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Unilateral global bifurcation for an overdetermined problem in $\mathbb{S}^N \times \mathbb{R}$ and $\mathbb{H}^N \times \mathbb{R}$

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Abstract. We establish the Dancer-type unilateral global bifurcation theorem for nonlinear operator equation of $u + f(\lambda, u) = F(\lambda, u) = 0$, where X is a real Banach space and $f : \mathbb{R} \times X \to X$ is completely continuous with $f(\lambda, 0) = 0$ and C^1 with respect to u at $u = 0$. We shall show that, if $\dim\mathrm{Ker}(D_u F(\mu, 0)) = 1$ for some $\mu \in \mathbb{R}$ and $D_u F(\lambda, 0)$ has an odd crossing number at $\lambda = \mu$, there exist two branches $\mathscr{C}_\mu^\nu$ ($\nu \in \{+, -\}$) emanating from $(\mu, 0)$, such that either $\mathscr{C}_\mu^+$ and $\mathscr{C}_\mu^-$ are both unbounded or $\mathscr{C}_\mu^+ \cap \mathscr{C}_\mu^- \neq \{(\mu, 0)\}$. As one of applications, we obtain the unilateral global bifurcation result for an overdetermined problem in $\mathbb{S}^N \times \mathbb{R}$ and $\mathbb{H}^N \times \mathbb{R}$.

Keywords: unilateral global bifurcation, odd crossing number, completely continuous, overdetermined problem.

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1 Introduction

Let X be real Banach space. We investigate the structure of the set of nontrivial solutions for the following nonlinear parameter-dependent problem

$$u + f(\lambda, u) = F(\lambda, u) = 0, \ (\lambda, u) \in \mathbb{R} \times X, \tag{1.1}$$

where $f : \mathbb{R} \times X \to X$ is completely continuous with $f(\lambda, 0) = 0$ for $\lambda \in \mathbb{R}$ and C^1 with respect to u at $u = 0$.

For $f(\lambda, u) = -\lambda Lu - H(\lambda, u)$ where $L : X \to X$ is a linear compact operator and $H : \mathbb{R} \times X \to X$ is completely continuous with $H = o(\|u\|)$ near $u = 0$ uniformly on bounded λ sets, Krasnosel'skii [12] has shown that all characteristic values of L which are of odd multiplicity are bifurcation points. Rabinowitz [20] has extended this result by showing that bifurcation has global consequences.

Rabinowitz [20] also established two unilateral global bifurcation theorems from simple eigenvalues, i.e., Theorem 1.27 and Theorem 1.40 of [20]. As pointed out by Dancer [6], the

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proofs of these two theorems both contain gaps. Dancer [6,7] constructed a counterexample to Theorem 1.40 of [20]. López-Gómez [15] also pointed out that the proof of Theorem 1.27 is insufficient. Moreover, Dancer [6] established the so-called Dancer-type unilateral global bifurcation theorem from simple eigenvalues. Concretely, if μ^{-1} is an eigenvalue of L of algebraic multiplicity 1, then there exist two sub-continua, $\mathscr{C}_\mu^+$ and $\mathscr{C}_\mu^-$, of $\mathscr{C}_\mu$ bifurcating from $(\mu,0)$, such that either $\mathscr{C}_\mu^+$ and $\mathscr{C}_\mu^-$ are both unbounded or $\mathscr{C}_\mu^+\cap\mathscr{C}_\mu^-\neq\{(\mu,0)\}$. Further, Dancer [7] also proved that the above beautiful unilateral global bifurcation result is also valid for the case of μ^{-1} being an eigenvalue of L with geometric multiplicity 1 and odd algebraic multiplicity. López-Gómez [15] also established a unilateral global bifurcation theorem. López-Gómez's result indicates that $\mathscr{C}_\mu^\nu$ with each $\nu\in\{+,-\}$ either satisfies Rabinowitz-type global alternative or contains a nontrivial element of complement of kernel space.

Following the Rabinowitz's reflection argument in the proof of [20, Theorem 1.27], Kielhofer [11, Theorem II.5.9] established a unilateral global bifurcation theorem via the conception of odd crossing number. To present the Kielhofer's unilateral global bifurcation theorem, we recall the conception of odd crossing number. Let 0 be an isolated eigenvalue of algebraic multiplicity m of $D_uF(\mu,0)$ for some $\mu\in\mathbb{R}$. It is well known that the number m is an invariant, i.e., the dimension of eigenspace is invariant under perturbation near μ. The set of all perturbed eigenvalues near 0 is called 0-*group*. Further, define $\sigma(\lambda)=1$ if there are no negative real eigenvalues in the 0-group of $D_uF(\lambda,0)$ and

$$\sigma(\lambda)=(-1)^{m_1+m_2+\cdots+m_k}$$

if $\mu_1,\mu_2,\ldots,\mu_k$ are all negative real eigenvalues in the 0-group having algebraic multiplicities $m_1,m_2,\ldots,m_k$, respectively. From now on, for simplicity, $\sigma(\lambda)$ is called 0-*group index* of λ. If $D_uF(\lambda,0)$ is regular for $\lambda\in(\mu-\delta,\mu)\cup(\mu,\mu+\delta)$ and if $\sigma(\lambda)$ changes at $\lambda=\mu$, then $D_uF(\lambda,0)$ has an *odd crossing number* at $\lambda=\mu$.

If $\dim\operatorname{Ker}(D_uF(\mu,0))=1$ and $D_uF(\lambda,0)$ has an odd crossing number at $\lambda=\mu$, Kielhofer [11, Theorem II.5.9] proved that there exist two sub-continua, $\mathscr{C}_\mu^+$ and $\mathscr{C}_\mu^-$, of $\mathscr{C}_\mu$ bifurcating from $(\mu,0)$, such that $\mathscr{C}_\mu^\nu$ with each $\nu\in\{+,-\}$ either satisfies Rabinowitz-type global alternative or contains a pair of points (λ,u) and $(\lambda,-u)$ with $u\neq0$. If $f(\lambda,u)=\lambda f(u)$ and 0 is the simple eigenvalue of $D_uF(\mu,0)$, Kielhofer's unilateral global bifurcation theorem is just the Rabinowitz's unilateral global bifurcation theorem of [20, Theorem 1.27]. So, Kielhofer's result extends Rabinowitz's unilateral global bifurcation theorem of [20, Theorem 1.27]. As the mentioned above, since the proof of Theorem 1.27 of [20] is insufficient, Kielhofer's argument is also insufficient because he also adopted the Rabinowitz's reflection argument.

The first aim of this work is to establish the Dancer-type unilateral global bifurcation theorem under the assumptions of Kielhofer, which fills the above gap by providing a corrected unilateral global bifurcation theorem. Let $\mathcal{S}$ be the closure of the set of nontrivial solutions of equation (1.1). The following Dancer-type unilateral global bifurcation theorem is our first main result.

Theorem 1.1. *Assume that* $\dim\operatorname{Ker}(D_uF(\mu,0))=1$ *for some* $\mu\in\mathbb{R}$ *and* $D_uF(\lambda,0)$ *has an odd crossing number at* $\lambda=\mu$. *Then* $\mathcal{S}$ *possesses two maximal sub-continua* $\mathscr{C}_\mu^\pm$ *emanating from* $(\mu,0)$, *such that either* $\mathscr{C}_\mu^+$ *and* $\mathscr{C}_\mu^-$ *are both unbounded or* $\mathscr{C}_\mu^+\cap\mathscr{C}_\mu^-\neq\{(\mu,0)\}$.

If $f(\lambda,u)$ has the form of $\lambda f(u)$ and 0 is the simple eigenvalue of $D_uF(\mu,0)$, the conclusion of Theorem 1.1 would degenerate to the famous Dancer-type unilateral global bifurcation theorem from simple eigenvalues [6]. While, $f(\lambda,u)$ has the form of $\lambda f(u)$ and 0 is odd algebraic multiplicity eigenvalue of $D_uF(\mu,0)$, Theorem 1.1 degenerates to the Dancer-type unilateral global bifurcation theorem from odd algebraic multiplicity eigenvalues [7]. As pointed out by Dancer [7], although the conditions are the same, Dancer-type unilateral global bifurcation conclusion is better than López-Gómez's [15, Theorem 6.4.3].

As one of applications for our unilateral global bifurcation theorem, we next investigate the unilateral global bifurcation phenomenon for an overdetermined elliptic problem in $\mathbb{S}^N \times \mathbb{R}$ and $\mathbb{H}^N \times \mathbb{R}$, where $\mathbb{S}^N$ is the N-dimensional sphere, and $\mathbb{H}^N$ is the N-dimensional hyperbolic space. We consider the following overdetermined elliptic problem

$$\begin{cases} \Delta_g u + \lambda u = 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \\ g(\nabla u, \nu) = \text{const.} & \text{on } \partial\Omega, \end{cases} \tag{1.2}$$

where $\Omega \subset \mathbb{M}^N \times \mathbb{R}$ with $\mathbb{M}^N = \mathbb{S}^N$ or $\mathbb{H}^N$, $N \geq 2$, g denotes the standard metric of $\mathbb{M}^N \times \mathbb{R}$ and Δ_g is the Laplace–Beltrami operator.

When $\mathbb{M}^N = \mathbb{R}^N$, Sicbaldi [24] constructed periodic solutions of (1.2) that are perturbations of a cylinder, which can be seen the first unbounded case counterexample to the following BCN Conjecture.

BCN Conjecture: If Ω is a smooth domain and $\mathbb{R}^N \setminus \overline{\Omega}$ is connected such that problem (1.2) exists a bounded solution, then Ω is either a ball, a half-space, a generalized cylinder $B^k \times \mathbb{R}^{N-k}$ where B^k is a ball in $\mathbb{R}^k$, or the complement of one of them.

In [23], it is shown that such new solutions belong in fact to a smooth local 1-parameter family. Generalizations of such results have been done in the Riemannian manifolds $\mathbb{S}^N \times \mathbb{R}$ and $\mathbb{H}^N \times \mathbb{R}$ [5,16], and in the Euclidean case for some functions f [22]. The boundaries of the domains Ω constructed in [24] are related to the Delaunay surfaces in $\mathbb{R}^N$. The analogy between constant mean curvatures surfaces and overdetermined elliptic problems has inspired a lot of works [9,14,21,25]. Such analogy motivated also the study of overdetermined elliptic problems in the Riemannian manifolds $\mathbb{S}^N \times \mathbb{R}$ and $\mathbb{H}^N \times \mathbb{R}$. It is well known that the Thurston's Geometrization Conjecture proved by G. Perelman in 2003 [17–19]. Since these two manifolds with $N = 2$ represent two of the eight Thurston's 3-dimensional geometries [1], the theory of constant mean curvature surfaces in such ambient spaces are extremely important.

While, we note that bifurcation results in [5,16] are all local. A natural question is whether these local bifurcation results can be extended to the global? Our second main result provides a positive answer to this question.

Theorem 1.2. *Let $C^{2,\alpha}_{\text{even},0}(\mathbb{R}/2\pi\mathbb{Z})$ be the space of even 2π-periodic $C^{2,\alpha}$ functions of mean zero. There exist two nontrivial branches $\mathscr{C}^+$ and $\mathscr{C}^-$ in $\mathcal{V} \times (0, T_0)$ emanating from $(0, T_*(N))$, such that $\mathscr{C}^\nu \cap (\{0\} \times (0, +\infty)) = \{(0, T_*(N))\}$ and for any $(v^\nu, T) \in \mathscr{C}^\nu \setminus \{(0, T_*(N))\}$ with $\nu \in \{+, -\}$, the overdetermined problem (1.2) has a positive T-periodic solution $u \in C^{2,\alpha}(\Omega^\nu)$ on the modified cylinder*

$$\Omega^\nu = \left\{ (x, t) \in \mathbb{M}^N \times \mathbb{R} : |x| < 1 + v^\nu \left(\frac{2\pi}{T} t \right) \right\},$$

where $\mathcal{V}$ is an open neighborhoods of the 0 in $\{v \in C^{2,\alpha}_{\text{even},0}(\mathbb{R}/2\pi\mathbb{Z}) : v > -1\}$ for some positive constant T_0. Moreover, $\mathscr{C}^\nu$ satisfies at least one of the following three properties:

(i) $\mathscr{C}^{\nu} \cap \partial\mathcal{O} \neq \emptyset$ with $\mathcal{O} = \mathcal{V} \times (0, T_0)$,

(ii) $\mathscr{C}^{\nu}$ *is unbounded,*

(iii) $\mathscr{C}^{\nu}$ *contains a point* $(T^*, 0) \in \mathcal{O}$ *with some* $T^* \neq T_*$.

The outline of the rest of this article is as follows. In Section 2, we mainly establish a new Dancer-type unilateral global bifurcation theorem via the so-called 0-group, which ends the proof of Theorem 1.1. In Section 3, we finish the proof of Theorem 1.2 by using the new Dancer-type unilateral global bifurcation Theorem 1.1.

2 Proof of Theorem 1.1

Under the assumptions of Theorem 1.1, the operator F can be rewritten as

$$F(\lambda, u) = u + D_u f(\lambda, 0)u + H(\lambda, u),$$

where $H(\lambda, u)$ is $o(\|u\|)$ for $u \in X$ near 0 uniformly on the bounded λ interval. Since $f : \mathbb{R} \times X \to X$ is a completely continuous operator, $D_u f(\lambda, 0)$ is also a completely continuous operator (see [8] or [13]). It further follows that $H : \mathbb{R} \times X \to X$ is a completely continuous operator.

Let $\mathbb{X} = \mathbb{R} \times X$. Given any $\iota \in \mathbb{R}$ and $0 < s < +\infty$, we consider an open neighborhood of $(\iota, 0)$ in $\mathbb{X}$ defined by

$$\mathbb{B}_s(\iota, 0) = \{(\lambda, u) \in \mathbb{X} : \|u\| + |\lambda - \iota| < s\}.$$

Let X_0 be a closed subspace of X such that

$$X = \operatorname{span}\{w_0\} \oplus X_0,$$

where w_0 is a nonzero element in $\operatorname{Ker}(D_u F(\mu, 0))$. Without loss of generality, we assume that $\|w_0\| = 1$. According to the Hahn–Banach theorem, there exists a linear functional $l \in X^*$ such that

$$l(w_0) = 1 \quad \text{and} \quad X_0 = \{u \in X : l(u) = 0\},$$

where X^* denotes the dual space of X. For any $0 < \eta < 1$, define

$$K_\eta = \{(\lambda, u) \in \mathbb{X} : |l(u)| > \eta\|u\|\}.$$

Obviously, K_η is an open subset of $\mathbb{X}$ consisting of two disjoint components K_η^+ and K_η^- with

$$K_\eta^+ = \{(\lambda, u) \in \mathbb{X} : l(u) > \eta\|u\|\},$$
$$K_\eta^- = \{(\lambda, u) \in \mathbb{X} : l(u) < -\eta\|u\|\}.$$

Clearly, both K_η^+ and K_η^- are convex cones, $K_\eta^- = -K_\eta^+$, and $\nu t w_0 \in K_\eta^\nu$ for every $t > 0$ and each $\nu \in \{+, -\}$.

Applying [15, Lemma 6.4.1] or [3, Lemma 2.2], we have the the following lemma, which localizes the possible solutions of problem (1.1) bifurcating from $(\mu, 0)$.

Lemma 2.1. *For every* $\eta \in (0, 1)$ *there exists a number* $\delta_0 > 0$ *such that for each* $0 < \delta < \delta_0$*, there holds*

$$(\mathcal{S} \setminus \{(\mu, 0)\}) \cap \bar{\mathbb{B}}_\delta(\mu, 0) \subseteq K_\eta,$$

and when

$$(\lambda, u) \in (\mathcal{S} \backslash \{(\mu, 0)\}) \cap \bar{\mathbb{B}}_\delta(\mu, 0)$$

there are $s \in \mathbb{R}$ *and a unique* $y \in X_0$ *such that* $u = sw_0 + y$ *and* $|s| > \eta \|u\|$. *Furthermore, for each* (λ, u), *there holds* $\lambda = \mu + o(1)$ *and* $y = o(s)$ *as* $s \to 0$.

It follows from [11, Theorem II.3.3] that $(\mu, 0)$ is a bifurcation point for equation (1.1), and $\mathcal{S}$ possesses a maximal continuum $\mathscr{C}_\mu$ such that $(\mu, 0) \in \mathscr{C}_\mu$ and $\mathscr{C}_\mu$ either meets at infinity in $\mathbb{R} \times X$, or meets at $(\hat{\mu}, 0)$ with some $\hat{\mu} \neq \mu$. Furthermore, by [15, Lemma 6.4.2] or [3, Lemma 2.3], we have that $\mathscr{C}_\mu$ possesses a subcontinuum in each of the cones $K_\eta^+ \cup \{(\mu, 0)\}$ and $K_\eta^- \cup \{(\mu, 0)\}$ each of which meets $(\mu, 0)$ and $\partial \bar{\mathbb{B}}_\varrho(\mu, 0)$ for all $\varrho > 0$ sufficiently small, which is the local unilateral bifurcation structure of $\mathscr{C}_\mu$.

Proof of Theorem 1.1. For any $\varepsilon > 0$ small enough, let $a = \mu - \varepsilon$ and $b = \mu + \varepsilon$. Since $D_u F(a, 0)$ and $D_u F(b, 0)$ are isomorphism, the isolated zero index formula is well-defined for $I + D_u f(a, 0)$ and $I + D_u f(b, 0)$, which are denoted by $i(I + D_u f(a, 0), 0)$ and $i(I + D_u f(b, 0), 0)$. From the definition of 0-group index we see that

$$i(I + D_u f(a, 0), 0) = \sigma(a)$$

and

$$i(I + D_u f(b, 0), 0) = \sigma(b).$$

Thus, we have that

$$i(I + D_u f(a, 0), 0) \neq i(I + D_u f(b, 0), 0).$$

That is to say

$$\deg(I + D_u f(a, 0), \mathfrak{B}_r(0), 0) \neq \deg(I + D_u f(a, 0), \mathfrak{B}_r(0), 0),$$

where $\mathfrak{B}_r(0) = \{u \in X : \|u\| < r\}$ is an isolating neighborhood of the trivial solution. Applying [4, Theorem 3.1], we obtain that $\mathcal{S}$ possesses two maximal sub-continua $\mathscr{C}_\mu^\pm$ emanating from $(\mu, 0)$, such that either $\mathscr{C}_\mu^+$ and $\mathscr{C}_\mu^-$ are both unbounded or $\mathscr{C}_\mu^+ \cap \mathscr{C}_\mu^- \neq \{(\mu, 0)\}$. □

Note that the unilateral global bifurcation result of [4, Theorem 3.1] is for multiparameter problem. Here we use its special case of single parameter. If f is not globally defined, it is not difficult to get the following result.

Corollary 2.2. *Assume that* $\mathcal{O}$ *is an open subset of* $\mathbb{R} \times X$ *and* F *is defined on* $\mathcal{O}$. *Under the assumptions of Theorem 1.1, either* $\mathscr{C}_\mu^+$ *and* $\mathscr{C}_\mu^-$ *satisfy the alternatives of Theorem 1.1 or at least one of them meets* $\partial \mathcal{O}$.

3 Proof of Theorem 1.2

Let k be the sectional curvature of the manifold $\mathbb{M}^N$ (i.e. $k = 1$ if $\mathbb{M}^N = \mathbb{S}^N$ and $k = -1$ if $\mathbb{M}^N = \mathbb{H}^N$). If we choose spherical coordinates (r, θ), with $\theta \in \mathbb{S}^{N-1}$ and $r \in [0, +\infty)$ if $k < 0$ and $r \in [0, \pi]$ if $k > 0$, the usual metric in $\mathbb{M}^N$ [2, Section II.5, Theorem 1] can be written as

$$g_{\mathbb{M}^N} = \mathrm{d}r^2 + S_k^2(r)\, \mathrm{d}\theta^2$$

where

$$S_k(r) = \begin{cases} \sinh r & \text{if } k = -1, \\ \sin r & \text{if } k = 1. \end{cases}$$

Consider the eigenvalue problem

$$\begin{cases} \Delta_{g_{\mathbb{M}^N}} u + \lambda u = 0 & \text{in } B_1, \\ u = 0 & \text{on } \partial B_1, \end{cases} \tag{3.1}$$

where B_1 is the unit geodesic ball of $\mathbb{M}^N$. It is well known that (3.1) possesses a unique principal eigenvalue λ_1. Let $\widetilde{\phi}_1$ be the positive eigenfunction associated to λ_1 normalized so that $\int_{B_1} \widetilde{\phi}_1^2 \,\mathrm{dvol}_{g_{\mathbb{M}^N}} = 1/2\pi$. Then, if g denotes the standard product metric of $\mathbb{M}^N \times \mathbb{R}$ and $r(x)$ denotes the geodesic distance of $x \in \mathbb{M}^N$ from a fixed point $0 \in \mathbb{M}^N$ (the origin), the function $\phi_1(x,t) = \widetilde{\phi}_1(x)$ is a solution of

$$\begin{cases} \Delta_g \phi_1 + \lambda_1 \phi_1 = 0 & \text{in } C_1^T, \\ \phi_1 = 0 & \text{on } \partial C_1^T, \end{cases}$$

where

$$C_1^T = \Big\{ (x,t) \in \mathbb{M}^N \times \mathbb{R}/T\mathbb{Z} : r(x) < 1 \Big\}.$$

It is easy to see that

$$\int_{C_1^{2\pi}} \phi_1^2 \,\mathrm{dvol}_g = 1. \tag{3.2}$$

For each $v \in \mathcal{C}^{2,\alpha}_{\mathrm{even},0}(\mathbb{R}/2\pi\mathbb{Z})$ with $v(t) > -1$, we set

$$C_{1+v}^T = \left\{ (x,t) \in \mathbb{M}^N \times \mathbb{R}/T\mathbb{Z} : r(x) < 1 + v\left(\frac{2\pi t}{T}\right) \right\}.$$

It follows from [2, 10] that there exists a unique positive function $\phi_v \in \mathcal{C}^{2,\alpha}\big(C_{1+v}^T\big)$ and a constant λ_v such that

$$\begin{cases} \Delta_g \phi_v + \lambda_v \phi_v = 0 & \text{in } C_{1+v}^T, \\ \phi_v = 0 & \text{on } \partial C_{1+v}^T \end{cases} \tag{3.3}$$

and

$$\int_{C_{1+v}^{2\pi}} \phi_v^2 \left(x, \frac{T}{2\pi} t \right) \mathrm{dvol}_g = 1.$$

Define the operator

$$N(v,T) = g(\nabla \phi_v, \omega) \mid_{\partial C_{1+v}^T} - \frac{1}{\mathrm{Vol}_g\big(\partial C_{1+v}^T\big)} \int_{\partial C_{1+v}^T} g(\nabla \phi_v, \omega) \,\mathrm{dvol}_g,$$

where ω denotes the unit normal vector field to ∂C_{1+v}^T. By the rotational symmetry of C_{1+v}^T, it is easy to show that N depends only on the variable t [16]. Set $F(v,T) = N(v,T)\big(\frac{T}{2\pi}t\big)$. Obviously, $F(0,T) = 0$ for any $T > 0$. From [5, Lemma 2.1] we know that F is a C^1 operator in a neighborhood of $(0,T)$ for any fixed $T > 0$.

Let ψ be the unique solution of

$$\begin{cases} \Delta_g \psi + \lambda_1 \psi = 0 & \text{in } C_1^T, \\ \psi = -\partial_r \phi_1 v\big(\frac{2\pi t}{T}\big) & \text{on } \partial C_1^T. \end{cases}$$

Define the function $\widetilde{H}_T(\cdot)$ as follows

$$\widetilde{H}_T(v) = \left(\partial_r \psi + \partial_r^2 \phi_1 v\left(\frac{2\pi}{T} t\right) \right) \Bigg|_{\partial C_1^T}$$

and set

$$H_T(v) = \widetilde{H}_T(v)\left(\frac{T}{2\pi}t\right).$$

It follows from [16, Proposition 3.2] that the linearization of F with respect to v at the point $(0, T)$ is just H_T.

If $v \in \mathcal{C}^{2,\alpha}_{\text{even},0}(\mathbb{R}/2\pi\mathbb{Z})$, its Fourier expansion is

$$v = \sum_{m \geq 1} a_m \cos(mt).$$

Let V_m be the space spanned by the function $\cos(mt)$. It follows from [16, Proposition 4.3] that H_T preserves the eigenspaces V_m. Let $\sigma_m(T)$ be the eigenvalue of H_T associated with the eigenfunction $\cos(mt)$. It is known (see [16]) that

$$\sigma_m(T) = \partial_r c_m(1) + \partial_r^2 \phi_1(1),$$

where c_m is the continuous solution on $[0, 1]$ of

$$\left(\partial_r^2 + (n-1)\frac{C_k(r)}{S_k(r)}\partial_r + \lambda_1\right)c - \left(\frac{2m\pi}{T}\right)^2 c = 0$$

with $c_m(1) = -\partial_r \phi_1(1)$, where

$$C_k(r) = \begin{cases} \cosh r & \text{if } k = -1, \\ \cos r & \text{if } k = 1. \end{cases}$$

It follows from [5, Proposition 2.1] that the function $\sigma_1(T)$ satisfies $\sigma_1'(T) < 0$ for any $T > 0$. Moreover, σ_1 has exactly one zero in $(0, +\infty)$, which is denoted by T_*. Furthermore, by [5, Proposition 2.2], we also know that the linearized operator

$$H_T = D_v F(0, T) : \mathcal{C}^{2,\alpha}_{\text{even},0}(\mathbb{R}/2\pi\mathbb{Z}) \longrightarrow \mathcal{C}^{1,\alpha}_{\text{even},0}(\mathbb{R}/2\pi\mathbb{Z})$$

is a formally self-adjoint, first order elliptic operator. It preserves the eigenspaces V_m for all m and $T > 0$. Moreover, the kernel of H_{T_m} is just V_m and the eigenvalue associated to the eigenspace V_m has a unique zero which is just T_m. Note that $\sigma_m(T) = \sigma_1(T/m)$, which indicates the property of σ_m can be deduced from the property of σ_1. So we next only consider the case of $m = 1$.

We now present the proof of Theorem 1.2.

Proof of Theorem 1.2. From the property of $\sigma_1(T)$ we know that there exists $T_0 > T_*$ such that $\sigma_1(T) > -1$ for any $T \in (0, T_0)$. We *claim* that $H_T + \text{Id}$ is invertible for any $T \in (0, T_0)$. For any $v \in \mathcal{C}^{2,\alpha}_{\text{even},0}(\mathbb{R}/2\pi\mathbb{Z})$ such that $H_T v + v = 0$, it follows from the Fourier expansion $v = \sum_{m \geq 1} a_m \cos(mt)$ and $\sigma_m(T) = \sigma(T/m)$ that

$$(\sigma(T) + 1)\int_{-\pi}^{\pi} v^2 \, \mathrm{d}t \leq \int_{-\pi}^{\pi} (H_T v^2 + v^2) \, \mathrm{d}t = 0.$$

It follows that $v \equiv 0$. Clearly, $H_T + \text{Id}$ is linear continuous. By Banach inverse operator theorem, $H_T + \text{Id}$ is an isomorphism for any $T \in (0, T_0)$.

Define $G : (0, T_0) \times \mathcal{V} \to \mathcal{W}$ by

$$G(T, v) = F(v, T) + v,$$

where $\mathcal{V} \subset \mathcal{C}^{2,\alpha}_{\text{even},0}(\mathbb{R}/2\pi\mathbb{Z})$ and $\mathcal{W} \subset \mathcal{C}^{1,\alpha}_{\text{even},0}(\mathbb{R}/2\pi\mathbb{Z})$ are open neighborhoods of the 0 function. Since the operator $H_T + \text{Id}$ is invertible for $T \in (0, T_0)$, $D_v G(T, 0)$ is an isomorphism for all $T \in (0, T_0)$. For $w \in \mathcal{W}$, there exists a unique $v \in \mathcal{V}$ such that $G(\lambda, v) = w$. Let $v = G^{-1}(w)$. Clearly, G^{-1} maps $\mathcal{W}$ into $\mathcal{V}$. Let $R(T, w) = w - G^{-1}(w)$, which maps $(0, T_0) \times \mathcal{W}$ into $\mathcal{W}$ because $\mathcal{V} \subset \mathcal{W}$. Since the embedding of $\mathcal{C}^{2,\alpha}_{\text{even},0}(\mathbb{R}/2\pi\mathbb{Z}) \hookrightarrow \mathcal{C}^{1,\alpha}_{\text{even},0}(\mathbb{R}/2\pi\mathbb{Z})$ is compact, $R : (0, T_0) \times \mathcal{W} \to \mathcal{W}$ is compact. Then $F(v, T) = 0$ is equivalent to $R(T, v) = 0$ for all $T \in (0, T_0)$. We see that $D_w R(T, 0)w = \mu w$ is equivalent to $H_\lambda(w) = \mu w/(1 - \mu)$ with $\mu < 1$. It follows that $D_w R(T, 0)$ has the same number of negative eigenvalues as H_T.

We have known that $\dim \text{Ker}(H_{T_*}) = 1$. So we also have that $\dim \text{Ker}(D_w R(T_*, 0)) = 1$. For any $\varepsilon > 0$ small enough, the property of $\sigma(T)$ implies that 0-group index $\sigma(T_* - \varepsilon) = (-1)^0 = 1$ and $\sigma(T_* + \varepsilon) = (-1)^1 = -1$. It further indicates that $D_w R(T, 0)$ has an odd crossing number at $T = T_*$. Applying Theorem 1.1 to $R(T, v) = 0$, we can conclude the desired unilateral global bifurcation result. □

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