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Positive solutions for a degenerate p-Kirchhoff problem with singular and sublinear nonlinearities

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Abstract. In this paper, we study the following p-Kirchhoff-type elliptic problem:

$$\begin{cases} -M\left(\int_\Omega |\nabla u|^p\right) \operatorname{div}\left(|\nabla u|^{p-2}\nabla u\right) = \dfrac{f(x)}{u^\gamma} + g(x)\, u^q & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

where $\Omega \subset \mathbb{R}^N$ is a bounded smooth domain, $1 < p < N$, $0 < \gamma \leq 1$, $0 < q < p - 1$ and $M : \mathbb{R}^+ \to \mathbb{R}^+$ is a continuous function satisfying additional hypotheses. The data f and g belonging to suitable Lebesgue space. Our approaches are based on an approximation scheme and the pseudomonotone operators theory.

Keywords: nonlocal elliptic equations, singular nonlinearity, pseudomonotone operators theory, weighted eigenvalue problems.

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1 Introduction

Let Ω be a bounded open subset in $\mathbb{R}^N$ with $1 < p < N$. We consider the following non local elliptic problem

$$\begin{cases} -M\left(\|u\|^p\right) \Delta_p u = \frac{f(x)}{u^\gamma} + g(x)\, u^q & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where $\|u\|^p = \int_\Omega |\nabla u|^p \, dx$, $0 < \gamma \leq 1$ and $M : \mathbb{R}^+ = [0, +\infty[\to \mathbb{R}^+$ is a continuous nondecreasing function satisfying the following structural assumption:

$$\exists\, m_0 > 0,\ \mu > 1 :\ M(t) \geq m_0\, t^{\mu-1},\ \forall t \in [0,1]. \tag{1.2}$$

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Observe that condition (1.2) is trivially satisfied in the non-degenerate case, i.e., when $M(0) > 0$. A prototypical example of a Kirchhoff-type function satisfying (1.2) is:

$$M(t) = a + brt^{r-1}, \qquad a,\, b \geq 0,\, a+b>0,\ r \geq 1,\ t \geq 0.$$

Such a condition has been previously utilized by Pucci et al. in [30].

We assume the source terms f and g satisfy the following conditions:

$$f, g \geq 0, \quad \underline{h} := \inf(f,g) \not\equiv 0, \tag{1.3}$$

$$f \in L^m(\Omega), \quad \text{with } m = \frac{pN}{N(p-1)+p+\gamma(N-p)} = \begin{cases} \left(\frac{p^*}{1-\gamma}\right)' & \text{if } 0<\gamma<1, \\ 1 & \text{if } \gamma = 1. \end{cases}, \tag{1.4}$$

$$g \in L^{\left(\frac{p*}{1+q}\right)'}(\Omega). \tag{1.5}$$

Due to the nonlocal term $M\left(\|u\|^p\right)$, the first equation in (1.1) is no longer a pointwise identity. This introduces significant mathematical challenges, making the study of such problems particularly interesting.

For $p = 2$, the operator $M\left(\|u\|^p\right)\Delta_p$ arises naturally in the Kirchhoff equation:

$$\rho\frac{\partial^2 u}{\partial t^2} - \left(\frac{P_0}{h} + \frac{E}{2L}\int_0^L \left|\frac{\partial u}{\partial x}\right|^2 dx\right)\frac{\partial^2 u}{\partial x^2} = 0,$$

which generalizes the classical D'Alembert wave equation for vibrating strings:

$$\frac{\partial^2 u}{\partial t^2} - c^2\frac{\partial^2 u}{\partial x^2} = 0, \qquad x \in (0, L)\,,\ t > 0.$$

For more on this physical background, see [11,23–25,27] and the references therein.

Before presenting the details of our results, we briefly review existing results for singular and nonlocal problems.

In the local case, i.e., $M \equiv 1$, the problem (1.1) is closely related to the foundational work in [10], where the authors showed the existence of a solution for the following nonlinear singular elliptic equation

$$\begin{cases} -\Delta u = \frac{\lambda}{u^\gamma} + u^q & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

where $\Omega \subset \mathbb{R}^N$ $(N \geq 3)$ is a bounded domain with smooth boundary $\partial\Omega$, $\lambda > 0$, and the exponent q of the sublinear term satisfies $0 < q < 1$.

Giacomoni, Schindler, and Takáč in [18] extended this analysis to the quasilinear setting:

$$\begin{cases} -\Delta_p u = \frac{\lambda}{u^\gamma} + u^q & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

where $0 < \gamma < 1$, $1 < p < N$ and $p-1 < q < p^*-1$, with $p^* = \frac{Np}{N-p}$ denotes the critical Sobolev exponent for the embedding $W_0^{1,p}(\Omega)$ in to $L^t(\Omega)$ for every $t \in [1, p^*]$.

In [15], the author considered:

$$\begin{cases} -\Delta_p u = \frac{f(x)}{u^\gamma} + g(x)u^q & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.6}$$

where $0 \leq q \leq p-1$, f and g are nonnegative functions belonging to suitable Lebesgue spaces. The author proved that

- If $q < p-1$, $0 < \gamma < 1$, $f \in \left(\frac{p^*}{1-\gamma}\right)'$, and $g \in L^{\left(\frac{p*}{1+q}\right)'}(\Omega)$, then the solution u belongs to $W_0^{1,p}(\Omega)$.
- if $\gamma = 1$, $f \in L^1(\Omega)$, and $g \in L^{\left(\frac{p*}{1+q}\right)'}(\Omega)$, then $u \in W_0^{1,p}(\Omega)$.

In the nonlocal setting, Corrêa et al. [12] studied:

$$\begin{cases} -M\left(\|u\|^p\right)\Delta_p u = \frac{f(x)}{u^\gamma} + u^q & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.7}$$

where $\Omega \subset \mathbb{R}^N$ is a bounded smooth domain, $2 \leq p \leq N$, $0 < \gamma, q < 1$, $f : \bar{\Omega} \to \mathbb{R}$ is continuous function with $f > 0$ on $\bar{\Omega}$, and $M : \mathbb{R}^+ \to \mathbb{R}^+$ is a continuous function satisfying

- there are $m_0 > 0$ and $\theta_1 > 0$ such that $M(t) \geq m_0\ \forall t \geq \theta_1$;
- $\theta_2 = \sup\{t > 0;\ M(t) = 0\} > 0$.

The authors consider an approximation of problem (1.7) by a regularized problem. They then prove that there is at least one solution in $W_0^{1,p}(\Omega)$ by employing the Galerkin method along with a variant of Brouwer's fixed point theorem [22, Lemme 4.3]. Subsequently, they pass to the limit to obtain a solution of (1.7).

Motivated by the above works, in this paper we investigate the existence and regularity of solutions to problem (1.1), extending some results from [15] to the degenerate nonlocal case. To the best of our knowledge, this work provides the first treatment of a degenerate Kirchhoff-type problem involving both singular and sublinear nonlinearities, under optimal integrability assumptions on the source terms f and g.

Before presenting our main results, we first define what we mean by a weak solution to (1.1). Following Giacomoni et al. [18], the solutions of (1.1) are understood in the following sense

Definition 1.1. A weak solution to problem (1.1) is a function $u \in W_0^{1,p}(\Omega)$ such that

$$\forall \omega \subset\subset \Omega,\ \exists c_\omega > 0 :\ u \geq c_\omega > 0 \text{ in } \omega, \tag{1.8}$$

and

$$M\left(\|u\|^p\right)\int_\Omega |\nabla u|^{p-2}\nabla u \nabla\varphi = \int_\Omega \left(\frac{f(x)}{u^\gamma} + g(x)\,u^q\right)\varphi, \qquad \forall \varphi \in C_0^\infty(\Omega). \tag{1.9}$$

Remark 1.2. If u is a weak solution to problem (1.1) in the sense of Definition 1.1, then for every test function $\varphi \in C_0^\infty(\Omega)$, the right-hand side of the weak formulation (1.9) is finite. Indeed, the local positivity condition (1.8) implies

$$\int_\Omega \frac{f\varphi}{u^\gamma} \leq \frac{\|\varphi\|_{L^\infty(\Omega)}}{c_w^\gamma} \int_\Omega f < +\infty.$$

Hence, $\frac{f(x)}{u^\gamma} \in L^1_{loc}(\Omega)$, and consequently,

$$\frac{f(x)}{u^\gamma} + g(x)\, u^q \in L^1_{loc}(\Omega).$$

Therefore, the weak formulation (1.9) is well-defined.

2 Statements of the main result

Here is the main result of this paper.

Theorem 2.1. *Let us assume that* (1.2)–(1.3) *hold true,* $g \in L^{\left(\frac{p*}{1+q}\right)'}(\Omega)$, *and* $f \in L^m(\Omega)$, *with*

$$m = \frac{pN}{N(p-1) + p + \gamma(N-p)}. \tag{2.1}$$

Then there exists at least one weak solution to problem (1.1).

Remark 2.2. Note that

$$\frac{pN}{N(p-1) + p + \gamma(N-p)} = \begin{cases} \left(\frac{p^*}{1-\gamma}\right)' & \text{if } 0 < \gamma < 1, \\ 1 & \text{if } \gamma = 1. \end{cases}$$

The structure of the rest of this paper is as follows. Section 3 introduces the necessary notations and preliminary results that will be used throughout our analysis. Section 4 establishes the existence of a solution to the approximated problem by employing pseudomonotone operator theory. Section 5 derives essential a priori estimates for the approximate solution, which play a crucial role in proving the main result. Section 6 proves the main result by taking the limit of the sequence of approximated problems. Finally, Section 7 examines additional regularity properties of the obtained solution.

3 Notations and mathematical background

We begin by establishing the necessary framework for our analysis. Let Ω be a bounded smooth domain in $\mathbb{R}^N$ with $1 < p < N$. The truncation function $T_k : \mathbb{R} \to \mathbb{R}$ at level $k > 0$ is defined as

$$T_k(s) = \max\{-k, \min\{s, k\}\}.$$

We now present fundamental results concerning the spectrum of the p-Laplacian operator with weights. For further details on this topic, the reader is referred to [3,17,21,28].

3.1 Weighted eigenvalue problems

Let $V(x) \geq 0$ be a measurable function, not identically zero, $V(x) \in L^\infty(\Omega)$. Let us consider the problem:

$$\begin{cases} u \in W_0^{1,p}(\Omega),\ u \not\equiv 0, \\ -\operatorname{div}\left(|\nabla u|^{p-2}\nabla u\right) = \lambda V(x)|u|^{p-2}u & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{P_λ}$$

We define the functionals $\mathcal{A}$ and $\mathcal{B}$ as follows:

$$\begin{cases} \mathcal{A}(w) = \dfrac{1}{p}\|w\|_{W_0^{1,p}(\Omega)}, \\ \mathcal{B}(w) = \dfrac{1}{p}\displaystyle\int_\Omega V(x)|w(x)|^p\,dx, \end{cases}$$

for all $w \in W_0^{1,p}(\Omega)$.

Definition 3.1 ([3]). A real number $\lambda \in \mathbb{R}$ is called an eigenvalue if there exists a function u such that the pair (u, λ) is a solution to problem (P_λ). Such a function (non-trivial by definition) is called an eigenfunction associated with λ. The first eigenvalue λ_1 is defined by:

$$\lambda_1 = \inf\left\{\mathcal{A}(w) :\ w \in W_0^{1,p}(\Omega), \text{ and } \mathcal{B}(w) = 1\right\}.$$

Lemma 3.2 ([14, 19, 33]). *Let $u \in W_0^{1,p}(\Omega)\setminus\{0\}$ be an eigenfunction associated to λ. Then $u \in L^\infty(\Omega)$.*

Lemma 3.3 ([3, Lemma 3.1]). *If u is an eigenfunction of (P_λ), then $u \in C^{1,\alpha}(\overline{\Omega})$ for some $\alpha \in (0,1)$. Moreover, if $u \geq 0$, then $u > 0$ in Ω and $\frac{\partial u}{\partial \nu} < 0$ on $\partial\Omega$, where ν denotes the unit exterior normal vector to $\partial\Omega$.*

Lemma 3.4 ([3, Lemma 3.3]). *Every eigenfunction u associated with λ_1 has constant sign, i.e., either $u > 0$ or $u < 0$ in Ω.*

Lemma 3.5 ([3, Theorem 3.1]). *The first eigenvalue λ_1 is simple, i.e., if u and v are two eigenfunctions associated with λ_1, then $u = \alpha v$ for some $\alpha \in \mathbb{R}$.*

3.2 Weak comparison principle

The following weak comparison principle will be essential for our subsequent analysis. For the proof, we refer to [28, Lemma A.0.7] or the more general version in [26, Lemma 3].

Lemma 3.6. *Let u_1, $u_2 \in W^{1,p}(\Omega)$ satisfying*

$$\begin{cases} -\Delta_p u_1 \leq -\Delta_p u_2, & \text{in } \Omega,\ (\text{in the weak sense}) \\ u_1 \leq u_2, & \text{on } \partial\Omega. \end{cases}$$

Then, $u_1 \leq u_2$ a.e. in Ω.

3.3 Pseudomonotone operator theory

The following operator classes are fundamental to our nonlinear analysis framework. We recall these standard definitions from [37, Definition 26.1].

Definition 3.7. Let $A : X \to X'$ be an operator on the real reflexive Banach space X.

1. A is said to be hemicontinuous iff the real function
$$t \mapsto \langle A(u+tv), w \rangle,$$
is continuous on $[0,1]$ for all u, v, $w \in X$.

2. A is said to be strongly continuous iff $u_n \rightharpoonup u$ implies $A(u_n) \to A(u)$.

3. A is called to be bounded iff A maps bounded sets into bounded sets.

4. A is said to be monotone iff
$$\langle A(u) - A(v), u - v \rangle \geq 0 \quad \text{for all } u,\ v \in X.$$

5. A is called coercive if
$$\lim_{\|u\| \to \infty} \frac{\langle A(u), u \rangle}{\|u\|} = +\infty.$$

The following properties of the p-Laplacian operator are well-known, see [32, Theorem 17.11].

Lemma 3.8. *The p-Laplacian operator $-\Delta_p : W_0^{1,p}(\Omega) \to W^{-1,p'}(\Omega)$ is monotone, coercif and continuous.*

The following definition, originally introduced by Brézis [8], plays a fundamental role in our existence proof.

Definition 3.9. [37, Definition 27.5] Let $A : X \to X'$ be an operator on the real reflexive Banach space X. The operator A is called pseudomonotone iff $u_n \rightharpoonup u$ as $n \to \infty$ and
$$\limsup_{n \to \infty} \langle A(u_n), u_n - u \rangle \leq 0,$$
implies that
$$\langle A(u), u - w \rangle \leq \liminf_{n \to \infty} \langle A(u_n), u_n - w \rangle.$$

The following result relates pseudomonotonicity to the sum of a monotone hemicontinuous operator and a strongly continuous operator, see [37, Proposition 27.6].

Lemma 3.10. *Let $A : X \to X'$ be an operator on the real reflexive Banach space X. If A is monotone and hemicontinuous and B is strongly continuous, then $A + B$ is pseudomonotone.*

For our existence proof, we rely on the following surjectivity result due to Brézis [8]. The proof can be found in [31, Theorem 2.6].

Lemma 3.11. *Let X be real, separable, and reflexive Banach space with $\dim X = \infty$. If $A : X \to X'$ is pseudomonotone, bounded, and coercive then A is surjective; this means, for any $f \in X'$, there is at least one solution to the equation*
$$A(u) = f.$$

3.4 Nemytskii's operators

We begin by recalling the definition of Nemytskii's operator.

Definition 3.12 ([1, 2]). Let $F : \Omega \times \mathbb{R} \to \mathbb{R}$. The Nemytskii operator associated to F is the map $\mathcal{N}_F$ defined on $\mathcal{M}(\Omega)$ (the set of all measurable real valued functions defined on Ω) by setting

$$\mathcal{N}_F(u(x)) = F(x, u(x)).$$

We recall the following fundamental result about Nemytskii operators [1, Theorem 2.2], with a more general version can be found in [2, Theorem 1.7].

Lemma 3.13. *Let α, $\beta \geq 1$, and $F : \Omega \times \mathbb{R} \to \mathbb{R}$ be a Carathéodory function. Assume that there exist a positive function $a_1 \in L^\beta(\Omega)$ and a constant $a_2 > 0$ such that*

$$|F(x,t)| \leq a_1(x) + a_2 |t|^{\frac{\alpha}{\beta}}, \quad a.e\ x \in \Omega,\ \forall t \in \mathbb{R}.$$

Then the Nemystskii's operator $\mathcal{N}_F$ associated to the function F is continuous from $L^\alpha(\Omega)$ to $L^\beta(\Omega)$.

4 The approximated problem

We approximate the problem (1.1) by the following non-singular and non-degenerate problem

$$\begin{cases} -M_n\left(\|u_n\|^p\right)\Delta_p u_n = \dfrac{f_n}{(u_n + \frac{1}{n})^\gamma} + g_n\, u_n^q & \text{in } \Omega, \\ u_n = 0 & \text{on } \partial\Omega, \end{cases} \tag{4.1}$$

where $f_n = T_n(f)$, $g_n = T_n(g)$ and

$$M_n(t) = M(t) + \frac{1}{n}, \qquad \forall t \in \mathbb{R}^+. \tag{4.2}$$

Lemma 4.1. *Assume that (1.2)–(1.5) hold. Then, problem (4.1) has at least one nonnegative weak solution $u_n \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$.*

Proof. Our proof relies on pseudomonotone operator theory. We define the operators $\mathcal{A}_n$, $\mathcal{F}_n$: $W_0^{1,p}(\Omega) \to W^{-1,p'}(\Omega)$ by

$$\langle \mathcal{A}_n(u), v\rangle = M_n\left(\|u\|^p\right)\int_\Omega |\nabla u|^{p-2}\nabla u \cdot \nabla v, \tag{4.3}$$

and

$$\langle \mathcal{F}_n(u), v\rangle = \int_\Omega \mathcal{N}_F(u) v\, dx, \tag{4.4}$$

where $\mathcal{N}_F$ is the Nemyskii operator associated to the function F defined by

$$F(x,t) := -\frac{f_n(x)}{(t^+ + \frac{1}{n})^\gamma} - g_n(x)(t^+)^q \quad \text{a.e. } x \in \Omega,\ \forall t \in \mathbb{R},$$

where $t^+ := \max(t, 0)$. Next, we set

$$\mathcal{B}_n(u) = \mathcal{A}_n(u) + \mathcal{F}_n(u), \tag{4.5}$$

We now verify that $\mathcal{B}_n$ satisfies the hypotheses of Lemma 3.11. The proof proceeds in several steps.

Step 1: Hemicontinuity of $\mathcal{A}_n$. This follows directly from the continuity of the p-Laplacian operator (see Lemma 3.8) and the continuity of the function M_n.

Step 2: Monotonicity of $\mathcal{A}_n$. Let $u, v \in W_0^{1,p}(\Omega)$. Then,

$$\begin{aligned}\langle \mathcal{A}_n(u) - \mathcal{A}_n(v), u - v \rangle &= \langle M_n(\|u\|^p)\Delta_p u + M_n(\|v\|^p)\Delta_p v, u - v \rangle \\ &= M_n(\|u\|^p)\int_\Omega |\nabla u|^{p-2}\nabla u \cdot \nabla(u - v) \\ &\quad - M_n(\|v\|^p)\int_\Omega |\nabla v|^{p-2}\nabla v \cdot \nabla(u - v) \\ &= M_n(\|u\|^p)\|u\|^p - M_n(\|u\|^p)\int_\Omega |\nabla u|^{p-2}\nabla u \cdot \nabla v \\ &\quad + M_n(\|v\|^p)\|v\|^p - M_n(\|v\|^p)\int_\Omega |\nabla v|^{p-2}\nabla v \cdot \nabla u.\end{aligned} \tag{4.6}$$

Applying Young's inequality:

$$\int_\Omega |\nabla u|^{p-2}\nabla u \cdot \nabla v \le \frac{1}{p'}\|u\|^p + \frac{1}{p}\|v\|^p, \tag{4.7}$$

and

$$\int_\Omega |\nabla v|^{p-2}\nabla v \cdot \nabla u \le \frac{1}{p'}\|v\|^p + \frac{1}{p}\|u\|^p. \tag{4.8}$$

Combining (4.6), (4.7) and (4.8) yields:

$$\begin{aligned}\langle \mathcal{A}_n(u) - \mathcal{A}_n(v), u - v \rangle &\ge M_n(\|u\|^p)\|u\|^p\left(1 - \frac{1}{p'}\right) - \frac{1}{p}M_n(\|u\|^p)\|v\|^p \\ &\quad + M_n(\|v\|^p)\|v\|^p\left(1 - \frac{1}{p'}\right) - \frac{1}{p}M_n(\|v\|^p)\|u\|^p \\ &= \frac{1}{p}\left(M_n(\|u\|^p) - M_n(\|v\|^p)\right)\left(\|u\|^p - \|v\|^p\right) \\ &= \frac{1}{p}\left(M(\|u\|^p) - M(\|v\|^p)\right)\left(\|u\|^p - \|v\|^p\right) \\ &\ge 0,\end{aligned} \tag{4.9}$$

where the last inequality holds because M is nondecreasing function.

Step 3: Strong continuity of $\mathcal{F}_n$. Note that F satisfies the growth condition

$$|F(x,t)| \le C(n)\left(1 + (t^+)^{(q+1)-1}\right), \tag{4.10}$$

where $C(n) > 0$ is a constant depending on n.

Let $(u_k)_{k\in\mathbb{N}} \subset W_0^{1,p}(\Omega)$ be a sequence converging weakly to $u \in W_0^{1,p}(\Omega)$. By the Sobolev compact embedding theorem, we have

$$u_k \to u \quad \text{strongly in } L^r(\Omega), \text{ for all } 1 \le r < p^*. \tag{4.11}$$

Since $q + 1 < p < p^*$, it follows that

$$u_k \to u \quad \text{strongly in } L^{q+1}(\Omega).$$

By Lemma 3.13, the Nemytskii operator $\mathcal{N}_F$ associated with the function F is continuous from $L^{q+1}(\Omega)$ into $L^{(q+1)'}(\Omega)$. Hence,

$$\mathcal{N}_F(u_k) \to \mathcal{N}_F(u) \quad \text{strongly in } L^{(q+1)'}(\Omega).$$

Moreover, since $(q+1)' > (p^*)'$ (due to $q+1 < p^*$), we further deduce

$$\mathcal{N}_F(u_k) \to \mathcal{N}_F(u) \quad \text{strongly in } L^{(p^*)'}(\Omega). \tag{4.12}$$

Now, let $\varphi \in W_0^{1,p}(\Omega)$. Applying Hölder's inequality and the Sobolev embedding theorem, we obtain

$$\begin{aligned}\Big|\langle \mathcal{F}_n(u_k) - \mathcal{F}_n(u), \varphi \rangle\Big| &\leq \int_\Omega |\mathcal{N}_F(u_k) - \mathcal{N}_F(u)||\varphi|\,dx \\ &\leq \|\mathcal{N}_F(u_k) - \mathcal{N}_F(u)\|_{L^{(p^*)'}(\Omega)}\|\varphi\|_{L^{p*}(\Omega)} \\ &\leq \mathcal{S}_p\|\mathcal{N}_F(u_k) - \mathcal{N}_F(u)\|_{L^{(p^*)'}(\Omega)}\|\varphi\|_{W_0^{1,p}(\Omega)},\end{aligned}$$

where $\mathcal{S}_p$ is the optimal Sobolev constant for the embedding $W_0^{1,p}(\Omega)$ into $L^{p*}(\Omega)$.

Consequently,

$$\|\mathcal{F}_n(u_k) - \mathcal{F}_n(u)\|_{W^{-1,p'}(\Omega)} \leq \mathcal{S}_p\|\mathcal{N}_F(u_k) - \mathcal{N}_F(u)\|_{L^{(p^*)'}(\Omega)}. \tag{4.13}$$

Combining (4.12) and (4.13), we conclude that

$$\mathcal{F}_n(u_k) \to \mathcal{F}_n(u) \quad \text{strongly in } W^{-1,p'}(\Omega).$$

Step 4: Pseudomonotonicity of $\mathcal{B}_n$. Since the operator $\mathcal{A}_n$ is monotone, hemicontinuous, and the operator $\mathcal{F}_n$ is strongly continuous, by Lemma 3.10, we deduce that $\mathcal{B}_n$ is pseudomonotone operator.

Step 5: Boundedness of $\mathcal{B}_n$. It suffices to prove that both $\mathcal{A}_n$ and $\mathcal{F}_n$ are bounded operators.

1. **Boundedness of $\mathcal{A}_n$.** Let $R > 0$ and consider $u \in W_0^{1,p}(\Omega)$ with $\|u\| \leq R$. For any test function $v \in W_0^{1,p}(\Omega)$, we apply assumptions (1.2), (4.2) and Hölder's inequality to obtain:

$$\begin{aligned}\Big|\langle \mathcal{A}_n(u), v \rangle\Big| &\leq (M(\|u\|^p) + 1)\int_\Omega |\nabla u|^{p-1}|\nabla v|\,dx \\ &\leq (M(R^p) + 1)\,R^{p-1}\|v\|_{W_0^{1,p}(\Omega)}.\end{aligned}$$

   Hence,

$$\|\mathcal{A}_n(u)\|_{W^{-1,p'}(\Omega)} \leq (M(R^p) + 1)\,R^{p-1}.$$

   This establishes that $\mathcal{A}_n$ maps bounded subsets of $W_0^{1,p}(\Omega)$ to bounded subsets of $W^{-1,p'}(\Omega)$.

2. **Boundedness of $\mathcal{F}_n$.** For any $u, v \in W_0^{1,p}(\Omega)$, we use the growth condition (4.10) and the fact that $p' < \frac{p}{q}$ (since $q < p-1$), we have

$$\begin{aligned}\Big|\langle \mathcal{F}_n(u), v \rangle\Big| &\leq C\int_\Omega (1 + |u|^q)\,v\,dx \\ &\leq C_1\|v\|_{L^p(\Omega)} + \left(\int_\Omega u^{p'q}\right)^{\frac{1}{p'}}\|v\|_{L^p(\Omega)} \\ &\leq C_1\|v\|_{L^p(\Omega)} + C_2\|u\|^q_{L^p(\Omega)}\|v\|_{L^p(\Omega)} \\ &\leq C_3\left(1 + \|u\|^q_{W_0^{1,p}(\Omega)}\right)\|v\|_{W_0^{1,p}(\Omega)}\end{aligned}$$

Consequently,

$$\|\mathcal{F}_n(u)\|_{W^{-1,p'}(\Omega)} \leq C_3 \left(1 + \|u\|^q_{W_0^{1,p}(\Omega)}\right).$$

Therefore, $\mathcal{F}_n$ maps bounded subsets of $W_0^{1,p}(\Omega)$ into bounded subsets of $W^{-1,p'}(\Omega)$.

Step 6: Coercivity of $\mathcal{B}_n$. Let $u \in W_0^{1,p}(\Omega)$. Using the growth condition (4.10), we obtain

$$\begin{aligned}
\langle \mathcal{B}_n(u), u \rangle &= \langle \mathcal{A}_n(u), u \rangle + \langle \mathcal{F}_n(u), u \rangle \\
&= M_n\left(\|u\|^p\right)\|u\|^p - \int_\Omega F(x,u)u \\
&\geq \frac{1}{n}\|u\|^p - C_1\|u\| - C_2\|u\|^{q+1} \\
&\geq \|u\|^p \left(\frac{1}{n} - C_1\|u\|^{1-p} - C_2\|u\|^{q+1-p}\right).
\end{aligned}$$

Since $q < p-1$, it follows that

$$\lim_{\|u\|\to\infty} \frac{\langle \mathcal{B}_n(u), u \rangle}{\|u\|} = +\infty.$$

The operator $\mathcal{B}$ satisfies all assumptions of Lemma 3.11. Consequently, there exists $u_n \in W_0^{1,p}(\Omega)$ such that $\mathcal{B}_n(u) = 0$, meaning u_n weakly solves:

$$\begin{cases} -M_n\left(\|u_n\|^p\right)\Delta_p u_n = \dfrac{f_n}{(u_n^+ + \frac{1}{n})^\gamma} + g_n\,(u_n^+)^q & \text{in } \Omega, \\ u_n = 0 & \text{on } \partial\Omega. \end{cases} \tag{4.14}$$

Non-negativity and regularity Choosing u_n^- as a test function in the weak formulation of (4.14) yields:

$$-\int_\Omega |\nabla u_n^-|^p = \frac{1}{M_n\left(\|u_n\|^p\right)} \int_\Omega \left(\frac{f_n}{(u_n^+ + \frac{1}{n})^\gamma} + g_n\,(u_n^+)^q\right) u_n^- \geq 0, \tag{4.15}$$

which implies $u_n^- = 0$ a.e. in Ω, and thus $u_n \geq 0$.

Furthermore, by [19, Proposition 1.2], we have u_n belongs to $L^t(\Omega)$ for any $t \in [1, +\infty)$. Since the right-hand side of (4.1) belongs to $L^\beta(\Omega)$ with $\beta > \frac{N}{p}$, the L^∞-regularity result of [34, Théorème 4.2] implies that $u_n \in L^\infty(\Omega)$. This completes the proof of Lemma 4.1. □

5 A priori estimates

Lemma 5.1. *Let $(u_n)_n$ be a sequence of nonnegative solutions to problem (4.1). Assume that*

$$g \in L^{\left(\frac{p*}{1+q}\right)'}(\Omega) \quad \text{and} \quad f \in L^m(\Omega),$$

where

$$m = \frac{pN}{N(p-1) + p + \gamma(N-p)}.$$

Then there exists a constant $L > 0$, independent of n, such that

$$\|u_n\|_{W_0^{1,p}(\Omega)} \leq L \quad \text{for all } n \in \mathbb{N}^*. \tag{5.1}$$

Proof. We argue by contradiction. Suppose that (5.1) does not hold. Then, there exists a subsequence, still denoted by (u_n), such that

$$\|u_n\|^p_{W_0^{1,p}(\Omega)} \to +\infty \quad \text{as } n \to +\infty. \tag{5.2}$$

Consequently, for n sufficiently large, we have

$$\|u_n\|^p_{W_0^{1,p}(\Omega)} \geq 1. \tag{5.3}$$

Since the function M is nondecreasing and satisfies assumption (1.2), it follows from (5.3) that

$$\|u_n\|^p_{W_0^{1,p}(\Omega)} \geq M(1) \geq m_0. \tag{5.4}$$

Testing (4.1) with u_n, and using (5.4), we obtain

$$m_0\|u_n\|^p_{W_0^{1,p}(\Omega)} \leq \int_\Omega \frac{f_n\, u_n}{(u_n+\frac{1}{n})^\gamma} + \int_\Omega g_n\, u_n^{1+q}. \tag{5.5}$$

We now distinguish two cases depending on the value of γ.

Case 1: $0 < \gamma < 1$. In this case, the exponent m satisfies $1 < m < \frac{N}{p}$.

Since $f_n \leq f$ and $g_n \leq g$, we apply Hölder inequality to both terms on the right-hand side of (5.5):

$$m_0\|u_n\|^p_{W_0^{1,p}(\Omega)} \leq \|f\|_{L^{\left(\frac{p*}{1-\gamma}\right)'}(\Omega)} \|u_n\|^{1-\gamma}_{L^{p*}(\Omega)} + \|g\|_{L^{\left(\frac{p*}{1+q}\right)'}(\Omega)} \|u_n\|^{q+1}_{L^{p*}(\Omega)}. \tag{5.6}$$

Applying Sobolev's inequality on the left-hand side, we derive:

$$\mathcal{S}m_0\|u_n\|^p_{L^{p*}(\Omega)} \leq \|f\|_{L^{\left(\frac{p*}{1-\gamma}\right)'}(\Omega)} \|u_n\|^{1-\gamma}_{L^{p*}(\Omega)} + \|g\|_{L^{\left(\frac{p*}{1+q}\right)'}(\Omega)} \|u_n\|^{q+1}_{L^{p*}(\Omega)}. \tag{5.7}$$

Since $0 < 1-\gamma < q+1 < p$, it follows from inequality (5.7) that the sequence (u_n) is uniformly bounded in $L^{p*}(\Omega)$.

Letting n tends to infinity in (5.6), we arrive at a contradiction: the right-hand side remains bounded, while the left-hand side diverges to infinity as a consequence of assumption (5.2).

Case 2: $\gamma = 1$. In this case $m = 1$. From (5.5) and $f_n \leq f$, we have:

$$m_0\|u_n\|^p_{W_0^{1,p}(\Omega)} \leq \|f\|_{L^1(\Omega)} + \int_\Omega g_n\, u_n^{1+q}. \tag{5.8}$$

Using the fact that $g_n \leq g$, and Hölder inequality on the right hand side of (5.8), we obtain

$$m_0\|u_n\|^p_{W_0^{1,p}(\Omega)} \leq \|f\|_{L^1(\Omega)} + \|g\|_{L^{\left(\frac{p*}{1+q}\right)'}(\Omega)} \|u_n\|^{q+1}_{L^{p*}(\Omega)}. \tag{5.9}$$

Again, applying Sobolev's inequality on the right-hand side of (5.9), we get

$$m_0\|u_n\|^p_{W_0^{1,p}(\Omega)} \leq \|f\|_{L^1(\Omega)} + \mathcal{S}^{q+1}\|g\|_{L^{\left(\frac{p*}{1+q}\right)'}(\Omega)} \|u\|^{q+1}_{W_0^{1,p}(\Omega)}. \tag{5.10}$$

Since $q+1 < p$, inequality (5.10) implies that the sequence (u_n) is uniformly bounded in $W_0^{1,p}(\Omega)$, contradicting (5.2).

In both cases, we reach a contradiction, which completes the proof of Lemma 5.1. $\square$

Lemma 5.2. *Let u_n be a nonnegative solution of problem (4.1). Then, for any $\omega \subset\subset \Omega$, there exists a constant $c_\omega > 0$ such that*

$$u_n \geq c_\omega \quad \text{in } \omega, \text{ for all } n \in \mathbb{N}^*. \tag{5.11}$$

Proof. Using (5.1) and the fact that M is a nondecreasing function, we obtain

$$M_n(\|u_n\|^p) \leq M(L) = M_\infty, \quad \forall n \in \mathbb{N}^*. \tag{5.12}$$

Thus, in the sense of distributions, we have

$$\begin{aligned} -\Delta_p u_n &\geq \frac{T_1(f)+T_1(g)}{M_\infty}\left(\frac{1}{(u_n+1)^\gamma}+u_n^q\right) \\ &\geq \frac{\min\left(T_1(f)+T_1(g)\right)}{M_\infty}\left(\frac{1}{(u_n+1)^\gamma}+u_n^q\right) \\ &\geq \frac{c_0}{M_\infty}T_1(\underline{h}) = V(x), \end{aligned} \tag{5.13}$$

where we have used the fact the function $\frac{1}{t^\gamma+1}+t^q$, attains a positive minimum on $[0,+\infty[$.

Now consider the problem:

$$\begin{cases} -\Delta_p v = V(x) & \text{in } \Omega, \\ v = 0 & \text{on } \partial\Omega, \end{cases} \tag{5.14}$$

and the weighted eigenvalue problem:

$$\begin{cases} -\Delta_p \phi_1 = \lambda_1 V(x)|\phi_1|^{p-2}\phi_1 & \text{in } \Omega, \\ \phi_1 = 0 & \text{on } \partial\Omega, \end{cases} \tag{5.15}$$

where λ_1 is the first eigenvalue of $-\Delta_p$ with weight V, and $\phi_1 \in W_0^{1,p}(\Omega)\cap L^\infty(\Omega)$ is its associated eigenfunction. By Lemma 3.3, $\phi_1 \in C^{1,\alpha}(\overline{\Omega})$ for some $\alpha \in (0,1)$ and $\phi_1(x) > 0$ for all $x \in \Omega$. Choose $\eta > 0$ sufficiently small so that

$$\lambda_1 \eta^{p-1}\|\phi_1\|^{p-1}_{L^\infty(\Omega)} \leq 1. \tag{5.16}$$

Then, $\underline{v} = \eta\phi_1$ is a subsolution for problem (5.14), since

$$-\Delta_p \underline{v} = -\Delta_p(\eta\phi_1) = \eta^{p-1}\lambda_1 V(x)\phi_1^{p-1} \leq V(x) \quad \text{in } \Omega.$$

By the weak comparison principle, we obtain

$$u_n(x) \geq v(x) \geq \eta\phi_1(x) \quad \text{a.e. in } \Omega. \tag{5.17}$$

Since $\phi_1 > 0$ in Ω and $\omega \subset\subset \Omega$, there exists $c_\omega = \min_\omega \phi_1 > 0$ such that

$$u_n(x) \geq c_\omega > 0 \quad \text{for a.e. } x \in \omega \text{ for all } n \in \mathbb{N}^*. \tag{5.18}$$

This completes the proof of Lemma 5.2. □

Lemma 5.3. *Let u_n be a nonnegative solution of (4.1). Then there exists a constant $\Lambda > 0$, independent of n, such that*

$$M\left(\|u_n\|^p\right) \geq \Lambda > 0 \quad \text{for all } n \in \mathbb{N}^*. \tag{5.19}$$

Proof. Let ϕ_1 be as in the proof of Lemma 5.2. By normalizing ϕ_1, we may assume that

$$\|\phi_1\|^p_{L^p(\Omega)} = \int_\Omega \phi_1^p \, dx = 1. \tag{5.20}$$

Using the normalization (5.20), the uniform positivity (5.17), and Poincaré's inequality, we obtain

$$\eta^p = \eta^p \int_\Omega \phi_1^p dx \le \int_\Omega u_n^p dx \le C_\Omega \|u_n\|^p_{W_0^{1,p}(\Omega)}.$$

This yields

$$\|u_n\|^p_{W_0^{1,p}(\Omega)} \ge \frac{\eta^p}{C_\Omega}. \tag{5.21}$$

Since M is nondecreasing, it follows from (5.21) that

$$M\left(\|u_n\|^p\right) \ge M\left(\frac{\eta^p}{C_\Omega}\right). \tag{5.22}$$

Next, we choose $\eta > 0$ sufficiently small so that it satisfies both (5.16) and

$$0 < \frac{\eta^p}{C_\Omega} \le 1.$$

Applying the growth condition (1.2) on M, we get

$$M\left(\frac{\eta^p}{C_\Omega}\right) \ge m_0 \left(\frac{\eta^p}{C_\Omega}\right)^{\mu-1}. \tag{5.23}$$

Combining (5.22) and (5.23), we conclude that

$$M\left(\|u_n\|^p\right) \ge m_0 \left(\frac{\eta^p}{C_\Omega}\right)^{\mu-1} = \Lambda.$$

This completes the proof. □

6 Proof of the main results

By Lemma 5.1, the sequence (u_n) is bounded in $W_0^{1,p}(\Omega)$. Consequently, there exists a function $u \in W_0^{1,p}(\Omega)$ such that, up to subsequence,

$$u_n \rightharpoonup u \quad \text{weakly in } W_0^{1,p}(\Omega),$$

and

$$u_n \to u \quad \text{almost everywhere in } \Omega.$$

Moreover, using (5.11), we deduce that u satisfies (1.8).

Furthermore, by (5.19) and (5.11), the sequence

$$\frac{1}{M_n\left(\|u_n\|^p\right)} \left(\frac{f_n}{(u_n + \frac{1}{n})^\gamma} + g_n \, u_n^q \right),$$

is bounded in $L^1_{\text{loc}}(\Omega)$. By Theorem 2.1 of [5], it follows that

$$\nabla u_n \to \nabla u \quad \text{almost everywhere in } \Omega.$$

We now prove that u satisfies (1.9). Let $\varphi \in C_0^\infty(\Omega)$. The inequality (5.11) implies

$$0 \le \left| \frac{f_n \varphi}{(u_n + \frac{1}{n})^\gamma} \right| \le \left| \frac{f_n \varphi}{u_n^\gamma} \right| \le \frac{\|\varphi\|_{L^\infty(\Omega)}}{c_\omega^\gamma} f,$$

where ω is the compact support of the function φ. By the Lebesgue dominated convergence theorem,

$$\lim_{n\to+\infty} \int_\Omega \frac{f_n \varphi}{(u_n + \frac{1}{n})^\gamma} = \int_\Omega \frac{f\varphi}{u^\gamma}. \tag{6.1}$$

Since $g \in L^{\frac{p*}{p^*-q-1}}(\Omega)$ and (u_n) is bounded in $L^{p^*}(\Omega)$, Vitali's theorem (see [36, p. 333]) ensures

$$\int_\Omega g_n\, u_n^q \varphi \to \int_\Omega g u^q \varphi. \tag{6.2}$$

Indeed, for any measurable subset $E \subset \Omega$, we have

$$\begin{aligned}
\int_E g_n\, u_n^q \varphi \, dx &\le \|\varphi\|_{L^\infty(\Omega)} \int_E |g(x)|\, |u_n|^q \, dx \\
&\le \|\varphi\|_{L^\infty(\Omega)} \|g\|_{L^{\frac{p*}{p^*-q-1}}(\Omega)} \left(\int_E |u_n|^{\frac{qp^*}{q+1}} \, dx \right)^{\frac{q+1}{p^*}} \\
&\le \|\varphi\|_{L^\infty(\Omega)} \|g\|_{L^{\frac{p*}{p^*-q-1}}(\Omega)} \|u_n\|_{L^{p^*}(\Omega)} |E|^{\frac{1}{p^*}} \to 0 \quad \text{as } |E| \to 0.
\end{aligned}$$

Thus, $\left(g_n\, u_n^q \varphi\right)_n$ is equi-integrable, and since it converges a.e. to $g\, u^q \varphi$, (6.2) holds by Vitali's theorem.

Combining (6.1) and (6.2), we obtain

$$\lim_{n\to+\infty} \int_\Omega \left(\frac{f_n}{(u_n + \frac{1}{n})^\gamma} + g_n\, u_n^q \right) \varphi = \int_\Omega \left(\frac{f}{u^\gamma} + g\, u^q \right) \varphi. \tag{6.3}$$

Recalling that u_n satisfies the identity

$$\int_\Omega |\nabla u_n|^{p-2} \nabla u_n \cdot \nabla \varphi = \frac{1}{M_n\left(\|u_n\|^p\right)} \int_\Omega \left(\frac{f_n}{(u_n + \frac{1}{n})^\gamma} + g_n\, u_n^q \right) \varphi, \tag{6.4}$$

for every $\varphi \in C_0^\infty(\Omega)$. By Lemma 5.3, the sequence (u_n) is bounded in $W_0^{1,p}(\Omega)$, so up to a subsequence,

$$\|u_n\| \longrightarrow \theta_0. \tag{6.5}$$

Passing to the limit in (6.4) using (6.3) and (6.5), we conclude

$$M(\theta_0^p) \int_\Omega |\nabla u|^{p-2} \nabla u \cdot \nabla \varphi = \int_\Omega \left(\frac{f}{u^\gamma} + g u^q \right) \varphi, \ \forall \varphi \in C_0^\infty(\Omega). \tag{6.6}$$

To complete the proof of (1.9), it remains to show that

$$M(\theta_0^p) = M\left(\|u\|^p\right).$$

Let $\varphi_\varepsilon \in C_0^\infty(\Omega)$ such that $\varphi_\varepsilon \to u$ strongly in $W_0^{1,p}(\Omega)$. Testing (6.6) with φ_ε and taking liminf (using Fatou's Lemma) yields

$$M(\theta_0^p)\|u\|^p \ge \int_\Omega f u^{1-\gamma} + \int_\Omega g u^{q+1}. \tag{6.7}$$

On the other hand, testing the weak formulation (4.1) with u_n gives

$$M_n\left(\|u_n\|^p\right)\|u_n\|^p = \int_\Omega \frac{f_n\, u_n}{(u_n+\frac{1}{n})^\gamma} + \int_\Omega g_n\, u_n^{q+1}. \tag{6.8}$$

To pass to the limit in the right side of (6.8), we distinguish two cases based on the value of γ.

Case 1: $0<\gamma<1$. Since $f \in L^{\frac{p^*}{p^*+\gamma-1}}(\Omega)$ and $g \in L^{\frac{p^*}{p^*-q-1}}(\Omega)$, it follows from [6, Theorem 2.5.7] that for every for every $\varepsilon>0$, there exists $\delta>0$ such that for every measurable set $E\subset\Omega$ with $|E|\le\delta$, we have

$$\int_E |f(x)|^{\frac{p^*}{p^*+\gamma-1}}\,dx \le \varepsilon^{\frac{p^*}{p^*+\gamma-1}}, \quad \int_E |g(x)|^{\frac{p^*}{p^*-q-1}}\,dx \le \varepsilon^{\frac{p^*}{p^*-q-1}},$$

Applying Hölder's inequality, we obtain

$$\begin{aligned}\int_E\left(\frac{f_n\, u_n}{(u_n+\frac{1}{n})^\gamma} + g_n\, u_n^{q+1}\right)dx &\le \|u_n\|^{1-\gamma}_{L^{p^*}(\Omega)}\left(\int_E |f(x)|^{\frac{p^*}{p^*+\gamma-1}}\,dx\right)^{\frac{p^*+\gamma-1}{p^*}}\\ &\quad + \|u_n\|^{q+1}_{L^{p^*}(\Omega)}\left(\int_E |g(x)|^{\frac{p^*}{p^*-q-1}}\,dx\right)^{\frac{p^*-q-1}{p^*}}\\ &\le C\varepsilon,\end{aligned} \tag{6.9}$$

where the constant $C>0$ is independent of n, thanks to the uniform boundedness of (u_n) in $W_0^{1,p}(\Omega)$ and the continuous Sobolev embedding.

From (6.9), we deduce that the sequences

$$\left(\frac{f_n\, u_n}{(u_n+\frac{1}{n})^\gamma}\right)_n \quad \text{and} \quad \left(g_n\, u_n^{q+1}\right)_n$$

are equi-integrable. Therefore, by Vitali's convergence theorem, we obtain

$$\int_\Omega \frac{f_n\, u_n}{(u_n+\frac{1}{n})^\gamma}\,dx \to \int_\Omega f u^{1-\gamma}, \tag{6.10}$$

and

$$\int_\Omega g_n\, u_n^{q+1}\,dx \to \int_\Omega g u^{q+1}\,dx. \tag{6.11}$$

Combining (6.10) and (6.11), it follows that

$$\int_\Omega \frac{f_n\, u_n}{(u_n+\frac{1}{n})^\gamma}\,dx + \int_\Omega g_n\, u_n^{q+1}\,dx \to \int_\Omega f u^{1-\gamma} + \int_\Omega g u^{q+1}\,dx. \tag{6.12}$$

Case 2: $\gamma=1$. In this case, observe that

$$\frac{f_n\, u_n}{u_n+\frac{1}{n}} \le f,$$

and

$$\frac{f_n\, u_n}{u_n+\frac{1}{n}} \to f \quad \text{almost everywhere in } \Omega.$$

Hence, by the Lebesgue dominated convergence theorem, we deduce

$$\int_\Omega \frac{f_n\, u_n}{u_n + \frac{1}{n}}\, dx \to \int_\Omega f\, dx. \tag{6.13}$$

Combining (6.13) with the already established convergence (6.11), we obtain

$$\int_\Omega \frac{f_n\, u_n}{u_n + \frac{1}{n}}\, dx + \int_\Omega g_n\, u_n^{q+1}\, dx \to \int_\Omega f\, dx + \int_\Omega g u^{q+1}\, dx. \tag{6.14}$$

Using (6.5) in conjunction with either (6.12) or (6.14) (depending on the value of γ), we can pass to the limit in (6.8). This leads to the following identity:

$$M\left(\theta_0^p\right)\theta_0^p = \int_\Omega f u^{1-\gamma} + \int_\Omega g\, u^{q+1}. \tag{6.15}$$

Comparing (6.7) and (6.15), we obtain $M(\theta_0^p)\|u\|^p \geq M\left(\theta_0^p\right)\theta_0^p$, which implies $\|u\| \geq \theta_0$, since $M(\theta_0) \geq \Lambda > 0$.

On the other hand, since u_n converges weakly to u in $W_0^{1,p}(\Omega)$, and given that $\|u_n\|$ converges to θ_0, we have

$$\theta_0 = \liminf_{n\to+\infty} \|u_n\| \geq \|u\|.$$

Thus, $\|u\| = \theta_0$, and the proof of Theorem 2.1 is complete.

7 Regularity results

In this section, we investigate how the regularity of a solution u to problem (1.1) depends on the summability of f and g.

We begin with a key lemma that extends the class of admissible test functions in the weak formulation (1.9) from $C_0^\infty(\Omega)$ to $W_0^{1,p}(\Omega)$.

Lemma 7.1. *Let $u \in W_0^{1,p}(\Omega)$ be a weak solution of the problem (1.1). Then, the formulation (1.9) holds for all test functions $W_0^{1,p}(\Omega)$, i.e.,*

$$M\left(\|u\|^p\right)\int_\Omega |\nabla u|^{p-2}\nabla u\cdot\nabla\varphi = \int_\Omega \left(\frac{f(x)}{u^\gamma} + g(x)u^q\right)\varphi, \tag{7.1}$$

for every $\varphi \in W_0^{1,p}(\Omega)$.

Proof. The proof follows the approach of [18, Lemma A.5] and also [16, Lemma 5.1]. Let $u \in W_0^{1,p}(\Omega)$ be a weak solution of the problem (1.1). Given $\varphi \in W_0^{1,p}(\Omega)$, there exists a sequence $(\varphi_\varepsilon)_{\varepsilon>0} \subset C_0^\infty(\Omega)$ such that

$$\varphi_\varepsilon \to \varphi \quad \text{strongly in } W_0^{1,p}(\Omega). \tag{7.2}$$

Define $\psi_\varepsilon = |\varphi_\varepsilon - \varphi|$. By density, for each fixed $\varepsilon > 0$, there exists a sequence of nonnegative functions $\left(\psi_{\varepsilon\eta}\right)_{\eta>0} \subset C_0^\infty(\Omega)$ such that

$$\psi_{\varepsilon\eta} \to \psi_\varepsilon \quad \text{in } W_0^{1,p}(\Omega) \text{ and a.e. in } \Omega.$$

Testing (1.9) with $\psi_{\varepsilon\eta}$ yields

$$M\left(\|u\|^p\right)\int_\Omega |\nabla u|^{p-2}\nabla u\cdot\nabla\psi_{\varepsilon\eta}=\int_\Omega\left(\frac{f(x)}{u^\gamma}+g(x)u^q\right)\psi_{\varepsilon\eta}.$$

Taking the lim inf as $\eta\to 0$ and applying Fatou's lemma to the right-hand side, we obtain

$$\int_\Omega\left(\frac{f(x)}{u^\gamma}+g(x)u^q\right)\psi_\varepsilon\le M\left(\|u\|^p\right)\int_\Omega |\nabla u|^{p-2}\nabla u\cdot\nabla\psi_\varepsilon \tag{7.3}$$

Using Hölder's inequality on the right-hand side of (7.3), we deduce

$$\int_\Omega\left(\frac{f(x)}{u^\gamma}+g(x)u^q\right)|\varphi_\varepsilon-\varphi|\le M\left(\|u\|^p\right)\|u\|^{p-1}\|\psi_\varepsilon\|.$$

Passing to the limit as $\varepsilon\to 0$, we conclude that

$$\int_\Omega\left(\frac{f(x)}{u^\gamma}+g(x)u^q\right)\varphi_\varepsilon\to\int_\Omega\left(\frac{f(x)}{u^\gamma}+g(x)u^q\right)\varphi. \tag{7.4}$$

From (7.2), it follows that

$$M\left(\|u\|^p\right)\int_\Omega |\nabla u|^{p-2}\nabla u\cdot\nabla\varphi_\varepsilon\to M\left(\|u\|^p\right)\int_\Omega |\nabla u|^{p-2}\nabla u\cdot\nabla\varphi. \tag{7.5}$$

Combining (7.4) and (7.5), we obtain the desired result. □

Theorem 7.2. *Assume that $f\in L^m(\Omega)$, with $m>\frac{N}{p}$, and $g\in L^\infty(\Omega)$. Let u be a positive solution of* (1.1)*. Then, $u\in L^\infty(\Omega)$.*

Proof. The proof follows the approach in [14,29,35]. For $k\ge 1$, define the truncation function

$$G_k(t)=t-T_k(t)=\begin{cases}k & \text{if } t>k,\\ t & \text{if } -k\le t\le k,\\ -k & \text{if } t<-k.\end{cases}$$

For $s>0$ and $h>0$, define the test function

$$w=G_k(u)\min\left\{G_k(u)^{ps},h^p\right\}.$$

Testing the weak formulation (7.1) with w, and using the uniform lower bound $M\left(\|u\|^p\right)\ge\Lambda$ from Lemma 5.3 (see (5.19)), we obtain

$$\begin{aligned}\Lambda\int_\Omega|\nabla G_k(u)|^p\min\left\{G_k(u)^{ps},h^p\right\}+ps\int_{\{0\le G_k(u)^s\le h\}}G_k(u)^{ps}|\nabla G_k(u)|^p\\ \le\int_\Omega\left(\frac{f}{u^\gamma}+g\,u^q\right)G_k(u)\min\left\{G_k(u)^{ps},h^p\right\}.\end{aligned}$$

Since $u\ge 1$ on the set $\{u\ge k\}$, we deduce that

$$\begin{aligned}\Lambda\int_\Omega\left|\nabla\Big(G_k(u)\min\left\{G_k(u)^s,h\right\}\Big)\right|^p\le&\int_\Omega fG_k(u)\min\left\{G_k(u)^{ps},h^p\right\}\\ &+\|g\|_{L^\infty(\Omega)}\int_\Omega G_k(u)^p\min\left\{G_k(u)^{ps},h^p\right\}.\end{aligned}$$

Since $f \in L^{\frac{N}{p}}(\Omega)$, the absolute continuity of the integral (see [6, Theorem 2.5.7]) implies that for every $\varepsilon > 0$, there exists $\delta > 0$ such that for any measurable set $E \subset \Omega$ with $|E| \leq \delta$, we have

$$\int_E |f(x)|^{\frac{N}{p}}\, dx \leq \varepsilon^{\frac{N}{p}}. \tag{7.6}$$

Now, by the Chebyshev inequality (see [7, Lemma 20.1])), for any $c > 0$,

$$|\{f(x) > c\}| \leq \frac{1}{c^{\frac{N}{p}}} \int_\Omega |f(x)|^{\frac{N}{p}}\, dx.$$

Then choosing c sufficiently large, we conclude that

$$|\{f > c\}| \leq \delta. \tag{7.7}$$

Combining (7.6) and (7.7), it follows that

$$\left(\int_{\{f \geq c\}} |f|^{\frac{N}{p}} \right)^{\frac{p}{N}} < \varepsilon.$$

Assume $G_k(u) \in L^{p(s+1)}(\Omega)$. Applying the Sobolev inequality on the left-hand side and the Hölder inequality on the right hand-side, we obtain

$$\begin{aligned} &S\Lambda \left(\int_\Omega \left| G_k(u) \min\{G_k(u)^s, h\} \right|^{p^*} \right)^{\frac{p}{p^*}} \\ &\quad \leq h^p |\Omega|^{\frac{p-1}{p^*}} \left(\int_{\{f \geq c\}} |f|^{\frac{N}{p}} \right)^{\frac{p}{N}} \|G_k(u)\|_{L^{p^*}(\Omega)} + \left(c + \|g\|_{L^\infty(\Omega)} \right) \int_\Omega G_k(u)^{p(s+1)} \\ &\quad \leq \varepsilon h^p |\Omega|^{\frac{p-1}{p^*}} \|G_k(u)\|_{L^{p^*}(\Omega)} + \left(c + \|g\|_{L^\infty(\Omega)} \right) \int_\Omega G_k(u)^{p(s+1)}. \end{aligned}$$

Choosing $\varepsilon < \frac{1}{h^p |\Omega|^{\frac{p-1}{p^*}}}$, we derive

$$\left(\int_\Omega \left| G_k(u) \min\{G_k(u)^s, h\} \right|^{p^*} \right)^{\frac{p}{p^*}} \leq \frac{1}{S\Lambda} \|G_k(u)\|_{L^{p^*}(\Omega)} + \frac{c + \|g\|_{L^\infty(\Omega)}}{S\Lambda} \|G_k(u)\|^{p(s+1)}_{L^{p(s+1)}(\Omega)}.$$

Taking $h \to \infty$, we obtain

$$\left(\int_\Omega |G_k(u)|^{p^*(s+1)} \right)^{\frac{p}{p^*}} \leq \frac{1}{S\Lambda} \|G_k(u)\|_{L^{p^*}(\Omega)} + \frac{c + \|g\|_{L^\infty(\Omega)}}{S\Lambda} \|G_k(u)\|^{p(s+1)}_{L^{p(s+1)}(\Omega)}. \tag{7.8}$$

Since $u \in W_0^{1,p}(\Omega)$, it follows from Sobolev's embedding theorem that $G_k(u) \in L^{p^*}(\Omega)$. Therefore, the bootstrap inequality (7.8) holds for $s = s_0 = 0$.

Now, set $s = s_1$ in (7.8) such that

$$p(s_1 + 1) = p^* = \frac{pN}{N - p},$$

which gives

$$s_1 + 1 = (s_0 + 1)N/(N - p).$$

This implies $G_k(u) \in L^{p^*(s_1+1)}(\Omega)$.

We now iterate this argument. Suppose $G_k(u) \in L^{p^*(s_{i-1}+1)}(\Omega)$; then, applying the same reasoning, we obtain

$$G_k(u) \in L^{p^*(s_i+1)}(\Omega), \quad \text{where } s_i + 1 = N(s_{i-1}+1)/(N-p).$$

The sequence $(s_i)_{i\in\mathbb{N}}$ is positive (since $s_1 > 0$) and increasing because $s_i + 1 = N(s_{i-1} + 1)/(N-p) > s_{i-1}+1$. Thus, $(s_i)_{i\in\mathbb{N}}$ approaches a limit l as $i \to \infty$. Suppose, for contradiction, that l is finite. Then, passing to the limit in the recurrence relation gives

$$l+1 = N(l+1)/(N-p),$$

which is impossible since $\frac{N}{N-p} > 1$. Hence $l = +\infty$, and we conclude that

$$G_k(u) \in L^\rho(\Omega) \quad \text{for all } \rho \in (1,\infty).$$

Next, we take $G_k(u)$ as a test function in the weak formulation (7.1). Using the uniform positivity estimate $M(\|u\|^p) \geq \Lambda$ from Lemma 5.3 (see (5.19)), we obtain

$$\Lambda \int_\Omega |\nabla G_k(u)|^p \leq \int_\Omega \left(\frac{f}{u^\gamma} + gu^q\right) G_k(u).$$

Since $u \geq k \geq 1$ on the set $\{u \geq k\}$, it follows that

$$\int_\Omega |\nabla G_k(u)|^p \leq \frac{1}{\Lambda} \int_\Omega (f + gG_k(u)^q)\, G_k(u).$$

Because $f + gG_k(u)^q \in L^\alpha(\Omega)$ for some $\alpha > \frac{N}{p}$, we can apply Stampacchia's L^∞-regularity result (see [34]) to conclude that $u \in L^\infty(\Omega)$. □

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