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Quadratic systems with two invariant real straight lines and an invariant parabola

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Abstract. After the linear differential systems in the plane the easiest ones are the quadratic polynomial differential systems. Due to their nonlinearity and also to their many applications these systems have been studied by many authors. Let **QS** denote the set of all planar quadratic polynomial differential systems, or simply *quadratic systems*, and let **QSR2P** denote the subset of **QS** having two finite invariant real straight lines and an invariant parabola. We classify the phase portraits in the Poincaré disc of all the quadratic systems in the class **QSR2P**.

Keywords: quardratic systems, planar polynomial differential systems, global phase portrait, Poincaré disc.

2020 Mathematics Subject Classification: 34C05, 34C20.

1 Introduction

The study of the quadratic systems started at the early twentieth century. Coppel in [10] mentions that in 1904 Büchel [3] published the first article on quadratic systems. Two classical surveys on quadratic systems were published in 1966 by Coppel [10], and in 1982 by Chicone and Tian [8].

Quadratic systems have been studied intensively during these last decades obtaining many good results on them, see for instance the books [1, 28] and the references cited therein. More than one thousand papers have been published on the quadratic systems, but we are still far from completely understanding them.

We consider the following class of planar polynomial differential systems

$$\dot{x} = P(x, y), \quad \dot{y} = Q(x, y), \tag{1.1}$$

where $P, Q \in \mathbb{R}[x, y]$, being $\mathbb{R}[x, y]$ the ring of real polynomials in the two variables x and y. System (1.1) is said to be quadratic if the highest degree of the polynomials P and Q is two, that is, if $\max\{\deg P, \deg Q\} = 2$. Such quadratic systems include a wide range of planar

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dynamical models, including classical examples such as certain Lotka–Volterra systems in population dynamics and Liénard-type systems in control theory.

In this work we restrict our attention to non-degenerate quadratic systems, meaning that the polynomials P and Q are coprime. This assumption guarantees that the quadratic system cannot be reduced to linear or constant differential systems.

The study of quadratic systems has been a central topic in the qualitative theory of dynamical systems, because they exhibit structurally rich dynamics despite their relatively simple algebraic form. These systems may admit multiple invariant algebraic curves, limit cycles, and different types of equilibria. The classification and characterisation of such systems remains an active area of research.

Quadratic systems having invariant conics, i.e., invariant algebraic curves of degree 2, have been studied by several authors, see [2, 5, 14, 16–18, 24–27]. Quadratic systems having invariant algebraic curves of degree 3 have been analysed in [4, 7, 12]. Quadratic systems having invariant straight lines with multiplicity 4 have been classified in [22, 23]. Quadratic systems having invariant straight lines with multiplicity 5 have been classified in [20, 21].

The objective of this paper is to classify the quadratic systems possessing two invariant real straight lines and an invariant parabola. Building on previous classifications of quadratic systems with invariant algebraic curves of degrees two and three [6, 12], this study forms part of a broader effort to investigate systems possessing degree-four invariant curves. Within this framework, we focus on a specific subclass and contribute to its classification. By deriving normal forms, analysing equilibria, and classifying their global phase portraits via Poincaré compactification, we integrate algebraic, geometric, and topological methods to provide all their different topological phase portraits in the Poincaré disc.

2 Preliminaries

We begin by recalling some fundamental notions used throughout the paper.

Let $X = P\frac{\partial}{\partial x} + Q\frac{\partial}{\partial y}$ be the polynomial vector field associated with the polynomial differential system (1.1).

2.1 First integrals and integrating factors.

Let U be an open subset of $\mathbb{R}^2$ and let $H : U \to \mathbb{R}$ be a non-constant C^1 function. We say that H is a *first integral* of the vector field X or of the system (1.1) if $H(x(t), y(t))$ is constant on all the solutions $(x(t), y(t))$ of system (1.1) contained in U. Note that H is a first integral if and only if

$$\frac{dH}{dt} = \frac{\partial H}{\partial x}P(x,y) + \frac{\partial H}{\partial y}Q(x,y) = 0.$$

Let $R : U \to \mathbb{R}$ be a C^1 non-zero function. The function R is an *integrating factor* of the vector field X or of the system (1.1) on U if one of the following three equivalent conditions holds on U:

$$\frac{\partial(RP)}{\partial x} = -\frac{\partial(RQ)}{\partial y}, \qquad \operatorname{div}(RP, RQ) = 0, \qquad XR = -R\operatorname{div}(P, Q).$$

Here $\operatorname{div}(P, Q) = \dfrac{\partial P}{\partial x} + \dfrac{\partial Q}{\partial y}$. The *first integral H associated to the integrating factor R* is given by

$$H(x, y) = \int R(x, y)P(x, y)\, dy + h(x),$$

where the function h is chosen such that $\frac{\partial H}{\partial x} = -RQ$. Then

$$\dot{x} = RP = \frac{\partial H}{\partial y}, \qquad \dot{y} = RQ = -\frac{\partial H}{\partial x}.$$

2.2 Invariant algebraic curves.

We recall that system (1.1) admits an *invariant algebraic curve* $f = f(x,y) = 0$ if

$$Xf = \frac{\partial f}{\partial x}P(x,y) + \frac{\partial f}{\partial y}Q(x,y) = K(x,y)f(x,y) = Kf, \tag{2.1}$$

where K is a polynomial called the *cofactor* of the invariant algebraic curve $f(x,y) = 0$. This terminology reflects the fact that if an orbit of the system has a point on the algebraic curve $f(x,y) = 0$, then the equality (2.1) forces that the entire orbit is contained in the algebraic curve $f(x,y) = 0$.

2.3 Extactic polynomial.

Let X be the vector field associated with the polynomial differential system (1.1) of degree m. The *n–th extactic polynomial of* X, $E_n(X)$, is defined by the determinant

$$\det \begin{pmatrix} v_1 & v_2 & \cdots & v_l \\ X(v_1) & X(v_2) & \cdots & X(v_l) \\ \vdots & \vdots & \ddots & \vdots \\ X^{l-1}(v_1) & X^{l-1}(v_2) & \cdots & X^{l-1}(v_l) \end{pmatrix},$$

where $v_1, v_2, \ldots, v_l$ form a basis of $\mathbb{R}_n[x,y]$, the $\mathbb{R}$–vector space of polynomials in $\mathbb{R}[x,y]$ of degree at most n, $l = (n+1)(n+2)/2$, $X^0(v_i) = v_i$ and $X^j(v_i) = X^{j-1}(X(v_i))$.

Note that the definition of the extactic polynomial is independent of the choice of basis in $\mathbb{R}_n[x,y]$.

Proposition 2.1 (Proposition 5.2 in [9])**.** *Let $f = 0$ be an invariant algebraic curve of degree n of the vector field X. Then the polynomial f divides the polynomial $E_n(X)$.*

If the invariant algebraic curve $f = f(x,y) = 0$ in Proposition 2.1 satisfies that f^k divides the polynomial $E_n(X)$ but f^{k-1} does not divide it, then by definition the *multiplicity* of the invariant algebraic curve f is k. Geometrically, this means that within the class of all polynomial vector fields of degree m, one can perturb the vector field X into a family X_ε, such that each X_ε admits k invariant algebraic curves, all of them tending to the curve $f = 0$ when $\varepsilon \to 0$. For further details see [9].

2.4 Poincaré compactification.

The Poincaré compactification is a classical tool for analysing the dynamics of planar polynomial differential systems in a neighbourhood of infinity.

Let X be a polynomial vector field of degree d and

$$\mathbb{S}^2 = \{(y_1, y_2, y_3) \in \mathbb{R}^3 : y_1^2 + y_2^2 + y_3^2 = 1\}$$

be the unit sphere in $\mathbb{R}^3$. Let π_0 be the plane tangent to $\mathbb{S}^2$ at $(0,0,1)$, then we identify the plane π_0 with $\mathbb{R}^2$. Consider the central projection $f : \mathbb{R}^2 \to \mathbb{S}^2$, which maps the point $(x,y,1)$ in plane π_0 to the two intersection points of the straight line through $(x,y,1)$ and $(0,0,0)$ with the sphere $\mathbb{S}^2$. Then $\mathbb{S}^1 = \{y \in \mathbb{S}^2 : y_3 = 0\}$ corresponds to the infinity of $\mathbb{R}^2$. Thus the differential Df maps the vector field X in $\mathbb{R}^2$ to X' in $\mathbb{S}^2 \setminus \mathbb{S}^1$. The new vector field X' consists of two symmetric copies of X with respect to the origin of coordinates.

Next we multiply X' by y_3^d to get a vector field $p(X)$ defined in the whole sphere $\mathbb{S}^2$. This process is called the *Poincaré compactification*. Furthermore, the dynamical behavior of $p(X)$ near $\mathbb{S}^1$ corresponds to behavior of X in a neighborhood of infinity.

Since $\mathbb{S}^2$ is a curved manifold we use local charts on the surface of $\mathbb{S}^2$ to facilitate the computations. More precisely, we map the surface of $\mathbb{S}^2$ into a group of local charts (U_i, ϕ_i) and (V_i, ψ_i) that are identical to $\mathbb{R}^2$ for $i = 1,2,3$, where

$$U_i = \{(y_1, y_2, y_3) \in \mathbb{S}^2 : y_i > 0\}, \quad V_i = \{(y_1, y_2, y_3) \in \mathbb{S}^2 : y_i < 0\},$$
$$\phi_i(y_1, y_2, y_3) = \psi_i(y_1, y_2, y_3) = \left(\frac{y_j}{y_i}, \frac{y_k}{y_i}\right), \text{ with } j < k, j, k \neq i.$$

The expressions of $p(X)$ in the local charts (U_1, ϕ_1), (U_2, ϕ_2), and (U_3, ϕ_3) are

$$\dot{u} = v^d \left[-uP\left(\frac{1}{v}, \frac{u}{v}\right) + Q\left(\frac{1}{v}, \frac{u}{v}\right)\right], \quad \dot{v} = -v^{d+1} P\left(\frac{1}{v}, \frac{u}{v}\right).$$
$$\dot{u} = v^d \left[P\left(\frac{u}{v}, \frac{1}{v}\right) - uQ\left(\frac{u}{v}, \frac{1}{v}\right)\right], \quad \dot{v} = -v^{d+1} Q\left(\frac{u}{v}, \frac{1}{v}\right),$$
$$\dot{u} = P(u,v), \quad \dot{v} = Q(u,v),$$

respectively. In the local chart (V_i, ψ_i) the expression of $p(X)$ is the same as in (U_i, ϕ_i) multiplied by $(-1)^{d-1}$ for $i = 1,2,3$. Furthermore, the straight line $v = 0$ in local charts U_1, U_2, V_1, V_2 corresponds to the equator $\mathbb{S}^1$ of $\mathbb{S}^2$, i.e., to the infinity of $\mathbb{R}^2$.

We note that an *infinite equilibrium* of X under the Poincaré compactification is an equilibrium of $p(X)$ in $\mathbb{S}^1$. Moreover, if y is an equilibrium in $\mathbb{S}^1$, then $-y$ is also an equilibrium. When the degree d of X is odd, these two equilibria have the same stability, and if d is even they have opposite stability. Due to the symmetry of $p(X)$ in $\mathbb{S}^2$, it is sufficient to investigate equilibria at infinity in the chart (U_1, ϕ_1) and checking if the origin of chart (U_2, ϕ_2) is also an equilibrium or not.

Since the integral curves in $\mathbb{S}^2$ are symmetric with respect to the origin, we only need to study the flow of $p(X)$ in the closed northern hemisphere, i.e., when $y_3 \geqslant 0$, projecting this part of $\mathbb{S}^2$ onto plane $y_3 = 0$ via a projection $\pi : (y_1, y_2, y_3) \mapsto (y_1, y_2, 0)$ gives the so-called *Poincaré disc* $\mathbb{D}$.

For more details on the Poincaré compactification see Chapter 5 of [11].

2.5 Topological equivalence of two polynomial vector fields.

Two Poincaré compactifications $p(X_1)$ and $p(X_2)$ on $\mathbb{D}$ are *topologically equivalent* if there exists a homomorphism $h : \mathbb{D} \to \mathbb{D}$ preserving $\mathbb{S}^1$ and mapping the orbits of $\pi(p(X_1))$ onto the orbits of $\pi(p(X_2))$ preserving or reversing orientation of all the orbits.

The set of separatrices Σ_X of a Poincaré compactification $\pi(p(X))$ consists of all the orbits on $\mathbb{S}^1$, finite equilibria, limit cycles, and the two orbits of the boundary of all hyperbolic sectors associated with the finite and infinite equilibria. For more details, see [13,15].

The open connected regions of $\mathbb{D} \setminus \Sigma_X$ are called *canonical regions* of $\pi(p(X))$. The set Σ_X and one orbit inside each canonical region form the *separatrix configuration* Σ'_X. We denote by S the number of separatrices in the phase portrait of $\pi(p(X))$ on $\mathbb{D}$, and by R the number of its canonical regions. When the circle at infinity in the Poincaré disc is filled with equilibria we do not compute them in the number S.

Two separatrix configurations Σ'_1 and Σ'_2 are *topologically equivalent* if and only if there exists a homomorphism $h : \Sigma'_1 \to \Sigma'_2$ such that $h(\Sigma'_1) = \Sigma'_2$. Hence by the following theorem which was proved by Markus [13], Neumann [15] and Peixoto [19] independently, it suffices to determine when two phase portraits in the Poincaré disc are topologically equivalent by analysing their separatrix configurations.

Theorem 2.2. *Two Poincaré compactified polynomial vector fields $\pi(p(X_1))$ and $\pi(p(X_2))$ with finitely many separatrices are topologically equivalent if and only if their separatrix configurations Σ'_{X_1} and Σ'_{X_2} are topologically equivalent.*

2.6 Quadratic systems with an invariant parabola

The following proposition provides the classification of the quadratic systems having an invariant parabola.

Lemma 2.3 (Proposition 4 in [6])**.** *A quadratic system having an invariant parabola after an affine change of coordinates and a rescaling of the time can be written in the following form*

$$\dot{x} = \frac{b}{2}xy - \frac{a}{2}\left(y - x^2\right) - (p + qx + ry), \quad \dot{y} = by^2 + c\left(y - x^2\right) - 2x(p + qx + ry), \tag{P}$$

where a, b, c, p, q, r are real parameters.

3 Main results

In the following two theorems, we present the normal forms of all planar quadratic systems possessing two invariant real straight lines and an invariant parabola, together with their corresponding phase portraits in the Poincaré disc.

Theorem 3.1. *By doing an affine change of variables and a rescaling of the time all the quadratic systems having either two invariant real straight lines and an invariant parabola $f_j = f_j(x, y) = 0$ with cofactors $K_j = K_j(x, y)$ for $j = 1, 2, 3$, or having one real straight line of multiplicity two and an invariant parabola, can be written as one of the following systems, where $H = H(x, y)$ is a first integral and $R = R(x, y)$ is an integrating factor of the corresponding quadratic system.*

$S(1)$ $\dot{x} = x + 1, \quad \dot{y} = x^2 + 2x + y,$
$f_1 = y - x^2, \quad K_1 = 1,$
$f_2 = x + 1,$ *with multiplicity two*, $K_2 = 1,$
$H = \frac{f_1}{f_2}.$

$S(2)$ $\dot{x} = -4x^2 - 2ax + 2y - a + \frac{1}{2}, \quad \dot{y} = -4xy + (1 - 2a)x - 4ay,$
$f_1 = y - x^2, \quad K_1 = -4(2x + a),$
$f_2 = 4x + 4y + 1, \quad K_2 = -4x - 4a + 2,$
$f_3 = 4ax + 4y + 2a - 1, \quad K_3 = -2(2x + a), \quad H = \frac{f_1}{f_3^2}.$

S(3) $\dot{x} = -x^2 + (a+b)\,x - ab, \quad \dot{y} = (2b-1)\,x^2 - 2xy + (2a+1)\,y - 2abx,$
$f_1 = y - x^2, \quad K_1 = -2x + 2a + 1,$
$f_2 = x - a, \quad K_2 = b - x,$
$f_3 = x - b, \quad K_3 = a - x, \quad H = f_1^{a-b}\, f_2\, f_3^{2b-2a-1},$
where $(2a - 2b + 1) \neq 0$.

S(4) $\dot{x} = 8(a+1)\,x^2 + 8xy + 2(4ab + 4a + 4b + 3)\,x + 4(2b+1)\,y + (4ab + 2a + 2b + 1),$
$\dot{y} = 8(a+2b+1)\,x^2 + 8(2a+2b+3)\,xy + 2(4ab+2a+2b+1)\,x + 4(4ab+2a+1)\,y + 16y^2,$
$f_1 = y - x^2, \quad K_1 = 16(a+1)\,x + 16y + 4(4ab + 2a + 1),$
$f_2 = 4x + 4y + 1, \quad K_2 = 16(a+b+1)\,x + 16y + 4(4ab + 2a + 2b + 1),$
$f_3 = 2x + 2b + 1, \quad K_3 = 8(a+1)\,x + 8y + 2(2a+1),$
$R = \frac{f_2^{(a-2b)/(2b)}}{f_1^{(a+b)/(2b)} f_3^2},$
where $ab \neq 0$.

S(5) $\dot{x} = 4(a-b+2)\,x^2 + 8xy + 2(b+3)\,x - 4(a-b-1)\,y + (b+1),$
$\dot{y} = 4(2-a+b)\,x^2 + 24xy + 16y^2 + 2(b+1)\,x + 4(a+1)\,y,$
$f_1 = y - x^2, \quad K_1 = 8(a-b+2)\,x + 16y + 4(a+1),$
$f_2 = 4x + 4y + 1, \quad K_2 = 4(4x + 4y + b + 1),$
$f_3 = 4x + 4y + b + 1, \quad K_3 = 4(4x + 4y + 1),$
$R = \frac{1}{f_1 \sqrt{f_2 f_3}},$
where $a - b \neq 0$.

S(6) $\dot{x} = 8(ab + 2a - b^2 + 4)\,x^2 + 8(b+2)^2\,xy - 2(b^2 - 4a - 12)\,x - 4(ab - 2b^2 - 6b + 2a - 4)\,y - (ab - 2a + 2b - 4),$
$\dot{y} = 16(2+b)\,x^2 + 8(ab + 2a + 6b + 12)\,xy + 16(b+2)^2\,y^2 - 2(ab - 2a + 2b - 4)\,x - 4(b^2 - 4a + 4b - 4)\,y,$
$f_1 = y - x^2, \quad K_1 = 16(ab + 2a - b^2 + 4)\,x + 16(b+2)^2\,y - 4(b^2 - 4a + 4b - 4),$
$f_2 = 4x + 4y + 1, \quad K_2 = 8(ab + 2a + 2b - b^2 + 8)\,x + 16(b+2)^2\,y - 4(ab - 2a + 2b - 4),$
$f_3 = 8x + 4(b+2)\,y + (2-b), \quad K_3 = 8(a+4)(b+2)\,x + 16(b+2)^2\,y + 8(a+2),$
$R = \frac{f_1^{(a-3b)/(2b-2a)} f_2^{b/(b-a)}}{f_3^2},$
where $(a-b)(b+2) \neq 0$.

S(7) $\dot{x} = 8(a+1)\,x^2 + 8xy + 2(ab + 2a + 2b + 3)\,x - 4(a-b-1)\,y + (ab + a + b + 1),$
$\dot{y} = 8(b+1)\,x^2 + 8(a+b+3)\,xy + 16y^2 + (2ab + 2a + 2b + 2)\,x + 4(ab + 2a + 1)\,y,$
$f_1 = y - x^2, \quad K_1 = 16(a+1)\,x + 16y + 4(ab + 2a + 1),$
$f_2 = 4x + 4y + 1, \quad K_2 = 8(a+b+2)\,x + 16y + 4(ab + a + b + 1),$
$f_3 = 4(b+1)\,x + 4y + (b+1)^2, \quad K_3 = 8(a+2)\,x + 16y + 4(a+1),$
$R = \frac{1}{\sqrt{f_1 f_2 f_3}},$
where $a(a-b) \neq 0$.

S(8) $\dot{x} = 8(a-b+1)\,x^2 + 8xy - 2(4b^2 - 4a - 3)\,x + 4(2b+1)\,y - (2a+1)(4b^2 - 1),$
$\dot{y} = 8(a+b+1)\,x^2 + 8(2a+3)\,xy + 16y^2 - 2(8ab^2 + 4b^2 - 2a - 1)\,x - 4(4b^2 + 2b - 2a - 1)\,y,$
$f_1 = y - x^2, \quad K_1 = 16(a-b+1)\,x + 16y - 4(4b^2 + 2b - 2a - 1),$
$f_2 = 4b^2 - 1 - 4x - 4y, \quad K_2 = 16(a+1)\,x + 16y + 4(2a+1),$

$$f_3 = 2x + 2b + 1, \quad K_3 = 8(a - b + 1)\,x + 8y - 2(4ab - 2a + 2b - 1),$$
$$R = f_1^{-(a+3b)/(2b)} f_2^{(a-3b)/(2b)} f_3,$$
where $b(a-b) \neq 0$.

Here $a, b \in \mathbb{R}$.

Theorem 3.2. *The global phase portrait of each system from Theorem 3.1 in the Poincaré disc is topologically equivalent to one of the phase portraits shown in Figure 3.1. Below each phase portrait in the Poincaré disc appear its name, the number of its separatrices S, the number of its canonical regions R, and the system associated with this phase portrait.*

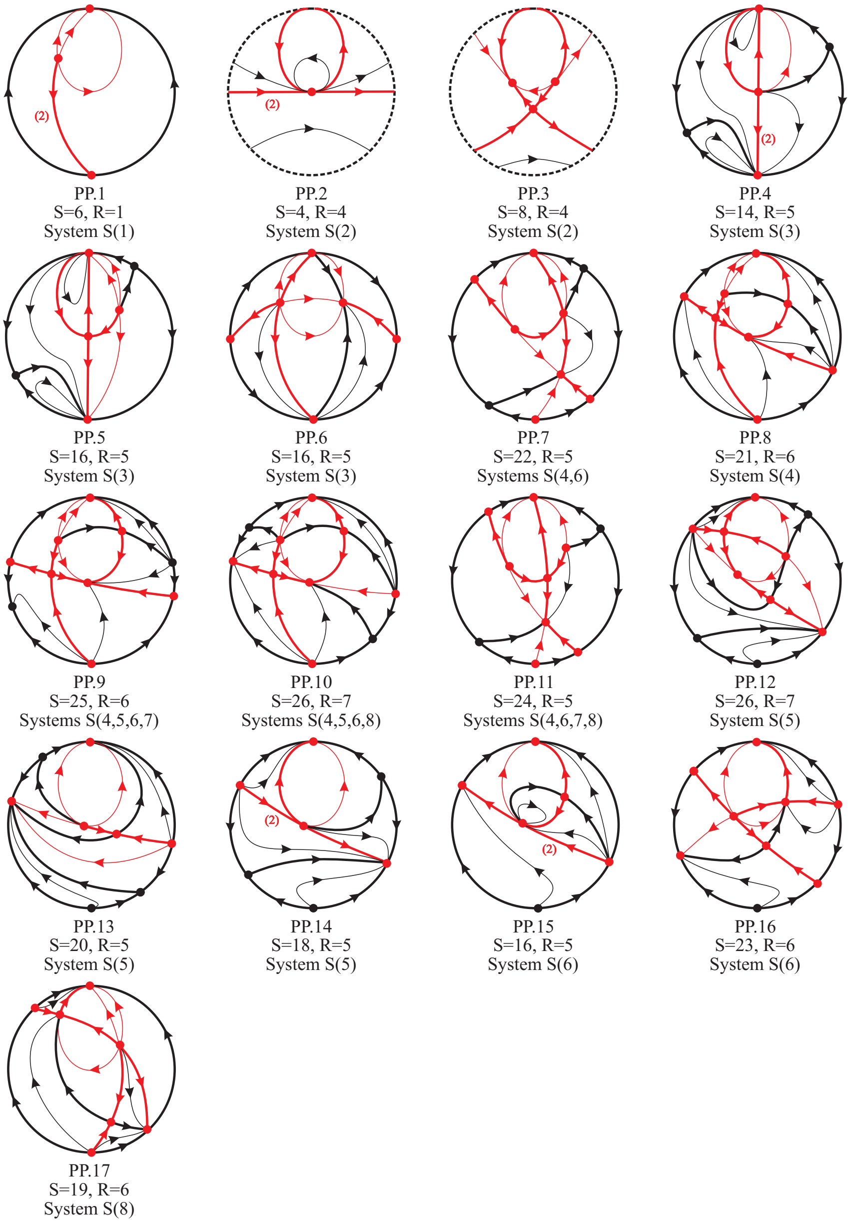

Figure 3.1: Phase portraits in the Poincaré disc of the systems in Theorem 3.1. Separatrices are represented by thick black or red lines, invariant curves are drawn in red, while trajectories within canonical regions are drawn with thin black lines.

4 Structure of the proof

To simplify the proofs of Theorems 3.1 and 3.2 we divide the original system (P) into two separate cases: $b = 0$ and $b = 1$. The overall structure of the analysis for each case is summarised in Figures 4.1 and 4.2.

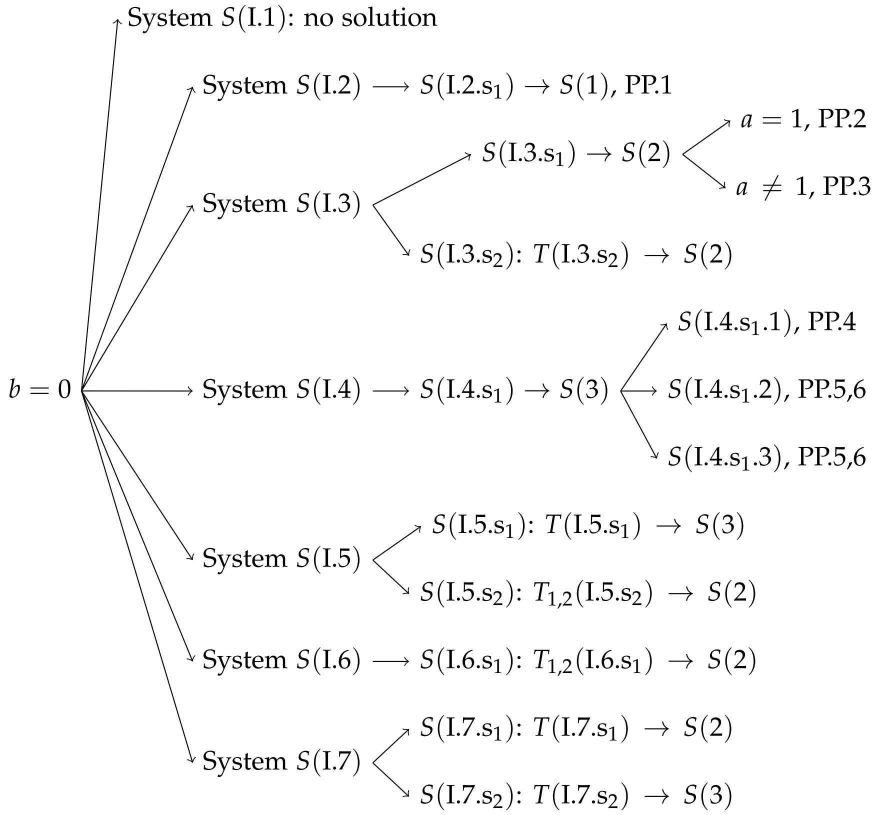

Figure 4.1: Structure of the proof of Theorem 3.1 when $b = 0$.

For each system in Theorem 3.1 we first impose the existence of one invariant straight line to the system, obtaining in total 12 systems, denoted by $S(\text{I}.i)$ and $S(\text{II}.i)$ for $b = 0$ and $b = 1$, respectively. Then, for each of these systems, we impose the existence of a second invariant straight line, obtaining new systems denoted by $S(\text{I}.i.\text{s}_j)$ and $S(\text{II}.i.\text{s}_j)$ which already have an invariant parabola and two invariant straight lines.

Sometimes one of those systems having an invariant parabola and two invariant straight lines is particular cases of another system having an invariant parabola and two invariant straight lines. For example, the notation

$$S(\text{I}.3.\text{s}_2) : T(\text{I}.3.\text{s}_2) \to S(2)$$

indicates that the system $S(\text{I}.3.\text{s}_2)$ can be obtained from system $S(2)$ via the transformation $T(\text{I}.3.\text{s}_2)$, and can therefore be regarded as a particular or an equivalent case. Consequently, when computing the phase portraits, it suffices to consider only the systems that cannot be obtained from other systems.

We shall identify only eight distinct systems whose phase portraits need to be studied. For each of these systems we list the relevant parameter conditions corresponding to different combinations of equilibria, along with the associated phase portrait labels. For instance, under system $S(2)$, the condition "$a = 1$, PP.2" means that when $a = 1$, the corresponding phase portrait is labelled as PP.2.

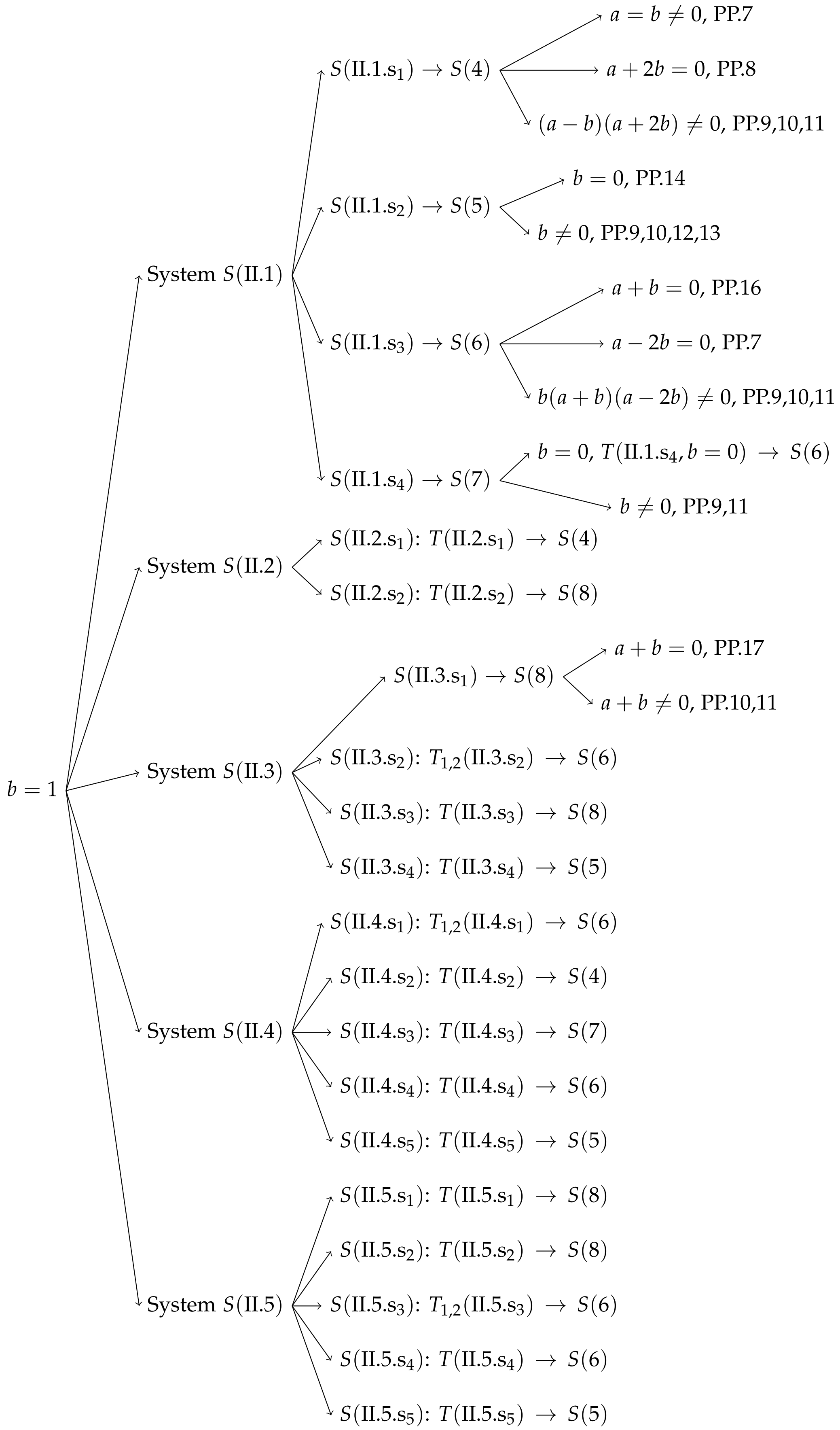

Figure 4.2: Structure of the proof of Theorem 3.1 when $b = 1$.

5 Proof of Theorem 3.1

Proof. Our objective is to determine the general forms of all systems in **QSR2P**. The proof begins with system (P) in Lemma 2.3, which admits an invariant parabola defined by $f_1 = y - x^2 = 0$.

The approach is first to force the existence of one additional invariant straight line $f_2 = 0$ in system (P). By applying the definition of invariant algebraic curve, we obtain all possible systems with one invariant straight line. After that, for each system, we force the existence of a second invariant straight line $f_3 = 0$. In this way, we obtain all systems that admit two invariant straight lines $f_2 = 0$ and $f_3 = 0$, and the invariant parabola $f_1 = 0$.

To facilitate the computations in the study of system (P) we consider the two distinct cases $b = 0$ and $b = 1$. Note that when $b \neq 0$, after rescaling the time, we can assume $b = 1$; the other parameters of the new system are divided by b, but we keep using the same letters to denote them.

Case I: $b = 0$. If $a = 0$, the existence of the invariant parabola requires $c \neq 0$. In this case, after performing the change of parameters $(p, q, r) \to (cp, cq, cr)$ together with the time rescaling $t \to \frac{t}{c}$, the system (P) simplifies to

$$\dot{x} = -qx - ry - p, \quad \dot{y} = (y - x^2) - 2x(qx + ry + p). \tag{P.1}$$

The existence of one invariant straight line $f_2 = 0$ in system (P.1) leads to two distinct subsystems, denoted by $S(\text{I}.k)$ for $k = 1, 2$.

If $a \neq 0$, after doing the change of parameters $(p, q, r, c) \to \left(\frac{ap}{2}, \frac{aq}{2}, \frac{ar}{2}, -\frac{ac}{2}\right)$ and the time rescaling $t \to \frac{-2t}{a}$ the system (P) reduces to

$$\dot{x} = y - x^2 + qx + ry + p, \quad \dot{y} = c(y - x^2) + 2x(qx + ry + p). \tag{P.2}$$

The existence of an invariant straight line $f_2 = 0$ in system (P.2) leads to five distinct subsystems, denoted by $S(\text{I}.k)$ for $k = 3, 4, 5, 6, 7$.

Case II: $b = 1$. The system (P) reduces to

$$\dot{x} = \frac{1}{2}xy - \frac{a}{2}(y - x^2) - (qx + ry + p), \quad \dot{y} = y^2 + c(y - x^2) - 2x(qx + ry + p). \tag{P.3}$$

Forcing the existence of the straight line $f_2 = 0$ produces five subsystems, denoted by $S(\text{II}.k)$ for $k = 1, 2, 3, 4, 5$.

Thus, we have obtained a total of twelve systems. To each of these systems we force the existence of a second invariant straight line $f_3 = 0$ in order to find all possible systems in the class **QSR2P**.

The proof is organised into the following fourteen steps:

- **Step 1**: Derivation of the systems $S(\text{I}.1)$ through $S(\text{I}.7)$,
- **Step 2**: Derivation of the systems $S(\text{II}.1)$ through $S(\text{II}.5)$,
- **Steps 3–14**: To each of the above twelve systems we force the existence of a second invariant straight line $f_3 = 0$ for generating all systems in **QSR2P**.

Step 1. Suppose that system (P.1) possesses an invariant straight line $f_2 = a_1x + a_2y + a_0 = 0$ with cofactor $K_2 = k_1x + k_2y + k_0$. Substituting f_2, K_2 and the system (P.1) into (2.1), we obtain two solutions

$$\mathrm{I.s}_1 = \begin{cases} a_0 = \dfrac{a_1^2}{4a_2},\ k_0 = \dfrac{a_1k_1}{2a_2} + 1,\ k_2 = 0,\ p = -\dfrac{a_1(2a_2 + a_1k_1)}{8a_2^2}, \\ q = -\dfrac{a_2 + a_1k_1}{2a_2},\ r = -\dfrac{k_1}{2} \end{cases},$$

$$\mathrm{I.s}_2 = \left\{ a_2 = 0,\ k_1 = 0,\ k_2 = 0,\ p = -\frac{a_0k_0}{a_1},\ q = -k_0,\ r = 0 \right\}.$$

Since $a_2 \neq 0$ in solution $(\mathrm{I.s}_1)$, substituting it into the system (P.1), after doing the transformation $a_1 \to 2a_1a_2$ and the time rescaling $t \to 2t$, we obtain the system

$$\begin{aligned} S(\mathrm{I.1})\quad & \dot{x} = (2a_1k_1 + 1)x + k_1y + a_1 + a_1^2k_1, \\ & \dot{y} = 4a_1k_1x^2 + 2k_1xy + (2a_1 + 2a_1^2k_1)x + 2y, \\ & f_2 = 2a_1x + y + a_1^2, \qquad K_2 = 2k_1x + 2k_1a_1 + 2. \end{aligned}$$

For the solution $(\mathrm{I.s}_2)$, we have $a_1k_0 \neq 0$. Indeed, if $k_0 = 0$, then we must have $p = q = r = 0$, which implies $\dot{x} = 0$. Substituting $(\mathrm{I.s}_2)$ into the system (P.1), after doing the transformation $a_0 \to \frac{a_1a_0}{k_0}$, we obtain the system

$$\begin{aligned} S(\mathrm{I.2})\quad & \dot{x} = k_0x + a_0, \qquad \dot{y} = (2k_0 - 1)x^2 + 2a_0x + y, \\ & f_2 = k_0x + a_0, \qquad K_2 = k_0. \end{aligned}$$

Suppose that the system (P.2) possesses an invariant straight line $f_2 = 0$ with cofactor K_2 as defined previously. To simplify the computations, we consider two cases for the parameter a_1 in f_2: $a_1 = 1$ and $a_1 = 0$. Of course, if $a_1 \neq 0$ by rescaling of the time we can set $a_1 = 1$.

Case 1. $a_1 = 1, f_2 = x + a_2y + a_0$. Substituting f_2, K_2, and the system (P.2) into (2.1), we obtain the following three solutions

$$\mathrm{I.s}_3 = \left\{ a_0 = \frac{1}{4a_2},\ c = \frac{2a_2k_0 - k_1 - 2}{2a_2},\ k_2 = 0,\ p = \frac{k_0}{4a_2},\ q = \frac{2a_2k_0 + k_1}{4a_2},\ r = \frac{k_1}{2} \right\},$$

$$\mathrm{I.s}_4 = \{ a_2 = 0,\ k_1 = -1,\ k_2 = 0,\ p = a_0k_0,\ q = k_0 - a_0,\ r = -1 \},$$

$$\mathrm{I.s}_5 = \left\{ c = -\frac{1}{a_2},\ k_1 = 2a_2k_0,\ k_2 = 0,\ p = a_0k_0,\ q = k_0,\ r = a_2k_0 \right\}.$$

For the solution $(\mathrm{I.s}_3)$, we have $a_2 \neq 0$. Substituting $(\mathrm{I.s}_3)$ into the system (P.2), after doing the transformation $(x,\ y,\ t,\ k_0) \to \left(\frac{x}{a_2},\ \frac{y}{a_2^2},\ 4a_2t,\ \frac{k_0}{a_2}\right)$, we obtain the system

$$\begin{aligned} S(\mathrm{I.3})\quad & \dot{x} = -4x^2 + (2k_0 + k_1)x + (2k_1 + 4)y + k_0, \\ & \dot{y} = (4k_1 + 4)x^2 + 2k_0x + 4k_1xy + (4k_0 - 2k_1 - 4)y, \\ & f_2 = 4x + 4y + 1, \qquad K_2 = 4(k_1x + k_0). \end{aligned}$$

Substituting the solution $(\mathrm{I.s}_4)$ into system the (P.2), we obtain the system $*S(\mathrm{I.4})$

$$\dot{x} = -x^2 + (k_0 - a_0)\,x + a_0k_0, \quad \dot{y} = (2k_0 - 2a_0 - c)\,x^2 + 2a_0k_0\,x - 2xy + cy.$$

Here we have $c + 2a_0 \neq 0$. Otherwise, if $c + 2a_0 = 0$, the system becomes

$$\dot{x} = -(x + a_0)(x - k_0), \quad \dot{y} = 2(x + a_0)(k_0x - y),$$

and can be reduced to a linear differential system.

Doing the transformation $(x,\ y,\ t,\ a_0,\ c,\ k_0) \to \left(c_1x,\ c_1^2y,\ \frac{t}{c_1},\ a_0c_1,\ c_1 - 2a_0,\ k_0c_1\right)$ with $c_1 \neq 0$ to system $*S(\mathrm{I.4})$ we obtain

$$S(\text{I.4})\ \ \dot{x} = -x^2 + (k_0 - a_0)\,x + a_0k_0, \quad \dot{y} = (2k_0 - 1)\,x^2 + 2a_0k_0\,x - 2xy - (2a_0 - 1)\,y,$$
$$f_2 = x + a_0, \quad K_2 = k_0 - x.$$

For the solution (I.s$_5$), we have $a_2 \neq 0$. Substituting (I.s$_5$) into the system (P.2), after doing the transformation $(x,\ y,\ t,\ k_0,\ a_0) \to (\frac{x}{a_2},\ \frac{y}{a_2^2},\ a_2t,\ \frac{k_0}{a_2},\ \frac{a_0}{a_2})$, we obtain the system

$$S(\text{I.5})\ \ \dot{x} = -x^2 + k_0x + (k_0 + 1)\,y + a_0k_0, \quad \dot{y} = (2k_0 + 1)\,x^2 + 2a_0k_0\,x + 2k_0xy - y,$$
$$f_2 = x + y + a_0, \quad K_2 = k_0(2x + 1).$$

Case 2. $a_1 = 0$, $f_2 = a_2y + a_0$. Substituting f_2, K_2, and the system (P.2) into (2.1), we obtain two solutions

$$\text{I.s}_6 = \left\{a_0 = 0,\ k_0 = c,\ k_2 = 0,\ p = 0,\ q = \frac{c}{2},\ r = \frac{k_1}{2}\right\},$$
$$\text{I.s}_7 = \left\{c = 0,\ k_0 = 0,\ k_2 = 0,\ p = \frac{a_0k_1}{2a_2},\ q = 0,\ r = \frac{k_1}{2}\right\}.$$

For the solution (I.s$_6$), we have $k_1 \neq 2$. Indeed, substituting (I.s$_6$) and $k_1 = 2$ into the system (P.2) we obtain a linear differential system. Taking (I.s$_6$) into (P.2), doing the parameter change $k_1 \to k_1 - 2$ and the time rescaling $t \to 2t$ we obtain

$$S(\text{I.6})\ \ \dot{x} = -2x^2 + cx + k_1y, \quad \dot{y} = (2k_1 - 4)\,xy + 2cy,$$
$$f_2 = y, \quad K_2 = (2k_1 - 4)\,x + 2c.$$

For the solution (I.s$_7$), we have $a_2 \neq 0$. Substituting (I.s$_7$) into the system (P.2), after doing the transformation $(x,\ y,\ t,\ a_0) \to (\frac{x}{a_2},\ \frac{y}{a_2^2},\ 2a_2t,\ \frac{a_0}{a_2k_1})$, we obtain the system

$$S(\text{I.7})\ \ \dot{x} = -2x^2 + (k_1 + 2)\,y + a_0, \quad \dot{y} = 2k_1xy + 2a_0x,$$
$$f_2 = k_1y + a_0, \quad K_2 = 2k_1x.$$

Step 2. Suppose that system (P.3) possesses an invariant straight line $f_2 = 0$ with cofactor K_2 as defined previously. In order to simplify the computations, we consider two cases for the parameter a_1 in f_2: $a_1 = 1$ and $a_1 = 0$.

Case 1. $a_1 = 1$, $f_2 = x + a_2y + a_0$. Substituting f_2, K_2, and the system (P.3) into (2.1), we obtain three solutions

$$\text{II.s}_1 = \left\{\begin{array}{l} a = 2a_2c - 2a_2k_0 + k_1,\ a_0 = \dfrac{1}{4a_2},\ k_2 = 1,\ p = -\dfrac{k_0}{4a_2},\ q = -\dfrac{2a_2k_0 + k_1}{4a_2}, \\ r = -\dfrac{2a_2k_1 + 1}{4a_2} \end{array}\right\},$$
$$\text{II.s}_2 = \left\{a = 2k_1,\ a_2 = 0,\ k_2 = \frac{1}{2},\ p = -a_0k_0,\ q = -k_0 - a_0k_1,\ r = -\frac{a_0 + 2k_1}{2}\right\},$$
$$\text{II.s}_3 = \left\{\begin{array}{l} a = \dfrac{2a_2^2c - 2a_0a_2 + 1}{a_2},\ k_1 = \dfrac{4a_2^2k_0 + 1}{2a_2},\ k_2 = 1,\ p = -a_0k_0, \\ q = -\dfrac{a_0 + 2a_2k_0}{2a_2},\ r = -\dfrac{2a_2^2k_0 + 1}{2a_2} \end{array}\right\}.$$

For the solution (II.s$_1$), we have $a_2 \neq 0$. Substituting (II.s$_1$) into the system (P.3), after doing the transformation $(x, y, t, k_0, k_1, c) \to (\frac{x}{a_2},\ \frac{y}{a_2^2},\ 4a_2^2t,\ \frac{2k_0}{a_2^2},\ \frac{k_1}{a_2},\ \frac{c}{a_2^2})$, we obtain the system

$$S(\text{II.1})\ \ \dot{x} = (4c - 8k_0 + 2k_1)\,x^2 + 2xy + (4k_0 + k_1)\,x + (8k_0 - 4c + 1)\,y + 2k_0,$$
$$\dot{y} = (8k_0 - 4c + 2k_1)\,x^2 + (2 + 4k_1)\,xy + 4y^2 + 4k_0x + 4cy,$$
$$f_2 = 4(x + y) + 1, \quad K_2 = 4(k_1x + y + 2k_0).$$

Substituting (II.s$_2$) into the system (P.3) and doing the time rescaling $t \to 2t$, it simplifies to system $*S(\text{II}.2)$

$$\dot{x} = 2k_1x^2 + xy + (2k_0 + 2a_0k_1)\,x + a_0y + 2a_0k_0,$$
$$\dot{y} = (4a_0k_1 + 4k_0 - 2c)\,x^2 + (2a_0 + 4k_1)\,xy + 2y^2 + 4a_0k_0x + 2c\,y.$$

Here we have $c - 2k_0 \neq 0$. If $c - 2k_0 = 0$, the system becomes

$$\dot{x} = (x + a_0)(2k_1x + y + 2k_0), \quad \dot{y} = 2(a_0x + y)(2k_1x + y + 2k_0),$$

and can be reduced to a linear differential system.

Doing the following transformation to system $*S(\text{II}.2)$

$$(x, y, t, a_0, c, k_0, k_1) \to \left(\sqrt{c_1}\,x,\ c_1y,\ \frac{t}{c_1},\ \sqrt{c_1}\,a_0,\ c_1(2k_0+1),\ c_1k_0,\ \sqrt{c_1}\,k_1\right)$$

yields

$$S(\text{II}.2)\quad \begin{aligned} &\dot{x} = 2k_1x^2 + xy + a_0y + (2k_0 + 2a_0k_1)\,x + 2a_0k_0,\\ &\dot{y} = (4a_0k_1 - 2)\,x^2 + (2a_0 + 4k_1)\,xy + 2y^2 + 4a_0k_0x + (2 + 4k_0)\,y,\\ &f_2 = x + a_0, \quad K_2 = 2k_1x + y + 2k_0. \end{aligned}$$

For the solution (II.s$_3$), we have $a_2 \neq 0$. Substituting (II.s$_3$) into the system (P.3), after doing the transformation $(x, y, t, k_0, c, a_0) \to \left(\frac{x}{a_2},\ \frac{y}{a_2^2},\ 2a_2^2t,\ \frac{k_0}{a_2^2},\ \frac{c}{a_2^2},\ \frac{a_0}{a_2}\right)$, it reduces to

$$S(\text{II}.3)\quad \begin{aligned} &\dot{x} = (2c - 2a_0 + 1)\,x^2 + xy + (a_0 + 2k_0)\,x + (2a_0 - 2c + 2k_0)\,y + 2a_0k_0,\\ &\dot{y} = (2a_0 - 2c + 4k_0)\,x^2 + (4k_0 + 2)\,xy + 2y^2 + 4a_0k_0x + 2cy,\\ &f_2 = x + y + a_0, \quad K_2 = (4k_0 + 1)\,x + 2y + 2k_0. \end{aligned}$$

Case 2. $a_1 = 0$, $f_2 = a_2y + a_0$. Since $a_2 \neq 0$, we set $a_2 = 1$ and then $f_2 = y + a_0$. Substituting f_2, K_2, and the system (P.3) into (2.1), we obtain two solutions

$$\text{II.s}_4 = \left\{a_0 = 0,\ k_0 = c,\ k_2 = 1,\ p = 0,\ q = -\frac{c}{2},\ r = -\frac{k_1}{2}\right\},$$
$$\text{II.s}_5 = \left\{c = a_0,\ k_0 = 0,\ k_2 = 1,\ p = -\frac{a_0k_1}{2},\ q = -\frac{a_0}{2},\ r = -\frac{k_1}{2}\right\}.$$

Substituting the solution (II.s$_4$) into the system (P.3) and doing the time rescaling $t \to 2t$ yields system $*S(\text{II}.4)$

$$\dot{x} = ax^2 + xy + cx + (k_1 - a)\,y, \qquad \dot{y} = 2k_1xy + 2y^2 + 2cy.$$

Here we have $k_1 - a \neq 0$. If $k_1 - a = 0$, the system reduces to

$$\dot{x} = x(ax + y + c), \quad \dot{y} = 2y(ax + y + c),$$

and can be reduced to a linear differential system.

Applying the transformation $(x, y, t, a, c, k_1) \to \left(kx,\ k^2y,\ \frac{t}{k^2},\ ak,\ ck^2,\ k(a+1)\right)$ to the system $*S(\text{II}.4)$ yields

$$S(\text{II}.4)\quad \begin{aligned} &\dot{x} = ax^2 + xy + cx + y, \quad \dot{y} = (2a + 2)\,xy + 2y^2 + 2cy,\\ &f_2 = y, \quad K_2 = 2(a + 1)\,x + 2y + 2c. \end{aligned}$$

Substituting the solution (II.s$_5$) into the system (P.3) and doing the time rescaling $t \to 2t$ yields system $*S$(II.5)

$$\dot{x} = ax^2 + xy + a_0 x + (k_1 - a)\, y + a_0 k_1, \qquad \dot{y} = 2k_1 xy + 2y^2 + 2a_0 k_1 x + 2a_0 y.$$

Here we have $a \neq 0$. If $a = 0$, the system becomes

$$\dot{x} = (x + k_1)(y + a_0), \quad \dot{y} = 2(y + a_0)(k_1 x + y),$$

and it can be reduced to a linear differential system.

Applying the transformation $(x, y, t, a_0, k_1) \to \left(ax,\ a^2 y,\ \frac{t}{a^2},\ a_0 a^2,\ k_1 a\right)$ to the system $*S$(II.5) yields

$$S(\text{II.5})\ \ \dot{x} = x^2 + xy + a_0 x + (k_1 - 1)\, y + a_0 k_1, \qquad \dot{y} = 2k_1 xy + 2y^2 + 2a_0 k_1 x + 2a_0 y,$$
$$f_2 = y + a_0, \quad K_2 = 2(k_1 x + y).$$

Step 3. Suppose that system S(I.1) possesses an invariant straight line $f_3 = b_1 x + b_2 y + b_0 = 0$ with cofactor $K_3 = l_1 x + l_2 y + l_0$. Substituting f_3, K_3, and the system S(I.1) into (2.1), we obtain the following four solutions

$$\begin{aligned}
\text{I.1.s}_1 &= \left\{b_0 = a_1^2 b_2,\ b_1 = 2a_1 b_2,\ l_0 = 2a_1 k_1 + 2,\ l_1 = 2k_1,\ l_2 = 0\right\},\\
\text{I.1.s}_2 &= \{b_0 = a_1 b_1,\ b_2 = 0,\ k_1 = 0,\ l_0 = 1,\ l_1 = 0,\ l_2 = 0\},\\
\text{I.1.s}_3 &= \left\{b_0 = \frac{a_1 b_1}{2},\ b_2 = \frac{b_1}{2a_1},\ k_1 = 0,\ l_0 = 2,\ l_1 = 0,\ l_2 = 0\right\},\\
\text{I.1.s}_4 &= \{a_1 = 0,\ b_0 = 0,\ b_2 = 0,\ k_1 = 0,\ l_0 = 1,\ l_1 = 0,\ l_2 = 0\}.
\end{aligned}$$

Solutions (I.1.s$_k$), $k = 2, 3, 4$ correspond to linear systems, so we discard them. Solution (I.1.s$_1$) corresponds to a system with a double invariant straight line. By requiring that f_2^2 be a factor of the extactic polynomial E_1 we obtain the following polynomial system

$$2a_1^3 k_1^2 + 4a_1^2 k_1 + a_1 = 0, \quad 6a_1^2 k_1^2 + 8a_1 k_1 + 1 = 0, \quad 6a_1 k_1^2 + 4k_1 = 0,\ 2k_1^2 = 0.$$

However, there are no solutions for the above polynomial system.

Step 4. Imposing an invariant straight line $f_3 = 0$ with cofactor K_3 on S(I.2), and substituting f_3, K_3, and S(I.2) into (2.1), we obtain the following three solutions

$$\begin{aligned}
\text{I.2.s}_1 &= \left\{a_0 = \frac{b_0 l_0}{b_1},\ b_2 = 0,\ k_0 = l_0,\ l_1 = 0,\ l_2 = 0\right\},\\
\text{I.2.s}_2 &= \left\{a_0 = \frac{b_1}{4b_2},\ b_0 = \frac{b_1^2}{4b_2},\ k_0 = \frac{1}{2},\ l_0 = 1,\ l_1 = 0,\ l_2 = 0\right\},\\
\text{I.2.s}_3 &= \left\{a_0 = 0,\ b_0 = 0,\ b_1 = 0,\ k_0 = \frac{1}{2},\ l_0 = 1,\ l_1 = 0,\ l_2 = 0\right\}.
\end{aligned}$$

Solutions (I.2.s$_2$), (I.2.s$_3$) correspond to linear systems. Solution (I.2.s$_1$) corresponds to a system with a double invariant straight line. By requiring that f_2^2 be a factor of the extactic polynomial E_1 we obtain the following polynomial equations

$$\frac{b_0^2 l_0 (l_0 - 1)}{a_1 b_1^2} = 0, \quad \frac{l_0 (1 - l_0)}{a_1} = 0,$$

and two solutions $l_0 = 0$ and $l_0 = 1$. If $l_0 = 0$, the system degenerates to

$$\dot{x} = 0, \quad \dot{y} = y - x^2.$$

If $l_0 = 1$, taking it into the solution (I.2.s_1) we obtain

$$\text{I.2.s}_{1.1} = \left\{ a_0 = \frac{b_0}{b_1},\ b_2 = 0,\ k_0 = 1,\ l_0 = 1,\ l_1 = 0,\ l_2 = 0 \right\}.$$

Substituting $(\text{I.2.s}_{1.1})$ into system $S(\text{I.2})$, we obtain the system $S(\text{I.2.s}_{1.1})$:

$$\dot{x} = x + \frac{b_0}{b_1}, \qquad \dot{y} = x^2 + \frac{2b_0 x}{b_1} + y.$$

If $b_0 \neq 0$, set $a \neq 0$. Doing the transformation $(x, y, b_0) \to (ax,\ a^2 y,\ ab_1)$, the system $S(\text{I.2.s}_{1.1})$ reduces to

$$\begin{aligned} S(1)\quad & \dot{x} = x + 1, \quad \dot{y} = x^2 + 2x + y, \\ & f_1 = y - x^2, \quad K_1 = 1, \\ & f_2 = x + 1, \quad K_2 = 1, \quad H = \frac{f_1}{f_2}. \end{aligned}$$

Here and in what follows, H denotes a first integral.

If $b_0 = 0$, applying the translation $(x, y) \to (x + 1,\ y - 1)$ to the system $S(\text{I.2.s}_{1.1})$, it reduces to $S(1)$.

Step 5. We impose that $S(\text{I.3})$ possesses an invariant straight line $f_3 = 0$ with cofactor K_3. To simplify the computations, we consider the parameter b_1 in f_3 in two cases: $b_1 = 1$ and $b_1 = 0$.

Case 1. $b_1 = 1$, so that $f_3 = x + b_2 y + b_0$. Substituting f_3, K_3, and $S(\text{I.3})$ into (2.1), eliminating the solutions corresponding to linear systems, and treating the solutions corresponding to systems with a double invariant straight line as in $S(\text{I.1})$ and $S(\text{I.2})$, we eventually obtain two solutions

$$\begin{aligned} \text{I.3.s}_1 &= \left\{ b_0 = \frac{1}{2} - \frac{b_2}{4},\ k_0 = \frac{1}{2} - \frac{1}{b_2},\ k_1 = -1,\ l_0 = -\frac{2}{b_2},\ l_1 = -4,\ l_2 = 0 \right\}, \\ \text{I.3.s}_2 &= \left\{ b_0 = \frac{1}{4b_2},\ k_0 = -\frac{1}{2b_2},\ k_1 = -1,\ l_0 = -2,\ l_1 = -4,\ l_2 = 0 \right\}. \end{aligned}$$

Substituting (I.3.s_1) into $S(\text{I.3})$, we obtain the system $S(\text{I.3.s}_1)$:

$$\dot{x} = -4x^2 - \frac{2}{b_2}x + 2y - \frac{1}{b_2} + \frac{1}{2}, \qquad \dot{y} = -4xy - \left(\frac{2}{b_2} - 1\right)x - \frac{4}{b_2}y.$$

Doing the transformation $b_2 \to \frac{1}{a}$, the system reduces to

$$\begin{aligned} S(2)\quad & \dot{x} = -4x^2 - 2ax + 2y - a + \frac{1}{2}, \quad \dot{y} = -4xy + (1 - 2a)x - 4ay, \\ & f_1 = y - x^2, \quad K_1 = -4(2x + a), \\ & f_2 = 4x + 4y + 1, \quad K_2 = -4x - 4a + 2, \\ & f_3 = 4ax + 4y + 2a - 1, \quad K_3 = -2(2x + a), \quad H = \frac{f_1}{f_3^2}. \end{aligned}$$

Substituting (I.3.s$_2$) into S(I.3), we obtain the system S(I.3.s$_2$):

$$\dot{x} = -4x^2 - \left(\frac{1}{b_2}+1\right)x + 2y - \frac{1}{2b_2}, \qquad \dot{y} = -4xy - \frac{1}{b_2}x - \left(\frac{2}{b_2}+2\right)y.$$

However, applying the parameter change $T(\text{I.3.s}_2) : a \to \frac{b_2+1}{2b_2}$ to system $S(2)$, the system reduces to S(I.3.s$_2$).

Case 2. $b_1 = 0$, so that $f_3 = b_2 y + b_0$. Following the same process as in **Case 1**, we obtain two solutions

$$\text{I.3.s}_3 = \{b_0 = 0,\ k_0 = 0,\ k_1 = -1,\ l_0 = -2,\ l_1 = -4,\ l_2 = 0\},$$
$$\text{I.3.s}_4 = \left\{b_0 = -\frac{b_2}{4},\ k_0 = \frac{1}{2},\ k_1 = -1,\ l_0 = 0,\ l_1 = -4,\ l_2 = 0\right\}.$$

However, the system associated with solution (I.3.s$_3$) is the case $b_2 = 2$ of system S(I.3.s$_1$), while the solution (I.3.s$_4$) arises when $b_2 = -1$ in system S(I.3.s$_2$).

Step 6. Substituting f_3, K_3, and S(I.4) into (2.1), and following the same process as in S(I.3), we obtain a solution

$$\text{I.4.s}_1 = \{b_0 = -k_0,\ b_2 = 0,\ l_0 = -a_0,\ l_1 = -1,\ l_2 = 0\}.$$

Substituting (I.4.s$_1$) into S(I.4) yields system S(I.4.s$_1$):

$$\dot{x} = -x^2 + (k_0 - a_0)x + a_0 k_0, \qquad \dot{y} = (2k_0 - 1)x^2 - 2xy + (1 - 2a_0)y + 2a_0 k_0 x.$$

Applying the parameter change $(a_0, k_0) \to (-a,\ b)$ to the above system yields

$S(3)$ $\dot{x} = -x^2 + (a+b)x - ab, \quad \dot{y} = (2b-1)x^2 - 2xy + (2a+1)y - 2abx,$
$f_1 = y - x^2, \quad K_1 = -2x + 2a + 1,$
$f_2 = x - a, \quad K_2 = b - x,$
$f_3 = x - b, \quad K_3 = a - x, \quad H = f_1^{a-b} f_2 f_3^{2b-2a-1}.$

Here we have $2a - 2b + 1 \neq 0$. If $2a - 2b + 1 = 0$, the system becomes

$$\dot{x} = -\frac{1}{2}(2x - 2a - 1)(x - a), \quad \dot{y} = (2x - 2a - 1)(ax - y),$$

and can be reduced to a linear differential system.

Step 7. Substituting f_3, K_3, and S(I.5) into (2.1), we obtain two solutions

$$\text{I.5.s}_1 = \{a_0 = b_0 - b_0^2,\ b_2 = 0,\ k_0 = -1,\ l_0 = b_0 - 1,\ l_1 = -1,\ l_2 = 0\},$$
$$\text{I.5.s}_2 = \left\{a_0 = \frac{2b_2 - 1}{4b_2^2},\ b_0 = \frac{1}{4b_2},\ k_0 = -\frac{1}{2},\ l_0 = \frac{1 - 2b_2}{2b_2},\ l_1 = -1,\ l_2 = 0\right\}.$$

Substituting (I.5.s$_1$) and (I.5.s$_2$) into S(I.5), we obtain systems S(I.5.s$_1$) and S(I.5.s$_2$):

$$\dot{x} = -x^2 - x + b_0^2 - b_0, \qquad \dot{y} = -x^2 - 2xy + (2b_0^2 - 2b_0)x - y.$$

$$\dot{x} = -x^2 - \frac{1}{2}x + \frac{1}{2}y + \frac{1}{8b_2^2} - \frac{1}{4b_2}, \qquad \dot{y} = -xy + \left(\frac{1}{4b_2^2} - \frac{1}{2b_2}\right)x - y,$$

respectively. Applying the transformations T(I.5.s$_1$) to system $S(3)$, it reduces to S(I.5.s$_1$), where

$$T(\text{I.5.s}_1) : (x, y, t, a, b) \to \left(a + \frac{x + b_0}{2b_0 - 1},\ a^2 + \frac{b_0^2 + 2a(b_0 + x) - y}{2b_0 - 1},\ (2b_0 - 1)^2 t,\ a,\ a + 1\right).$$

For the system $S(\text{I.5.s}_2)$, if $2b_2 - 1 \neq 0$, then applying the transformation $T_1(\text{I.5.s}_2)$ to system $S(2)$ yields $S(\text{I.5.s}_2)$ with $b_2 \neq \frac{1}{2}$; if $2b_2 - 1 = 0$, then applying the transformation $T_2(\text{I.5.s}_2)$ to system $S(2)$ yields $S(\text{I.5.s}_2)$ with $b_2 = \frac{1}{2}$, where

$$T_1(\text{I.5.s}_2): \quad (x, y, t, a) \to \left(\frac{b_2 x}{2b_2 - 1}, \frac{b_2^2 y}{(2b_2 - 1)^2}, \frac{(2b_2 - 1)t}{4b_2}, \frac{b_2}{2b_2 - 1}\right).$$

$$T_2(\text{I.5.s}_2): \quad (x, y, t, a) \to \left(\frac{x}{2}, \frac{y}{4}, \frac{t}{2}, \frac{1}{2}\right).$$

Step 8. Substituting f_3, K_3, and $S(\text{I.6})$ into (2.1), we obtain a solution

$$\text{I.6.s}_1 = \{b_0 = 0,\ c = -b_2,\ k_1 = 1,\ l_0 = -b_2,\ l_1 = -2,\ l_2 = 0\}.$$

Substituting (I.6.s_1) into $S(\text{I.6})$, the system $S(\text{I.6})$ simplifies to $S(\text{I.6.s}_1)$:

$$\dot{x} = -2x^2 - b_2 x + y, \qquad \dot{y} = -2xy - 2b_2 y.$$

If $b_2 \neq 0$, then applying the transformation $T_1(\text{I.6.s}_1)$ to $S(2)$ yields $S(\text{I.6.s}_1)$; if $b_2 = 0$, then applying the transformation $T_2(\text{I.6.s}_1)$ to $S(2)$ yields $S(\text{I.6.s}_1)$ with $b_2 = 0$, where

$$T_1(\text{I.6.s}_1): \quad (x, y, t, a) \to \left(\frac{x}{2b_2}, \frac{y}{4b_2^2}, b_2 t, \frac{1}{2}\right).$$

$$T_2(\text{I.6.s}_1): \quad (x, y, t, a) \to \left(-x - \frac{1}{2},\ x + y + \frac{1}{4},\ -\frac{t}{2},\ 1\right).$$

Step 9. Substituting f_3, K_3, and $S(\text{I.7})$ into (2.1), we obtain two solutions

$$\text{I.7.s}_1 = \left\{a_0 = \frac{b_2^2}{4},\ b_0 = \frac{b_2}{4},\ k_1 = -1,\ l_0 = b_2,\ l_1 = -2,\ l_2 = 0\right\},$$
$$\text{I.7.s}_2 = \{a_0 = 2b_0^2,\ b_2 = 0,\ k_1 = -2,\ l_0 = 2b_0,\ l_1 = -2,\ l_2 = 0\}.$$

Substituting (I.7.s_1) and (I.7.s_2) into $S(\text{I.7})$, we obtain the systems $S(\text{I.7.s}_1)$ and $S(\text{I.7.s}_2)$

$$\dot{x} = -2x^2 + y + \frac{b_2^2}{4}, \qquad \dot{y} = -2xy + \frac{b_2^2}{2}x,$$

$$\dot{x} = -2x^2 + 2b_0^2, \qquad \dot{y} = -4xy + 4b_0^2 x,$$

respectively. Applying the transformation $T(\text{I.7.s}_1)$ to system $S(2)$ yields $S(\text{I.7.s}_1)$, where

$$T(\text{I.7.s}_1): \quad (x, y, t, a) \to \left(-x + \frac{b_2}{2} - \frac{1}{2},\ (1 - b_2)x + y + \frac{1}{4}(b_2 - 1)^2,\ -\frac{t}{2},\ 1 - b_2\right).$$

Similarly, applying the transformation $T(\text{I.7.s}_2)$ to $S(3)$ yields $S(\text{I.7.s}_2)$, where

$$T(\text{I.7.s}_2): \quad (x, y, t, a, b) \to \left(a + \frac{x + b_0}{2b_0},\ a^2 - b_0^2 + \frac{(2a + 1)(x + b_0)}{2b_0} + y,\ -4b_0 t,\ a,\ a + 1\right).$$

Step 10. Substituting f_3, K_3, and $S(\text{II.1})$ into (2.1), we obtain four solutions

$$
\begin{aligned}
&\text{II.1.s}_1 = \left\{ b_2 = 0,\ c = b_0(l_0 - \frac{1}{2}) + \frac{1}{4},\ k_1 = b_0 + l_0,\ k_0 = \frac{l_0 b_0}{2},\ l_1 = 2l_0 + 1,\ l_2 = 2 \right\},\\
&\text{II.1.s}_2 = \{ b_0 = 2k_0,\ b_2 = 1,\ k_1 = 1,\ l_0 = 1,\ l_1 = 4,\ l_2 = 4 \},\\
&\text{II.1.s}_3 = \left\{ \begin{array}{l} b_0 = \dfrac{2-b_2}{4},\ k_0 = \dfrac{(2-b_2)(4c+1)}{16},\ k_1 = \dfrac{4b_2^2 c + b_2^2 - 2b_2 + 4}{4b_2},\\ l_0 = 2c + \dfrac{1}{2},\ l_1 = \dfrac{b_2^2 + 4b_2^2 c + 2}{b_2},\ l_2 = 4 \end{array} \right\},\\
&\text{II.1.s}_4 = \left\{ \begin{array}{l} b_0 = \dfrac{1}{4b_2},\ c = \dfrac{8b_2k_0 + 8b_2^2 k_0 - 1}{4b_2},\ k_1 = \dfrac{8b_2^2 k_0 + 1}{2b_2},\ l_0 = 8b_2k_0,\\ l_1 = 16b_2k_0 + 2,\ l_2 = 4 \end{array} \right\}.
\end{aligned}
$$

Substituting (II.1.k) into $S(\text{II.1})$, we obtain the systems $S(\text{II.1.k})$ for $k = \text{s}_1,\ \text{s}_2,\ \text{s}_3,\ \text{s}_4$.

$$
\begin{aligned}
\dot{x} &= (2l_0+1)\,x^2 + 2xy + (b_0 + l_0 + 2b_0l_0)\,x + 2b_0y + b_0l_0,\\
\dot{y} &= (4b_0 + 2l_0 - 1)\,x^2 + (4b_0 + 4l_0 + 2)\,xy + 4y^2 + 2b_0l_0x + (4b_0l_0 - 2b_0 + 1)\,y.
\end{aligned} \qquad (S(\text{II.1.s}_1))
$$

$$
\begin{aligned}
\dot{x} &= 2(2c - 4k_0 + 1)\,x^2 + 2xy + (4k_0+1)\,x + (8k_0 - 4c + 1)\,y + 2k_0,\\
\dot{y} &= 2(4k_0 - 2c + 1)\,x^2 + 6xy + 4y^2 + 4k_0x + 4cy.
\end{aligned} \qquad (S(\text{II.1.s}_2))
$$

$$
\begin{aligned}
\dot{x} &= \left(\frac{2}{b_2} + b_2 + 4b_2c - 2\right) x^2 + 2xy + \left(\frac{1}{b_2} + 2c\right) x - \left(\frac{b_2}{2} + 2b_2c - 2\right) y\\
&\quad + \left(c - \frac{b_2c}{2} - \frac{b_2}{8} + \frac{1}{4}\right),\\
\dot{y} &= \frac{2}{b_2}x^2 + \left(\frac{4}{b_2} + b_2 + 4b_2c\right) xy + \left(2c - \frac{b_2}{4} - b_2c + \frac{1}{2}\right) x + 4y^2 + 4cy.
\end{aligned} \qquad (S(\text{II.1.s}_3))
$$

$$
\begin{aligned}
\dot{x} &= 16b_2k_0\,x^2 + 2xy + \left(4k_0 + 4b_2k_0 + \frac{1}{2b_2}\right) x + \left(\frac{1}{b_2} - 8b_2k_0 + 1\right) y + 2k_0,\\
\dot{y} &= \frac{2}{b_2}x^2 + \left(\frac{2}{b_2} + 16b_2k_0 + 2\right) xy + 4y^2 + 4k_0x + \left(8k_0 + 8b_2k_0 - \frac{1}{b_2}\right) y.
\end{aligned} \qquad (S(\text{II.1.s}_4))
$$

Applying the parameter change $(b_0, l_0) \to \left(b + \frac{1}{2},\ a + \frac{1}{2}\right)$ and the time rescaling $t \to 4t$ to $S(\text{II.1.s}_1)$, we obtain

S(4) $\dot{x} = 8(a+1)\,x^2 + 8xy + 2(4ab + 4a + 4b + 3)\,x + 4(2b+1)\,y + (4ab + 2a + 2b + 1),$
$\dot{y} = 8(a + 2b + 1)\,x^2 + 8(2a + 2b + 3)\,xy + 2(4ab + 2a + 2b + 1)\,x + 4(4ab + 2a + 1)\,y + 16y^2,$
$f_1 = y - x^2, \quad K_1 = 16(a+1)\,x + 16y + 4(4ab + 2a + 1),$
$f_2 = 4x + 4y + 1, \quad K_2 = 16(a + b + 1)\,x + 16y + 4(4ab + 2a + 2b + 1),$
$f_3 = 2x + 2b + 1, \quad K_3 = 8(a+1)\,x + 8y + 2(2a+1),$
$R = \dfrac{f_2^{(a-2b)/(2b)}}{f_1^{(a+b)/(2b)} f_3^2}.$

Here R is an integrating factor of the system and $ab \neq 0$. If $a = 0$, the system becomes

$$\dot{x} = (4x + 4y + 1)(2x + 2b + 1), \quad \dot{y} = 2(4x + 4y + 1)((2b+1)x + 2y).$$

If $b = 0$, the system becomes

$$\dot{x} = (4(a+1)x + 4y + 2a + 1)(2x+1), \quad \dot{y} = 2(4(a+1)x + 4y + 2a + 1)(x + 2y).$$

In both cases these systems can be reduced to linear differential systems.

Applying the parameter change $(c, k_0) \to \left(\frac{1+a}{4}, \frac{1+b}{8}\right)$ and the time rescaling $t \to 4t$ to $S(\text{II.1.s}_2)$, we obtain

S(5) $\dot{x} = 4(a-b+2)\,x^2 + 8xy + 2(b+3)\,x - 4(a-b-1)\,y + (b+1)$,
$\dot{y} = 4(2-a+b)\,x^2 + 24xy + 16y^2 + 2(b+1)\,x + 4(a+1)\,y$,
$f_1 = y - x^2, \quad K_1 = 8(a-b+2)\,x + 16y + 4(a+1)$,
$f_2 = 4x + 4y + 1, \quad K_2 = 4(4x + 4y + b + 1)$,
$f_3 = 4x + 4y + b + 1, \quad K_3 = 4(4x + 4y + 1)$,
$R = \dfrac{1}{f_1 \sqrt{f_2}\, f_3}$.

Here we have $a - b \neq 0$. If $a - b = 0$, the system becomes

$$\dot{x} = (4x + 4y + b + 1)(2x+1), \quad \dot{y} = 2(4x + 4y + b + 1)(x + 2y),$$

and the system can be reduced to a linear differential system.

Applying the parameter change $(c, b_2) \to \left(\frac{a+2}{(b+2)^2} - \frac{1}{4}, \frac{b+2}{2}\right)$ and the time rescaling $t \to 4(b+2)^2 t$ to $S(\text{II.1.s}_3)$, we obtain

S(6) $\dot{x} = 8(ab + 2a - b^2 + 4)\,x^2 + 8(b+2)^2\,xy - 2(b^2 - 4a - 12)\,x - 4(ab - 2b^2 - 6b + 2a - 4)\,y - (ab - 2a + 2b - 4)$,
$\dot{y} = 16(2+b)\,x^2 + 8(ab + 2a + 6b + 12)\,xy + 16(b+2)^2\,y^2 - 2(ab - 2a + 2b - 4)\,x - 4(b^2 - 4a + 4b - 4)\,y$,
$f_1 = y - x^2, \quad K_1 = 16(ab + 2a - b^2 + 4)\,x + 16(b+2)^2\,y - 4(b^2 - 4a + 4b - 4)$,
$f_2 = 4x + 4y + 1, \quad K_2 = 8(ab + 2a + 2b - b^2 + 8)\,x + 16(b+2)^2\,y - 4(ab - 2a + 2b - 4)$,
$f_3 = 8x + 4(b+2)\,y + (2-b), \quad K_3 = 8(a+4)(b+2)\,x + 16(b+2)^2\,y + 8(a+2)$,
$R = \dfrac{f_1^{(a-3b)/(2b-2a)}\, f_2^{b/(b-a)}}{f_3^2}$.

Here we have $(a-b)(b+2) \neq 0$. Indeed, if $a - b = 0$, the system becomes

$$\dot{x} = (a+2)(8x + 4(a+2)y - a + 2)(2x+1), \quad \dot{y} = 2(a+2)(8x + 4(a+2)y - a + 2)(x + 2y).$$

If $b + 2 = 0$, the system becomes

$$\dot{x} = 4(a+2)(2x+1), \quad \dot{y} = 8(a+2)(x + 2y).$$

Applying the parameter change $(k_0, b_2) \to \left(\frac{(b+1)(a+1)}{8}, \frac{1}{b+1}\right)$ and the time rescaling $t \to 4t$ to $S(\text{II.1.s}_4)$, we obtain

S(7) $\dot{x} = 8(a+1)\,x^2 + 8xy + 2(ab + 2a + 2b + 3)\,x - 4(a - b - 1)\,y + (ab + a + b + 1)$,
$\dot{y} = 8(b+1)\,x^2 + 8(a + b + 3)\,xy + 16y^2 + (2ab + 2a + 2b + 2)\,x + 4(ab + 2a + 1)\,y$,
$f_1 = y - x^2, \quad K_1 = 16(a+1)\,x + 16y + 4(ab + 2a + 1)$,
$f_2 = 4x + 4y + 1, \quad K_2 = 8(a + b + 2)\,x + 16y + 4(ab + a + b + 1)$,
$f_3 = 4(b+1)\,x + 4y + (b+1)^2, \quad K_3 = 8(a+2)\,x + 16y + 4(a+1)$,
$R = \dfrac{1}{\sqrt{f_1}\, f_2\, f_3}$.

Here we have $a(a-b)\neq 0$. If $a=0$ the system becomes

$$\dot{x}=(4x+4y+1)(2x+b+1),\quad \dot{y}=2(4x+4y+1)((b+1)x+2y).$$

If $a-b=0$, the system becomes

$$\dot{x}=(4(a+1)x+4y+(a+1)^2)(2x+1),\quad \dot{y}=2(4(a+1)x+4y+(a+1)^2)(x+2y).$$

Step 11. Substituting f_3, K_3, and $S(\text{II}.2)$ into (2.1), we obtain two solutions

$$\text{II}.2.\text{s}_1=\left\{\begin{matrix} b_0=\dfrac{1}{4b_2},\ k_0=\dfrac{2a_0b_2+4b_2^2-1}{8b_2^2(2a_0b_2-1)},\ k_1=\dfrac{2b_2(a_0+b_2)-1}{2b_2(2a_0b_2-1)},\\ l_0=\dfrac{a_0(2a_0b_2+4b_2^2-1)}{b_2(2a_0b_2-1)},\ l_1=\dfrac{4(1+a_0^2)b_2^2-1}{b_2(2a_0b_2-1)},\ l_2=2 \end{matrix}\right\},$$

$$\text{II}.2.\text{s}_2=\left\{\begin{matrix} b_0=a_0-a_0^2b_2,\ k_0=\dfrac{(a_0b_2-1)(a_0-2b_2-2a_0^2b_2)}{2b_2(2a_0b_2-1)},\ k_1=\dfrac{2b_2(a_0+b_2)-1}{2b_2(2a_0b_2-1)},\\ l_0=\dfrac{2b_2+2a_0^2b_2-a_0}{b_2(2a_0b_2-1)},\ l_1=\dfrac{4(a_0^2+1)b_2^2-1}{b_2(2a_0b_2-1)},\ l_2=2 \end{matrix}\right\}.$$

Substituting $(\text{II}.2.k)$ into $S(\text{II}.2)$, we obtain the systems $S(\text{II}.2.k)$ for $k=\text{s}_1,\ \text{s}_2$.

$$\begin{aligned}\dot{x}=&\frac{1}{4b_2^2(2a_0b_2-1)}\left(4b_2(2a_0b_2+2b_2^2-1)\,x^2+4b_2^2(2a_0b_2-1)\,xy+4a_0b_2^2(2a_0b_2-1)\,y\right.\\&\left.+(8a_0^2b_2^2+8a_0b_2^3+4b_2^2-2a_0b_2-1)\,x+a_0(2a_0b_2+4b_2^2-1)\right),\\ \dot{y}=&\frac{1}{4b_2^2(2a_0b_2-1)}\left(8(2a_0^2b_2^2-a_0b_2+b_2^2)\,x^2+8b_2(2a_0^2b_2^2+a_0b_2+2b_2^2-1)\,xy\right.\\&\left.+8b_2^2(2a_0b_2-1)\,y^2+2a_0(2a_0b_2+4b_2^2-1)\,x+2(8a_0b_2^3+2a_0b_2-1)\,y\right).\end{aligned}\tag{$S(\text{II}.2.\text{s}_1)$}$$

$$\begin{aligned}\dot{x}=&\frac{1}{b_2(2a_0b_2-1)}\left((2a_0b_2+2b_2^2-1)\,x^2+b_2(2a_0b_2-1)\,xy+a_0b_2(2a_0b_2-1)\,y\right.\\&\left.-(2a_0^3b_2^2-5a_0^2b_2+2a_0-2b_2)\,x-(2a_0^4b_2^2-3a_0^3b_2+2a_0^2b_2^2+a_0^2-2a_0b_2)\right),\\ \dot{y}=&\frac{1}{b_2(2a_0b_2-1)}\left(2(2a_0^2b_2-a_0+b_2)\,x^2+(4a_0^2b_2^2+2a_0b_2+4b_2^2-2)\,xy\right.\\&+2(2a_0b_2^2-b_2)\,y^2-(4a_0^4b_2^2+4a_0^2b_2^2-6a_0^3b_2-4a_0b_2+2a_0^2)\,x\\&\left.-(4a_0^3b_2^2-6a_0^2b_2+2a_0-2b_2)\,y\right).\end{aligned}\tag{$S(\text{II}.2.\text{s}_2)$}$$

Doing the transformation $T(\text{II}.2.\text{s}_1)$ to system $S(4)$ yields $S(\text{II}.2.\text{s}_1)$, where

$$T(\text{II}.2.\text{s}_1):\quad (x,y,t,a,b)\to\left(b_2x,\ b_2^2y,\ \frac{t}{8b_2^2},\ \frac{2b_2^2}{2a_0b_2-1},\ a_0b_2-\frac{1}{2}\right).$$

Applying the transformation

$$T_1(\text{II}.2.\text{s}_2):\quad (x,y,a_0)\to\left(x-\frac{2ab_2+1}{2b_2},\ \frac{(b_2-1)\,x}{b_2}+y+\frac{(1+2ab_2)^2}{4b_2^2},\ \frac{2ab_2+1}{2b_2}\right),$$

to $S(\text{II}.2.\text{s}_2)$ we obtain the system $S(\text{II}.2.\text{s}_2.1)$.

$$\dot{x}=\frac{a+1}{a}\,x^2+xy-2x,\qquad \dot{y}=\frac{2a^2+a+1}{a}\,x^2+\frac{2a^2+3a+2}{a}\,xy+2y^2-2y.$$

In **Step 12** we will prove that this system is a special case of system $S(8)$.
Step 12. Substituting f_3, K_3, and $S(\text{II}.3)$ into (2.1), we obtain four solutions

$$\text{II.3.s}_1 = \left\{ \begin{aligned} & a_0 = (1-b_0)b_0,\ b_2 = 0,\ c = \frac{b_0}{2} - b_0^2 + k_0,\ l_0 = 2(1-b_0)k_0, \\ & l_1 = 1 - b_0 + 2k_0,\ l_2 = 1 \end{aligned} \right\},$$

$$\text{II.3.s}_2 = \left\{ \begin{aligned} & a_0 = \frac{2b_2-1}{4b_2^2},\ b_0 = \frac{1}{4b_2},\ c = 2k_0 - \frac{1}{4b_2^2},\ l_0 = \left(4 - \frac{2}{b_2}\right)k_0, \\ & l_1 = 4k_0 - \frac{1}{b_2} + 2,\ l_2 = 2 \end{aligned} \right\},$$

$$\text{II.3.s}_3 = \left\{ \begin{aligned} & a_0 = -\frac{(2b_2k_0-1)(2b_2k_0+b_2-1)}{b_2^2},\ b_0 = 2k_0(1-2b_2k_0), \\ & c = \frac{b_2 + 8b_2k_0 - 8b_2^2k_0^2 - 2}{2b_2^2},\ l_0 = \frac{b_2 + 2b_2k_0 - 1}{b_2^2}, \\ & l_1 = 4k_0 - \frac{1}{b_2} + 2,\ l_2 = 2 \end{aligned} \right\},$$

$$\text{II.3.s}_4 = \left\{ b_0 = \frac{1}{4},\ b_2 = 1,\ k_0 = \frac{1}{4},\ l_0 = 2a_0,\ l_1 = 2,\ l_2 = 2 \right\}.$$

Substituting $(\text{II}.3.k)$ into $S(\text{II}.3)$, we obtain the systems $S(\text{II}.3.k)$ for $k = \text{s}_1,\ \text{s}_2,\ \text{s}_3,\ \text{s}_4$.

$$\begin{aligned} \dot{x} &= (2k_0 - b_0 + 1)\,x^2 + xy + b_0 y - (b_0^2 - b_0 - 2k_0)\,x - 2b_0k_0(b_0-1), \\ \dot{y} &= (2k_0 + b_0)\,x^2 + (4k_0+2)\,xy + 2y^2 - 4b_0k_0(b_0-1)\,x - (2b_0^2 - b_0 - 2k_0)\,y. \end{aligned} \tag{$S(\text{II}.3.\text{s}_1)$}$$

$$\begin{aligned} \dot{x} &= \frac{1}{4b_2^2}\left(4b_2(4b_2k_0 + b_2 - 1)\,x^2 + 4b_2^2\,xy - 4b_2(2b_2k_0-1)\,y + (8b_2^2k_0 + 2b_2 - 1)\,x \right. \\ &\quad \left. + 2k_0(2b_2-1)\right), \\ \dot{y} &= \frac{1}{4b_2^2}\left(4b_2\,x^2 + 8b_2^2(2k_0+1)\,xy + 8b_2^2\,y^2 + 4k_0(2b_2-1)\,x + 2(8b_2^2k_0 - 1)\,y\right). \end{aligned} \tag{$S(\text{II}.3.\text{s}_2)$}$$

$$\begin{aligned} \dot{x} &= \frac{1}{b_2^2}\left(b_2(4b_2k_0 + b_2 - 1)\,x^2 + b_2^2\,xy - b_2(2b_2k_0-1)\,y - (4b_2^2k_0^2 - 4b_2k_0 - b_2 + 1)\,x \right. \\ &\quad \left. - 8b_2^2k_0^3 - 4b_2^2k_0^2 + 8b_2k_0^2 + 2b_2k_0 - 2k_0\right), \\ \dot{y} &= \frac{1}{b_2^2}\left(b_2\,x^2 + 2b_2^2(2k_0+1)\,xy + 2b_2^2\,y^2 - 4k_0(4b_2^2k_0^2 + 2b_2^2k_0 - 4b_2k_0 - b_2 + 1)\,x \right. \\ &\quad \left. - (8b_2^2k_0^2 - 8b_2k_0 - b_2 + 2)\,y\right). \end{aligned} \tag{$S(\text{II}.3.\text{s}_3)$}$$

$$\begin{aligned} \dot{x} &= -(2a_0 - 2c - 1)\,x^2 + xy + \left(a_0 + \frac{1}{2}\right)x + \left(2a_0 - 2c + \frac{1}{2}\right)y + \frac{a_0}{2}, \\ \dot{y} &= (2a_0 - 2c + 1)\,x^2 + 3xy + 2y^2 + a_0 x + 2cy. \end{aligned} \tag{$S(\text{II}.3.\text{s}_4)$}$$

Applying the parameter change $(b_0, k_0) \to \left(b + \frac{1}{2},\ \frac{2a+1}{4}\right)$ and the time rescaling $t \to 8t$ to $S(\text{II}.3.\text{s}_1)$, we obtain

S(8) $\dot{x} = 8(a - b + 1)\,x^2 + 8xy - 2(4b^2 - 4a - 3)\,x + 4(2b+1)\,y - (2a+1)(4b^2-1)$,
$\dot{y} = 8(a + b + 1)\,x^2 + 8(2a+3)\,xy + 16y^2 - 2(8ab^2 + 4b^2 - 2a - 1)\,x - 4(4b^2 + 2b - 2a - 1)\,y$,
$f_1 = y - x^2, \quad K_1 = 16(a - b + 1)\,x + 16y - 4(4b^2 + 2b - 2a - 1)$,

$$f_2 = 4b^2 - 1 - 4x - 4y, \quad K_2 = 16(a+1)\,x + 16y + 4(2a+1),$$
$$f_3 = 2x + 2b + 1, \quad K_3 = 8(a-b+1)\,x + 8y - 2(4ab - 2a + 2b - 1),$$
$$R = f_1^{-(a+3b)/(2b)} f_2^{(a-3b)/(2b)} f_3.$$

Here we have $b(a-b) \neq 0$. If $b = 0$ the system becomes

$$\dot{x} = (4(a+1)x + 4y + 2a + 1)(2x+1), \quad \dot{y} = 2(4(a+1)x + 4y + 2a + 1)(x + 2y).$$

If $a - b = 0$, the system becomes

$$\dot{x} = (4x + 4y - 4a^2 + 1)(2x + 2a + 1), \quad \dot{y} = 2(4x + 4y - 4a^2 + 1)((2a+1)x + 2y).$$

Applying transformation $T_2(\text{II.2.s}_2)$ to $S(8)$ yields $S(\text{II.2.s}_2.1)$.

$$T_2(\text{II.2.s}_2): \quad (x,y,t,a,b) \to \left(x - a - \frac{1}{2},\ y + \frac{1}{4}(2a+1)^2,\ \frac{t}{8},\ \frac{1}{a} + a,\ a\right).$$

For solution (II.3.s_2), if $2b_2 - 1 \neq 0$, then applying the transformation $T_1(\text{II.3.s}_2)$ to $S(6)$ yields $S(\text{II.3.s}_2)$; if $2b_2 - 1 = 0$, then applying the transformation $T_2(\text{II.3.s}_2)$ to $S(6)$ yields $S(\text{II.3.s}_2)$ with $b_2 = \frac{1}{2}$, where

$$T_1(\text{II.3.s}_2): \quad (x,y,t,a,b) \to$$
$$\left(x + \frac{1-b_2}{2b_2},\ y + \frac{(b_2-1)(b_2-4b_2x-1)}{4b_2^2},\ \frac{(2b_2-1)^2 t}{32b_2^2},\ \frac{2b_2(4k_0-1)}{2b_2-1},\ \frac{2(1-b_2)}{2b_2-1}\right).$$

$$T_2(\text{II.3.s}_2): \quad (x,y,t,a,b) \to \left(\frac{x}{2},\ \frac{y}{4},\ \frac{t}{32},\ 8k_0 - 2,\ 2\right).$$

Applying the transformation $T(\text{II.3.s}_3)$ to $S(8)$ yields $S(\text{II.3.s}_3)$, where

$$T(\text{II.3.s}_3): \quad (t,a,b) \to \left(\frac{t}{8},\ 2k_0 - \frac{1}{2},\ \frac{1-2b_2k_0}{b_2} - \frac{1}{2}\right).$$

For the solution (II.3.s_4) applying the transformation $T(\text{II.3.s}_4)$ to $S(5)$ yields $S(\text{II.3.s}_4)$, where

$$T(\text{II.3.s}_4): \quad (t,a,b) \to \left(\frac{t}{8},\ 4c - 1,\ 4a_0 - 1\right).$$

Step 13. Substituting f_3, K_3, and $S(\text{II.4})$ into (2.1), we obtain five solutions

$$\text{II.4.s}_1 = \left\{a = \frac{1}{b_2} - 2,\ b_0 = 0,\ c = -\frac{1}{b_2},\ l_0 = -\frac{1}{b_2},\ l_1 = \frac{1}{b_2} - 2,\ l_2 = 2\right\},$$
$$\text{II.4.s}_2 = \{a = l_1,\ b_0 = 1,\ b_2 = 0,\ c = l_1,\ l_0 = 0,\ l_2 = 1\},$$
$$\text{II.4.s}_3 = \left\{a = \frac{1}{b_2} - 2,\ b_0 = \frac{1}{4b_2},\ c = \frac{1-2b_2}{4b_2^2},\ l_0 = 0,\ l_1 = \frac{1}{b_2} - 2,\ l_2 = 2\right\},$$
$$\text{II.4.s}_4 = \{a = -2,\ b_0 = 0,\ c = 0,\ l_0 = 0,\ l_1 = -2,\ l_2 = 2\},$$
$$\text{II.4.s}_5 = \left\{a = -1,\ c = \frac{b_0}{b_2},\ l_0 = 0,\ l_1 = 0,\ l_2 = 2\right\}.$$

Substituting $(\text{II.4.}k)$ into $S(\text{II.4})$, we obtain the systems $S(\text{II.4.}k)$ for $k = \text{s}_1,\ \text{s}_2,\ \text{s}_3,\ \text{s}_4,\ \text{s}_5$. In system $S(\text{II.4.s}_5)$ we replace $\frac{b_0}{b_2}$ by c.

$$\dot{x} = -\frac{2b_2 - 1}{b_2}\,x^2 + xy - \frac{1}{b_2}x + y, \qquad \dot{y} = \frac{-2(b_2-1)}{b_2}\,xy + 2y^2 - \frac{2}{b_2}y. \qquad (S(\text{II.4.s}_1))$$

$$\dot{x} = (x+1)(l_1 x + y), \qquad \dot{y} = 2y\left((l_1+1)\,x + y + l_1\right). \tag{S(II.4.s$_2$)}$$

$$\dot{x} = -\frac{2b_2-1}{b_2}\,x^2 + xy - \frac{2b_2-1}{4b_2^2}\,x + y, \qquad \dot{y} = \frac{-2(b_2-1)}{b_2}\,xy + 2y^2 - \frac{2b_2-1}{2b_2^2}\,y. \tag{S(II.4.s$_3$)}$$

$$\dot{x} = -2x^2 + xy + y, \qquad \dot{y} = -2xy + 2y^2. \tag{S(II.4.s$_4$)}$$

$$\dot{x} = -x^2 + xy + cx + y, \qquad \dot{y} = 2y^2 + 2cy. \tag{S(II.4.s$_5$)}$$

Applying the following transformations T(II.4.s$_2$) to $S(4)$, T(II.4.s$_3$) to $S(7)$, T(II.4.s$_4$) to $S(6)$, and T(II.4.s$_5$) to $S(5)$ yields S(II.4.s$_2$), S(II.4.s$_3$), S(II.4.s$_4$), and S(II.4.s$_5$), respectively.

$$T(\text{II.4.s}_2): \quad (x,y,t,b,a) \to \left(-x-\frac{1}{2},\ x+y+\frac{1}{4},\ \frac{t}{8},\ -1,\ -l_1\right).$$

$$T(\text{II.4.s}_3): \quad (x,y,t,a,b) \to \left(\frac{1}{2}(x-1),\ -\frac{1}{4}(2x-y-1),\ \frac{t}{2},\ \frac{1}{2b_2}-1,\ \frac{1}{2b_2}\right).$$

$$T(\text{II.4.s}_4): \quad (x,y,t,a,b) \to \left(\frac{1}{2}(x-1),\ -\frac{1}{4}(2x-y-1),\ \frac{t}{8},\ -2,\ 0\right).$$

$$T(\text{II.4.s}_5): \quad (x,y,t,a,b) \to \left(\frac{1}{2}(x-1),\ -\frac{1}{4}(2x-y-1),\ \frac{t}{2},\ c-1,\ c\right).$$

For solution (II.4.s$_1$), if $2b_2+1 \neq 0$, then applying the transformation T_1(II.4.s$_1$) to system $S(6)$ yields S(II.4.s$_1$); if $2b_2+1=0$, then applying the transformation T_2(II.4.s$_1$) to $S(6)$ yields S(II.4.s$_1$) with $b_2 = -\frac{1}{2}$, where

$$T_1(\text{II.4.s}_1): \quad (x,y,t,a,b) \to \left(\frac{1}{2}(x-1),\ -\frac{1}{4}(2x-y-1),\ \frac{(2b_2+1)^2 t}{32b_2^2},\ -2,\ -\frac{2}{2b_2+1}\right).$$

$$T_2(\text{II.4.s}_1): \quad (x,y,t,a,b) \to \left(-\frac{1}{2}(x+1),\ \frac{1}{4}(2x+y+1),\ \frac{t}{2},\ 0,\ -1\right).$$

Step 14. Substituting f_3, K_3, and S(II.5) into (2.1), we obtain five solutions

$$\begin{aligned}
\text{II.5.s}_1 &= \left\{a_0 = -b_0^2,\ b_2 = 0,\ k_1 = b_0+1,\ l_0 = -b_0(b_0+1),\ l_1 = 1,\ l_2 = 1\right\},\\
\text{II.5.s}_2 &= \left\{a_0 = -\frac{(b_2-1)^2}{4b_2^2},\ b_0 = \frac{1-b_2^2}{4b_2},\ k_1 = \frac{b_2+1}{2b_2},\ l_0 = \frac{b_2-1}{2b_2^2},\ l_1 = 1,\ l_2 = 2\right\},\\
\text{II.5.s}_3 &= \left\{a_0 = -\frac{1}{4b_2^2},\ b_0 = \frac{1}{4b_2},\ k_1 = \frac{b_2+1}{2b_2},\ l_0 = -\frac{b_2+1}{2b_2^2},\ l_1 = 1,\ l_2 = 2\right\},\\
\text{II.5.s}_4 &= \left\{a_0 = 0,\ b_0 = 0,\ k_1 = \frac{1}{2},\ l_0 = 0,\ l_1 = 1,\ l_2 = 2\right\},\\
\text{II.5.s}_5 &= \left\{b_0 = 0,\ k_1 = 0,\ l_0 = 2a_0,\ l_1 = 0,\ l_2 = 2\right\}.
\end{aligned}$$

Substituting (II.5.k) into S(II.5), we obtain systems S(II.5.k) for $k = \text{s}_1,\ \text{s}_2,\ \text{s}_3,\ \text{s}_4,\ \text{s}_5$.

$$\begin{aligned}
\dot{x} &= x^2 + xy - b_0^2 x + b_0 y - b_0^2(b_0+1),\\
\dot{y} &= 2(b_0+1)\,xy + 2y^2 - 2b_0^2(b_0+1)\,x - 2b_0^2 y.
\end{aligned} \tag{S(II.5.s$_1$)}$$

$$\begin{aligned}
\dot{x} &= \frac{1}{8b_2^3}\left(8b_2^3x^2 + 8b_2^3xy - 2b_2(b_2-1)^2x - 4b_2^2(b_2-1)y - b_2^3 + b_2^2 + b_2 - 1\right),\\
\dot{y} &= \frac{1}{8b_2^3}\left(8b_2^2(b_2+1)\,xy + 16b_2^3y^2 - 2(b_2-1)^2(b_2+1)\,x - 4b_2(b_2-1)^2y\right).
\end{aligned} \tag{S(II.5.s$_2$)}$$

$$\dot{x} = \frac{1}{8b_2^3}\left(8b_2^3x^2 + 8b_2^3xy - 2b_2x - 4b_2^2(b_2-1)\,y - (b_2+1)\right),$$
$$\dot{y} = \frac{1}{8b_2^3}\left(8b_2^2(b_2+1)\,xy + 16b_2^3y^2 - 2(b_2+1)\,x - 4b_2y\right). \qquad (S(\text{II.5.s}_3))$$

$$\dot{x} = x^2 - \frac{1}{2}y + xy, \qquad \dot{y} = xy + 2y^2. \qquad (S(\text{II.5.s}_4))$$

$$\dot{x} = x^2 + xy + a_0x - y, \qquad \dot{y} = 2y^2 + 2a_0y. \qquad (S(\text{II.5.s}_5))$$

Applying the following transformations $T(\text{II.5.s}_1)$ to $S(8)$, $T(\text{II.5.s}_2)$ to $S(8)$, $T(\text{II.5.s}_4)$ to $S(6)$, and $T(\text{II.5.s}_5)$ to $S(5)$ yields $S(\text{II.5.s}_1)$, $S(\text{II.5.s}_2)$, $S(\text{II.5.s}_4)$, and $S(\text{II.5.s}_5)$, respectively.

$$T(\text{II.5.s}_1): \quad (x,y,t,a,b) \to \left(-x - \frac{1}{2},\ x + y + \frac{1}{4},\ \frac{t}{8},\ -b_0 - 1,\ -b_0\right).$$

$$T(\text{II.5.s}_2): \quad (x,y,t,a,b) \to$$
$$\left(\frac{x}{2} + \frac{1-2b_2}{4b_2},\ \frac{1-2b_2}{4b_2}x + \frac{y}{4} + \frac{(2b_2-1)^2}{16b_2^2},\ \frac{t}{2},\ \frac{b_2-2}{4b_2},\ -\frac{1}{4}\right).$$

$$T(\text{II.5.s}_4): \quad (x,y,a,b) \to \left(-x - \frac{1}{2},\ x + y + \frac{1}{4},\ \frac{t}{32},\ -2,\ 0\right).$$

$$T(\text{II.5.s}_5): \quad (x,y,t,a,b) \to \left(-\frac{1}{2}(x+1),\ \frac{1}{4}(2x+y+1),\ \frac{t}{2},\ a_0 - 1,\ a_0\right).$$

For the solution (II.5.s$_3$), if $2b_2 - 1 \neq 0$, then applying the transformation $T_1(\text{II.5.s}_3)$ to system $S(6)$ yields $S(\text{II.5.s}_3)$; if $2b_2 - 1 = 0$, then applying the transformation $T_2(\text{II.5.s}_3)$ to $S(6)$ yields $S(\text{II.5.s}_3)$ with $b_2 = \frac{1}{2}$, where

$$T_1(\text{II.5.s}_3): \quad (x,y,t,a,b) \to$$
$$\left(\frac{x}{2} - \frac{2b_2-1}{4b_2},\ \frac{1-2b_2}{4b_2}x + \frac{y}{4} + \frac{(2b_2-1)^2}{16b_2^2},\ \frac{(2b_2-1)^2t}{32b_2^2},\ \frac{2(b_2+1)}{2b_2-1},\ \frac{2}{2b_2-1}\right).$$

$$T_2(\text{II.5.s}_3): \quad (x,y,t,a,b) \to \left(-\frac{x}{2} - 1,\ x + \frac{y}{4} + 1,\ \frac{t}{2},\ -\frac{3}{2},\ -1\right).$$

This finishes the proof. □

6 Proof of Theorem 3.2

Proof. **System** $S(1)$**.** The system together with its invariant curves is

$$\dot{x} = x + 1, \quad \dot{y} = x^2 + 2x + y.$$

$$f_1 = y - x^2, \quad f_2 = x + 1.$$

It has only one finite equilibrium at $P_1 = (-1,1)$ with eigenvalues $1,1$. In what follows, we use the notation $P_1 = (-1,1) : 1,1$ to denote the equilibrium point and its corresponding eigenvalues.

The Poincaré compactification of this system in the chart U_1 (with coordinates (u,v)) is

$$\dot{u} = -uv^2 + 2v + 1, \quad \dot{v} = -v^3 - v^2.$$

Hence, there are no infinite equilibria in the chart U_1.

The Poincaré compactification of this system in the chart U_2 is

$$\dot{u} = -u^3 - 2u^2v + v^2, \quad \dot{v} = -u^2v - 2uv^2 - v^2.$$

The infinite equilibrium point O_2 at the origin of the local chart U_2 is linearly zero. Doing the vertical blow-up $(u, v) = (u_1, u_1v_1)$ and a rescaling of the time we eliminate the common factor u_1 between $\dot{u}_1$ and $\dot{v}_1$ obtaining the system

$$\dot{u}_1 = -2u_1^2v_1 + u_1v_1^2 - u_1^2, \qquad \dot{v}_1 = -v_1^3 - v_1^2.$$

It has two infinite equilibria on the straight line $u_1 = 0$:

$$(0, 0) : 0, 0; \quad (0, -1) : -1, 1.$$

However, the origin in the coordinates (u_1, v_1) is again a linearly zero equilibrium. Since $u_1 = 0$ is a characteristic direction of the equilibrium $(0, 0)$, we do the twist $(u_1, v_1) = (u_2 - v_2, v_2)$ in order to avoid the loss of information when we do a vertical blow-up, in this way we obtain the system

$$\dot{u}_2 = -4v_2^3 + 5u_2v_2^2 - 2u_2^2v_2 - u_2^2 + 2u_2v_2 - 2v_2^2, \qquad \dot{v}_2 = -v_2^3 - v_2^2.$$

Doing the vertical blow-up $(u_2, v_2) = (u_3, u_3v_3)$ and a rescaling of time we eliminate the common factor u_3 between $\dot{u}_3$ and $\dot{v}_3$, obtaining the system

$$\begin{aligned}\dot{u}_3 &= -4u_3^2v_3^3 + 5u_3^2v_3^2 - 2u_3^2v_3 - 2u_3v_3^2 + 2u_3v_3 - u_3,\\ \dot{v}_3 &= 4u_3v_3^4 - 6u_3v_3^3 + 2u_3v_3^2 + 2v_3^3 - 3v_3^2 + v_3.\end{aligned}$$

This system has three equilibria on the straight line $u_3 = 0$:

$$(0, 0) : -1, 1; \quad \left(0, \frac{1}{2}\right) : -\frac{1}{2}, -\frac{1}{2}; \quad (0, 1) : -1, 1.$$

Doing the blowing-down $B(\text{I.2.s}_1)$ (Figure 6.1) we obtain the local phase portrait of the infinite equilibrium O_2.

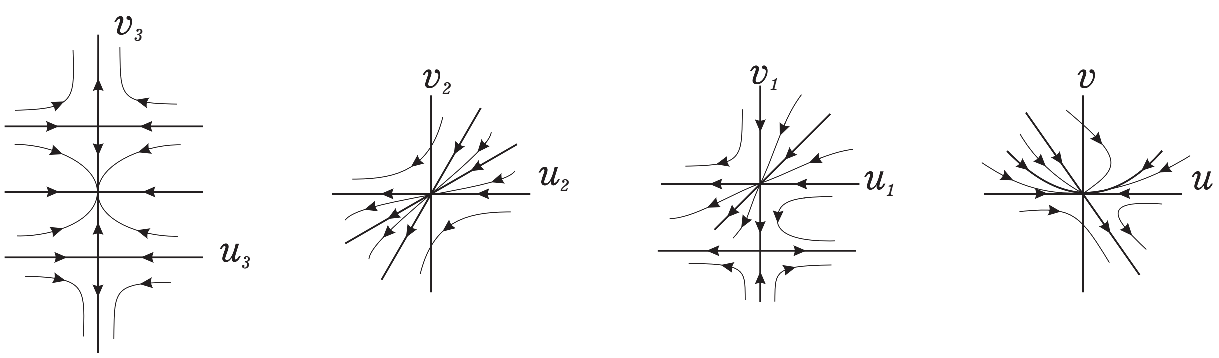

Figure 6.1: $B(\text{I.2.s}_1)$: Blowing-down at the origin O_2 of the chart U_2 for system $S(1)$.

Since $y - x^2 = 0$ and $x + 1 = 0$ are invariant algebraic curves of system $S(1)$, we obtain the configuration $C(\text{I.2.s}_1)$ (see Figure 6.2).

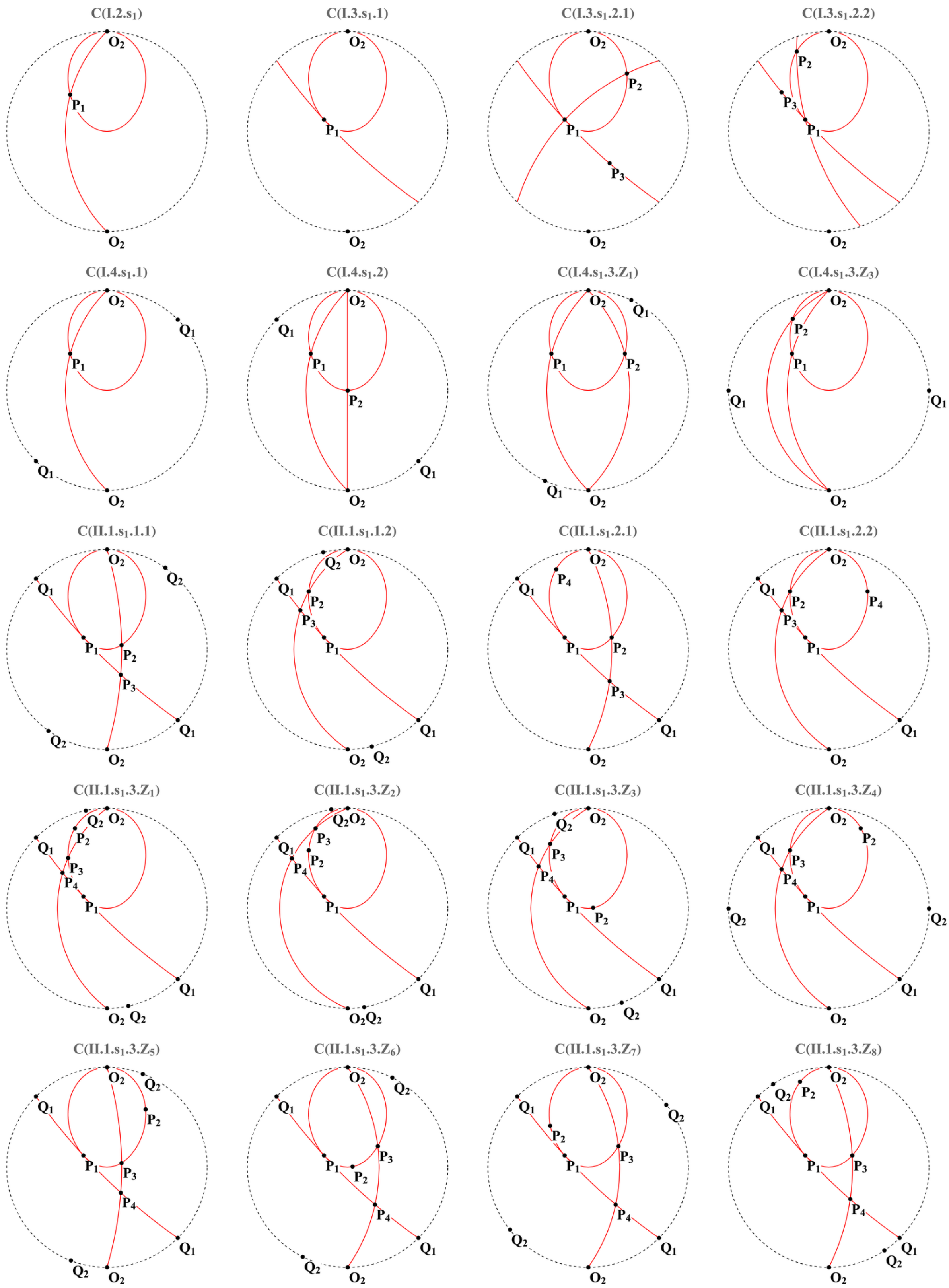

Figure 6.2: Configurations of the invariant algebraic curves of the systems of Theorem 3.1. Here P_i for $i = 1, 2, 3, 4$ are equilibria in the plane $\mathbb{R}^2$, Q_i for $i = 1, 2$ are equilibria in the chart U_1 and O_2 is the origin of the chart U_2.

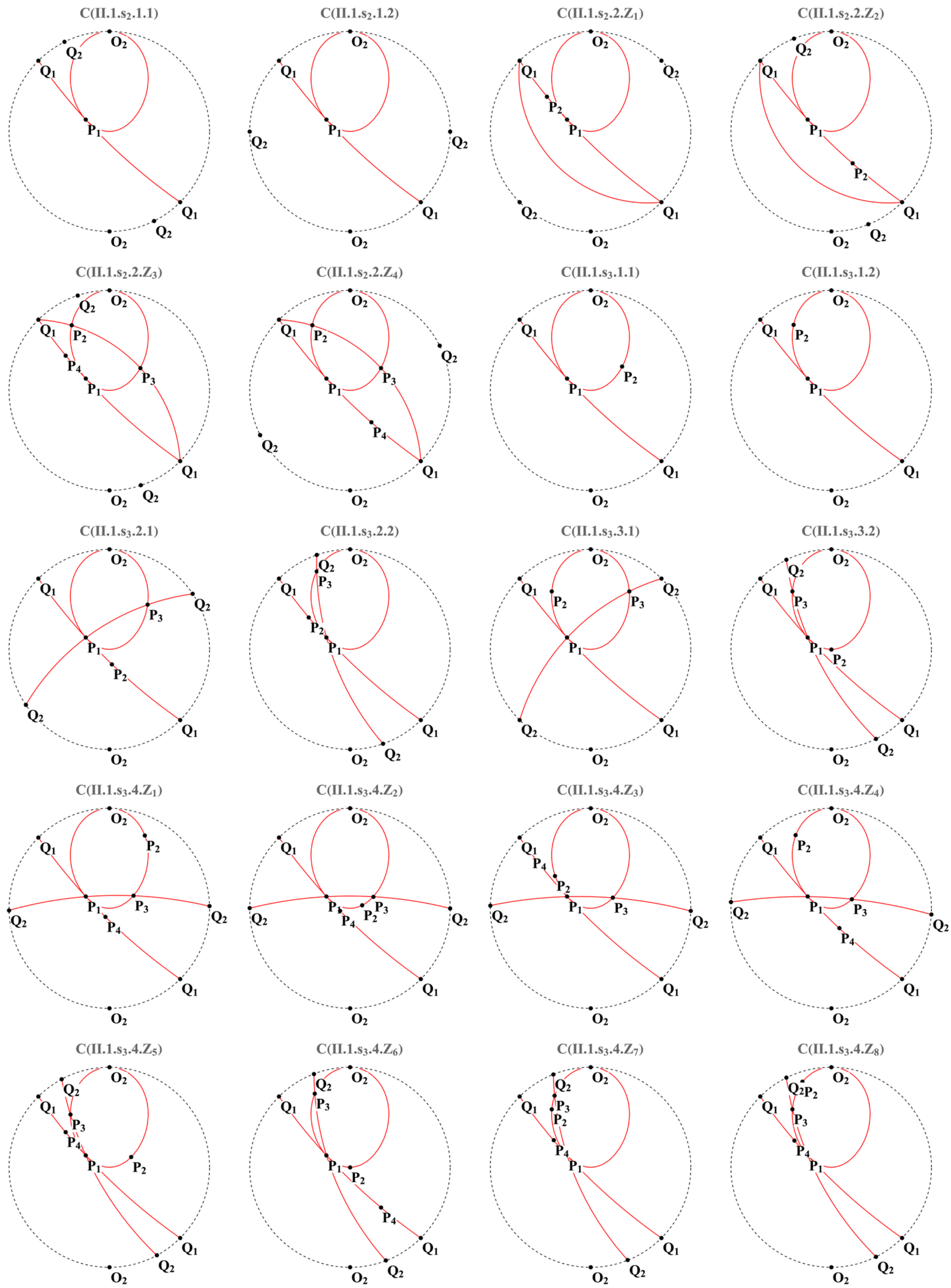

Figure 6.3: Configurations of the invariant algebraic curves of the systems of Theorem 3.1. Here P_i for $i = 1, 2, 3, 4$ are equilibria in the plane $\mathbb{R}^2$, Q_i for $i = 1, 2$ are equilibria in the chart U_1 and O_2 is the origin of the chart U_2.

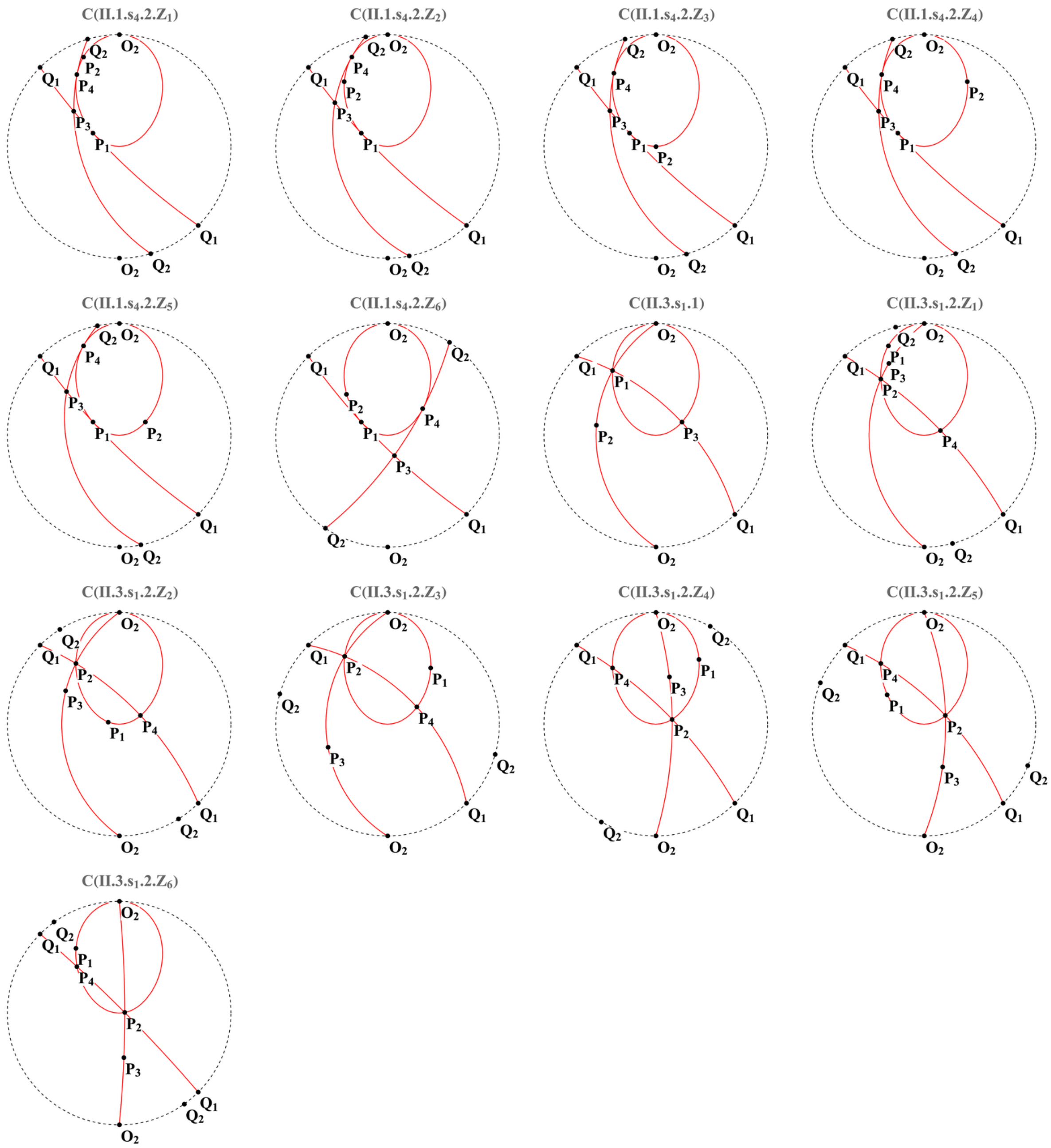

Figure 6.4: Configurations of the invariant algebraic curves of the systems of Theorem 3.1. Here P_i for $i = 1, 2, 3, 4$ are equilibria in the plane $\mathbb{R}^2$, Q_i for $i = 1, 2$ are equilibria in the chart U_1 and O_2 is the origin of the chart U_2.

Taking into account the blowing-down $B(\mathrm{I}.2.\mathrm{s}_1)$ and the configuration $C(\mathrm{I}.2.\mathrm{s}_1)$, we obtain the phase portrait PP.1 of Figure 3.1.

System $S(2)$**.** The system together with its invariant curves is

$$\dot{x} = -4x^2 - 2ax + 2y - a + \frac{1}{2}, \quad \dot{y} = -4xy + (1-2a)\,x - 4ay.$$

$$f_1 = y - x^2, \quad f_2 = 4x + 4y + 1, \quad f_3 = 4ax + 4y + 2a - 1.$$

We divide the study of the phase portraits of system $S(2)$ into two cases: $a = 1$ and $a \neq 1$.

Case 1. $a = 1$**.** Then system $S(2)$ reduces to the system

$$\dot{x} = -4x^2 - 2x + 2y - \frac{1}{2}, \qquad \dot{y} = -4xy - x - 4y.$$

This system has a finite equilibrium at $P_1 = \left(-\frac{1}{2}, \frac{1}{4}\right) : 0, 0$. It is a nilpotent equilibrium and satisfies condition (4.iii.iii2) of Theorem 3.5 in [11], then its phase portrait is formed by one hyperbolic and one elliptic sector.

The Poincaré compactification of this system in the chart U_1 is

$$\dot{u} = \frac{1}{2}uv^2 - 2u^2v - 2uv - v, \qquad \dot{v} = \frac{1}{2}v^3 - 2uv^2 + 2v^2 + 4v.$$

Every point at infinity is an equilibrium. Removing the common factor v between $\dot{u}$ and $\dot{v}$, we obtain the system

$$\dot{u} = -2u^2 + \frac{1}{2}uv - 2u - 1, \qquad \dot{v} = \frac{1}{2}v^2 - 2uv + 2v + 4.$$

It has no equilibria on $v = 0$.

The Poincaré compactification in the chart U_2 is

$$\dot{u} = u^2v + 2uv - \frac{1}{2}v^2 + 2v, \qquad \dot{v} = uv^2 + 4uv + 4v^2.$$

Removing the common factor v between $\dot{u}$ and $\dot{v}$, we obtain the system

$$\dot{u} = u^2 + 2u - \frac{1}{2}v + 2, \qquad \dot{v} = uv + 4u + 4v.$$

There is an orbit tangent to the line at infinity at the origin O_2.

Since $y - x^2 = 0$ and $4x + 4y + 1 = 0$ are invariant curves of this system, we obtain the configuration $C(\mathrm{I}.3.\mathrm{s}_{1.1})$ shown in Figure 6.2. Based on this configuration and the local phase portrait of the equilibrium P_1, we obtain the phase portrait PP.2 of Figure 3.1.

Case 2. $a \neq 1$**.** System $S(2)$ has three finite equilibria and their corresponding eigenvalues are

$$P_1 = \left(-\frac{1}{2}, \frac{1}{4}\right) : -4(a-1), -2(a-1);$$

$$P_2 = \left(-\frac{1}{2}(2a-1), \frac{1}{4}(2a-1)^2\right) : 2(a-1), 4(a-1);$$

$$P_3 = \left(-\frac{a}{2}, \frac{1}{4}(2a-1)\right) : -2(a-1), 2(a-1).$$

The Poincaré compactification of this system in the chart U_1 is

$$\dot{u} = \frac{2a-1}{2}uv^2 - 2u^2v - 2auv + (1-2a)v, \qquad \dot{v} = -2uv^2 + 2av^2 + \frac{2a-1}{2}v^3 + 4v.$$

Every point at infinity is an equilibrium.

The Poincaré compactification of this system in the chart U_2 is

$$\dot{u} = (2a-1)u^2v + 2auv + \left(\frac{1}{2} - a\right)v^2 + 2v, \qquad \dot{v} = (2a-1)uv^2 + 4av^2 + 4uv.$$

Removing the common factor v, we find an orbit tangent to the line at infinity at the origin O_2. Since $y - x^2 = 0$, $4x + 4y + 1 = 0$ and $4ax + 4y + 2a - 1 = 0$ are invariant algebraic curves of the system, we obtain the configurations $C(\mathrm{I.3.s}_{1.1})$ and $C(\mathrm{I.3.s}_{1.2})$, as shown in Figure 6.2.

Table 6.1 shows the local phase portraits of the finite equilibria and the corresponding configurations and phase portraits in the Poincaré disc according to the parameter a.

Zone	P_1	P_2	P_3	Configuration	PP
$a > 1$	Ns	Nu	S	$C(\mathrm{I.3.s}_1.2.2)$	PP.3
$a < 1$	Nu	Ns	S	$C(\mathrm{I.3.s}_1.2.1)$	PP.3

Table 6.1: Types of the finite and infinite equilibria, together with their corresponding local phase portraits, configurations, and global phase portraits in the Poincaré disc for Case 2 of the system $S(2)$. The abbreviations “Ns”, “Nu”, and “S” denote “stable node”, “unstable node” and “saddle”, respectively.

System $S(3)$**.** The system together with its invariant curves is

$$\dot{x} = -x^2 + (a+b)\,x - ab, \quad \dot{y} = (2b-1)\,x^2 - 2xy + (2a+1)\,y - 2abx,$$

$$f_1 = y - x^2, \quad f_2 = x - a, \quad f_3 = x - b,$$

where $(2a - 2b + 1) \neq 0$.

We consider the following four cases: 1. $a = b \neq 0$; 2. $a = b = 0$; 3. $ab = 0$ and $a^2 + b^2 \neq 0$; 4. $ab(a-b) \neq 0$.

Case 1. $a = b \neq 0$ and $(2a - 2b + 1) \neq 0$. Let $c \neq 0$. The transformation $(x,\, y,\, t,\, a,\, b) \to \left(-\frac{x}{c},\, \frac{y}{c^2},\, \frac{t}{c},\, \frac{1}{c},\, \frac{1}{c}\right)$ reduces the system to $S(\mathrm{I.4.s}_{1.1})$:

$$\dot{x} = (x+1)^2, \quad \dot{y} = (2-c)x^2 + 2xy + 2x + (c+2)y, \quad f_1 = y - x^2, \quad f_2 = x + 1.$$

We now analyse the phase portraits of the system $S(\mathrm{I.4.s}_{1.1})$. There is a finite equilibrium $P_1 = (-1,\, 1) : 0,\, c$. According to Theorem 2.19 of [11], the point is a saddle-node.

The Poincaré compactification of this system in the chart U_1 is

$$\dot{u} = -uv^2 + cuv + u + 2v - c + 2, \qquad \dot{v} = -v^3 - 2v^2 - v.$$

There is an equilibrium $Q_1 = (c+2,\, 0) : -1,\, 1$.

The Poincaré compactification of this system in the chart U_2 is

$$\dot{u} = (c-2)u^3 - 2u^2v - u^2 - cuv + v^2, \qquad \dot{v} = (c-2)u^2v - 2uv^2 - 2uv - (c+2)v^2.$$

The origin O_2 is linearly zero. Doing a vertical blow-up $(u, v) \to (u_1, u_1v_1)$ on the system, followed by a time rescaling $(\dot{u}_1, \dot{v}_1) \to \left(\frac{\dot{u}_1}{u_1}, \frac{\dot{v}_1}{u_1}\right)$, we obtain the system:

$$\dot{u}_1 = cu_1^2 - 2u_1^2v_1 + u_1v_1^2 - (c+2)u_1v_1 - u_1, \qquad \dot{v}_1 = -v_1^3 - 2v_1^2 - v_1.$$

There are two equilibria on $u_1 = 0$:

$$(0, -1) : 0,\ c; \quad (0, 0) : -1,\ -1.$$

According to Theorem 2.19 of [11], $(0, -1)$ is a saddle-node.

By carrying out the blow-downs $B(\text{I.4.s}_{1.1}.k)$ for $k = 1, 2$, we obtain the local phase portrait of O_2.

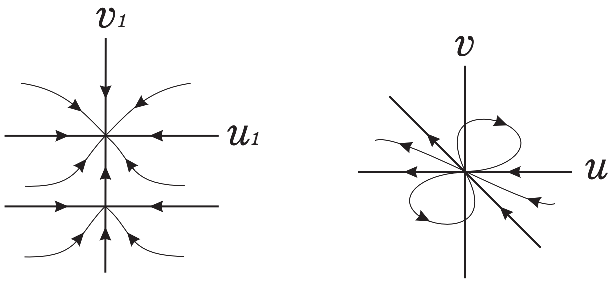

Figure 6.5: $B(\text{I.4.s}_1.1.1)$: Blowing-down at O_2 of the system $S(\text{I.4.s}_{1.1})$ with $c < 0$.

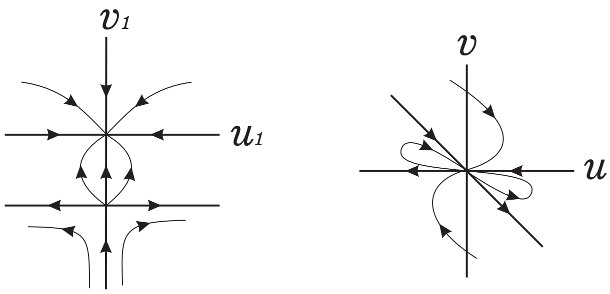

Figure 6.6: $B(\text{I.4.s}_1.1.2)$: Blowing-down at O_2 of the system $S(\text{I.4.s}_{1.1})$ with $c > 0$.

Table 6.2 shows the local phase portraits of the finite and infinite equilibria. By analysing the local behaviour at O_2, we obtain the phase portraits in the Poincaré disc.

Zone	P_1	P_2	Q_1	O_2	Blow-up	Configuration	PP
$c < 0$	SN	S	S	LN	$B(\text{I.4.s}_1.1.1)$	$C(\text{I.4.s}_1.1)$	PP.4
$c > 0$	SN	S	S	LN	$B(\text{I.4.s}_1.1.2)$	$C(\text{I.4.s}_1.1)$	PP.4

Table 6.2: Types of the finite and infinite equilibria, together with their corresponding local phase portraits, configurations, and global phase portraits in the Poincaré disc for Case 1 of the system $S(3)$. The abbreviations "SN" and "LN" denote "saddle-node" and "linearly zero equilibrium", respectively.

Case 2. $a = b = 0$ and $(4a - 2b + 1)(2a + 1) \neq 0$. Then system $S(3)$ simplifies to

$$\dot{x} = -x^2, \quad \dot{y} = -x^2 - 2xy + y.$$

Taking $c = -1$ in the system $S(\text{I.4.s}_1.1)$ and applying the transformation $(x, y) \to (x - 1, y - 2x + 1)$, we obtain the above system. Thus, it is a particular case of Case 1 with $a = b = -1$.
Case 3. $ab = 0$ and $(a^2 + b^2)(2a - 2b + 1) \neq 0$. If $b \neq 0$, then system $S(3)$ for $a = 0$ simplifies to $S(\text{I.4.s}_1.a = 0)$

$$\dot{x} = -x(x - b), \quad \dot{y} = (2b - 1)x^2 - 2xy + y.$$

Let $c \neq 0$, doing the transformation $(x,y,t,b) \to \left(-\frac{x}{c},\ \frac{y}{c^2},\ ct,\ \frac{1}{c}\right)$ to $S(\text{I.4.s}_1.a=0)$ yields $S(\text{I.4.s}_1.2)$:

$$\dot{x} = x(x+1), \quad \dot{y} = (2-c)x^2 + 2xy + cy,$$
$$f_1 = y - x^2, \quad f_2 = x, \quad f_3 = x+1.$$

Since $2a - 2b + 1 \neq 0$, $a = 0$ and $b = \dfrac{1}{c}$, here we have $c(c-2) \neq 0$.

If $b = 0$ then system $S(3)$ simplifies to $S(\text{I.4.s}_1.b=0)$

$$\dot{x} = -x(x-a), \quad \dot{y} = -x^2 - 2xy + (2a+1)y.$$

Let $c - 2 \neq 0$, doing the transformation $(x,y,t,a) \to \left(\frac{-x}{c-2},\ \frac{y}{(c-2)^2},\ (c-2)t,\ \frac{1}{c-2}\right)$ to $S(\text{I.4.s}_1.b = 0)$ yields $S(\text{I.4.s}_1.2)$. Thus, we only have to consider system $S(\text{I.4.s}_1.2)$ in this case.

The system $S(\text{I.4.s}_1.2)$ has two finite equilibria

$$P_1 = (-1,\ 1) : -1,\ c-2; \quad P_2 = (0,\ 0) : 1,\ c.$$

The Poincaré compactification of this system in the chart U_1 is

$$\dot{u} = (c-1)uv + u - c + 2, \qquad \dot{v} = -v^2 - v.$$

Hence, there is an infinite equilibrium $Q_1 = (c-2, 0) : -1, 1$ in the chart U_1.

The Poincaré compactification of this system in the chart U_2 is

$$\dot{u} = (c-2)u^3 - u^2 + (1-c)uv, \qquad \dot{v} = (c-2)u^2v - 2uv - cv^2.$$

The origin O_2 is linearly zero. Since $u = 0$ is a characteristic direction of the equilibrium, we perform the twist $(u,v) \to (u_1 - v_1, v_1)$, and obtain the following system:

$$\dot{u}_1 = (c-2)u_1^3 + 2(2-c)u_1^2v_1 + (c-2)u_1v_1^2 - u_1^2 + (1-c)u_1v_1,$$
$$\dot{v}_1 = (c-2)u_1^2v_1 - 2(c-2)u_1v_1^2 + (c-2)v_1^3 - 2u_1v_1 + (2-c)v_1^2.$$

By doing the vertical blow-up $(u_1, v_1) \to (u_2, u_2v_2)$ and eliminating the common factor u_2 between $\dot{u}_2$ and $\dot{v}_2$ doing a time rescaling, we obtain the system

$$\dot{u}_2 = (c-2)u_2^2v_2^2 - 2(c-2)u_2^2v_2 + (1-c)u_2v_2 + (c-2)u_2^2 - u_2, \qquad \dot{v}_2 = v_2^2 - v_2.$$

This system has two equilibria on the straight line $u_2 = 0$

$$(0,\ 0) : -1,\ -1; \quad (0,\ 1) : 1,\ -c.$$

By doing the blow-downs $B(\text{I.4.s}_1.2.k)$ for $k = 1,2,3$, we obtain the local phase portrait of O_2.

Since $y - x^2 = 0$, $x = 0$, and $x + 1 = 0$ are invariant algebraic curves of the system, we obtain the configuration $C(\text{I.4.s}_1.2)$.

Table 6.3 shows the local phase portraits of the finite and infinite equilibria, the process of blowing-down at O_2, and the corresponding phase portraits in the Poincaré disc.

Case 4. $ab(a-b)(2a-2b+1) \neq 0$. Assume that $c(c-1)(d-2c) \neq 0$, doing the following transformation to system $S(3)$

$$(x,y,t,a,b) \to \left(\frac{-x}{d-2c},\ \frac{y}{(d-2c)^2},\ (d-2c)t,\ \frac{c}{d-2c},\ \frac{1}{d-2c}\right),$$

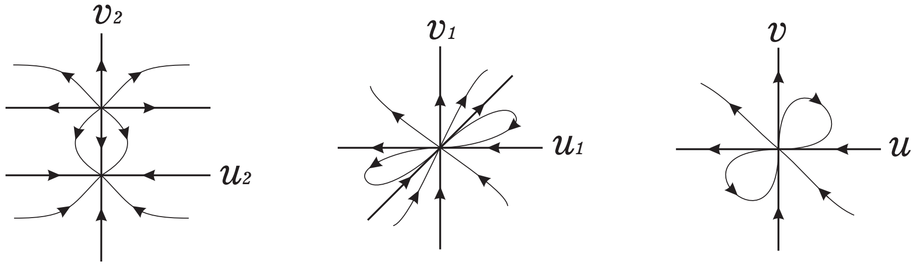

Figure 6.7: $B(\mathrm{I.4.s_1.2.1})$: Blowing-down at O_2 of the system $S(\mathrm{I.4.s_1.2})$ with $c < 0$.

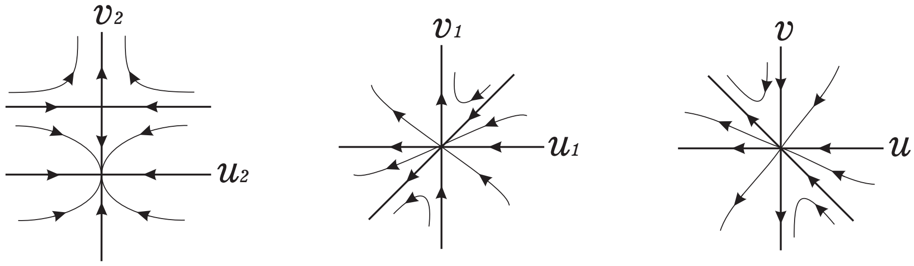

Figure 6.8: $B(\mathrm{I.4.s_1.2.2})$: Blowing-down at O_2 of the system $S(\mathrm{I.4.s_1.2})$ with $0 < c < 2$.

we obtain the system $S(\mathrm{I.4.s_1.3})$

$$\dot{x} = (x+1)(x+c), \quad \dot{y} = (2c-d+2)\,x^2 + 2xy + 2cx + dy,$$
$$f_1 = y - x^2, \quad f_2 = x + c, \quad f_3 = x + 1.$$

Since $(2a - 2b + 1) \neq 0$ in system $S(3)$, it follows that $(d-2) \neq 0$ in $S(\mathrm{I.4.s_1.3})$. This further implies that the full parameter condition is $c(c-1)(d-2c)(d-2) \neq 0$.

The system has two finite equilibria

$$P_1 = (-1,\ 1) : c-1,\ d-2; \quad P_2 = (-c,\ c^2) : 1-c,\ d-2c.$$

The Poincaré compactification of this system in the chart U_1 is

$$\dot{u} = -cuv^2 - (c-d+1)uv + 2cv + u + 2c - d + 2, \qquad \dot{v} = -cv^3 - (c+1)v^2 - v.$$

Hence, there is an infinite equilibrium $Q_1 = (d-2c-2, 0) : (-1, 1)$ in the local chart U_1.

The Poincaré compactification of this system in the chart U_2 is

$$\dot{u} = (d-2c-2)u^3 - 2cu^2v - u^2 + (c-d+1)uv + cv^2,$$
$$\dot{v} = (d-2c-2)u^2v - 2cuv^2 - 2uv - dv^2.$$

The origin is linearly zero. By doing the vertical blow-up $(u, v) \to (u_1, u_1v_1)$ and eliminating the common factor u_1 between $\dot{u}_1$ and $\dot{v}_1$ doing a time rescaling, we obtain the system

$$\dot{u}_1 = (d-2c-2)u_1^2 - 2cu_1^2v_1 + cu_1v_1^2 + (c-d+1)u_1v_1 - u_1,$$
$$\dot{v}_1 = -cv_1^3 - (c+1)v_1^2 - v_1.$$

This system has three equilibria on the straight line $u_1 = 0$

$$(0,\ -1) : 1-c,\ d-2; \quad (0,\ 0) : -1,\ -1; \quad \left(0,\ -\frac{1}{c}\right) : \frac{c-1}{c},\ \frac{d-2c}{c}.$$

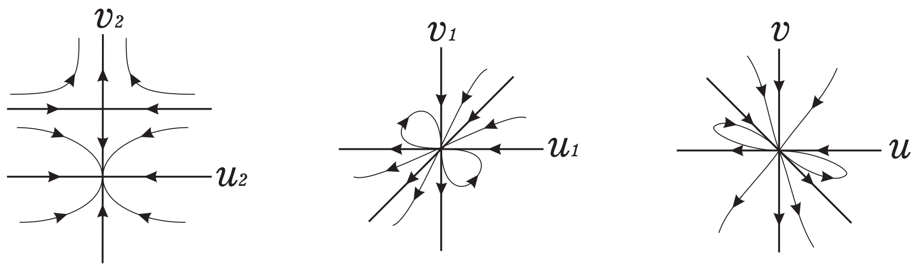

Figure 6.9: $B(\mathrm{I.4.s_1.2.3})$: Blowing-down at O_2 of the system $S(\mathrm{I.4.s_1.2})$ with $c > 2$.

Zone	P_1	P_2	Q_1	O_2	Blow-up	Configuration	PP
$c < 0$	Ns	S	S	LN	$B(\mathrm{I.4.s_1.2.1})$	$C(\mathrm{I.4.s_1.2})$	PP.5
$0 < c < 2$	Ns	Nu	S	LN	$B(\mathrm{I.4.s_1.2.2})$	$C(\mathrm{I.4.s_1.2})$	PP.6
$c > 2$	S	Nu	S	LN	$B(\mathrm{I.4.s_1.2.3})$	$C(\mathrm{I.4.s_1.2})$	PP.5

Table 6.3: Types of the finite and infinite equilibria, together with their corresponding local phase portraits, configurations, and global phase portraits in the Poincaré disc for Case 3 of the system $S(3)$.

According to the expressions of the eigenvalues, and under the condition $c(c-1)(2c-d)(d-2) \neq 0$, we divide the parameter space (c,d) into nine regions Z_k for $k = 1,2,\ldots,9$, as shown in Figure 6.10.

By doing the blow-downs $B(\mathrm{I.4.s_1.3.}Z_k)$ for $k = 1,2,4,5,8,9$, we obtain the local phase portrait at O_2.

Since $y - x^2 = 0, x + c = 0$, and $x + 1 = 0$ are invariant curves of the system, we obtain the configurations $C(\mathrm{I.4.s_1.3.}Z_1)$ and $C(\mathrm{I.4.s_1.3.}Z_3)$.

Table 6.4 shows the types of finite and infinite equilibria, the process of blowing-down at O_2, and the corresponding global phase portraits in different regions.

Zone	P_1	P_2	Q_1	O_2	Blow-up	Configuration	PP
Z_1	S	Nu	S	LN	$B(\mathrm{I.4.s_1.3.}Z_1)$	$C(\mathrm{I.4.s_1.3.}Z_1)$	PP.5
Z_2	S	Nu	S	LN	$B(\mathrm{I.4.s_1.3.}Z_2)$	$C(\mathrm{I.4.s_1.3.}Z_1)$	PP.5
Z_3	Nu	S	S	LN	$B(\mathrm{I.4.s_1.3.}Z_2)$	$C(\mathrm{I.4.s_1.3.}Z_3)$	PP.5
Z_4	Nu	Ns	S	LN	$B(\mathrm{I.4.s_1.3.}Z_4)$	$C(\mathrm{I.4.s_1.3.}Z_3)$	PP.6
Z_5	S	Ns	S	LN	$B(\mathrm{I.4.s_1.3.}Z_5)$	$C(\mathrm{I.4.s_1.3.}Z_3)$	PP.5
Z_6	Ns	S	S	LN	$B(\mathrm{I.4.s_1.3.}Z_5)$	$C(\mathrm{I.4.s_1.3.}Z_1)$	PP.5
Z_7	Ns	Nu	S	LN	$B(\mathrm{I.4.s_1.3.}Z_4)$	$C(\mathrm{I.4.s_1.3.}Z_1)$	PP.6
Z_8	Ns	S	S	LN	$B(\mathrm{I.4.s_1.3.}Z_8)$	$C(\mathrm{I.4.s_1.3.}Z_1)$	PP.5
Z_9	Ns	Nu	S	LN	$B(\mathrm{I.4.s_1.3.}Z_9)$	$C(\mathrm{I.4.s_1.3.}Z_1)$	PP.6

Table 6.4: Types of the finite and infinite equilibria, together with their corresponding local phase portraits, configurations, and global phase portraits in the Poincaré disc for Case 4 of system $S(3)$.

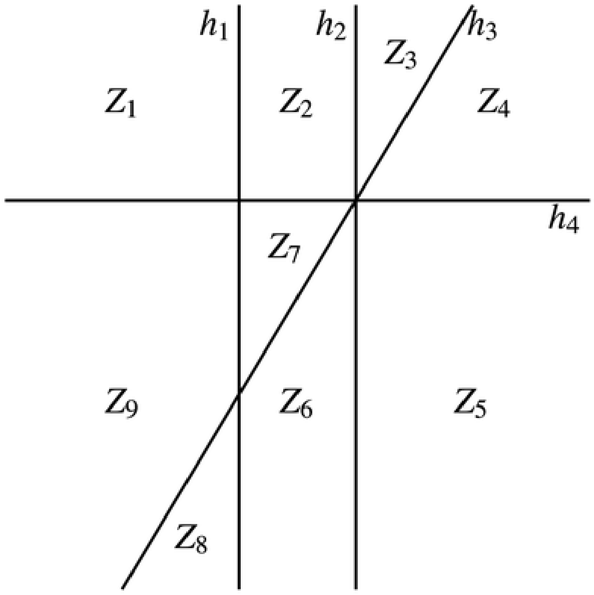

Figure 6.10: Zones of the parameter space (c, d), bounded by the straight lines $h_1 : c = 0$, $h_2 : c = 1$, $h_3 : d = 2c$, and $h_4 : d = 2$.

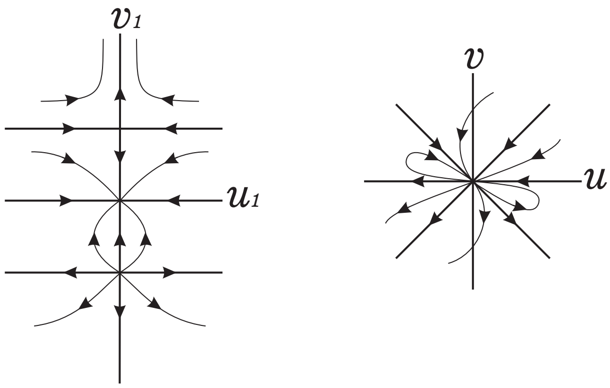

Figure 6.11: $B(\text{I}.4.\text{s}_1.3.Z_1)$: Blowing-down at O_2 of the system $S(\text{I}.4.\text{s}_1.3)$ with (c, d) in Z_1.

System $S(4)$**.** The system together with its invariant curves is

$$\begin{aligned}\dot{x} &= 8(a+1)\,x^2 + 8xy + 2(4ab+4a+4b+3)\,x + 4(2b+1)\,y + (4ab+2a+2b+1),\\ \dot{y} &= 8(a+2b+1)\,x^2 + 8(2a+2b+3)\,xy + 2(4ab+2a+2b+1)\,x + 4(4ab+2a+1)\,y + 16y^2,\end{aligned}$$

$$f_1 = y - x^2, \quad f_2 = 4x + 4y + 1, \quad f_3 = 2x + 2b + 1,$$

where $ab \neq 0$.

We consider the following three cases: 1. $a = b \neq 0$; 2. $a + 2b = 0$; 3. $(a - b)(a + 2b) \neq 0$.

Case 1. $a = b \neq 0$. The system simplifies to

$$\begin{aligned}\dot{x} &= 8(b+1)x^2 + 8xy + 2(2b+1)(2b+3)x + 4(2b+1)y + (2b+1)^2,\\ \dot{y} &= 8(3b+1)x^2 + 8(4b+3)xy + 2(2b+1)^2x + 4(4b^2+2b+1)y + 16y^2.\end{aligned}$$

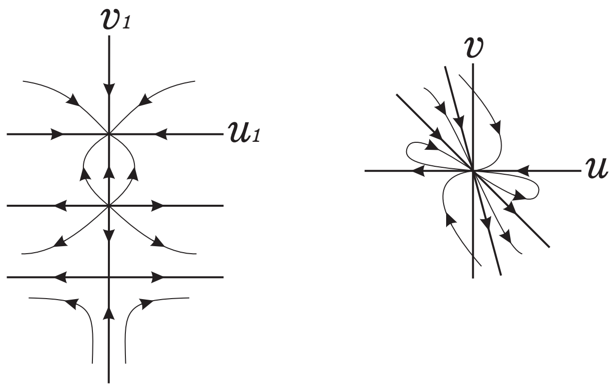

Figure 6.12: $B(\text{I.4.s}_1.3.Z_2)$: Blowing-down of the origin on chart U_2 for system $S(\text{I.4.s}_1.3)$ with (c,d) in Z_2, Z_3.

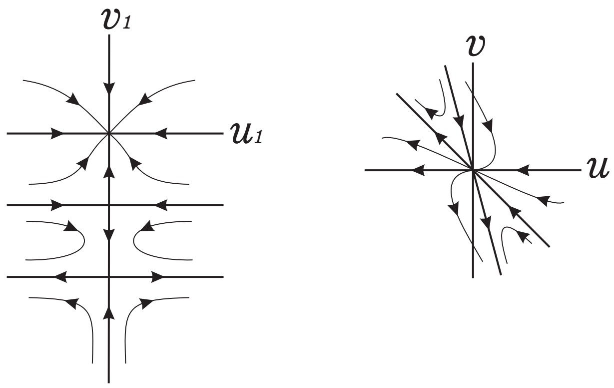

Figure 6.13: $B(\text{I.4.s}_1.3.Z_4)$: Blowing-down at O_2 of the system $S(\text{I.4.s}_1.3)$ with (c,d) in Z_4, Z_7.

There are three equilibria

$$P_1 = \left(-\frac{1}{2}, \frac{1}{4}\right) : 8b^2,\ 16b^2;\quad P_2 = \left(-\frac{1}{2}(2b+1),\ \frac{1}{4}(2b+1)^2\right) : 0,\ 16b^2,$$
$$P_3 = \left(-b-\frac{1}{2}, \frac{b}{4}\right) : -16b^2,\ -8b^2.$$

The Poincaré compactification in the chart U_1 is

$$\begin{aligned}\dot u =& 8u^2 - 4(2b+1)u^2v + 2(4b^2-4b-1)uv - (2b+1)^2uv^2 + 8(3b+2)u\\ &+ 2(2b+1)^2v + 8 + 24b,\\ \dot v =& -(2b+1)^2v^3 - 4(2b+1)uv^2 - 2(2b+1)(2b+3)v^2 - 8uv - 8(b+1)v.\end{aligned}$$

There are two equilibria:

$$Q_1 = (-1,\ 0) : -8b,\ 24b;\quad Q_2 = (-3b-1,\ 0) : -24b,\ 16b.$$

The Poincaré compactification in the chart U_2 is

$$\begin{aligned}\dot u =& -8(3b+1)u^3 - 2(2b+1)^2u^2v - 8(3b+2)u^2 - 2(4b^2-4b-1)uv - 8u\\ &+ (2b+1)^2v^2 + 4(2b+1)v,\\ \dot v =& -8(3b+1)u^2v - 2(2b+1)^2uv^2 - 8(4b+3)uv - 4(4b^2+2b+1)v^2 - 16v.\end{aligned}$$

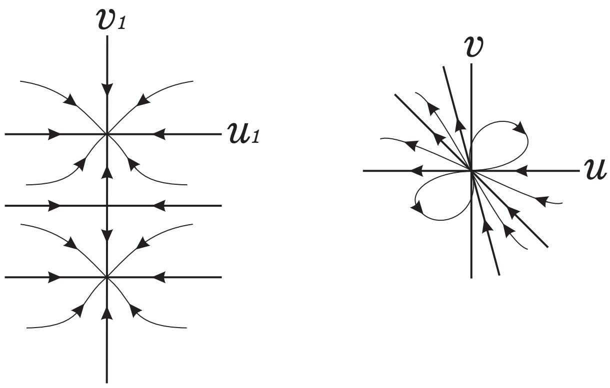

Figure 6.14: $B(\text{I.4.s}_1.3.Z_5)$: Blowing-down at O_2 of the system $S(\text{I.4.s}_1.3)$ with (c,d) in Z_5, Z_6.

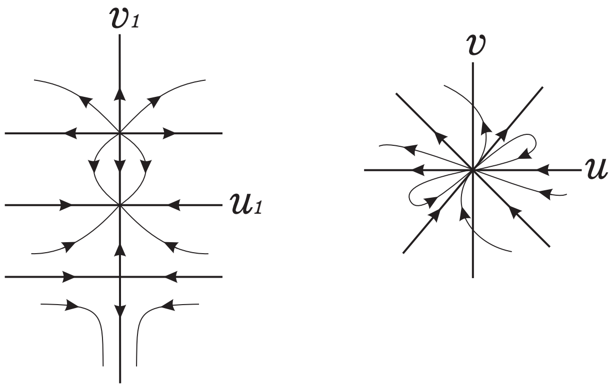

Figure 6.15: $B(\text{I.4.s}_1.3.Z_8)$: Blowing-down at O_2 of the system $S(\text{I.4.s}_1.3)$ with (c,d) in Z_8.

The origin is an equilibrium with eigenvalues: $-16, -8$.

Since $y - x^2 = 0, 4x + 4y + 1 = 0$, and $2x + 2b + 1 = 0$ are invariant curves of the system, we obtain the configurations $C(\text{II.1.s}_1.1.1)$ and $C(\text{II.1.s}_1.1.2)$.

Table 6.5 shows the different types of finite and infinite equilibria, the corresponding configurations, and the associated global phase portraits.

Zone	P_1	P_2	P_3	Q_1	O_2	Configuration	PP
$a = b < 0$	Nu	SN	Ns	S	Ns	$C(\text{II.1.s}_1.1.1)$	PP.7
$a = b > 0$	Nu	SN	Ns	S	Ns	$C(\text{II.1.s}_1.1.2)$	PP.7

Table 6.5: Types of the finite and infinite equilibria, together with their corresponding local phase portraits, configurations, and global phase portraits in the Poincaré disc for Case 1 of the system $S(4)$.

Case 2. $a + 2b = 0$ and $ab \neq 0$. The system simplifies to

$$\dot{x} = -8(2b-1)x^2 + 8xy - 2(8b^2 + 4b - 3)x + 4(2b+1)y - (2b+1)(4b-1),$$
$$\dot{y} = 8x^2 - 8(2b-3)xy - 2(2b+1)(4b-1)x - 4(8b^2 + 4b - 1)y + 16y^2.$$

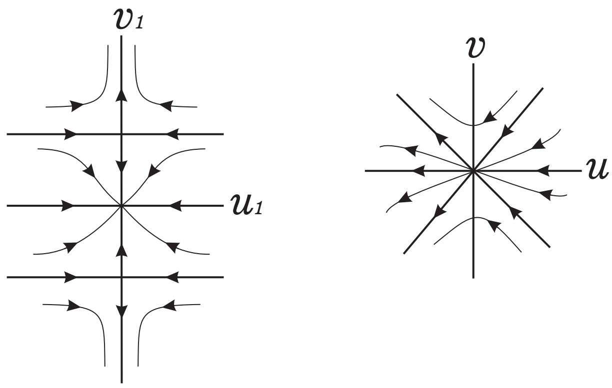

Figure 6.16: $B(\text{I}.4.\text{s}_1.3.Z_9)$: Blowing-down at O_2 of the system $S(\text{I}.4.\text{s}_1.3)$ with (c,d) in Z_9.

This system has four equilibria

$$P_1 = \left(-\frac{1}{2}, \frac{1}{4}\right) : -32b^2,\ -16b^2;$$
$$P_2 = \left(-\frac{1}{2}(2b+1), \frac{1}{4}(2b+1)^2\right) : 16b^2,\ 24b^2;$$
$$P_3 = \left(-b-\frac{1}{2}, b+\frac{1}{4}\right) : -16b^2,\ 16b^2;$$
$$P_4 = \left(-\frac{1}{2}(1-4b), \frac{1}{4}(1-4b)^2\right) : -32b^2,\ 48b^2.$$

The Poincaré compactification in the chart U_1 is

$$\begin{aligned}\dot{u} &= -4(2b+1)u^2v + (2b+1)(4b-1)uv^2 - 2(8b^2+4b+1)uv + 8u^2 + 16u\\ &\quad -2(2b+1)(4b-1)v + 8,\\ \dot{v} &= (2b+1)(4b-1)v^3 - 4(2b+1)uv^2 + 2(8b^2+4b-3)v^2 - 8uv + 8(2b-1)v.\end{aligned}$$

There is an equilibrium $Q_1 = (-1,0) : 0, 16b$.

The Poincaré compactification in the chart U_2 is

$$\begin{aligned}\dot{u} &= -8u^3 + 2(2b+1)(4b-1)u^2v - 16u^2 + 2(8b^2+4b+1)uv - 8u\\ &\quad -(2b+1)(4b-1)v^2 + 4(2b+1)v,\\ \dot{v} &= -8u^2v + 8(2b-3)uv + 2(2b+1)(4b-1)uv^2 + 4(8b^2+4b-1)v^2 - 16v.\end{aligned}$$

The origin is an equilibrium with eigenvalues: $-16, -8$.

Since $y - x^2 = 0, 4x + 4y + 1 = 0$, and $2x + 2b + 1 = 0$ are invariant curves of the system, we obtain the configurations $C(\text{II}.1.\text{s}_1.2.1)$ and $C(\text{II}.1.\text{s}_1.2.2)$.

Table 6.6 shows the types of finite and infinite equilibria, the corresponding configurations, and the associated global phase portraits.

Zone	P_1	P_2	P_3	P_4	Q_1	O_2	Configuration	PP
$a = -2b > 0$	Ns	Nu	S	S	SN	Ns	$C(\text{II}.1.\text{s}_1.2.1)$	PP.8
$a = -2b < 0$	Ns	Nu	S	S	SN	Ns	$C(\text{II}.1.\text{s}_1.2.2)$	PP.8

Table 6.6: Types of the finite and infinite equilibria, together with their corresponding local phase portraits, configurations, and global phase portraits in the Poincaré disc for Case 2 of the system $S(4)$.

Case 3. $ab(a-b)(a+2b) \neq 0$. Then system $S(4)$ has four finite equilibria

$$\begin{aligned}
P_1 &= \left(-\frac{1}{2}, \frac{1}{4}\right) : 8ab,\ 16ab;\\
P_2 &= \left(-\frac{1}{2}(2a+1),\ \frac{1}{4}(2a+1)^2\right) : 16ab,\ 8a(a-b);\\
P_3 &= \left(-\frac{1}{2}(2b+1),\ \frac{1}{4}(2b+1)^2\right) : 8b(b-a),\ 16b^2;\\
P_4 &= \left(-\frac{1}{2}(2b+1),\ \frac{1}{4}(4b+1)\right) : -8ab,\ -16b^2.
\end{aligned}$$

The Poincaré compactification in the chart U_1 is

$$\begin{aligned}
\dot{u} &= -4(2b+1)u^2v - (2a+1)(2b+1)uv^2 + 2(4ab-4b-1)uv + 2(2a+1)(2b+1)v,\\
&\quad +8u^2 + 8(a+2b+2)u + 8(a+2b+1)\\
\dot{v} &= -4(2b+1)uv^2 - (2a+1)(2b+1)v^3 - 2(4ab+4a+4b+3)v^2 - 8uv - 8(a+1)v.
\end{aligned}$$

The system in the chart U_1 has two infinite equilibria

$$Q_1 = (-1,\ 0) : -8a,\ 8(a+2b); \quad Q_2 = (-a-2b-1,\ 0) : -8(a+2b),\ 16b.$$

The Poincaré compactification in the chart U_2 is

$$\begin{aligned}
\dot{u} &= -8(a+2b+1)u^3 - 2(2a+1)(2b+1)u^2v - 8(a+2b+2)u^2 + 2(4b+1-4ab)uv\\
&\quad -8u + (2a+1)(2b+1)v^2 + 4(2b+1)v,\\
\dot{v} &= -2(2a+1)(2b+1)uv^2 - 4(4ab+2a+1)v^2 - 8(a+2b+1)u^2v\\
&\quad -8(2a+2b+3)uv - 16v.
\end{aligned}$$

The origin O_2 is an infinite equilibrium with eigenvalues $-16, -8$.

According to the expressions of the eigenvalues, and under the condition $ab(a-b)(a+2b) \neq 0$, we divide the parameter space (a, b) into eight regions Z_k for $k = 1, \ldots, 8$, as shown in Figure 6.17.

Since $y - x^2 = 0$, $4x + 4y + 1 = 0$, and $2x + 2b + 1 = 0$ are invariant algebraic curves of the system, we obtain the configurations $C(\text{II}.1.\text{s}_1.3.Z_k)$ for $k = 1, 2, \ldots, 8$.

Table 6.7 shows the types of finite and infinite equilibria, the corresponding configurations, and the corresponding phase portraits in the Poincaré disc of Figure 3.1.

System $S(5)$. The system together with its invariant curves is

$$\begin{aligned}
\dot{x} &= 4(a-b+2)\,x^2 + 8xy + 2(b+3)\,x - 4(a-b-1)\,y + (b+1),\\
\dot{y} &= 4(2-a+b)\,x^2 + 24xy + 16y^2 + 2(b+1)\,x + 4(a+1)\,y,
\end{aligned}$$

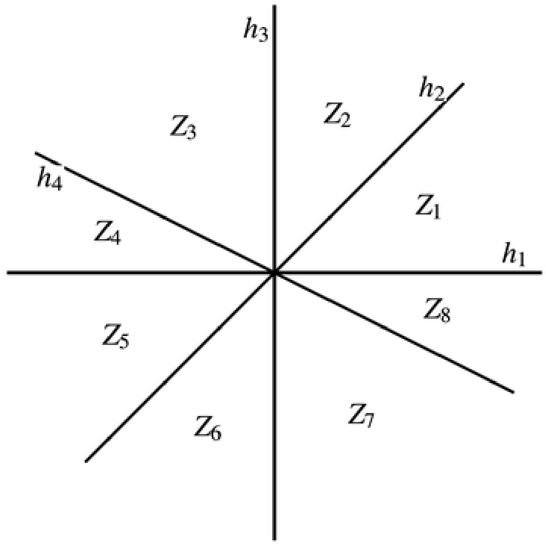

Figure 6.17: Zones of the parameter space (a, b), bounded by the straight lines $h_1 : b = 0$, $h_2 : a - b = 0$, $h_3 : a = 0$, and $h_4 : a + 2b = 0$.

$$f_1 = y - x^2, \quad f_2 = 4x + 4y + 1, \quad f_3 = 4x + 4y + b + 1,$$

where $a - b \neq 0$.

We consider two cases: 1. $b = 0$; 2. $b \neq 0$.

Case 1. $b = 0$ and $a - b \neq 0$. System $S(5)$ simplifies to

$$\begin{aligned}\dot{x} &= 4(a+2)x^2 + 8xy + 6x - 4(a-1)y + 1,\\ \dot{y} &= -4(a-2)x^2 + 24xy + 2x + 4(1+a)y + 16y^2.\end{aligned}$$

There is a finite equilibrium $P_1 = \left(-\frac{1}{2}, \frac{1}{4}\right) : (0, 0)$. It satisfies condition (4.i.i2) in Theorem 3.5 of [11] and is therefore a saddle-node.

The Poincaré compactification in the chart U_1 is

$$\begin{aligned}\dot{u} &= 4(a-1)u^2v + 2(2a-1)uv - 4(a-4)u + 8u^2 - uv^2 + 2v - 4(a-2),\\ \dot{v} &= 4(a-1)uv^2 - 8uv - 4(a+2)v - 6v^2 - v^3.\end{aligned}$$

Hence, the system has two infinite equilibria in the chart U_1

$$P_1 = (-1, 0) : -4a, -4a; \quad P_2 = \left(\frac{1}{2}(a-2), 0\right) : -8a, 4a.$$

The Poincaré compactification in the chart U_2 is

$$\begin{aligned}\dot{u} &= 4(a-2)u^3 + 4(a-4)u^2 - 2u^2v - 2(2a-1)uv - 8u + v^2 - 4(a-1)v,\\ \dot{v} &= 4(a-2)u^2v - 2uv^2 - 24uv - 4(a+1)v^2 - 16v.\end{aligned}$$

The origin O_2 is an infinite equilibrium with eigenvalues: $-16, -8$.

Since $y - x^2 = 0$, $4x + 4y + 1 = 0$ are invariant curves of the system, we obtain the configurations $C(\text{II.1.s}_2.1.1)$ and $C(\text{II.1.s}_2.1.2)$.

Zone	P_1	P_2	P_3	P_4	Q_1	Q_2	O_2	Configuration	PP
Z_1	Nu	Nu	S	Ns	S	S	Ns	C(II.1.s1.3.Z_1)	PP.11
Z_2	Nu	S	Nu	Ns	S	S	Ns	C(II.1.s1.3.Z_2)	PP.11
Z_3	Ns	S	Nu	S	Nu	S	Ns	C(II.1.s1.3.Z_3)	PP.10
Z_4	Ns	S	Nu	S	S	Nu	Ns	C(II.1.s1.3.Z_4)	PP.9
Z_5	Nu	Nu	S	Ns	S	S	Ns	C(II.1.s1.3.Z_5)	PP.11
Z_6	Nu	S	Nu	Ns	S	S	Ns	C(II.1.s1.3.Z_6)	PP.11
Z_7	Ns	S	Nu	S	Ns	S	Ns	C(II.1.s1.3.Z_7)	PP.10
Z_8	Ns	S	Nu	S	S	Ns	Ns	C(II.1.s1.3.Z_8)	PP.9

Table 6.7: Types of the finite and infinite equilibria, together with their corresponding local phase portraits, configurations, and global phase portraits in the Poincaré disc for Case 3 of the system $S(4)$.

Zone	P_1	Q_1	Q_2	O_2	Configuration	PP
$b = 0, a < 0$	Ni	Nu	S	Ns	C(II.1.s$_2$.1.1)	PP.14
$b = 0, a > 0$	Ni	Ns	S	Ns	C(II.1.s$_2$.1.2)	PP.14

Table 6.8: Types of the finite and infinite equilibria, together with their corresponding local phase portraits, configurations, and global phase portraits in the Poincaré disc for Case 1 of the system $S(5)$. Here "Ni" stands for "nilpotent".

Table 6.8 shows the phase portraits of the finite and infinite equilibria, the corresponding configurations, and the associated global phase portraits in Figure 3.1.

Case 2. $b(a-b) \neq 0$. If $b < 0$, then system $S(5)$ has four finite equilibria

$$\begin{aligned}
P_1 &= \left(-\frac{1}{2}, \frac{1}{4}\right) : 2b,\ 4b;\\
P_2 &= \left(-\frac{1}{2}\left(\sqrt{-b}+1\right), \frac{1}{4}\left(1+2\sqrt{-b}-b\right)\right) : -4(a-b)\sqrt{-b},\ -4b;\\
P_3 &= \left(\frac{1}{2}\left(\sqrt{-b}-1\right), \frac{1}{4}\left(1-2\sqrt{-b}-b\right)\right) : 4(a-b)\sqrt{-b},\ -4b;\\
P_4 &= \left(\frac{-a}{2(a-b)}, \frac{a+b}{4(a-b)}\right) : -2b,\ 4b.
\end{aligned}$$

If $b > 0$, then the system $S(5)$ has only two finite equilibria P_1 and P_4.

The Poincaré compactification in the chart U_1 is

$$\begin{aligned}
\dot{u} &= 4(a-b-1)u^2v - (b+1)uv^2 + 2(2a-b-1)uv + 8u^2 - 4(a-b-4)u\\
&\quad +2(b+1)v - 4(a-b-2),\\
\dot{v} &= 4(a-b-1)uv^2 - 8uv - (b+1)v^3 - (2b+6)v^2 - 4(a-b+2)v.
\end{aligned}$$

There are two equilibria

$$P_1 = (-1, 0) : 4(b-a),\ 4(b-a); \quad P_2 = \left(\frac{a-b-2}{2}, 0\right) : 8(b-a),\ 4(a-b).$$

The Poincaré compactification in the chart U_2 is

$$\dot{u} = 4(a-b-2)u^3 - 2(b+1)u^2v + 4(a-b-4)u^2 - 2(2a-b-1)uv - 8u \\ +(b+1)v^2 - 4(a-b-1)v,$$
$$\dot{v} = 4(a-b-2)u^2v - 2(b+1)uv^2 - 24uv - 4(a+1)v^2 - 16v.$$

The origin is an equilibrium with eigenvalues: $-16, -8$.

According to the expressions of the eigenvalues, and under the condition $b \neq 0$, we divide the parameter space (a, b) into four regions Z_k for $k = 1, 2, 3, 4$, as shown in Figure 6.18.

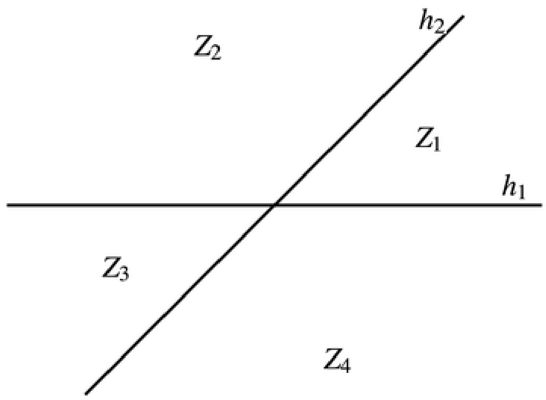

Figure 6.18: Zones of the parameter space (a, b), bounded by the straight lines $h_1 : b = 0$, $h_2 : a - b = 0$.

Since $y - x^2 = 0, 4x + 4y + 1 = 0$, and $4x + 4y + b + 1 = 0$ are invariant curves of the system, we obtain the configurations $C(\text{II.1.s}_2.1.Z_k)$ for $k = 1, 2, 3, 4$.

Table 6.9 shows the types of finite and infinite equilibria, the corresponding configurations, and the associated global phase portraits.

Zone	P_1	P_2	P_3	P_4	Q_1	Q_2	O_2	Configuration	PP
Z_1	Nu	–	–	S	Ns	S	Ns	$C(\text{II.1.s}_2.2.Z_1)$	PP.13
Z_2	Nu	–	–	S	Nu	S	Ns	$C(\text{II.1.s}_2.2.Z_2)$	PP.13
Z_3	Ns	Nu	S	S	Nu	S	Ns	$C(\text{II.1.s}_2.2.Z_3)$	PP.9,10,12
Z_4	Ns	S	Nu	S	Ns	S	Ns	$C(\text{II.1.s}_2.2.Z_4)$	PP.9,10,12

Table 6.9: Types of the finite and infinite equilibria, together with their corresponding local phase portraits, configurations, and global phase portraits in the Poincaré disc for Case 2 of the system $S(5)$.

System $S(6)$**.** The system together with its invariant curves is

$$\begin{aligned}\dot{x} &= 8(ab+2a-b^2+4)\,x^2+8(b+2)^2\,xy-2(b^2-4a-12)\,x\\ &\quad -4(ab-2b^2-6b+2a-4)\,y-(ab-2a+2b-4),\\ \dot{y} &= 16(2+b)\,x^2+8(ab+2a+6b+12)\,xy+16(b+2)^2\,y^2-2(ab-2a+2b-4)\,x\\ &\quad -4(b^2-4a+4b-4)\,y,\end{aligned}$$

$$f_1 = y-x^2, \quad f_2 = 4x+4y+1, \quad f_3 = 8x+4(b+2)\,y+(2-b),$$

where $(a-b)(b+2) \neq 0$.

We distinguish four cases: 1.$b=0$; 2.$a+b=0$; 3.$a-2b=0$; 4.$b(a+b)(a-2b)\neq 0$.

Case 1. $b=0$ and $(a-b)(b+2)\neq 0$. System $S(6)$ simplifies to

$$\begin{aligned}\dot{x} &= 16(a+2)x^2+32xy+8(a+3)x+8(2-a)y+2a+4,\\ \dot{y} &= 32x^2+16(a+6)xy+4(a+2)x+16(a+1)y+64y^2.\end{aligned}$$

This system has two finite equilibria

$$P_1 = \left(-\frac{1}{2}, \frac{1}{4}\right) : 0,\ 0; \quad P_2 = \left(-\frac{1}{4}(a+2),\ \frac{1}{16}(a+2)^2\right) : -4a^2,\ 2a^2.$$

The point P_1 satisfies the condition (4.iii.iii2) in Theorem 3.5 of [11]. Therefore, its phase portrait is formed by one hyperbolic and one elliptic sector.

The Poincaré compactification of this system in the chart U_1 is

$$\begin{aligned}\dot{u} &= 8(a-2)u^2v-2(a+2)uv^2+8(a-1)uv+32u^2+64u+4(a+2)v+32,\\ \dot{v} &= -2(a+2)v^3+8(a-2)uv^2-8(a+3)v^2-32uv-16(a+2)v.\end{aligned}$$

Hence, there is an infinite equilibrium $Q_1 = (-1, 0) : -16a, 0$ in the chart U_1.

The Poincaré compactification of the system in the chart U_2 is

$$\begin{aligned}\dot{u} &= -32u^3-4(a+2)u^2v-64u^2-8(a-1)uv-32u+2(a+2)v^2-8(a-2)v,\\ \dot{v} &= -32u^2v-4(a+2)uv^2-16(a+6)uv-16(a+1)v^2-64v.\end{aligned}$$

The origin O_2 is an infinite equilibrium with eigenvalues $-64, -32$.

Since $y-x^2=0$, $4x+4y+1=0$ are invariant algebraic curves for this system, we obtain the configurations $C(\text{II.1.s}_3.1.1)$ and $C(\text{II.1.s}_3.1.2)$ and the phase portraits in the Poincaré disc indicated in Table 6.10.

Zone	P_1	P_2	Q_1	O_2	Configuration	PP
$b=0,\ a<0$	Ni	S	SN	Ns	$C(\text{II.1.s}_3.1.1)$	PP.15
$b=0,\ a>0$	Ni	S	SN	Ns	$C(\text{II.1.s}_3.1.2)$	PP.15

Table 6.10: Types of the finite and infinite equilibria, together with their corresponding local phase portraits, configurations, and global phase portraits in the Poincaré disc for Case 1 of the system $S(6)$.

Case 2. $a+b=0$ and $b(a-b)(b+2)\neq 0$. System $S(6)$ simplifies to

$$\begin{aligned}\dot{x} &= -16(b-1)(b+2)x^2+8(b+2)^2xy-2(b-2)(b+6)x+4(b+2)(3b+2)y+(b-2)^2,\\ \dot{y} &= 16(b+2)x^2-8(b-6)(b+2)xy+2(b-2)^2x-4(b^2+8b-4)y+16(b+2)^2y^2.\end{aligned}$$

Since $b+2 \neq 0$, this system has three finite equilibria

$$P_1 = \left(-\frac{1}{2}, \frac{1}{4}\right) : 8b^2,\ 16b^2; \quad P_2 = \left(-\frac{b+6}{6(b+2)}, -\frac{b-6}{12(b+2)}\right) : -8b^2,\ \frac{16b^2}{3};$$
$$P_3 = \left(\frac{b-2}{2(b+2)}, \frac{(b-2)^2}{4(b+2)^2}\right) : -16b^2,\ 0.$$

The Poincaré compactification of this system in the chart U_1 is

$$\begin{aligned} \dot{u} &= -4(3b+2)(b+2)u^2v - (b-2)^2uv^2 - 2(b^2+12b+4)uv + 8(b+2)^2u^2, \\ &\quad +8(b+2)(b+4)u + 2(b-2)^2v + 16(b+2) \\ \dot{v} &= -4(b+2)(3b+2)uv^2 - 8(b+2)^2uv - (b-2)^2v^3 + 2(b-2)(b+6)v^2 \\ &\quad +16(b+2)(b-1)v. \end{aligned}$$

Since $b+2 \neq 0$, the system has two infinite equilibria in the chart U_1

$$Q_1 = (-1,\ 0) : 24b(b+2),\ -8b(b+2); \quad Q_2 = \left(\frac{-2}{b+2},\ 0\right) : 16b(b+2),\ 8b(b+2).$$

The Poincaré compactification of the system in the chart U_2 is

$$\begin{aligned} \dot{u} &= -16(b+2)u^3 - 2(b-2)^2u^2v - 8(b+2)(b+4)u^2 + 2(b^2+12b+4)uv \\ &\quad -8(b+2)^2u + (b-2)^2v^2 + 4(b+2)(3b+2)v, \\ \dot{v} &= -16(b+2)u^2v - 2(b-2)^2uv^2 + 8(b-6)(b+2)uv + 4(b^2+8b-4)v^2 - 16(b+2)^2v. \end{aligned}$$

The origin O_2 is an infinite equilibrium with eigenvalues: $-16(b+2)^2, -8(b+2)^2$.

Since $y-x^2=0$, $4x+4y+1=0$, and $8x+8y+4by-b+2=0$ are invariant algebraic curves of the system, we obtain the configurations $C(\text{II.1.s}_3\text{.2.1})$ and $C(\text{II.1.s}_3\text{.2.2})$.

Table 6.11 shows the local phase portraits of the finite and infinite equilibria, the corresponding configurations and phase portraits in the Poincaré disc.

Zone	P_1	P_2	P_3	Q_1	Q_2	O_2	Configuration	PP
$b<-2$ or $b>0$	Nu	S	SN	S	Nu	Ns	$C(\text{II.1.s}_3\text{.2.1})$	PP.16
$-2<b<0$	Nu	S	SN	S	Ns	Ns	$C(\text{II.1.s}_3\text{.2.2})$	PP.16

Table 6.11: Types of the finite and infinite equilibria, together with their corresponding local phase portraits, configurations, and global phase portraits in the Poincaré disc for Case 2 of the system $S(6)$.

Case 3. $a-2b=0$ and $b(a-b)(b+2)\neq 0$. System $S(6)$ simplifies to

$$\begin{aligned} \dot{x} &= 8(b+2)^2x^2 + 8(b+2)^2xy - 2(b^2-8b-12)x + 8(b+2)y - 2(b-2)(b+1), \\ \dot{y} &= 16(b+2)x^2 + 16(b+2)(b+3)xy - 4(b-2)(b+1)x - 4(b^2-4b-4)y + 16(b+2)^2y^2. \end{aligned}$$

Since $b+2 \neq 0$, this system has three finite equilibria

$$P_1 = \left(-\frac{1}{2}, \frac{1}{4}\right) : -8b^2,\ -4b^2; \quad P_2 = \left(-\frac{b+1}{b+2}, \frac{(b+1)^2}{(b+2)^2}\right) : -4b^2,\ 6b^2;$$
$$P_3 = \left(\frac{b-2}{2(b+2)}, \frac{(b-2)^2}{4(b+2)^2}\right) : 8b^2,\ 12b^2,$$

The Poincaré compactification of this system in the chart U_1 is

$$\begin{aligned}\dot{u} &= -8(b+2)u^2v + 2(b-2)(b+1)uv^2 - 2(b^2+4)uv + 8(b+2)^2u^2 + 8(b+2)(b+4)u \\ &\quad -4(b-2)(b+1)v + 16(b+2), \\ \dot{v} &= -8(b+2)^2uv - 8(b+2)uv^2 + 2(b-2)(b+1)v^3 + 2(b^2-8b-12)v^2 - 8(b+2)^2v.\end{aligned}$$

Since $b+2 \neq 0$, there are two infinite equilibria in the chart U_1

$$Q_1 = (-1,\ 0) : -8b(b+2),\ 0; \quad Q_2 = \left(-\frac{2}{b+2},\ 0\right) : -8b(b+2),\ 8b(b+2).$$

The Poincaré compactification of this system in the chart U_2 is

$$\begin{aligned}\dot{u} &= -16(b+2)u^3 + 4(b-2)(b+1)u^2v - 8(b+2)(b+4)u^2 + 2(b^2+4)uv \\ &\quad -8(b+2)^2u - 2(b-2)(b+1)v^2 + 8(b+2)v, \\ \dot{v} &= -16(b+2)u^2v + 4(b-2)(b+1)uv^2 - 16(b+2)(b+3)uv + 4(b^2-4b-4)v^2 \\ &\quad -16(b+2)^2v.\end{aligned}$$

The origin O_2 is an infinite equilibrium with eigenvalues: $-16(b+2)^2, -8(b+2)^2$.

Since $y - x^2 = 0$, $4x+4y+1 = 0$, and $8x+8y+4by-b+2 = 0$ are invariant algebraic curves of the system, we obtain the configurations $C(\text{II.1.s}_3.3.1)$ and $C(\text{II.1.s}_3.3.2)$.

Table 6.12 shows the local phase portraits of the finite and infinite equilibria, the corresponding configurations and phase portraits in the Poincaré disc.

Zone	P_1	P_2	P_3	Q_1	Q_2	O_2	Configuration	PP
$b > 0$ or $b < -2$	Ns	S	Nu	SN	S	Ns	$C(\text{II.1.s}_3.3.1)$	PP.7
$-2 < b < 0$	Ns	S	Nu	SN	S	Ns	$C(\text{II.1.s}_3.3.2)$	PP.7

Table 6.12: Types of the finite and infinite equilibria, together with their corresponding local phase portraits, configurations, and global phase portraits in the Poincaré disc for Case 3 of the system $S(6)$.

Case 4. $b(a-2b)(a+b)(a-b)(b+2) \neq 0$. Then system $S(6)$ has four finite equilibria

$$\begin{aligned}P_1 &= \left(-\frac{1}{2},\ \frac{1}{4}\right) : 8b(b-a),\ 4b(b-a); \\ P_2 &= \left(-\frac{a+2}{2(b+2)},\ \frac{(a+2)^2}{4(b+2)^2}\right) : -4(a-b)^2,\ 2(a-b)(a+b); \\ P_3 &= \left(\frac{b-2}{2(b+2)},\ \frac{(b-2)^2}{4(b+2)^2}\right) : 4b(a+b),\ 8(a-b)b; \\ P_4 &= \left(\frac{b(b+4)-2a}{2(a-2b)(b+2)},\ -\frac{a(b-2)+4b}{4(a-2b)(b+2)}\right) : -\frac{4(a-b)^2b}{a-2b},\ 4(a-b)b,\end{aligned}$$

The Poincaré compactification in the chart U_1 is

$$\begin{aligned}\dot{u} &= 4(a-2b-2)(b+2)u^2v + (a+2)(b-2)uv^2 - 2(b^2-4a+8b+4)uv \\ &\quad +8(b+2)^2u^2 + 8(b+2)(b+4)u - 2(a+2)(b-2)v + 16(b+2), \\ \dot{v} &= (a+2)(b-2)v^3 + 2(b^2-4a-12)v^2 + 4(a-2b-2)(b+2)uv^2 \\ &\quad -8(a-b+2)(b+2)v - 8(b+2)^2uv.\end{aligned}$$

Since $b+2 \neq 0$, there are two equilibria

$$Q_1 = (-1,\ 0) : 8(2b-a)(b+2),\ -8b(b+2);$$
$$Q_2 = \left(-\frac{2}{b+2},\ 0\right) : 8(b-a)(b+2),\ 8b(b+2),$$

The Poincaré compactification in the chart U_2 is

$$\begin{aligned}\dot{u} &= -8(b+2)^2u - 8(b+2)(b+4)u^2 - 16(b+2)u^3 - 4(a-2b-2)(b+2)v \\ &\quad +2(b^2-4a+8b+4)uv - (a+2)(b-2)v^2 + 2(a+2)(b-2)u^2v, \\ \dot{v} &= -16(b+2)u^2v - 8(a+6)(b+2)uv + 2(a+2)(b-2)uv^2 \\ &\quad +4(b^2-4a+4b-4)v^2 - 16(b+2)^2v.\end{aligned}$$

The origin O_2 is an infinite equilibrium with eigenvalues: $-16(b+2)^2, -8(b+2)^2$.

According to the expressions of the eigenvalues, and under the condition $b(a+b)(a-b)(a+2b)(b+2) \neq 0$, we divide the parameter space (a,b) into twelve regions Z_k for $k = 1,2,\ldots,12$, as shown in Figure 6.19.

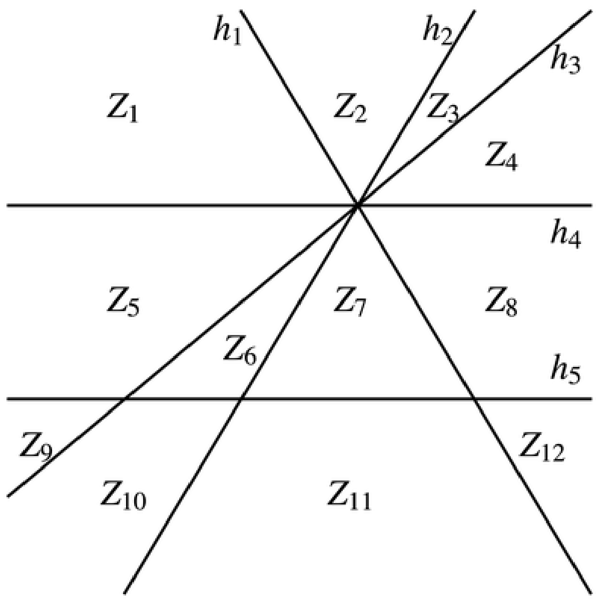

Figure 6.19: Zones of the parameter space (a,b), bounded by the straight lines $h_1 : a+b=0$, $h_2 : a-b=0$, $h_3 : a-2b=0$, $h_4 : b=0$, and $h_5 : b+2=0$.

Since $y-x^2=0$, $4x+4y+1=0$, and $8x+8y+4by-b+2=0$ are invariant algebraic curves of the system, we obtain the configurations $C(\text{II.1.s}_3.4.Z_k)$ for $k=1,2,\ldots,8$.

Table 6.13 shows the local phase portraits of the finite and infinite equilibria, the corresponding configurations, and phase portraits in the Poincaré disc.

System $S(7)$**.** The system together with its invariant curves is

$$\begin{aligned}\dot{x} &= 8(a+1)\,x^2 + 8xy + 2(ab+2a+2b+3)\,x - 4(a-b-1)\,y + (ab+a+b+1), \\ \dot{y} &= 8(b+1)\,x^2 + 8(a+b+3)\,xy + 16y^2 + (2ab+2a+2b+2)\,x + 4(ab+2a+1)\,y,\end{aligned}$$

$$f_1 = y - x^2, \quad f_2 = 4x+4y+1, \quad f_3 = 4(b+1)\,x + 4y + (b+1)^2,$$

Zone	P_1	P_2	P_3	P_4	Q_1	Q_2	O_2	Configuration	PP
Z_1	Nu	S	Ns	S	S	Nu	Ns	$C(\text{II}.1.s_3.4.Z_1)$	PP.9
Z_2	Nu	Ns	S	S	S	Nu	Ns	$C(\text{II}.1.s_3.4.Z_2)$	PP.9
Z_3	Ns	S	Nu	Nu	S	S	Ns	$C(\text{II}.1.s_3.4.Z_3)$	PP.11
Z_4	Ns	S	Nu	S	Ns	S	Ns	$C(\text{II}.1.s_3.4.Z_4)$	PP.10
Z_5	Ns	S	Nu	S	Nu	S	Ns	$C(\text{II}.1.s_3.4.Z_5)$	PP.10
Z_6	Ns	S	Nu	Nu	S	S	Ns	$C(\text{II}.1.s_3.4.Z_6)$	PP.11
Z_7	Nu	Ns	S	S	S	Ns	Ns	$C(\text{II}.1.s_3.4.Z_7)$	PP.9
Z_8	Nu	S	Ns	S	S	Ns	Ns	$C(\text{II}.1.s_3.4.Z_8)$	PP.9
Z_9	Ns	S	Nu	S	Ns	S	Ns	$C(\text{II}.1.s_3.4.Z_4)$	PP.10
Z_{10}	Ns	S	Nu	Nu	S	S	Ns	$C(\text{II}.1.s_3.4.Z_3)$	PP.11
Z_{11}	Nu	Ns	S	S	S	Nu	Ns	$C(\text{II}.1.s_3.4.Z_2)$	PP.9
Z_{12}	Nu	S	Ns	S	S	Nu	Ns	$C(\text{II}.1.s_3.4.Z_1)$	PP.9

Table 6.13: Types of the finite and infinite equilibria, together with their corresponding local phase portraits, configurations, and global phase portraits in the Poincaré disc for Case 4 of the system $S(6)$.

where $a(a-b)\neq 0$.

We divide the study of this system into two cases: $1. b=0$; $2. b\neq 0$.

Case 1. $b=0$ and $a(a-b)\neq 0$. System $S(7)$ simplifies to

$$\begin{aligned}\dot{x}&=8(a+1)x^2+8xy+2(2a+3)x-4(a-1)y+a+1,\\ \dot{y}&=8x^2+8(a+3)xy+2(a+1)x+16y^2+4(2a+1)y.\end{aligned}$$

Doing the transformation $(a,b,t)\to\left(2a,\ 0,\ \frac{t}{4}\right)$ on $S(6)$ we obtain the above system.

Case 2. $ab(a-b)\neq 0$. Then system $S(7)$ has four finite equilibria

$$\begin{aligned}&P_1=\left(-\frac{1}{2},\frac{1}{4}\right):2ab,\ 4ab;\quad P_2=\left(-\frac{1}{2}(a+1),\ \frac{1}{4}(a+1)^2\right):-4a(a-b),\ 2a(a-b);\\ &P_3=\left(-\frac{1}{4}(b+2),\ \frac{b+1}{4}\right):-2ab,\ 2(a-b)b;\\ &P_4=\left(-\frac{1}{2}(b+1),\ \frac{1}{4}(b+1)^2\right):4b(b-a),\ 2b(b-a),\end{aligned}$$

The Poincaré compactification of system $S(7)$ in the chart U_1 is

$$\begin{aligned}\dot{u}&=4(a-b-1)u^2v-(a+1)(b+1)uv^2+2(ab+2a-2b-1)uv+8u^2+8(b+2)u\\ &\quad+2(a+1)(b+1)v+8(b+1),\\ \dot{v}&=-(a+1)(b+1)v^3+4(a-b-1)uv^2-2(ab+2a+2b+3)v^2-8uv-8(a+1)v,\end{aligned}$$

Hence, there are two infinite equilibria in the chart U_1

$$Q_1=(-1,\ 0):-8a,\ 8b;\quad Q_2=(-b-1,\ 0):-8(a-b),\ -8b.$$

The Poincaré compactification of system $S(7)$ in the chart U_2 is

$$\begin{aligned}\dot{u}&=-8(b+1)u^3-8(b+2)u^2-2(a+1)(b+1)u^2v-2(ab+2a-2b-1)uv-8u\\ &\quad+(a+1)(b+1)v^2+4(b-a+1)v,\\ \dot{v}&=-8(b+1)u^2v-2(a+1)(b+1)uv^2-8(a+b+3)uv-4(ab+2a+1)v^2-16v.\end{aligned}$$

The origin O_2 is an infinite equilibrium with eigenvalues: $-16, -8$.

According to the expressions of the eigenvalues, and under the condition $ab(a-b) \neq 0$, we divide the parameter space (a, b) into six regions Z_k for $k = 1, 2, \ldots, 6$, as shown in Figure 6.20.

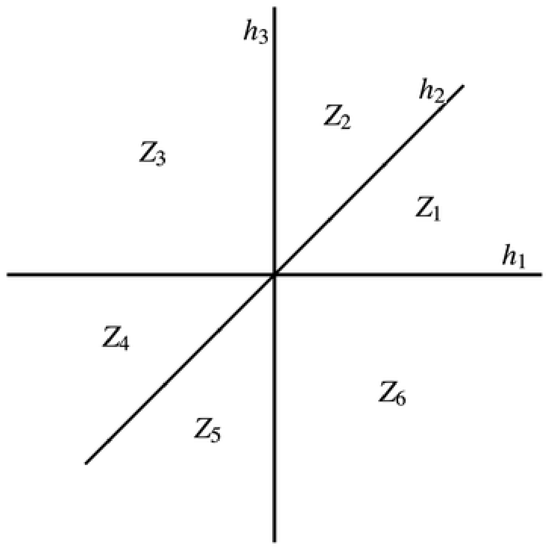

Figure 6.20: Zones of parameter space (a, b), bounded by the straight lines h_1 : $b = 0$, $h_2 : a - b = 0$, and $h_3 : a = 0$.

Since $y - x^2 = 0$, $4x + 4y + 1 = 0$, and $b^2 + 4x + 4bx + 4y + 2b + 1 = 0$ are invariant algebraic curves of the system, we obtain the configurations $C(\text{II}.1.s_4.2.Z_k)$ for $k = 1, \ldots, 6$.

Table 6.14 shows the local phase portraits of the finite and infinite equilibria, the corresponding configurations, and phase portraits in the Poincaré disc.

Zone	P_1	P_2	P_3	P_4	Q_1	Q_2	O_2	Configuration	PP
Z_1	Nu	S	S	Ns	S	Ns	Ns	$C(\text{II}.1.s_4.2.Z_1)$	PP.9
Z_2	Nu	S	Ns	Nu	S	S	Ns	$C(\text{II}.1.s_4.2.Z_2)$	PP.11
Z_3	Ns	S	S	Nu	Nu	S	Ns	$C(\text{II}.1.s_4.2.Z_3)$	PP.9
Z_4	Nu	S	S	Ns	S	Nu	Ns	$C(\text{II}.1.s_4.2.Z_4)$	PP.9
Z_5	Nu	S	Ns	Nu	S	S	Ns	$C(\text{II}.1.s_4.2.Z_5)$	PP.11
Z_6	Ns	S	S	Nu	Ns	S	Ns	$C(\text{II}.1.s_4.2.Z_6)$	PP.9

Table 6.14: Types of the finite and infinite equilibria, together with their corresponding local phase portraits, configurations, and global phase portraits in the Poincaré disc for Case 2 of the system $S(7)$.

System $S(8)$. The system together with its invariant curves is

$$\dot{x} = 8(a-b+1)\,x^2 + 8xy - 2(4b^2 - 4a - 3)\,x + 4(2b+1)\,y - (2a+1)(4b^2-1),$$
$$\dot{y} = 8(a+b+1)\,x^2 + 8(2a+3)\,xy + 16y^2 - 2(8ab^2 + 4b^2 - 2a - 1)\,x$$
$$-4(4b^2 + 2b - 2a - 1)\,y,$$

$$f_1 = y - x^2, \quad f_2 = 4x + 4y - 4b^2 + 1, \quad f_3 = 2x + 2b + 1,$$

where $b(a-b) \neq 0$.

For studying this system we consider the two cases: $1.a+b=0$; $\quad 2.a+b \neq 0$.

Case 1. $a+b=0$ and $b(a-b) \neq 0$. The system simplifies to:

$$\begin{aligned}\dot{x} &= -8(2b-1)\,x^2 + 8xy - 2(2b-1)(2b+3)\,x + 4(2b+1)\,y + (2b-1)^2(2b+1),\\ \dot{y} &= 8x^2 - 8(2b-3)\,xy + 2(2b-1)^2(2b+1)\,x - 4(4b^2+4b-1)\,y + 16y^2.\end{aligned}$$

The system has three finite equilibria

$$\begin{aligned}P_1 &= \left(-\frac{1}{2}(2b+1),\ \frac{1}{4}(2b+1)^2\right) : 32b^2,\ 32b^2;\\ P_2 &= \left(-\frac{1}{2}(2b+1),\ \frac{1}{4}(1+4b-4b^2)\right) : -32b^2,\ 16b^2;\\ P_3 &= \left(\frac{1}{2}(2b-1),\ \frac{1}{4}(2b-1)^2\right) : -32b^2,\ 0,\end{aligned}$$

The Poincaré compactification of this system in the chart U_1 is

$$\begin{aligned}\dot{u} &= -4(2b+1)\,u^2v - (2b-1)^2(2b+1)\,uv^2 - 2(2b+1)^2\,uv + 2(2b-1)^2(2b+1)\,v\\ &\quad +8u^2 + 16u + 8,\\ \dot{v} &= -(2b-1)^2(2b+1)\,v^3 - 4(2b+1)\,uv^2 + 2(2b-1)(2b+3)\,v^2 - 8uv + 8(2b-1)\,v.\end{aligned}$$

Hence, there is an infinite equilibrium $Q_1 = (-1,0) : 0, 16b$ in the local chart U_1. According to Theorem 2.19 of [11], this equilibrium is a saddle–node.

The Poincaré compactification of this system in the chart U_2 is

$$\begin{aligned}\dot{u} &= -8u^3 - 2(2b-1)^2(2b+1)\,u^2v + 2(2b+1)^2\,uv + (2b-1)^2(2b+1)\,v^2 - 16u^2\\ &\quad -8u + 4(2b+1)\,v,\\ \dot{v} &= -8u^2v - 2(2b-1)^2(2b+1)\,uv^2 + 8(2b-3)\,uv + 4(4b^2+4b-1)\,v^2 - 16v.\end{aligned}$$

The origin O_2 is an infinite equilibrium with eigenvalues: $-16, -8$.

Since $y - x^2 = 0$, $4x + 4y - 4b^2 + 1 = 0$, and $2x + 2b + 1 = 0$ are invariant algebraic curves of the system, we obtain the configurations $C(\text{II}.3.\text{s}_1.1)$.

Table 6.15 shows the local phase portraits of the finite and infinite equilibria, the corresponding configurations, and phase portraits in the Poincaré disc.

Zone	P_1	P_2	P_3	Q_1	O_2	Configuration	PP
$a+b=0$	Nu	S	SN	SN	Ns	$C(\text{II}.3.\text{s}_1.1)$	PP.17

Table 6.15: Types of the finite and infinite equilibria, together with their corresponding local phase portraits, configurations, and global phase portraits in the Poincaré disc for Case 1 of the system $S(8)$.

Case 2. $b(a-b)(a+b) \neq 0$. Then system $S(8)$ has four finite equilibria

$$\begin{aligned}
P_1 &= \left(-\frac{1}{2}(2a+1),\ \frac{1}{4}(2a+1)^2\right) : 16(a-b)b,\ 8(a-b)(a+b);\\
P_2 &= \left(-\frac{1}{2}(2b+1),\ \frac{1}{4}(2b+1)^2\right) : 16b(b-a),\ 16b(b-a);\\
P_3 &= \left(-\frac{1}{2}(2b+1),\ \frac{1}{4}(4ab+4b+1)\right) : -8b(a-b),\ 16b(a-b);\\
P_4 &= \left(\frac{1}{2}(2b-1),\ \frac{1}{4}(2b-1)^2\right) : 16(a-b)b,\ 16b(a+b).
\end{aligned}$$

The Poincaré compactification of this system in the chart U_1 is

$$\begin{aligned}
\dot{u} &= -4(2b+1)\,u^2v + (4b^2-1)(2a+1)\,uv^2 - 2(2b+1)^2\,uv + 8u^2 + 8(a+b+2)\,u\\
&\quad - 2(4b^2-1)(2a+1)\,v + 8(a+b+1),\\
\dot{v} &= (4b^2-1)(2a+1)\,v^3 - 4(2b+1)\,uv^2 + 2(4b^2-4a-3)\,v^2 - 8uv - 8(a-b+1)\,v.
\end{aligned}$$

Hence, there are two infinite equilibria in the local chart U_1

$$Q_1 = (-1,\ 0) : 8(b-a),\ 8(a+b); \quad Q_2 = (-a-b-1,\ 0) : -8(a+b),\ 16b,$$

The Poincaré compactification of this system in the chart U_2 is

$$\begin{aligned}
\dot{u} &= -8(a+b+1)\,u^3 + 2(2a+1)(4b^2-1)\,u^2v - 8(a+b+2)\,u^2\\
&\quad + 2(2b+1)^2\,uv - 8u - (4b^2-1)(2a+1)\,v^2 + 4(2b+1)\,v,\\
\dot{v} &= -8(a+b+1)\,u^2v + 2(2a+1)(4b^2-1)\,uv^2 - 8(2a+3)\,uv\\
&\quad + 4(4b^2-2a+2b-1)\,v^2 - 16v.
\end{aligned}$$

The origin O_2 is an infinite equilibrium with eigenvalues: $-16, -8$.

According to the expressions of the eigenvalues, and under the condition $b(a+b)(a-b) \neq 0$, we divide the parameter space (a,b) into six regions Z_k for $k = 1, 2, \ldots, 6$, as shown in Figure 6.21.

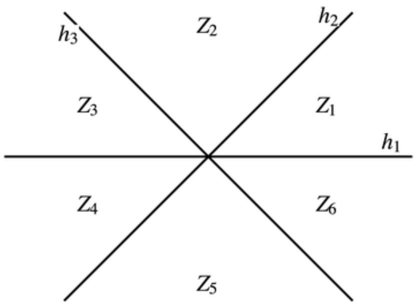

Figure 6.21: Zones of the parameter space (a,b), bounded by the straight lines $h_1 : b = 0$, $h_2 : a - b = 0$, and $h_3 : a + b = 0$.

Since $y - x^2 = 0$, $4x + 4y - 4b^2 + 1 = 0$, and $2x + 2b + 1 = 0$ are invariant algebraic curves of the system, we obtain the configurations $C(\text{II}.3.\text{s}_1.2.Z_k)$ for $k = 1, \ldots, 6$.

Table 6.16 shows the local phase portraits of the finite and infinite equilibria, the corresponding configurations, and phase portraits in the Poincaré disc.

Zone	P_1	P_2	P_3	P_4	Q_1	Q_2	O_2	Configuration	PP
Z_1	Nu	Ns	S	Nu	S	S	Ns	$C(\text{II}.3.\text{s}_1.2.Z_1)$	PP.11
Z_2	Ns	Nu	S	S	Nu	S	Ns	$C(\text{II}.3.\text{s}_1.2.Z_2)$	PP.10
Z_3	S	Nu	S	Ns	S	Nu	Ns	$C(\text{II}.3.\text{s}_1.2.Z_3)$	PP.10
Z_4	Nu	Ns	S	Nu	S	S	Ns	$C(\text{II}.3.\text{s}_1.2.Z_4)$	PP.11
Z_5	Ns	Nu	S	S	Ns	S	Ns	$C(\text{II}.3.\text{s}_1.2.Z_5)$	PP.10
Z_6	S	Nu	S	Ns	S	Ns	Ns	$C(\text{II}.3.\text{s}_1.2.Z_6)$	PP.10

Table 6.16: Types of the finite and infinite equilibria, together with their corresponding local phase portraits, configurations, and global phase portraits in the Poincaré disc for Case 2 of the system $S(8)$.

This completes the proof. □

7 Conclusion

This study systematically classifies the phase portraits in the Poincaré disc of the quadratic systems with one or two invariant real straight lines taking into account that their total multiplicity be two and an invariant parabola. The distinct phase portraits in the Poincaré disc of these class of quadratic systems are given in Theorem 3.2. The results contribute to a broader understanding of quadratic systems with algebraic invariant curves, offering new insights into their global dynamics and bifurcations.

As a natural extension of this work, future research will focus on the classification of phase portraits for quadratic systems with one or two invariant real straight lines taking into account that their total multiplicity be two and an invariant ellipse or hyperbola. This will ultimately lead to a complete classification of all planar quadratic differential systems possessing two invariant straight lines and an irreducible conic. This classification will provide a unified framework for analysing the global dynamics of this important class of quadratic systems.

8 Data and code availability

All symbolic computations supporting Theorems 3.1 and 3.2 are openly available at the following public repository:

- Llibre, J. & Ou, H. (2025). *Mathematica computations for the parabola paper (Theorems 3.1 and 3.2)*. Repository: `https://github.com/Ovason/R2P` (accessed on December 4, 2025).

The repository contains the Mathematica notebooks ordered exactly as in the manuscript.

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