\begin{cases} (-\Delta)^{s_1}_{p_1} u_1 + \mu_1 (-\Delta)^{t_1}_{q_1} u_1 \in F_1(x, u_1, u_2, D^{r_1} u_1, D^{r_2} u_2) & \text{in } \Omega, \\ (-\Delta)^{s_2}_{p_2} u_2 + \mu_2 (-\Delta)^{t_2}_{q_2} u_2 \in F_2(x, u_1, u_2, D^{r_1} u_1, D^{r_2} u_2) & \text{in } \Omega, \\ u_1 = u_2 = 0 & \text{in } \mathbb{R}^N \setminus \Omega, \end{cases} \tag{1.1}$$

where

(H_1) $0 < t_i < r_i < s_i \leq 1$ and $1 < q_i < p_i < \frac{N}{s_i}$ for each $i = 1, 2$.

The symbol $(-\Delta)^s_p$, with $p > 1$ and $0 < s < 1$, denotes the fractional p-Laplacian, defined by setting, provided u is smooth enough,

$$(-\Delta)^s_p u(x) := 2 \lim_{\varepsilon \to 0^+} \int_{\mathbb{R}^N \setminus B_\varepsilon(x)} \frac{|u(x) - u(y)|^{p-2}(u(x) - u(y))}{|x - y|^{N+ps}} \, \mathrm{d}y, \quad x \in \mathbb{R}^N,$$

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with $B_\varepsilon(x) := \{z \in \mathbb{R}^N : \|z - x\|_{\mathbb{R}^N} < \varepsilon\}$. When $s = 1$ it becomes the classical p-Laplacian, namely

$$-\Delta_p u := -\operatorname{div}(|\nabla u|^{p-2}\nabla u).$$

Moreover, $D^s u$ indicates the *distributional Riesz fractional gradient* of u in the sense of [19,20]. If u appropriately decays and is sufficiently smooth then, setting

$$c_{N,s} := -\frac{2^s\,\Gamma\left(\frac{N+s+1}{2}\right)}{\pi^{\frac{N}{2}}\,\Gamma\left(\frac{1-s}{2}\right)},$$

one has [20, pp. 289 and 298]

$$D^s u(x) := c_{N,s} \lim_{\varepsilon \to 0^+} \int_{\mathbb{R}^N \setminus B_\varepsilon(x)} \frac{u(x) - u(y)}{|x-y|^{N+s}} \frac{x-y}{|x-y|} \mathrm{d}y, \quad x \in \mathbb{R}^N.$$

The right-hand sides F_1 and F_2 satisfy the conditions below, where, to avoid cumbersome formulae, we shall write

$$y := (y_1, y_2), \quad z := (z_1, z_2), \quad p_i^* := \frac{Np_i}{N - s_i p_i}, \quad i = 1, 2. \tag{1.2}$$

(H$_2$) $x \mapsto F_i(x, y, z)$ is measurable on Ω for all $(y, z) \in \mathbb{R}^2 \times \mathbb{R}^{2N}$ and $(y, z) \mapsto F_i(x, y, z)$ is upper semi-continuous for almost every $x \in \Omega$.

(H$_3$) There exist $m_i > 0$, $\delta_i \in L^{(p_i^*)'}(\Omega)$, $i = 1, 2$, such that

$$\sup_{w_i \in F_i(x,y,z)} |w_i| \le m_i \left(|y_1|^{\frac{p_1^*}{(p_i^*)'}} + |y_2|^{\frac{p_2^*}{(p_i^*)'}} + |z_1|^{\frac{p_1}{(p_i^*)'}} + |z_2|^{\frac{p_2}{(p_i^*)'}} \right) + \delta_i(x)$$

a.e. in Ω and for all $(y, z) \in \mathbb{R}^2 \times \mathbb{R}^{2N}$.

(H$_4$) There are $M_i, M_i' > 0$, $\sigma_i \in L^1(\Omega)_+$, $i = 1, 2$, fulfilling

$$w_i y_i \le M_i(|y_1|^{p_1} + |y_2|^{p_2}) + M_i'(|z_1|^{p_1} + |z_2|^{p_2}) + \sigma_i(x)$$

a.e. in Ω and for all $(y, z) \in \mathbb{R}^2 \times \mathbb{R}^{2N}$, $w_i \in F_i(x, y, z)$.

The involved differential operators are of the type

$$A_\mu(u) := (-\Delta)_p^s u + \mu(-\Delta)_q^t u, \quad u \in W_0^{s,p}(\Omega),$$

where $\mu \in \mathbb{R}$, $0 < t \le r \le s \le 1$, $1 < q < p < \frac{N}{s}$, while convection comes from the presence of fractional gradients $D^r u$ at right-hand sides. A_μ exhibits different behaviors depending on the values of $t, s \in (0, 1]$. Precisely, if $t = 1$, then $t = r = s = 1$. Problem (1.1) falls inside the local framework, which has already been investigated in some recent works; see, e.g., [9] for single-valued reactions and [4, 18] as regards multi-valued ones. Moreover, the nature of A_μ drastically changes depending on μ. When $\mu > 0$, the operator A_μ is basically patterned after the (possibly) fractional (p, q)-Laplacian, which is non-homogeneous because $p \ne q$. If $\mu = 0$ it coincides with the fractional p-Laplacian. Both cases have been widely studied, and meaningful results are by now available in the literature. On the other hand, for $\mu < 0$ the operator A_μ contains the *difference* between the fractional p- and q-Laplacians. It is usually called competitive and, as already pointed out in [14, 17], does not satisfy any ellipticity or monotonicity condition. In fact, given $u_0 \in W_0^{s,p}(\Omega) \setminus \{0\}$ and chosen $u := \tau u_0$, $\tau > 0$, the expression

$$\langle A_\mu(u), u\rangle = \tau^p \langle (-\Delta)_p^s u_0, u_0\rangle + \mu\, \tau^q \langle (-\Delta)_q^t u_0, u_0\rangle$$

turns out negative for τ small and positive when τ is large, because

$$u_0 \neq 0 \quad \Longrightarrow \quad \langle (-\Delta)_p^s u_0, u_0\rangle > 0,\ \langle (-\Delta)_q^t u_0, u_0\rangle > 0;$$

cf. Section 2. Hence, nonlinear regularity theory, comparison principles, as well as existence theorems for pseudo-monotone maps cannot be employed. Moreover, since the reactions are multi-valued and contain the fractional gradient of the solutions, also variational techniques are no longer directly usable. To overcome these difficulties we first exploit Galerkin's method, thus working with a sequence $\{E_n\}$ of finite dimensional functional spaces. For each integer $n \geq 1$, an *approximate solution* $(u_{1,n}, u_{2,n}) \in E_n$ to (1.1) is obtained via a suitable version (see Proposition 2.3) of a classical surjectivity result. Next, letting $n \to +\infty$ yields a solution in a generalized sense (cf. Definitions 3.3 and 3.5), which turns out weak sense once $\min\{\mu_1, \mu_2\} \geq 0$.

Fractional gradients were first introduced more than sixty years ago by Horváth [12], but they garnered significant interest especially after the works of Shieh and Spector [5,19,20]. The operator $D^s u$ appears as a natural non-local version of ∇u, to which $D^s u$ formally converges when $s \to 1^-$. It possesses favorable geometric and physical properties [2,21], like invariance under translations or rotations, homogeneity of order s, continuity, etc.

Section 2 contains some auxiliary results and the functional framework needed for handling both fractional gradients and the fractional p-Laplacian. The existence of (generalized, strong generalized, or weak) solutions to (1.1) is established in Section 3.

2 Preliminaries

Let X, Y be two nonempty sets. A multifunction $\Phi : X \to 2^Y$ is a map from X into the family of all nonempty subsets of Y. A function $\varphi : X \to Y$ is called a selection of Φ when $\varphi(x) \in \Phi(x)$ for every $x \in X$. Given $B \subseteq Y$, put $\Phi^-(B) := \{x \in X \mid \Phi(x) \cap B \neq \emptyset\}$. If X, Y are topological spaces and $\Phi^-(B)$ turns out closed in X for all closed sets $B \subseteq Y$ then we say that Φ is upper semi-continuous. Suppose $(X, \mathcal{F})$ is a measurable space and Y is a topological space. The multifunction Φ is called measurable when $\Phi^-(B) \in \mathcal{F}$ for every open set $B \subseteq Y$. The result below, stated in [1, p. 215], will be repeatedly useful.

Proposition 2.1. *Let $F : \Omega \times \mathbb{R}^h \to 2^{\mathbb{R}}$ be a closed-valued multifunction such that:*

- *$x \mapsto F(x, \xi)$ is measurable for all $\xi \in \mathbb{R}^h$;*
- *$\xi \mapsto F(x, \xi)$ is upper semi-continuous for almost every $x \in \Omega$.*

Let $w : \Omega \to \mathbb{R}^h$ be measurable. Then the multifunction $x \mapsto F(x, w(x))$ admits a measurable selection.

Let $(X, \|\cdot\|)$ be a real normed space with topological dual X^* and duality brackets $\langle\cdot,\cdot\rangle$. Given a nonempty set $A \subseteq X$, define $|A| := \sup_{x\in A} \|x\|$. We say that $\varphi : X \to X^*$ is *monotone* when

$$\langle \varphi(x) - \varphi(z), x - z\rangle \geq 0 \quad \forall\, x, z \in X,$$

and *of type* $(\mathrm{S})_+$ provided

$$x_n \rightharpoonup x \quad \text{in } X, \quad \limsup_{n\to+\infty} \langle \varphi(x_n), x_n - x\rangle \le 0 \implies x_n \to x \quad \text{in } X.$$

The next elementary result [8, Proposition 2.1] will ensure that condition $(\mathrm{S})_+$ holds true for the differential operators we deal with.

Proposition 2.2. *Let $\varphi : X \to X^*$ be of type $(\mathrm{S})_+$ and let $\psi : X \to X^*$ be monotone. Then $\varphi + \psi$ satisfies condition $(\mathrm{S})_+$.*

A multifunction $\Phi : X \to 2^{X^*}$ is called coercive provided

$$\lim_{\|x\|\to\infty} \frac{\inf\{\langle x^*, x\rangle \mid x \in X,\, x^* \in \Phi(x)\}}{\|x\|} = +\infty\,.$$

The following result is a direct consequence of [11, Proposition 3.2.33].

Theorem 2.3. *Let X be a finite-dimensional normed space and let $\Phi : X \to 2^{X^*}$ be a convex compact-valued multifunction. Suppose Φ is upper semi-continuous and coercive. Then there exists $\hat{x} \in X$ satisfying $0 \in \Phi(\hat{x})$.*

Hereafter, if X and Y are two topological spaces, the symbol $X \hookrightarrow Y$ means that X continuously embeds in Y. Given $p > 1$, put $p' := \frac{p}{p-1}$, denote by $\|\cdot\|_p$ the usual norm of $L^p(\Omega)$, and indicate with $\|\cdot\|_{1,p}$ the norm on $W_0^{1,p}(\Omega)$ arising from Poincaré's inequality, namely

$$\|u\|_{1,p} := \|\nabla u\|_p\,, \quad u \in W_0^{1,p}(\Omega).$$

If $u \in W_0^{1,p}(\Omega)$, we set $u(x) = 0$ on $\mathbb{R}^N \setminus \Omega$; cf. [6, Section 5]. Fix $s \in (0,1)$. The Gagliardo seminorm of a measurable function $u : \mathbb{R}^N \to \mathbb{R}$ is

$$[u]_{s,p} := \left(\int_{\mathbb{R}^N\times\mathbb{R}^N} \frac{|u(x)-u(y)|^p}{|x-y|^{N+ps}}\mathrm{d}x\mathrm{d}y\right)^{\frac{1}{p}},$$

while $W^{s,p}(\mathbb{R}^N)$ denotes the fractional Sobolev space

$$W^{s,p}(\mathbb{R}^N) := \left\{u \in L^p(\mathbb{R}^N) : \ [u]_{s,p} < \infty\right\},$$

endowed with the norm

$$\|u\|_{W^{s,p}(\mathbb{R}^N)} := \left(\|u\|^p_{L^p(\mathbb{R}^N)} + [u]^p_{s,p}\right)^{\frac{1}{p}}.$$

As usual, on the space

$$W_0^{s,p}(\Omega) := \{u \in W^{s,p}(\mathbb{R}^N) : u = 0 \text{ a.e. in } \mathbb{R}^N \setminus \Omega\}$$

we will consider the equivalent norm

$$\|u\|_{s,p} := [u]_{s,p}, \quad u \in W_0^{s,p}(\Omega).$$

Let $W^{-s,p'}(\Omega) := (W_0^{s,p}(\Omega))^*$ and let p_s^* be the fractional Sobolev critical exponent, i.e., $p_s^* = \frac{Np}{N-sp}$ when $sp < N$, $p_s^* = +\infty$ otherwise. Thanks to Propositions 2.1–2.2, Theorem 6.7, and Corollary 7.2 of [6] one has

Proposition 2.4. *If $1 \le p < +\infty$ then:*

(a) $0 < s' \le s'' \le 1 \implies W_0^{s'',p}(\Omega) \hookrightarrow W_0^{s',p}(\Omega)$.

(b) $sp < N \implies W_0^{s,p}(\Omega) \hookrightarrow L^r(\Omega)$ *for all* $r \in [1, p_s^*]$.

(c) *The embedding in* (b) *is also compact once* $r \in [1, p_s^*)$.

However, contrary to the non-fractional case, we know [15] that

$$1 \le q < p \le +\infty \quad \not\Longrightarrow \quad W_0^{s,p}(\Omega) \subseteq W_0^{s,q}(\Omega).$$

Define, for every $u, v \in W_0^{s,p}(\Omega)$,

$$\langle (-\Delta)_p^s u, v \rangle := \int_{\mathbb{R}^N \times \mathbb{R}^N} \frac{|u(x)-u(y)|^{p-2}(u(x)-u(y))(v(x)-v(y))}{|x-y|^{N+ps}} \mathrm{d}x\mathrm{d}y\,.$$

The operator $(-\Delta)_p^s$ is called (negative) s-fractional p-Laplacian. It possesses the following properties.

(p_1) $(-\Delta)_p^s : W_0^{s,p}(\Omega) \to W^{-s,p'}(\Omega)$ turns out monotone, continuous, and of type $(\mathrm{S})_+$; vide, e.g., [7, Lemma 2.1].

(p_2) One has

$$\|(-\Delta)_p^s u\|_{W^{-s,p'}(\Omega)} \le \|u\|_{s,p}^{p-1} \quad \forall\, u \in W_0^{s,p}(\Omega).$$

Hence, $(-\Delta)_p^s$ maps bounded sets into bounded sets.

(p_3) The first eigenvalue $\lambda_{1,p,s}$ of $(-\Delta)_p^s$ is given by (cf. [13])

$$\lambda_{1,p,s} = \inf_{u \in W_0^{s,p}(\Omega), u \ne 0} \frac{\|u\|_{s,p}^p}{\|u\|_p^p}\,.$$

To deal with distributional fractional gradients, we first introduce the Bessel potential spaces $L^{\alpha,p}(\mathbb{R}^N)$, where $\alpha > 0$. Set, for every $x \in \mathbb{R}^N$,

$$g_\alpha(x) := \frac{1}{(4\pi)^{\frac{\alpha}{2}} \Gamma\left(\frac{\alpha}{2}\right)} \int_0^{+\infty} e^{\frac{-\pi|x|^2}{\delta}} e^{\frac{-\delta}{4\pi}} \delta^{\frac{\alpha-N}{2}} \frac{\mathrm{d}\delta}{\delta}\,.$$

On account of [16, Section 7.1] one can assert that:

1) $g_\alpha \in L^1(\mathbb{R}^N)$ and $\|g_\alpha\|_{L^1(\mathbb{R}^N)} = 1$.

2) g_α enjoys the semi-group property, i.e., $g_\alpha * g_\beta = g_{\alpha+\beta}$ for any $\alpha, \beta > 0$, with $*$ being the convolution operator.

We consider

$$L^{\alpha,p}(\mathbb{R}^N) := \{u : \; u = g_\alpha * \tilde{u} \text{ for some } \tilde{u} \in L^p(\mathbb{R}^N)\}.$$

If $u = g_\alpha * \tilde{u} = g_\alpha * \overline{u}$, with $\tilde{u}, \overline{u} \in L^p(\mathbb{R}^N)$, then a standard argument ensures that $\tilde{u} = \overline{u}$ because $g_\alpha > 0$. So, we can define

$$\|u\|_{L^{\alpha,p}(\mathbb{R}^N)} = \|\tilde{u}\|_{L^p(\mathbb{R}^N)} \quad \text{whenever} \quad u = g_\alpha * \tilde{u}.$$

Using 1) and 2) easily entails

$$0 < \alpha < \beta \implies L^{\beta,p}(\mathbb{R}^N) \subseteq L^{\alpha,p}(\mathbb{R}^N) \subseteq L^p(\mathbb{R}^N).$$

Moreover, by [19, Theorem 2.2], one has

Theorem 2.5. *If $1 < p < +\infty$ and $0 < \varepsilon < \alpha$ then*

$$L^{\alpha+\varepsilon,p}(\mathbb{R}^N) \hookrightarrow W^{\alpha,p}(\mathbb{R}^N) \hookrightarrow L^{\alpha-\varepsilon,p}(\mathbb{R}^N).$$

Finally, given $s \in (0,1)$, set

$$L_0^{s,p}(\Omega) := \{u \in L^{s,p}(\mathbb{R}^N) : u = 0 \text{ a.e. in } \mathbb{R}^N \setminus \Omega\}$$

and thanks to Theorem 2.5 we infer

$$L_0^{s+\varepsilon,p}(\Omega) \hookrightarrow W_0^{s,p}(\Omega) \hookrightarrow L_0^{s-\varepsilon,p}(\Omega) \quad \forall\, \varepsilon \in (0,s). \tag{2.1}$$

The next basic notion is taken from [19]. For $0 < \alpha < N$, let

$$\gamma(N,\alpha) := \frac{\Gamma\left(\frac{N-\alpha}{2}\right)}{\pi^{\frac{N}{2}} 2^\alpha \Gamma\left(\frac{\alpha}{2}\right)}, \quad I_\alpha(x) := \frac{\gamma(N,\alpha)}{|x|^{N-\alpha}}, \quad x \in \mathbb{R}^N \setminus \{0\}.$$

If $u \in L^p(\mathbb{R}^N)$ and $I_{1-s} * u$ makes sense then the vector

$$D^s u := \left(\frac{\partial}{\partial x_1}(I_{1-s} * u), \ldots, \frac{\partial}{\partial x_N}(I_{1-s} * u)\right),$$

where partial derivatives are understood in a distributional sense, is called distributional Riesz s-fractional gradient of u. Theorem 1.2 in [19] ensures that

$$D^s u = I_{1-s} * Du \quad \forall\, u \in C_c^\infty(\mathbb{R}^N).$$

Further, $D^s u$ looks like the natural extension of ∇u to the fractional framework, Indeed, it exhibits analogous properties and, roughly speaking, $D^s u \to \nabla u$ when $s \to 1^-$; see, e.g., [10, Section 2].

According to [19, Definition 1.5], $X^{s,p}(\mathbb{R}^N)$ denotes the completion of $C_c^\infty(\mathbb{R}^N)$ with respect to the norm

$$\|u\|_{X^{s,p}(\mathbb{R}^N)} := \left(\|u\|^p_{L^p(\mathbb{R}^N)} + \|D^s u\|^p_{L^p(\mathbb{R}^N)}\right)^{\frac{1}{p}}.$$

Since, by [19, Theorem 1.7], $X^{s,p}(\mathbb{R}^N) = L^{s,p}(\mathbb{R}^N)$ we can deduce many facts about $X^{s,p}(\mathbb{R}^N)$ from the existing literature on $L^{s,p}(\mathbb{R}^N)$. Moreover, if

$$X_0^{s,p}(\Omega) := \{u \in X^{s,p}(\mathbb{R}^N) : u = 0 \text{ a.e. in } \mathbb{R}^N \setminus \Omega\},$$

then $X_0^{s,p}(\Omega) = L_0^{s,p}(\Omega)$.

3 Existence results

To shorten notation, for $i = 1,2$, we set $U_i := W_0^{s_i,p_i}(\Omega)$ and denote by $\langle \cdot, \cdot \rangle_i$ the duality brackets of U_i. Lemma 2.6 in [3] guarantees that

$$U_i \hookrightarrow W_0^{t_i,q_i}(\Omega). \tag{3.1}$$

Hence, the differential operator $u \mapsto (-\Delta)^{s_i}_{p_i} u + \mu_i (-\Delta)^{t_i}_{q_i} u$ turns out well-defined on U_i. Let $A_i : U_i \to U_i^*$ be given by

$$\begin{aligned}\langle A_i(u), v\rangle_i := &\int_{\mathbb{R}^N\times\mathbb{R}^N} \frac{|u(x)-u(y)|^{p_i-2}(u(x)-u(y))(v(x)-v(y))}{|x-y|^{N+p_i s_i}}\,\mathrm{d}x\,\mathrm{d}y \\ &+ \mu_i \int_{\mathbb{R}^N\times\mathbb{R}^N} \frac{|u(x)-u(y)|^{q_i-2}(u(x)-u(y))(v(x)-v(y))}{|x-y|^{N+q_i t_i}}\,\mathrm{d}x\,\mathrm{d}y\end{aligned}$$

for every $u, v \in U_i$. Thanks to properties (p_1)–(p_2) stated in Section 2, A_i is bounded and continuous. Consequently,

Lemma 3.1. *Under* (H_1)*, the operator* $A : U_1 \times U_2 \to U_1^* \times U_2^*$ *defined by*

$$A(u_1, u_2) := (A_1(u_1), A_2(u_2)) \quad \forall\, (u_1, u_2) \in U_1 \times U_2$$

maps bounded sets into bounded sets and is continuous.

Next, put, provided $(u_1, u_2) \in U_1 \times U_2$,

$$\mathcal{S}_{F_1,F_2}(u_1, u_2) := \{(w_1, w_2) \in L^{(p_1^*)'}(\Omega) \times L^{(p_2^*)'}(\Omega) : \\ w_i(\cdot) \in F_i(\cdot, u_1, u_2, D^{r_1}u_1, D^{r_2}u_2) \text{ a.e. in } \Omega,\ i = 1, 2\},$$

with p_i^* as in (1.2).

Lemma 3.2. *Let* (H_1)–(H_3) *be satisfied. Then:*

(a_1) $\mathcal{S}_{F_1,F_2}(u_1, u_2)$ *turns out nonempty, convex, closed for all* $(u_1, u_2) \in U_1 \times U_2$.

(a_2) *The multifunction* $\mathcal{S}_{F_1,F_2} : U_1 \times U_2 \to 2^{L^{(p_1^*)'}(\Omega)\times L^{(p_2^*)'}(\Omega)}$ *is bounded and strongly-weakly upper semi-continuous.*

Proof. Since $r_i < s_i$, if $\varepsilon \in (0, s_i - r_i)$, combining Proposition 2.4 with (2.1) yields

$$U_i \hookrightarrow W_0^{r_i+\varepsilon, p_i}(\Omega) \hookrightarrow L_0^{r_i, p_i}(\Omega).$$

Thus,

$$(u_1, u_2) \in U_1 \times U_2 \implies (D^{r_1}u_1, D^{r_2}u_2) \in L^{p_1}(\Omega) \times L^{p_2}(\Omega).$$

Now, pick $(u_1, u_2) \in U_1 \times U_2$. Via (H_2) and Proposition 2.1 we see that $F_i(\cdot, u_1, u_2, D^{r_1}u_1, D^{r_2}u_2)$ admits a measurable selection $w_i : \Omega \to \mathbb{R}$. By (H_3) one has

$$\begin{aligned} \|w_1\|_{(p_1^*)'}^{(p_1^*)'} &\le \int_\Omega \left[m_1 \left(|u_1|^{p_1^*-1} + |u_2|^{\frac{p_2^*}{(p_1^*)'}} + |D^{r_1}u_1|^{\frac{p_1}{(p_1^*)'}} + |D^{r_2}u_2|^{\frac{p_2}{(p_1^*)'}} \right) + \delta_1 \right]^{(p_1^*)'} dx \\ &\le c \left(\|\delta_1\|_{(p_1^*)'}^{(p_1^*)'} + \|u_1\|_{p_1^*}^{p_1^*} + \|u_2\|_{p_2^*}^{p_2^*} + \|D^{r_1}u_1\|_{p_1}^{p_1} + \|D^{r_2}u_2\|_{p_2}^{p_2} \right) \end{aligned}$$

for some $c > 0$, whence $\|w_1\|_{(p_1^*)'} < \infty$. Similarly, $\|w_2\|_{(p_2^*)'} < \infty$. So, $\mathcal{S}_{F_1,F_2}(u_1, u_2) \neq \emptyset$. This proves (a_1), because convexity and closedness follow at once from the analogous properties of F_i. Let us next verify (a_2). The above inequalities also guarantee that $\mathcal{S}_{F_1,F_2}$ maps bounded sets into bounded sets. If B is a nonempty weakly closed subset of $L^{(p_1^*)'}(\Omega) \times L^{(p_2^*)'}(\Omega)$ while $\{(u_{1,n}, u_{2,n})\} \subseteq \mathcal{S}^-_{F_1,F_2}(B)$ converges to (u_1, u_2) in $U_1 \times U_2$, then $\{(u_{1,n}, u_{2,n})\} \subseteq U_1 \times U_2$ turns out bounded. The same holds true concerning the set

$$\bigcup_{n\in\mathbb{N}} \mathcal{S}_{F_1,F_2}(u_{1,n}, u_{2,n}) \subseteq L^{(p_1^*)'}(\Omega) \times L^{(p_2^*)'}(\Omega).$$

Thus, up to sub-sequences, there exists $(w_{1,n}, w_{2,n}) \in \mathcal{S}_{F_1,F_2}(u_{1,n}, u_{2,n}) \cap B$, $n \in \mathbb{N}$, such that

$$(w_{1,n}, w_{2,n}) \rightharpoonup (w_1, w_2) \quad \text{in} \quad L^{(p_1^*)'}(\Omega) \times L^{(p_2^*)'}(\Omega).$$

One evidently has $(w_1, w_2) \in B$, because B is weakly closed. Mazur's principle provides a sequence $\{(\tilde{w}_{1,n}, \tilde{w}_{2,n})\}$ of convex combinations of $\{(w_{1,n}, w_{2,n})\}$ satisfying

$$(\tilde{w}_{1,n}, \tilde{w}_{2,n}) \to (w_1, w_2) \quad \text{in } L^{(p_1^*)'}(\Omega) \times L^{(p_2^*)'}(\Omega).$$

By (H_2), after passing to a sub-sequence which converges a.e., this easily entails

$$w_i(x) \in F_i(x, u_1(x), u_2(x), D^{r_1}u_1(x), D^{r_2}u_2(x)) \quad \text{for almost every} \quad x \in \Omega,\ i = 1, 2.$$

Consequently, $(w_1, w_2) \in \mathcal{S}_{F_1,F_2}(u_1, u_2) \cap B$, i.e., $(u_1, u_2) \in \mathcal{S}^-_{F_1,F_2}(B)$, as desired. □

Our existence result can be established after introducing some suitable constants and the notion of generalized solution to (1.1). Since $r_i < s_i$, $i = 1, 2$, embeddings (2.1) produce

$$\|D^{r_1} u_1\|_{p_1}^{p_1} \le \hat{c}_1 \|u_1\|_{s_1,p_1}^{p_1} \quad \forall\, u_1 \in U_1, \qquad \|D^{r_2} u_2\|_{p_2}^{p_2} \le \hat{c}_2 \|u_2\|_{s_2,p_2}^{p_2} \quad \forall\, u_2 \in U_2, \tag{3.2}$$

with appropriate $\hat{c}_i > 0$. Via (3.1) and its analogue for couples (s_2, p_2)–(t_2, q_2) we next have

$$\|u_1\|_{t_1,q_1}^{q_1} \le \tilde{c}_1 \|u_1\|_{s_1,p_1}^{p_1} \quad \forall\, u_1 \in U_1, \qquad \|u_2\|_{t_2,q_2}^{q_2} \le \tilde{c}_2 \|u_2\|_{s_2,p_2}^{p_2} \quad \forall\, u_2 \in U_2, \tag{3.3}$$

where $\tilde{c}_i > 0$. Finally, given $(T_1, T_2) \in U_1^* \times U_2^*$, set

$$\langle (T_1, T_2), (u_1, u_2) \rangle := \langle T_1, u_1 \rangle_1 + \langle T_2, u_2 \rangle_2, \quad (u_1, u_2) \in U_1 \times U_2.$$

Observe also that, by (b) in Proposition 2.4,

$$L^{(p_1^*)'}(\Omega) \times L^{(p_2^*)'}(\Omega) \hookrightarrow U_1^* \times U_2^*.$$

Hence, every $w \in L^{(p_1^*)'}(\Omega) \times L^{(p_2^*)'}(\Omega)$, $w = (w_1, w_2)$, defines a functional $T_w \in U_1^* \times U_2^*$ through

$$T_w(u_1, u_2) := \int_\Omega (u_1 w_1 + u_2 w_2) \mathrm{d}x \quad \forall\, (u_1, u_2) \in U_1 \times U_2. \tag{3.4}$$

Definition 3.3. We say that $(u_1, u_2) \in U_1 \times U_2$ is a generalized solution of (1.1) if there exist two sequences $(u_{1,n}, u_{2,n}) \in U_1 \times U_2$ and $w_n \in \mathcal{S}_{F_1,F_2}(u_{1,n}, u_{2,n})$, $w_n = (w_{1,n}, w_{2,n})$, fulfilling:

(i) $(u_{1,n}, u_{2,n}) \rightharpoonup (u_1, u_2)$ in $U_1 \times U_2$;

(ii) $A(u_{1,n}, u_{2,n}) - T_{w_n} \rightharpoonup 0$ in $U_1^* \times U_2^*$;

(iii) $\langle A(u_{1,n}, u_{2,n}), (u_{1,n} - u_1, u_{2,n} - u_2) \rangle - T_{w_n}(u_{1,n} - u_1, u_{2,n} - u_2) \to 0$.

Theorem 3.4. *Suppose* (H_1)–(H_4) *are satisfied and, moreover,*

$$\frac{M_1 + M_2}{\lambda_{1,p_i,s_i}} + \hat{c}_i (M_1' + M_2') + \tilde{c}_i |\mu_i| < 1, \quad i = 1, 2. \tag{3.5}$$

Then Problem (1.1) *admits a generalized solution.*

Proof. The space $U_1 \times U_2$ is separable, therefore it possesses a Galerkin's basis, namely a sequence $\{E_n\}$ of linear sub-spaces of $U_1 \times U_2$ such that:

(i_1) $\dim(E_n) < \infty \quad \forall\, n \in \mathbb{N}$;

(i_2) $E_n \subseteq E_{n+1} \quad \forall\, n \in \mathbb{N}$;

(i_3) $\overline{\cup_{n=1}^\infty E_n} = U_1 \times U_2$.

Pick $n \in \mathbb{N}$. Consider the following problem: Find $(u_1, u_2) \in E_n$, $(w_1, w_2) \in \mathcal{S}_{F_1,F_2}(u_1, u_2)$ fulfilling

$$\langle A(u_1, u_2), (v_1, v_2) \rangle - T_w(v_1, v_2) = 0 \quad \forall\, (v_1, v_2) \in E_n^*, \tag{3.6}$$

with T_w as in (3.4). Thanks to Lemma 3.2 the multifunction $\Phi : E_n \to 2^{E_n^*}$ defined by

$$\Phi(u_1, u_2) := \{ (A(u_1, u_2) - T_w)\lfloor_{E_n} : w \in \mathcal{S}_{F_1,F_2}(u_1, u_2), w = (w_1, w_2) \}, \quad (u_1, u_2) \in E_n,$$

takes nonempty, convex, closed values and maps bounded sets into bounded sets. We claim that Φ is upper semi-continuous. In fact, let $B \subseteq E_n^*$ closed. If $\{(u_{1,k}, u_{2,k})\} \subseteq \Phi^-(B)$ and $(u_{1,k}, u_{2,k}) \to (u_1, u_2)$ in E_n then there exists a sequence $w_k \in \mathcal{S}_{F_1,F_2}(u_{1,k}, u_{2,k})$, $w_k = (w_{1,k}, w_{2,k})$, such that

$$(A(u_{1,k}, u_{2,k}) - T_{w_k})\lfloor_{E_n} \in B \quad \forall k \in \mathbb{N}. \tag{3.7}$$

The same argument used in the proof of Lemma 3.2 gives $w \in \mathcal{S}_{F_1,F_2}(u_1, u_2)$, $w = (w_1, w_2)$, satisfying $(w_{1,k}, w_{2,k}) \rightharpoonup (w_1, w_2)$ in $L^{(p_1^*)'}(\Omega) \times L^{(p_2^*)'}(\Omega)$. Since $\dim(E_n) < \infty$, one has $T_{w_k}\lfloor_{E_n} \to T_w\lfloor_{E_n}$ in E_n^*. Thus, from Lemma 3.1 and (3.7) it follows

$$(A(u_1, u_2) - T_w)\lfloor_{E_n} \in B, \quad \text{i.e.,} \quad (u_1, u_2) \in \Phi^-(B).$$

This shows that $\Phi^-(B)$ is closed. As B was arbitrary, the multifunction Φ turns out upper semi-continuous. Next, if $(u_1, u_2) \in U_1 \times U_2$ and $w \in \mathcal{S}_{F_1,F_2}(u_1, u_2)$ then, thanks to (H_4), we have

$$\begin{aligned}\langle A(u_1, u_2), (u_1, u_2)\rangle - T_w(u_1, u_2) \geq \|u_1\|_{s_1,p_1}^{p_1} + \|u_2\|_{s_2,p_2}^{p_2} - |\mu_1|\|u_1\|_{t_1,q_1}^{q_1} - |\mu_2|\|u_2\|_{t_2,q_2}^{q_2} \\ - \int_\Omega \left[M_1(|u_1|^{p_1} + |u_2|^{p_2}) + M_1'(|D^{r_1}u_1|^{p_1} + |D^{r_2}u_2|^{p_2}) + \sigma_1\right] \mathrm{d}x \\ - \int_\Omega \left[M_2(|u_1|^{p_1} + |u_2|^{p_2}) + M_2'(|D^{r_1}u_1|^{p_1} + |D^{r_2}u_2|^{p_2}) + \sigma_2\right] \mathrm{d}x.\end{aligned}$$

Exploiting (p_3) yields

$$\begin{aligned}&\langle A(u_1, u_2), (u_1, u_2)\rangle - T_w(u_1, u_2) \\ &\quad\geq \left(1 - \frac{M_1 + M_2}{\lambda_{1,p_1,s_1}}\right)\|u_1\|_{s_1,p_1}^{p_1} + \left(1 - \frac{M_1 + M_2}{\lambda_{1,p_2,s_2}}\right)\|u_2\|_{s_2,p_2}^{p_2} - |\mu_1|\|u_1\|_{t_1,q_1}^{q_1} - |\mu_2|\|u_2\|_{t_2,q_2}^{q_2} \\ &\quad\quad - \int_\Omega (M_1' + M_2')|D^{r_1}u_1|^{p_1}\mathrm{d}x - \int_\Omega (M_1' + M_2')|D^{r_2}u_2|^{p_2}\mathrm{d}x - \|\sigma_1\|_1 - \|\sigma_2\|_1,\end{aligned}$$

whence, on account of (3.2),

$$\begin{aligned}&\langle A(u_1, u_2), (u_1, u_2)\rangle - T_w(u_1, u_2) \\ &\quad\geq \left[1 - \frac{M_1 + M_2}{\lambda_{1,p_1,s_1}} - \hat{c}_1(M_1' + M_2')\right]\|u_1\|_{s_1,p_1}^{p_1} + \left[1 - \frac{M_1 + M_2}{\lambda_{1,p_2,s_2}} - \hat{c}_2(M_1' + M_2')\right]\|u_2\|_{s_2,p_2}^{p_2} \\ &\quad\quad - |\mu_1|\|u_1\|_{t_1,q_1}^{q_1} - |\mu_2|\|u_2\|_{t_2,q_2}^{q_2} - \|\sigma_1\|_1 - \|\sigma_2\|_1.\end{aligned}$$

Finally, through (3.3) we obtain

$$\begin{aligned}&\langle A(u_1, u_2), (u_1, u_2)\rangle - T_w(u_1, u_2) \\ &\quad\geq \left[1 - \frac{M_1 + M_2}{\lambda_{1,p_1,s_1}} - \hat{c}_1(M_1' + M_2') - |\mu_1|\tilde{c}_1\right]\|u_1\|_{s_1,p_1}^{p_1} \\ &\quad\quad + \left[1 - \frac{M_1 + M_2}{\lambda_{1,p_2,s_2}} - \hat{c}_2(M_1' + M_2') - |\mu_2|\tilde{c}_2\right]\|u_2\|_{s_2,p_2}^{p_2} - \|\sigma_1\|_1 - \|\sigma_2\|_1 \\ &\quad\geq \min_{i=1,2}\left[1 - \frac{M_1 + M_2}{\lambda_{1,p_i,s_i}} - \hat{c}_i(M_1' + M_2') - |\mu_i|\tilde{c}_i\right]\left(\|u_1\|_{s_1,p_1}^{p_1} + \|u_2\|_{s_2,p_2}^{p_2}\right) - \|\sigma_1\|_1 - \|\sigma_2\|_1,\end{aligned}$$

namely

$$\langle A(u_1, u_2), (u_1, u_2)\rangle - T_w(u_1, u_2) \geq \alpha\left(\|u_1\|_{s_1,p_1}^{p_1} + \|u_2\|_{s_2,p_2}^{p_2}\right) - \beta, \tag{3.8}$$

where
$$\alpha := \min_{i=1,2}\left[1-\frac{M_1+M_2}{\lambda_{1,p_i,s_i}}-\hat{c}_i(M_1'+M_2')-|\mu_i|\tilde{c}_i\right], \quad \beta := \|\sigma_1\|_1+\|\sigma_2\|_1.$$
Since (3.5) holds, the multifunction Φ turns out coercive. Now, Theorem 2.3 can be applied, and there exists a solution $(u_{1,n}, u_{2,n}) \in E_n$, $w_n \in \mathcal{S}_{F_1,F_2}(u_{1,n}, u_{2,n})$, $w_n = (w_{1,n}, w_{2,n})$, to Problem (3.6), i.e.,
$$\langle A(u_{1,n}, u_{2,n}), (v_1, v_2)\rangle - T_{w_n}(v_1, v_2) = 0, \quad (v_1, v_2) \in E_n^*. \tag{3.9}$$
From (3.8), written with $(u_1, u_2) := (u_{1,n}, u_{2,n})$, and (3.9) it follows
$$0 \geq \alpha\left(\|u_{1,n}\|_{s_1,p_1}^{p_1}+\|u_{2,n}\|_{s_2,p_2}^{p_2}\right)-\beta \quad \forall n \in \mathbb{N}.$$
Thus, $\{(u_{1,n}, u_{2,n})\} \subseteq U_1 \times U_2$ is bounded. By reflexivity one has $(u_{1,n}, u_{2,n}) \rightharpoonup (u_1, u_2)$ in $U_1 \times U_2$, taking a sub-sequence when necessary. Consequently, (i) of Definition 3.3 holds. Through Lemma 3.2 we next infer that $\{(w_{1,n}, w_{2,n})\} \subseteq L^{(p_1^*)'}(\Omega) \times L^{(p_3^*)'}(\Omega)$ turns out bounded. Therefore, always up to sub-sequences,
$$A(u_{1,n}, u_{2,n}) - T_{w_n} \rightharpoonup T \quad \text{in } U_1^* \times U_2^*. \tag{3.10}$$
Given any $(v_1, v_2) \in \cup_{k=1}^{\infty} E_k$, Property (i_2) and (3.9) yield
$$T(v_1, v_2) = \lim_{n\to\infty}\left(\langle A(u_{1,n}, u_{2,n}), (v_1, v_2)\rangle - T_{w_n}(v_1, v_2)\right) = 0.$$
Because of (i_3) this forces $T = 0$, namely condition (ii) is true. Moreover, using (3.9)–(3.10) entails
$$\begin{aligned}&\langle A(u_{1,n}, u_{2,n}), (u_{1,n}-u_1, u_{2,n}-u_2)\rangle - T_{w_n}(u_{1,n}-u_1, u_{2,n}-u_2)\\ &\quad = -\langle A(u_{1,n}, u_{2,n}), (u_1, u_2)\rangle + T_{w_n}(u_1, u_2) \to 0,\end{aligned} \tag{3.11}$$
which shows (iii) in Definition 3.3. Summing up, the pair (u_1, u_2) turns out to be a generalized solution to (1.1). □

If we strengthen (H_3) as follows:

$(\mathrm{H}_3)'$ For every $i = 1, 2$ there exist $\rho_i, \sigma_i \in (1, p_i^*)$, $m_i > 0$, and $\delta_i \in L^{\sigma_i'}(\Omega)$ such that
$$|F_i(x, y_1, y_2, z_1, z_2)| \leq m_i\left(|y_1|^{\frac{p_1^*}{\rho_i'}}+|y_2|^{\frac{p_2^*}{\rho_i'}}+|z_1|^{\frac{p_1}{\rho_i'}}+|z_2|^{\frac{p_2}{\rho_i'}}\right)+\delta_i(x)$$
a.e. in Ω and for all $(y_1, y_2, z_1, z_2) \in \mathbb{R}^2 \times \mathbb{R}^{2N}$,

then the next notion of strongly generalized solution can be given. Obviously, $(\mathrm{H}_3)'$ implies (H_3), because $\rho_i < p_i^*$ forces
$$\frac{\kappa}{\rho_i'} < \frac{\kappa}{(p_i^*)'} \quad \forall \kappa \in \{p_1, p_2, p_1^*, p_2^*\}.$$

Definition 3.5. We say that $(u_1, u_2) \in U_1 \times U_2$ is a strongly generalized solution to (1.1) if there are two sequences $(u_{1,n}, u_{2,n}) \in U_1 \times U_2$ and $w_n \in \mathcal{S}_{F_1,F_2}(u_{1,n}, u_{2,n})$, $w_n = (w_{1,n}, w_{2,n})$, satisfying (i) and (ii) of Definition 3.3 and, moreover,

(iii)$'$ $\lim_{n\to\infty}\langle A(u_{1,n}, u_{2,n}), (u_{1,n}-u_1, u_{2,n}-u_2)\rangle = 0$.

Theorem 3.6. *Under assumptions* (H_1)–(H_2)*,* $(\mathrm{H}_3)'$*,* (H_4)*, and* (3.5)*, Problem* (1.1) *admits a strongly generalized solution.*

Proof. Reasoning as in the proof of Theorem 3.4 yields both $(u_1, u_2) \in U_1 \times U_2$ and two sequences $(u_{1,n}, u_{2,n}) \in U_1 \times U_2$, $(w_{1,n}, w_{2,n}) \in \mathcal{S}_{F_1,F_2}(u_{1,n}, u_{2,n})$ that comply with (i)–(ii) in Definition 3.3 as well as (3.11). Thus, it remains to show (iii)$'$. By $(\mathrm{H}_3)'$ and Hölder's inequality we have

$$\begin{aligned}
&\left| \int_\Omega w_{i,n}(u_{i,n} - u_i) \mathrm{d}x \right| \\
&\quad \leq m_i \int_\Omega \left(|u_{1,n}|^{\frac{p_1^*}{\rho_i'}} + |u_{2,n}|^{\frac{p_2^*}{\rho_i'}} + |\nabla u_{1,n}|^{\frac{p_1}{\rho_i'}} + |\nabla u_{2,n}|^{\frac{p_2}{\rho_i'}} \right) |u_{i,n} - u_i| \mathrm{d}x + \int_\Omega \delta_i |u_{i,n} - u_i| \mathrm{d}x \\
&\quad \leq m_i \left(\|u_{1,n}\|_{p_1^*}^{p_1^*/\rho_i'} + \|u_{2,n}\|_{p_2^*}^{p_2^*/\rho_i'} + \|u_{1,n}\|_{1,p_1}^{p_1/\rho_i'} + \|u_{2,n}\|_{1,p_2}^{p_2/\rho_i'} \right) \|u_{i,n} - u_i\|_{\rho_i} + \|\delta_i\|_{\sigma_i'} \|u_{i.n} - u_i\|_{\sigma_i} \\
&\quad \leq C \|u_{i,n} - u_i\|_{\rho_i} + \|\delta_i\|_{\sigma_i'} \|u_{i,n} - u_i\|_{\sigma_i} \quad \forall n \in \mathbb{N},
\end{aligned}$$

with $C > 0$, because $\{u_{i,n}\} \subseteq U_i$ turns out bounded. The condition $\max\{\rho_i, \sigma_i\} < p_i^*$ forces $u_{i,n} \to u_i$ in $L^{\rho_i}(\Omega) \cap L^{\sigma_i}(\Omega)$, where a sub-sequence is considered if necessary; see Proposition 2.4. Hence,

$$\lim_{n\to\infty} \int_\Omega w_{i,n}(u_{i,n} - u_i) \mathrm{d}x = 0, \quad i = 1, 2. \tag{3.12}$$

Through (3.11)–(3.12), we arrive at

$$\lim_{n\to\infty} \langle A(u_{1,n}, u_{2,n}), (u_{1,n} - u_1, u_{2,n} - u_2) \rangle = 0,$$

namely (iii)$'$ of Definition 3.5 also holds. □

Finally, recall that $(u_1, u_2) \in U_1 \times U_2$ is called a *weak solution* to (1.1) when there exists $(w_1, w_2) \in \mathcal{S}_{F_1,F_2}(u_1, u_2)$ such that

$$A(u_1, u_2) = (w_1, w_2) \quad \text{in } U_1^* \times U_2^*. \tag{3.13}$$

Corollary 3.7. *Let the hypotheses of Theorem* 3.6 *be satisfied and let* $\min\{\mu_1, \mu_2\} \geq 0$*. Then Problem* (1.1) *possesses a weak solution.*

Proof. Keep the same notation of the previous proof. Since $\mu_i \geq 0$, gathering (p_1) with Proposition 2.2 together ensures that A_i is of type $(\mathrm{S})_+$. Therefore, from (iii)$'$ it follows $(u_{1,n}, u_{2,n}) \to (u_1, u_2)$ in $U_1 \times U_2$. On the other hand, (a_2) in Lemma 3.2 produces, up to subsequences, $(w_{1,n}, w_{2,n}) \rightharpoonup (w_1, w_2)$ in $U_1^* \times U_2^*$. Now, through (ii) and Lemma 3.1 we easily infer (3.13). □

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