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Multiple solutions for a fractional (p, t)-Laplacian system with logarithmic nonlinearities

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Abstract. This paper is concerned with the existence and multiplicity of a ground state solution for a class of fractional (p, t)-type systems involving logarithmic nonlinearities with sign-changing coefficients, which are obtained by variational methods. More precisely, by combining Nehari manifold and fibering map method. The paper proves two results: The first one provides the existence of a ground-state solution for the proposed problem. Under a different set of hypotheses with respect to the first result, a second result is obtained, which provides the existence of at least two nontrivial ground state solutions.

Keywords: logarithmic nonlinearity, fibering map, Nehari manifold, (p, t) fractional operator, multiplicity, mountain pass theorem.

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1 Introduction and main results

The study of elliptic equations and systems involving fractional operators (see [14]), as well as, fractional (p, t) Laplacians has energized many applications in nonlinear mathematical and physical analysis. In recent years, much attention has been paid to research on very delicate readings about the existence and multiplicity of ground state solutions for such elliptic problems with different types of nonlinearities. This paper deals with nonhomogeneous systems of equations involving fractional operators of type (p, t) with logarithmic coupling terms with coefficients changing sign, specifically we study the following equation

$$\begin{cases} (-\Delta)_p^{s_1} u + (-\Delta)_t^{s_2} u = \lambda H_1(x)|u|^{\theta-2}u \ln |u| + \dfrac{q}{r+q} R_1(x)|v|^r|u|^{q-2}u & \text{in } \Omega, \\ (-\Delta)_p^{s_1} v + (-\Delta)_t^{s_2} v = \mu H_2(x)|v|^{\theta-2}v \ln |v| + \dfrac{r}{r+q} R_2(x)|u|^r|v|^{q-2}v & \text{in } \Omega, \\ u = v = 0, \qquad \mathbb{R}^N \setminus \Omega, \end{cases} \tag{P}$$

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where $\Omega \subset \mathbb{R}^N$ is a bounded domain with Lipschitz boundary, $s \in (0,1)$, $N > ps$, $\lambda, \mu > 0$, $0 < s_2 < s_1 < 1 < t < p < \theta < p^*_{s_1} = \frac{pN}{N-ps_1}$, $\lambda, \mu > 0$ are two parameters, $r > 1, q > 1$ satisfies $q + r < p^*_{s_1}$ and the additional weights $H_1, H_2, R_1, R_2 \in C(\bar{\Omega})$ are such that: $R_1(x), R_2(x)$ are positive functions and $H_1(x), H_2(x)$ are sign-changing functions. The operators $(-\Delta)^{s_1}_p$ and $(-\Delta)^{s_2}_t$ represents, both, fractional p-Laplacian operator, a generalization for the fractional Laplacian $(-\Delta)^s, 0 < s_i < 1$ for $p = 2$, defined in a integral way as

$$(-\Delta)^{s_i} u(x) := \frac{c(n,s_i)}{2} \int_{\mathbb{R}^N} \frac{2u(x) - (x+y) - u(x-y)}{|y|^{n+2s_i}} dy, \qquad x \in \mathbb{R}^N,$$

where $c(n,s)$ is a positive normalizing constant, and another fractional operator.

Specifically, the operators $(-\Delta)^i_{p_i}$ are defined, up to a normalization constant, by the formula

$$(-\Delta)^{s_i}_{p_i} u(x) := \lim_{\varepsilon \to 0^+} 2 \int_{\mathbb{R}^N \setminus B_\varepsilon(x)} \frac{|u(x) - u(y)|^{p_i - 2}(u(x) - u(y))}{|x-y|^{N+s_i p_i}} dy,$$

where $u \in u \in C^\infty_0(\mathbb{R}^N)$ and $B_\varepsilon(x)$ denotes ball in $\mathbb{R}^N$ with centre x and radius ε. When $s_1 = s_2$ the equation reduces to a (p,t)-type Laplacian problem which appears in a more general reaction-diffusion system

$$u_t = \nabla \cdot (f(u)\nabla u) + h(x,u) \tag{1.1}$$

where $f(u) = |\nabla u|^{p-2}\nabla u + |\nabla u|^{t-2}\nabla u$. Such problems have a wide range of applications in physics and related sciences such as biophysics, plasma physics, and chemical reaction design, etc. where u describes a concentration, and the first term on the right-hand side of (1.1) corresponds to a diffusion with a diffusion coefficient $f(u)$; the term $g(x,u)$ stands for the reaction, related to sources and energy-loss processes. On the other hand, since logarithmic nonlinearities play a significant role in depicting the mathematical and physical phenomena, they have received much attention in PDEs (see [19,20,22] and references therein).

In [19], the authors studied the existence and multiplicity of a class of fractional Laplacian systems with logarithmic nonlinearity in which three types of weights with certain regularity are involved.

$$\begin{cases} (-\Delta)^s u = \lambda g_1(x) u \ln(|u|) + \dfrac{p}{p+q} b(x)|v|^q |u|^{p-2} u & \text{in } \Omega, \\ (-\Delta)^t v = \mu g_2(x) v \ln(|v|) + \dfrac{r}{q+r} b(x)|u|^r |v|^{q-2} v & \text{in } \Omega, \\ \quad u = v = 0, \quad \text{in } \mathbb{R}^N \setminus \Omega \end{cases} \tag{1.2}$$

where $\Omega \subset \mathbb{R}^N$ is a bounded domain with Lipschitz boundary, $s, t \in (0,1)$, $N > \max\{2s, 2t\}$, $\lambda, \mu > 0$, $2 < q + p < \min\left\{\frac{2N}{N-2s}, \frac{2N}{N-2t}\right\}$, and the additional weights $g_1, g_2, b \in C(\overline{\Omega})$ are such that: $b(x)$ are positive functions and $g_1(x), g_2(x)$ are sign-changing functions.

In [20], the authors studied a class of systems of equations where they showed the existence and multiplicity of solutions for a mixed local-nonlocal system with logarithmic nonlinearities

$$\begin{cases} (-\Delta)^s u_j + u_j = \lambda_j a_j(x) u_j \ln|u_j| + \sum_{i \neq j} \beta_{ij} |u_j|^{q-2} u_j & \text{in } \Omega, \\ \quad u = v = 0, \quad \text{in } \mathbb{R}^N \setminus \Omega \end{cases} \tag{1.3}$$

where $\Omega \subset \mathbb{R}^N$ is a bounded domain with Lipschitz boundary, $s \in (0,1)$, $N > 4$, λ_j are parameters, $\beta_{ij} > 0$ for all $1 \le i < j \le k$, $\beta_{ij} = \beta_{ji}$ for all $i, j = 1, \ldots, k$, $a_j \in C(\overline{\Omega})$. When

$1 < 2q < 2 < 2^* = \frac{2N}{N-2}$ and a_j they are functions that change sign, they obtained two different solutions using Nehari Method. When $2 < q < 2 < 2^* = \frac{2N}{N-2}$ and a_j are positive constant functions, the existence of the ground state solution is obtained using the minimization method.

In [13], the authors studied a class of fractional Laplacian systems where they showed the existence of solutions using the Nehari method and multiplicity of solutions using the Lusternik–Schnirelmann category, with polynomial nonlinearity.

$$\begin{cases} (-\Delta)^s u = \lambda|u|^{p-2}u + \dfrac{2\alpha}{\alpha+\beta}|v|^{\beta}|u|^{\alpha-2}u & \text{in } \Omega, \\ (-\Delta)^s v = \mu|u|^{p-2}u + \dfrac{2\beta}{\alpha+\beta}|u|^{\alpha}|v|^{\beta-2}v & \text{in } \Omega, \\ \quad u = v = 0, \quad \text{in } \mathbb{R}^N \setminus \Omega \end{cases} \tag{1.4}$$

where $\Omega \subset \mathbb{R}^N$ is a bounded domain, $s \in (0,1)$, $N > 4s$, $\lambda, \mu > 0$ are parameters, $\alpha + \beta = \frac{2N}{N-2s}$ is the critical Sobolev exponent.

In [5], the authors showed the existence of solutions for the fractional critical system (p,q) Laplacian using variational method

$$\begin{cases} (-\Delta)^{s_1}_p u + (-\Delta)^{s_2}_q u = \lambda Q(x)|u|^{r-2}u + \dfrac{2\alpha}{\alpha+\beta}|v|^{\beta}|u|^{\alpha-2}u & \text{in } \Omega, \\ (-\Delta)^{s_1}_p v + (-\Delta)^{s_2}_q v = \mu Q(x)|u|^{r-2}u + \dfrac{2\beta}{\alpha+\beta}|u|^{\alpha}|v|^{\beta-2}v & \text{in } \Omega, \\ \quad u = v = 0, \quad \text{in } \mathbb{R}^N \setminus \Omega \end{cases} \tag{1.5}$$

where $\Omega \subset \mathbb{R}^N$ is a bounded domain, $0 < s_2 < s_1 < 1$, $1 < q < p < r < p^*_{s_1}$, $N > ps_1$, $\lambda, \mu > 0$ are parameters and $\alpha > 1$ $\beta > 1$ satisfy $\alpha + \beta = p^*_{s_1}$ with $\frac{Np}{N-ps_1}$ is the critical Sobolev exponent, and $(-\Delta)^s_t$ is the fractional t-Laplacian operator.

In [22] was concerned with the existence and asymptotic behavior of normalized ground states solutions for the following coupled Schrödinger system with logarithmic terms:

$$\begin{cases} -\Delta u_1 + \omega_1 u_1 = \mu_1 u_1 \log u_1^2 + \dfrac{p}{p+q}|u_2|^q|u_1|^{p-2}u_1, & \text{in } \Omega, \\ -\Delta u_2 + \omega_2 u_2 = \mu_2 u_2 \log u_2^2 + \dfrac{q}{p+q}|u_1|^p|u_2|^{q-2}u_2, & \text{in } \Omega, \\ \int_\Omega |u_i|^2 dx = \rho_i, \qquad i = 1,2, \end{cases} \tag{1.6}$$

where $\Omega = \mathbb{R}^N$ or $\Omega \subset \mathbb{R}^N (N \geq 3)$ is a bounded smooth domain, $\omega_i \in \mathbb{R}, \mu_i, \rho_i > 0, i = 1,2$. Moreover, $p,q \geq 1, 2 \leq p+q \leq 2^*$ where $2^* := \frac{2N}{N-2}$.

In [11] the authors demonstrate existence of solutions for fractional (p,q) Laplacian systems

$$\begin{cases} (-\Delta)^{s_1}_p u + (-\Delta)^{s_2}_q u = \lambda|u|^{r-2} + \dfrac{\alpha\eta}{\eta+\nu}|u|^{\eta-2}|v|^{\nu}u & \text{in } \Omega, \\ (-\Delta)^{s_1}_p v + (-\Delta)^{s_2}_q v = \mu|v|^{r-2} + \dfrac{\alpha\nu}{\nu+\eta}|v|^{\nu-2}|u|^{\eta}v & \text{in } \Omega, \\ u = v = 0, \quad x \in \mathbb{R}^N \setminus \Omega, \end{cases} \tag{1.7}$$

where $\Omega \subset \mathbb{R}^N$ is a bounded smooth domain, $0 < s_2 < s_1 < 1 < q < r < p < p^*_{s_1} = \frac{pN}{N-ps_1}$, $\lambda, \mu > 0$ are two parameters, $r > 1, q > 1$ satisfies $q + r = p^*_{s_1}$.

In [21] the authors studied the existence of least energy solutions to the following fractional Kirchhoff problem with logarithmic nonlinearity

$$\begin{cases} M([u]_{s,t}^{p})(-\Delta)_p^s u = h(x)|u|^{\theta p-2}u\ln|u| + \lambda|u|^{q-2}u, & x\in\Omega \\ u=0, & x\in\mathbb{R}^N\setminus\Omega, \end{cases} \tag{1.8}$$

where $s\in(0,1)$, $1<p<Np$, $\Omega\subset\mathbb{R}^N$ is a bounded domain with Lipschitz boundary, $M([u]_{s,p}^p)=[u]_{s,p}^{(\theta-1)p}$, with $\theta\geq 1$ and $u,h\in C(\overline{\Omega})$ may change sign, $\lambda>0$ and $q\in(1,p_{s_1}^*)$.

On the other hand, parabolic and hyperbolic type equations with logarithmic nonlinearity have been studied extensively in recent years. Here we only refer some results involving fractional Laplacian. [2] considered the Cauchy problem of the following Schrödinger equation

$$iu_t-(-\Delta)^s u+u\log|u|^2=0, \qquad x\in\Omega,\ t>0. \tag{1.9}$$

The existence of global solutions was obtained by using a compactness method. Moreover, the author obtained the existence of ground states by the Nehari manifold approach.

The main tool used in this paper is the so-called fibering method introduced by Pohozaev [15], [16] and [17], and applied to a single equation of p-Laplacian type by Drábek and Pohozaev in [12]. Bozhkov and Mitidieri [3] used this method to study the existence for multiple solution for a quasilinear system. Fibering method is very used to proof existence of multiple solution for a large class of equations. For example, Brown and Wu [4] used this method to show the existence of at least two positive solutions for the semilinear elliptic boundary-value problem. For more recent applying of fibering method we indicate [1], used to show existence of multiple solution for a class of Schrödinger equations involving indefinite weight functions, and other interesting work involving the fractional operator is [7].

Inspired by references [9, 10, 13, 19, 20, 22], in this article, we investigate the existence and multiplicity of ground state solutions for problem (P) in both subcritical and critical cases. One goal of this present work is to establish the existence of a nontrivial ground state solution using variational methods. We remark that there are several types of difficulty in deriving the existence results:

(1) Furthermore, we need to deal with essential difficulties due to the fact that the logarithmic term $f(u)=|u|^{\theta-2}u\ln|u|$ neither satisfies the monotonicity condition nor the Ambrosetti–Rabinowitz (A-R) condition, we manage to overcome these difficulties by combining the Brezis–Lieb lemma with some delicate estimates of this logarithmic term.

(2) Since the weights $H_1,H_2\in C(\bar{\Omega})$ change sign, the study of this type of problem becomes difficult, to overcome this difficulty we use the fibers in the Nehari manifold.

(3) The operator $(-\Delta)_p^{s_1}+(-\Delta)_t^{s_2}$ does not have homogeneity properties, which makes it difficult to solve.

(4) This system is a broader class of the system studied in [9].

The main results of this paper is writing as follows:

Theorem 1.1. *Problem (P) has a nontrivial ground state solution in W for $0<s_2<s_1<1<t<p<\theta<p_{s_1}^*$ and H_1,H_2 being positive functions in $C(\overline{\Omega})$.*

Theorem 1.2. *Problem (P) has a nontrivial ground state solution in W for $0 < s_2 < s_1 < 1 < t < p < \theta < p^*_{s_1}$ and H_1, H_2 two sign-changing functions in $C(\overline{\Omega})$.*

The paper is organized as follows. In Section 2 we study the variational framework. In Section 3 we study Nehari Manifold and fibering map analysis. In Section 4 we prove Theorem 1.1. In Section 5 we prove Theorem 1.2.

2 The variational framework

This paper starts with an introduction to the fractional Sobolev space, where the solutions for problem (P) lies. After this introduction, some technical results that will be use to proof the next results that will be shown in the others chapters.

For this, let Ω be a bounded domain in $\mathbb{R}^N$ with smooth boundary, for $p \in (1, +\infty)$, $s_1 \in (0,1)$ and $N > s_1 p$, we define the fractional Sobolev space.

$$W^{s_1,p}(\Omega) := \left\{ u \in L^p(\Omega) : \frac{|u(x)-u(y)|^p}{|x-y|^{N+ps_1}} \in L^p(\Omega) \times L^p(\Omega) \right\}. \tag{2.1}$$

The space $W^{s_1,p}(\Omega)$ is an intermediate Banach space between $L^p(\Omega)$ and $W^{1,p}(\Omega)$, endowed with the natural norm

$$\|u\|_{W^{s_1,p}} = \left(\int_\Omega |u|^p dx \right)^{1/p} + [u]^p_{s_1},$$

where

$$[u]^p_{s_1,p} = \int_{\Omega\times\Omega} \frac{|u(x)-u(y)|^p}{|x-y|^{N+ps_1}} dxdy$$

is the so-called Gagliardo seminorm of u.

We set $\Omega^c = \mathbb{R}^N \setminus \Omega$ and $\mathcal{Q}^c = \mathbb{R}^N \setminus \Omega^c \times \Omega^c$. We define the following set

$$X^{s_1,p}(\Omega) := \left\{ u : \mathbb{R}^N \to \mathbb{R} \text{ is measurable, } u|_\Omega \in L^p(\Omega), \text{ and } \int_{\mathcal{Q}} \frac{|u(x)-u(y)|^p}{|x-y|^{N+s_1 p}} dxdy < +\infty \right\}.$$

The space $X^{s_1,p}(\Omega)$ is endowed with the norm defined by

$$\|u\|_{X^{s_1,p}(\Omega)} = \left(\int_\Omega |u|^p dx \right)^{1/p} + \left(\int_{\mathcal{Q}} \frac{|u(x)-u(y)|^p}{|x-y|^{N+ps_1}} dxdy \right)^{1/p}.$$

Also, we define the space

$$X_0^{s_1,p}(\Omega) := \{ u \in X^{s_1,p}(\Omega) : \; u = 0 \; a.e \text{ in } \Omega^c \},$$

or equivalently as $\overline{C_0^\infty(\Omega)}^{X^{s_1,p}(\Omega)}$. It is well-known that for $p > 1$, $X_0^{s_1,p}(\Omega)$ is a uniformly convex Banach space endowed with the norm

$$\|u\|_{s_1,p} = \left(\int_{\mathcal{Q}} \frac{|u(x)-u(y)|^p}{|x-y|^{N+ps_1}} dxdy \right)^{1/p}.$$

Since $u = 0$ in $\mathbb{R}^N \setminus \{0\}$, the above integral can be extended to all of $\mathbb{R}^N$.

The embedding $X_0^{s_1,p}(\Omega) \hookrightarrow L^\kappa(\Omega)$ is continuous for any $\kappa \in [0, p^*_{s_1}]$ and compact for $\kappa \in [1, p^*_{s_1})$.

Lemma 2.1 ([8]). *Let $1 \le t \le p$, $0 < s_2 < s_1 < 1$. Then, $W^{s_1,p}(\Omega) \subset W^{s_2,t}(\Omega)$.*

Lemma 2.2 ([8]). *Let $1 \le t \le p$, $0 < s_2 < s_1 < 1$ and and Ω be a smooth bounded domain in $\mathbb{R}^N$ where $N > s_1 p$. Then, $X_0^{s_1,p}(\Omega) \subset X_0^{s_2,t}(\Omega)$ and there exists $C = C(\Omega, t, p, s_1, s_2)$ such that*

$$\|u\|_{s_2,t} \le C\|u\|_{s_1,p}, \qquad \forall u \in X_0^{s_1,p}(\Omega). \tag{2.2}$$

Let $W_p := X_0^{s_1,p}(\Omega) \times X_0^{s_1,p}(\Omega)$ and $W_t := X_0^{s_2,t}(\Omega) \times X_0^{s_2,t}(\Omega)$ be reflexive Banach spaces equipped, respectively, with the norms

$$\|(u,v)\|_{s_1,p} = \left(\|u\|_{s_1,p}^p + \|v\|_{s_1,p}^p\right)^{1/p}, \qquad \|(u,v)\|_{s_2,t} = \left(\|u\|_{s_2,t}^t + \|v\|_{s_2,t}^t\right)^{1/t}.$$

In this paper, we work over the space $W := W_p \bigcap W_t$ endowed the norm $\|(u,v)\| := \|(u,v)\|_{s_1,p} + \|(u,v)\|_{s_2,t}$.

The following lemma is very important.

Lemma 2.3 ([8, Theorem 6.7]). *W is continuously embedding on $L^{s_1,p}(\Omega) \times L^{s_2,t}(\Omega)$.*

Lemma 2.4. *Let $S_{s_1,p}$ the best fractional critical Sobolev constant of embedding $X_0^{s_1,p}(\Omega)$ to $L^{p^*_{s_1}}(\Omega)$*

$$S_{s_1,p} := \inf_{u \in X_0^{s,p}(\Omega)\setminus\{0\}} \left\{ \frac{\|u\|_{s_1,p}^p}{\left(\int_\Omega |u|^{p_s^*} dx\right)^{p/p_s^*}} \right\}. \tag{2.3}$$

and the definition of $S_{q,r}$ is as follows:

$$S_{q,r} := \inf_{(u,v) \in W\setminus\{0\}} \left\{ \frac{\|(u,v)\|_{s_1,p}^p}{\left(\int_\Omega |u|^q |v|^r dx\right)^{p/p^*_{s_1}}} \right\}, \tag{2.4}$$

Then,

$$S_{q,r} := \left[\left(\frac{q}{r}\right)^{r/(q+r)} + \left(\frac{r}{q}\right)^{q/(q+r)} \right] S_{s_1,p}.$$

Lemma 2.5. *Let $\xi, \eta \in \mathbb{R}^N$. Then,*

$$|\xi - \eta|^p \le \begin{cases} 2^p(|\xi|^{p-2}\xi + |\eta|^{p-2}\eta)(\xi - \eta), & \text{for } p \ge 2, \\ \dfrac{1}{(p-1)} \dfrac{\left[(|\xi|^{p-2}\xi + |\eta|^{p-2}\eta)(\xi - \eta)\right]^{p/2}}{(|\xi|^p + |\eta|^p)^{(p-2)/p}} & \text{for } 1 < p < 2. \end{cases} \tag{2.5}$$

The proof of this lemma can be found in [18, Lemma 6].

Now we will use some technical results that involve the logarithmic term. This result represents an important part of the continuation of this paper

Lemma 2.6 ([10, Lemma 2.6]). *Let ρ be a positive real number. Then we have the following inequalities*

$$t^p \ln t \le \frac{t^{p+\rho}}{e\rho} \quad \text{for all } 1 \le t \tag{2.6}$$

and

$$|t^p \ln t| \le \frac{1}{e\rho} \quad \text{for all } 0 < t < 1, \tag{2.7}$$

where e is Euler logarithm basis.

Proceeding with the introduction, we will now make some definitions about the variational framework.

Definition 2.7. We say that $(u,v) \in W$ is a weak solution of problem (P) if

$$\begin{aligned}
&\int_{\Omega\times\Omega} \frac{|u(x)-u(y)|^{p-2}(u(x)-u(y))(\varphi(x)-\varphi(y))}{|x-y|^{N+ps_1}}dxdy \\
&\quad + \int_{\Omega\times\Omega} \frac{|v(x)-v(y)|^{p-2}(v(x)-v(y))(\varphi(x)-\varphi(y))}{|x-y|^{N+ps_1}}dxdy \\
&\quad + \int_{\Omega\times\Omega} \frac{|u(x)-u(y)|^{p-2}(u(x)-u(y))(\phi(x)-\phi(y))}{|x-y|^{N+ts_2}}dxdy \\
&\quad + \int_{\Omega\times\Omega} \frac{|v(x)-v(y)|^{p-2}(v(x)-v(y))(\phi(x)-\phi(y))}{|x-y|^{N+ts_2}}dxdy \\
&= \lambda \int_\Omega H_1(x)|u|^{\theta-2}u\ln|u|\varphi dx + \mu\int_\Omega H_2(x)|v|^{\theta-2}v\ln|v|\phi dx \\
&\quad + \frac{q}{q+r}\int_\Omega R_1(x)|v|^r|u|^{q-2}u\varphi dx + \frac{r}{q+r}\int_\Omega R_2(x)|u|^q|v|^{r-2}v\phi dx,
\end{aligned}$$

for any $(\varphi,\phi) \in W$.

Because of the above definition, we consider the energy functional for problem (P), $E : W \to \mathbb{R}$ defined by

$$\begin{aligned}
E(u,v) = &\frac{1}{p}\|(u,v)\|^p_{s_1,p} + \frac{1}{t}\|(u,v)\|^t_{s_2,t} - \frac{\lambda}{\theta}\int_\Omega H_1(x)|u|^\theta \ln|u|dx + \frac{\lambda}{\theta^2}\int_\Omega H_1(x)|u|^\theta dx \\
&- \frac{\mu}{\theta}\int_\Omega H_2(x)|v|^\theta\ln|v|dx + \frac{\mu}{\theta^2}\int_\Omega H_2(x)|v|^\theta dx - \frac{2}{q+r}\int_\Omega (R_1+R_2)(x)|u|^q|v|^r dx.
\end{aligned}$$

We also consider the auxiliary functional

$$\begin{aligned}
I(u,v) = &\|(u,v)\|^p_{s_1,p} + \|(u,v)\|^t_{s_2,t} - \lambda\int_\Omega H_1(x)|u|^\theta\ln|u|dx \\
&- \mu\int_\Omega H_2(x)|v|^\theta\ln|v|dx - \int_\Omega (R_1+R_2)(x)|u|^q|v|^r dx.
\end{aligned}$$

As a consequence of embedding on Remark 2.3 and Lemma 2.4, we obtain that the functional I is well-defined and $E, I \in C^1(W,\mathbb{R})$. Furthermore, we have that the derivatives I' and E' in the Fréchet sense are given as

$$\begin{aligned}
E'(u,v)(\varphi,\phi) = &\int_{\Omega\times\Omega} \frac{|u(x)-u(y)|^{p-2}(u(x)-u(y))(\varphi(x)-\varphi(y))}{|x-y|^{N+ps_1}}dxdy \\
&+ \int_{\Omega\times\Omega} \frac{|v(x)-v(y)|^{p-2}(v(x)-v(y))(\varphi(x)-\varphi(y))}{|x-y|^{N+ps_1}}dxdy \\
&+ \int_{\Omega\times\Omega} \frac{|u(x)-u(y)|^{p-2}(u(x)-u(y))(\phi(x)-\phi(y))}{|x-y|^{N+ts_2}}dxdy \\
&+ \int_{\Omega\times\Omega} \frac{|v(x)-v(y)|^{p-2}(v(x)-v(y))(\phi(x)-\phi(y))}{|x-y|^{N+ts_2}}dxdy \\
&- \lambda\int_\Omega H_1(x)|u|^{\theta-2}u\ln|u|\varphi dx - \mu\int_\Omega H_2(x)|v|^{\theta-2}v\ln|v|\phi dx \\
&- \frac{q}{q+r}\int_\Omega R_1(x)|v|^r|u|^{q-2}u\varphi dx - \frac{r}{q+r}\int_\Omega R_2(x)|u|^q|v|^{r-2}v\phi dx
\end{aligned}$$

and

$$\begin{aligned}
I'(u,v)(\varphi,\phi) = p\int_{\Omega\times\Omega} &\frac{|u(x)-u(y)|^{p-2}(u(x)-u(y))(\varphi(x)-\varphi(y))}{|x-y|^{N+ps_1}}dxdy \\
&+ p\int_{\Omega\times\Omega} \frac{|v(x)-v(y)|^{p-2}(v(x)-v(y))(\varphi(x)-\varphi(y))}{|x-y|^{N+ps_1}}dxdy \\
&+ t\int_{\Omega\times\Omega} \frac{|u(x)-u(y)|^{p-2}(u(x)-u(y))(\phi(x)-\phi(y))}{|x-y|^{N+t_2}}dxdy \\
&+ t\int_{\Omega\times\Omega} \frac{|v(x)-v(y)|^{p-2}(v(x)-v(y))(\phi(x)-\phi(y))}{|x-y|^{N+ts_2}}dxdy \\
&- \lambda\theta\int_\Omega H_1(x)|u|^{\theta-2}u\varphi\ln|u|dx - \lambda\int_\Omega H_1(x)|u|^{\theta-2}u\varphi dx \\
&- \mu\theta\int_\Omega H_2(x)|v|^{\theta-2}v\phi\ln|v|dx - \mu\int_\Omega H_2(x)|v|^{\theta-2}v\phi dx \\
&- (q+r)\Big[\int_\Omega R_1(x)|v|^r|u|^{q-2}u\varphi dx + \int_\Omega R_2(x)|u|^q|v|^{r-2}v\phi\Big]dx
\end{aligned}$$

for all $(\varphi,\phi)\in W$.

We now prove the Lebesgue dominated convergence theorem for the logarithm function.

Lemma 2.8 ([9, Lemma 2.6]). *Assume that (u_n,v_n) is bounded in W such that u_n converges to u a.e. and v_n converges to v in Ω . Then*

$$\lim_{n\to\infty}\int_\Omega H_1(x)|u_n|^\theta\ln|u_n|dx = \int_\Omega H_1(x)|u|^\theta\ln|u|\,dx, \quad \text{for } \theta\in(2,p^*_{s_1}),$$

and

$$\lim_{n\to\infty}\int_\Omega H_2(x)|v_n|^\theta\ln|v_n|dx = \int_\Omega H_2(x)|v|^\theta\ln|v|\,dx, \quad \text{for } \theta\in(2,p^*_{s_1}).$$

Lemma 2.9 ([6, Lemma 3.2]). *Let the sequences (u_n) and (v_n) be in $W_0^{s_1,p}(\Omega)$ such that $u_n\rightharpoonup u$, $v_n\rightharpoonup v$ in $W_0^{s_1,p}(\Omega)$ and $u_n(x)\to u(x)$, $v_n(x)\to v(x)$ a.e. in Ω. Then,*

$$\lim_{n\to\infty}\int_\Omega (|u_n|^q|v_n|^r - |u_n-u|^q|v_n-v|^r)\,dx = \int_\Omega |u|^q|v|^r\,dx.$$

Lemma 2.10. *Let $(u,v)\in W\setminus\{(0,0)\}$. Then*

$$\begin{aligned}
\int_\Omega &(\lambda H_1(x)|u|^\theta\ln|u| + \mu H_2(x)|v|^\theta\ln|v|)\,dx \\
&\le (\lambda C_{H_1}+\mu C_{H_2})L\left(\|(u,v)\|^p_{s_1,p} + \|(u,v)\|^t_{s_2,t}\right) \\
&\quad + \ln\|(u,v)\|\int_\Omega(\lambda H_1(x)|u|^\theta + \mu H_2(x)|v|^\theta)\,dx
\end{aligned}$$

where

$$L := \frac{|\Omega|}{e\theta} + \frac{1}{e(p^*_{s_1}-\theta)}S_{p^*_{s_1}}^{p_{p^*_{s_1}}}, \qquad C_{H_1} := \max_{x\in\overline{\Omega}}|H_1(x)|, \qquad C_{H_2} := \max_{x\in\overline{\Omega}}|H_2(x)|$$

and $S>0$ denote the best constants of embeddings from $W_0^{s_1,p}(\Omega)$ to $L^{p^*_{s_1}}(\Omega)$.

Proof. Let us consider $\Omega = \Omega_1 \cup \Omega_2$, using integration properties over Ω, where $\Omega_1 = \{x \in \Omega : |u(x)| \leq \|(u,v)\|\}$ and $\Omega_2 = \{x \in \Omega : |u(x)| > \|(u,v)\|\}$. Then

$$\int_\Omega H_1(x)|u|^\theta \ln \frac{|u(x)|}{\|(u,v)\|}dx = \int_{\Omega_1} H_1(x)|u|^\theta \ln \frac{|u(x)|}{\|(u,v)\|}dx + \int_{\Omega_2} H_1(x)|u|^\theta \ln \frac{|u(x)|}{\|(u,v)\|}dx.$$

Using (2.7) by direct calculation gives that

$$\int_{\Omega_1} H_1(x)|u|^\theta \ln \frac{|u(x)|}{\|(u,v)\|}dx \leq C_{H_1}\|(u,v)\|^\theta \int_\Omega \frac{1}{e\theta}dx = C_{H_1}\|(u,v)\|^\theta |\Omega| \frac{1}{e\theta}.$$

By Lemma 2.6 for $\rho = p^*_{s_1} - \theta > 0$ and direct calculation

$$\begin{aligned}\int_{\Omega_2} H_1(x)|u|^\theta \ln \frac{|u(x)|}{\|(u,v)\|}dx &\leq \frac{C_{H_1}}{e(p^*_{s_1}-\theta)} \int_\Omega |u|^\theta \left[\frac{|u(x)|}{\|(u,v)\|}\right]^{p^*_{s_1}-\theta} dx \\ &\leq \frac{C_{H_1}}{e(p^*_{s_1}-\theta)} \frac{1}{\|(u,v)\|^{p^*_{s_1}-\theta}} \int_\Omega |u|^{p^*_{s_1}}dx \\ &\leq \frac{C_{H_1}}{e(p^*_{s_1}-\theta)} \frac{1}{\|(u,v)\|^{p^*_{s_1}-\theta}} S^{p^*_{s_1}}_{p^*_{s_1}} \|u\|^{p^*_{s_1}}_{p^*_{s_1}} \\ &\leq \frac{C_{H_1}}{e(p^*_{s_1}-\theta)} S^{p^*_{s_1}}_{p^*_{s_1}} \|(u,v)\|^\theta.\end{aligned}$$

Consequently, we get

$$\int_\Omega |u|^\theta \ln \left[\frac{|u(x)|}{\|(u,v)\|}\right]dx \leq C_{H_1}\left[\frac{|\Omega|}{e\theta} + \frac{S^{p^*_{s_1}}_{p^*_{s_1}}}{e(p^*_{s_1}-\theta)}\right]\|(u,v)\|^\theta.$$

On the other hand,

$$\begin{aligned}&\lambda \int_\Omega H_1(x)|u|^\theta \ln |u| dx \\ &\quad = \lambda \int_\Omega H_1(x)|u|^\theta \ln \left[\frac{|u(x)|}{\|(u,v)\|}\right]dx + \ln(\|(u,v)\|) \int_\Omega \lambda H_1(x)|u|^\theta dx \\ &\quad \leq \lambda C_{H_1}\left[\frac{|\Omega|}{e\theta} + \frac{S^{p^*_{s_1}}}{e(p^*_{s_1}-\theta)}\right]\|(u,v)\|^\theta + C_{H_1}\ln(\|(u,v)\|)\int_\Omega \lambda|u|^\theta dx. \end{aligned} \tag{2.8}$$

Similarly,

$$\mu \int_\Omega H_2(x)|v|^\theta \ln |v| dx \leq \mu C_{H_2}\left[\frac{|\Omega|}{e\theta} + \frac{S^{p^*_{s_1}}}{e(p^*_t-\theta)}\right]\|(u,v)\|^\theta + C_{H_2}\ln(\|(u,v)\|)\int_\Omega \mu|v|^\theta dx. \tag{2.9}$$

Combining (2.8) and (2.9) follows the result. □

3 Nehari manifold and fibering map analysis

Continuing with the paper, for The main tool used in this paper is the so-called fibering method, introduced by Pohozaev [15], [16] and [17]. In this section, we assume $\lambda, \mu > 0$, and the functions $H_1, H_2, R_1, R_2 \in C(\overline{\Omega})$.

We define the Nehari Manifolds as:

$$\mathcal{N} := \{(u,v) \in W \setminus \{(0,0)\} \mid I(u,v) = 0\}.$$

For all $(u,v) \in \mathcal{N}$, we have $(u,v) \neq 0$ and

$$\|(u,v)\|_{s_1,p}^p + \|(u,v)\|_{s_2,t}^t = \lambda \int_\Omega H_1(x)|u|^\theta \ln|u|dx + \mu \int_\Omega H_2(x)|v|^\theta \ln|v|dx + \int_\Omega (R_1(x) + R_2(x))|u|^q|v|^r dx.$$

Now we define the fibering map $\Phi_{(u,v)} : (0,\infty) \to \mathbb{R}$, given by $\Phi_{(u,v)}(k) := E(k(u,v))$, $k > 0$, it is

$$\begin{aligned}\Phi_{(u,v)}(k) = {} & \frac{k^p}{p}\|(u,v)\|_{s_1,p}^p + \frac{k^t}{t}\|(u,v)\|_{s_2,t}^t - \frac{\lambda k^\theta}{\theta}\ln|k|\int_\Omega H_1(x)|u|^\theta dx - \frac{\lambda k^\theta}{\theta}\int_\Omega H_1(x)|u|^\theta \ln|u|dx \\ & + \frac{\lambda k^\theta}{\theta^2}\int_\Omega H_1(x)|u|^\theta dx - \frac{\mu k^\theta}{\theta}\ln|k|\int_\Omega H_2(x)|v|^\theta dx - \frac{\mu k^\theta}{\theta}\int_\Omega H_2(x)|v|^\theta \ln|v|dx \\ & + \frac{\mu k^\theta}{\theta^2}\int_\Omega H_2(x)|v|^\theta dx - \frac{k^{p^*_{s_1}}}{p^*_{s_1}}\int_\Omega (R_1(x) + R_2(x))|u|^q|v|^r dx. \end{aligned} \tag{3.1}$$

Differentiating (3.1) with respect to t, we have

$$\begin{aligned}\Phi'_{(u,v)}(k) = {} & k^{p-1}\|(u,v)\|_{s_1,p}^p + k^{t-1}\|(u,v)\|_{s_2,t}^t - \lambda k^{\theta-1}\int_\Omega H_1(x)|u|^\theta \ln|ku|dx \\ & - \mu k^{\theta-1}\int_\Omega H_2(x)|v|^\theta \ln|kv|dx - k^{p^*_{s_1}-1}\int_\Omega (R_1(x) + R_2(x))|u|^q|v|^r dx. \end{aligned} \tag{3.2}$$

Again, differentiating (3.2) with respect to t, we get

$$\begin{aligned}\Phi''_{(u,v)}(k) = {} & (p-1)k^{p-2}\|(u,v)\|_{s_1,p}^p + (t-1)k^{t-2}\|(u,v)\|_{s_2,t}^t - (\theta-1)k^{\theta-2}\lambda\int_\Omega H_1(x)|u|^\theta \ln|ku|dx \\ & - (\theta-1)k^{\theta-2}\mu\int_\Omega H_2(x)|v|^\theta \ln|kv|dx - \lambda k^{\theta-2}\int_\Omega H_1(x)|u|^\theta dx \\ & - \mu k^{\theta-2}\int_\Omega H_2(x)|v|^\theta dx - (p^*_{s_1}-1)k^{p^*_{s_1}-2}\int_\Omega (R_1(x) + R_2(x))|u|^q|v|^r dx. \end{aligned}$$

Thus, one can easily verify that for all $(u,v) \in \mathcal{N}$, $\Phi'_{(u,v)}(1) = 0$ and

$$\begin{aligned}\Phi''_{(u,v)}(1) = {} & (p-1)\|(u,v)\|_{s_1,p}^p + (t-1)\|(u,v)\|_{s_2,t}^t - \lambda(\theta-1)\int_\Omega H_1(x)|u|^\theta \ln|u|dx \\ & - \mu(\theta-1)\int_\Omega H_2(x)|v|^\theta \ln|v|dx - \lambda\int_\Omega H_1(x)|u|^\theta dx - \mu\int_\Omega H_2(x)|v|^\theta dx \\ & - (p^*_{s_1}-1)\int_\Omega (R_1(x) + R_2(x))|u|^q|v|^r dx, \end{aligned}$$

which implies that

$$\begin{aligned}\Phi''_{(u,v)}(1) = {} & (p-p^*_{s_1})\|(u,v)\|_{s_1,p}^p + (t-p^*_{s_1})\|(u,v)\|_{s_2,t}^t - (\theta-p^*_{s_1})\lambda\int_\Omega H_1(x)|u|^\theta \ln|u|dx \\ & - (\theta-p^*_{s_1})\mu\int_\Omega H_2(x)|v|^\theta \ln|v|dx - \lambda\int_\Omega H_1(x)|u|^\theta dx - \mu\int_\Omega H_2(x)|v|^\theta dx. \end{aligned}$$

As a consequence of the previously calculus, it's make possible to rewrite the Nehari manifold $\mathcal{N}$ as

$$\mathcal{N} = \mathcal{N}^+ \cup \mathcal{N}^0 \cup \mathcal{N}^-, \tag{3.3}$$

where

$$\mathcal{N}^+ = \Big\{(u,v)\in\mathcal{N} \ \mid \ \Phi''_{(u,v)}(1)>0\Big\} = \Big\{(tu,tv)\in W\setminus(0,0) \ \mid \ \Phi'_{(u,v)}(t)=0, \ \Phi''_{(u,v)}(t)>0\Big\};$$

$$\mathcal{N}^0 = \Big\{(u,v)\in\mathcal{N} \ \mid \ \Phi''_{(u,v)}(1)=0\Big\} = \Big\{(tu,tv)\in W\setminus(0,0) \ \mid \ \Phi'_{(u,v)}(t)=0, \ \Phi''_{(u,v)}(t)=0\Big\};$$

$$\mathcal{N}^- = \Big\{(u,v)\in\mathcal{N} \ \mid \ \Phi''_{(u,v)}(1)<0\Big\} = \Big\{(tu,tv)\in W\setminus(0,0) \ \mid \ \Phi'_{(u,v)}(t)=0, \ \Phi''_{(u,v)}(t)<0\Big\}.$$

Lemma 3.1. *Assume $(u,v)\in W\setminus\{(0,0)\}$ and $k>0$. Then $(ku,kv)\in\mathcal{N}$ if, and only if, $\Phi'_{(u,v)}(k)=0$.*

Proof. If $k(u,v)\in\mathcal{N}$, for $k>0$ we have $I(ku,kv)=0$. Then, we get

$$0 = k^p\|(u,v)\|^p_{s_1,p} + k^t\|(u,v)\|^t_{s_2,t} - \lambda k^\theta\int_\Omega H_1(x)|u|^\theta\ln|ku|dx - \mu k^{p^*_{s_1}}\int_\Omega (R_1+R_2)(x)|u|^q|v|^r dx.$$

Dividing the above equation for $k>0$, we have

$$\begin{aligned}0 &= k^{p-1}\|(u,v)\|^p_{s_1,p} + k^{t-1}\|(u,v)\|^t_{s_2,t} - \lambda k^{\theta-1}\int_\Omega H_1(x)|u|^\theta\ln|ku|dx\\ &\quad - \mu k^{\theta-1}\int_\Omega H_2(x)|u|^\theta\ln|kv|dx - k^{p^*_{s_1}-1}\int_\Omega (R_1+R_2)(x)|u|^q|v|^r dx = \Phi'(k).\end{aligned}$$

This completes the proof. □

Lemma 3.2. *If (u,v) is a local minimizer for the functional E on $\mathcal{N}$, with $(u,v)\notin\mathcal{N}^0$. Then, $E'(u,v)=0$.*

Proof. By the assumption for $u\in\mathcal{N}$, applying Lagrange's multipliers, there exists $\gamma\in\mathbb{R}$ such that

$$E'(u,v)(u,v) = \gamma I'(u,v)(u,v). \tag{3.4}$$

But because of $(u,v)\in\mathcal{N}$, we get

$$\begin{aligned}E'(u,v)(u,v) &= \|(u,v)\|^p_{s_1,p} + \|(u,v)\|^t_{s_2,t} - \lambda\int_\Omega H_1(x)|u|^\theta\ln|u|dx - \mu\int_\Omega H_2(x)|v|^\theta\ln|v|dx\\ &\quad - \int_\Omega (R_1+R_2)(x)|u|^q|v|^r dx = 0.\end{aligned}$$

Thus, $\gamma I'(u,v)(u,v)=0$. Now since $(u,v)\in\mathcal{N}^0$, we get

$$\begin{aligned}I'(u,v)(u,v) &= (p-p^*_{s_1})\|(u,v)\|^p_{s_1,p} + (t-p^*_{s_1})\|(u,v)\|^t_{s_2,t}\\ &\quad - (\theta-p^*_{s_1})\lambda\int_\Omega H_1(x)|u|^\theta\ln|u|dx - (\theta-p^*_{s_1})\mu\int_\Omega H_2(x)|v|^\theta\ln|v|dx\\ &\quad - \lambda\int_\Omega H_1(x)|u|^\theta dx - \mu\int_\Omega H_2(x)|v|^\theta dx = \Phi''_{(u,v)}(1).\end{aligned}$$

Thus, $\gamma\neq 0$, and $E'(u,v)=0$. □

Lemma 3.3. *Let $0<\lambda C_{H_1}+\mu C_{H_2}<\frac{1}{L}$ and $2<q+r<p^*_{s_1}$. If $R_1,R_2\in L^\infty(\Omega)$ are non-negative functions satisfying $C_B<K$, where $C_B:=\max_{x\in\overline{\Omega}}|R_1(x)+R_2(x)|$ then, for any $(u,v)\in W\setminus\{(0,0)\}$, we get*

1) If $\lambda\int_\Omega H_1(x)|u|^\theta + \mu\int_\Omega H_2(x)|v|^\theta dx \geq 0$, then there exists a unique $k_{(u,v)}>0$ such that $\Phi'_{(u,v)}(k_{(u,v)})=0$ and $k_{(u,v)}(u,v)\in\mathcal{N}^-$. Moreover,

$$E(k_{(u,v)}u, k_{(u,v)}v) = \sup_{k>0} E(ku,kv).$$

2) If $\lambda \int_\Omega H_1(x)|u|^\theta + \mu \int_\Omega H_2(x)|v|^\theta dx < 0$, then there exists $k_1, k_2 > 0$ such that $0 < k_1 < k_{\max} < k_2 < \infty$ where $\Phi'_{(u,v)}(k_1) = 0 = \Phi'_{(u,v)}(k_2)$ witch $(k_1 u, k_1 v) \in \mathcal{N}^+$ and $(k_2 u, k_2 v) \in \mathcal{N}^-$. Moreover,

$$E(k_1 u, k_1 v) = \inf_{0<k<k_{\max}} E(ku, kv) \quad and \quad E(k_2 u, k_2 v) = \sup_{k>0} E(ku, kv).$$

Proof. For $k > 0$, we defined a function $f : \mathbb{R} \to W$ given by

$$f(k) = k^{p-p^*_{s_1}} \Big[\|(u,v)\|^p_{s_1,p} + \|(u,v)\|^t_{s_2,t} - \lambda \ln|k| \int_\Omega H_1(x)|u|^\theta dx - \lambda \int_\Omega H_1(x)|u|^\theta \ln|u| dx$$
$$- \mu \ln|k| \int_\Omega H_2(x)|v|^\theta dx - \mu \int_\Omega H_2(x)|v|^\theta \ln|v| dx \Big].$$

We can rewrite $f(k)$, for $k > 0$, as

$$\begin{aligned} f(k) = {} & k^{p-p^*_{s_1}} \left(\|(u,v)\|^p_{s_1,p} + \|(u,v)\|^t_{s_2,t} \right) - k^{p-p^*_{s_1}} \ln|k| \lambda \int_\Omega H_1(x)|u|^\theta dx \\ & - k^{p-p^*_{s_1}} \lambda \int_\Omega H_1(x)|u|^\theta \ln|u| dx - k^{p-p^*_{s_1}} \ln|k| \lambda \int_\Omega H_2(x)|v|^\theta dx \\ & - k^{p-p^*_{s_1}} \lambda \int_\Omega H_2(x)|v|^\theta \ln|v| dx. \end{aligned}$$

A direct computation shows that

$$\begin{aligned} f'(k) = k^{p-p^*_{s_1}-1} \Big[& (p-p^*_{s_1}) \left(\|(u,v)\|^p_{s_1,p} + \|(u,v)\|^t_{s_2,t} \right) - \lambda (p-p^*_{s_1}) \int_\Omega H_1(x)|u|^\theta dx \\ & - \lambda (p-p^*_{s_1}) \int_\Omega H_1(x)|u|^\theta \ln|u| dx \\ & - \mu \int_\Omega H_2(x)|v|^\theta dx - \mu (p-p^*_{s_1}) \int_\Omega H_2(x)|v|^\theta \ln|v| dx \\ & - (p-p^*_{s_1}) \ln|k| \Big(\lambda \int_\Omega H_1(x)|u|^\theta dx + \mu \int_\Omega H_2(x)|v|^\theta dx \Big) \Big] \end{aligned}$$

Now we analyze all the possibilities:

i) If $\lambda \int_\Omega H_1(x)|u|^\theta dx + \mu \int_\Omega H_2(x)|v|^\theta dx > 0$, then $f \in C(0,\infty)$ and because $p^*_{s_1} > p > t$, we have

$$\lim_{k \to 0^+} f(k) = +\infty \quad \text{and} \quad \lim_{k \to \infty} f(k) = 0.$$

Then there exists a unique minimum point $k_{\min} > 0$ such that $f'(k_{\min}) = 0$. Because $k_{\min} > 0$,

$$\begin{aligned} (p-p^*_{s_1}) & \ln|k_{\min}| \Big[\lambda \int_\Omega H_1(x)|u|^\theta dx + \mu \int_\Omega H_2(x)|v|^\theta dx \Big] \\ & = (p-p^*_{s_1}) \left[\|(u,v)\|^p_{s_1,p} + \|(u,v)\|^t_{s_2,t} \right] \\ & \quad - \lambda \int_\Omega H_1(x)|u|^\theta dx - \lambda (p-p^*_{s_1}) \int_\Omega H_1(x)|u|^\theta \ln|u| dx \\ & \quad - \mu \int_\Omega H_2(x)|u|^\theta dx - \mu (p-p^*_{s_1}) \int_\Omega H_2(x)|v|^\theta \ln|v| dx. \end{aligned}$$

This implies that

$$k_{\min} = \exp \left[\frac{\|(u,v)\|^p_{s_1,p} + \|(u,v)\|^t_{s_2,t} - \int_\Omega [\lambda H_1(x)|u|^\theta \ln|u| + \mu H_2(x)|v|^\theta \ln|v|] dx}{\lambda \int_\Omega H_1(x)|u|^\theta dx + \mu \int_\Omega H_2(x)|v|^\theta dx} - \frac{1}{p-p^*_{s_1}} \right]. \tag{3.5}$$

Obviously, f is decreasing on $(0, k_{\min})$ and increasing on $(k_{\min}, \infty)$. Then, because of equation (3.5), the fact that $k_{\min} > 0$ and that

$$\lambda \int_\Omega H_1(x)|u|^\theta dx + \mu \int_\Omega H_2(x)|v|^\theta dx > 0,$$

we get

$$\begin{aligned} f(k_{\min}) = k_{\min}^{p-p^*_{s_1}} \Bigg[& \|(u,v)\|^p_{s_1,p} + \|(u,v)\|^t_{s_2,t} \\ & - \ln|k_{\min}| \left(\lambda \int_\Omega H_1(x)|u|^\theta dx + \mu \int_\Omega H_2(x)|v|^\theta dx \right) \\ & - \left(\lambda \int_\Omega H_1(x)|u|^\theta \ln|u| dx + \mu \int_\Omega H_2(x)|v|^\theta \ln|v| dx \right) \Bigg] \\ = k_{\min}^{p-p^*_{s_1}} & \left[\frac{1}{p - p^*_{s_1}} \left(\lambda \int_\Omega H_1(x)|u|^\theta dx + \mu \int_\Omega H_2(x)|v|^\theta dx \right) \right] < 0. \end{aligned}$$

Since we have

$$\int_\Omega (R_1+R_2)(x)|u|^q|v|^r dx > 0, \quad \text{and} \quad \lambda \int_\Omega H_1(x)|u|^\theta \ln|u| dx + \mu \int_\Omega H_2(x)|v|^\theta \ln|v| dx > 0,$$

and because

$$\begin{aligned} \Phi'_{(u,v)}(k) = k^{p^*_{s_1}+\theta-p-1} \Big[& f(k) - k^{2p-p^*_{s_1}-\theta} \|(u,v)\|^p_{s_1,p} + k^{2t-p^*_{s_1}-\theta} \|(u,v)\|^t_{s_1,t} \\ & - k^{p-p^*_{s_1}} \left(\|(u,v)\|^p_{s_1,p} + \|(u,v)\|^t_{s_2,t} \right) - k^{p-\theta} \int_\Omega (R_1+R_2)(x)|u|^q|v|^r dx \Big], \end{aligned}$$

there exists a unique $k_{(u,v)}$ such that $0 < k_{(u,v)} < k_{\min}$ such that

$$\begin{aligned} f(k_{(u,v)}) = k_{(u,v)}^{p-p^*_{s_1}} \Bigg[& \|(u,v)\|^p_{s_1,p} + \|(u,v)\|^t_{s_2,t} - \lambda \ln|k_{(u,v)}| \int_\Omega H_1(x)|u|^\theta dx \\ & - \lambda \int_\Omega H_1(x)|u|^\theta \ln|u| dx - \mu \ln|k_{(u,v)}| \int_\Omega H_2(x)|v|^\theta dx \\ & - \mu \int_\Omega H_2(x)|v|^\theta \ln|v| dx \Bigg] \end{aligned}$$

and $f'(k_{(u,v)}) < 0$, we get $k_{(u,v)}(u,v) \in \mathcal{N}^-$. Moreover, it follows from $f(k) < f(k_{(u,v)})$, for all $k > k_{(u,v)}$ and $f(k) > f(k_{(u,v)})$, for all $k < k_{(u,v)}$, that

$$E(k_{(u,v)}(u,v)) = \sup_{k>0} E(ku, kv).$$

ii) If $\lambda \int_\Omega H_1(x)|u|^\theta dx + \mu \int_\Omega H_2(x)|v|^\theta dx = 0$, it follow from Lemma 2.10 there exists a unique $k_{(u,v)} > 0$ such that $k_{(u,v)}(u,v) \in \mathcal{N}^-$, and

$$E(k_{(u,v)}) = \sup_{k>0} E(ku, kv).$$

iii) If $\lambda \int_\Omega H_1(x)|u|^\theta dx + \mu \int_\Omega H_2(x)|v|^\theta dx < 0$, then $f \in C(0,\infty)$, $\lim_{k\to 0^+} f(k) = -\infty$ and $\lim_{k\to\infty} f(k) = 0$. Then f has a unique maximum point $k_{\max} > 0$ which is given by

$$k_{\max}=\exp\left[\frac{\|(u,v)\|_{s_1,p}^p+\|(u,v)\|_{s_2,t}^t-\int_\Omega[\lambda H_1(x)|u|^\theta \ln|u|+\mu H_2(x)|v|^\theta \ln|v|]dx}{\lambda \int_\Omega H_1(x)|u|^\theta dx + \mu \int_\Omega H_2(x)|v|^\theta dx}-\frac{1}{p-p_{s_1}^*}\right].$$

Moreover, f is increasing on $(0,k_{\max})$ and decreasing on $(k_{\max},\infty)$. By Lemma 2.10, we get

$$k_{\max} \geq \exp\left[\frac{(1-(\lambda C_{H_1}+\mu C_{H_2})L)\left(\|(u,v)\|_{s_1,p}^p+\|(u,v)\|_{s_2,t}^t\right)}{\lambda \int_\Omega H_1(x)|u|^\theta dx + \mu \int_\Omega H_2(x)|v|^\theta dx} - \ln\|(u,v)\| - \frac{1}{p-p_{s_1}^*}\right].$$

So,

$$k_{\max}^{p-p_{s_1}^*} \geq \exp\left[(p-p_{s_1}^*)\frac{(1-(\lambda C_{H_1}+\mu C_{H_2})L)\left(\|(u,v)\|_{s_1,p}^p+\|(u,v)\|_{s_2,t}^t\right)}{\lambda \int_\Omega H_1(x)|u|^\theta dx + \mu \int_\Omega H_2(x)|v|^\theta dx} - 1\right]\|(u,v)\|^{p_{s_1}^*}.$$

It's known that the following inequality holds:

$$\exp(k-1) \geq k, \qquad \forall\, k \geq 0.$$

Then

$$k_{\max}^{p-p_{s_1}^*} \geq (p-p_{s_1}^*)\frac{1-(\lambda C_{H_1}+\mu C_{H_2})L}{\lambda \int_\Omega H_1(x)|u|^\theta dx + \mu \int_\Omega H_2(x)|v|^\theta dx}\|(u,v)\|^{p_{s_1}^*}.$$

Therefore

$$\begin{aligned} f(k_{\max}) &= k_{\max}^{p-p_{s_1}^*}\frac{\lambda \int_\Omega H_1(x)|u|^\theta dx + \mu \int_\Omega H_2(x)|v|^\theta dx}{p-p_{s_1}^*} \\ &\geq (1-(\lambda C_{H_1}+\mu C_{H_2})L)\|(u,v)\|^{p_{s_1}^*}. \end{aligned}$$

Because $\left(k_{(u,v)}^{2p-p_{s_1}^*-\theta}+k_{(u,v)}^{p-p_{s_1}^*}\right)\left(\|(u,v)\|_{s_1,p}^p+\|(u,v)\|_{s_2,t}^t\right)+k_{(u,v)}^{p-\theta}\int_\Omega(R_1+R_2)(x)|u|^q|v|^r dx > 0$, by Hölder's inequality,

$$\int_\Omega (R_1+R_2)(x)|u|^p|v|^r dx \leq C_B S_{q,r}^{\frac{p}{p_{s_1}^*}}\|(u,v)\|^{p_{s_1}^*},$$

for all $(u,v)\in W$, where $C_B = \max_{x\in\overline{\Omega}}|R_1(x)+R_2(x)|$.

Then

$$(1-(\lambda C_{H_1}+\mu C_{H_2})L)S_{q,r}^{\frac{p}{p_{s_1}^*}} > C_B,$$

it implies that

$$\begin{aligned} f(k_{\max}) &> k_{(u,v)}^{2p-p_{s_1}^*-\theta}\|(u,v)\|_{s_1,p}^p + k_{(u,v)}^{2t-p_{s_1}^*-\theta}\|(u,v)\|_{s_2,t}^t + k_{(u,v)}^{p-\theta}\int_\Omega (R_1+R_2)(x)|u|^q|v|^r dx \\ &> 0 = \lim_{k\to\infty} f(k), \end{aligned}$$

and it shows us that there exists k_1, k_2 for which

$$0 < k_1 < k_{\max} < k_2 < \infty,$$

such that

$$f(k_1) = k_{(u,v)}^{2p-p^*_{s_1}-\theta}\|(u,v)\|^p_{s_1,p} + k_{(u,v)}^{2t-p^*_{s_1}-\theta}\|(u,v)\|^t_{s_2,t} + k_{(u,v)}^{p-\theta}\int_\Omega (R_1+R_2)(x)|u|^q|v|^r dx$$
$$> 0 = \lim_{k\to\infty} f(k) = f(k_2).$$

It shows us that $\Phi'_{(u,v)}(k_1) = 0 = \Phi'_{(u,v)}(k_2)$. Moreover, f is increasing on $(0, k_{\max})$ and decreasing on $(k_{\max}, \infty)$. So

$$k_1(u,v) \in \mathcal{N}^+ \quad \text{and} \quad k_2(u,v) \in \mathcal{N}^-.$$

Furthermore, we have for all $k \in [k_1, k_2]$, that

$$f(k) \geq k_{(u,v)}^{2p-p^*_{s_1}-\theta}\|(u,v)\|^p_{s_1,p} + k_{(u,v)}^{2t-p^*_{s_1}-\theta}\|(u,v)\|^t_{s_2,t} + k_{(u,v)}^{p-\theta}\int_\Omega (R_1+R_2)(x)|u|^q|v|^r dx$$
$$> 0 = \lim_{k\to\infty} f(k),$$

and

$$f(k) < k_{(u,v)}^{2p-p^*_{s_1}-\theta}\|(u,v)\|^p_{s_1,p} + k_{(u,v)}^{2t-p^*_{s_1}-\theta}\|(u,v)\|^t_{s_2,t} + k_{(u,v)}^{p-\theta}\int_\Omega (R_1+R_2)(x)|u|^q|v|^r dx,$$

for all $k \in \mathbb{R}^*_+ \setminus [k_1, k_2]$.

Thus

$$E(k_1(u,v)) = \inf_{0<k\leq k_{\max}} E(k(u,v)) \qquad \text{and,} \qquad E(k_2(u,v)) = \sup_{k>0} E(k(u,v)).$$

The proof is complete. □

The next remark will help in understanding the next results.

Remark 3.4. In what follows, we define

$$M := \frac{|\Omega| S_{q,r}^{\frac{p^*_{s_1}}{p}}}{p^*_{s_1} - p}, \quad \text{and} \quad K := \frac{1 - (\lambda C_{H_1} + \mu C_{H_2})(L+M)}{S_{q,r}^{\frac{p^*_{s_1}}{p}}}.$$

Lemma 3.5. *If $0 < \lambda C_{H_1} + \mu C_{H_2} < \frac{1}{L+M}$ and $C_B < K$, then $\mathcal{N}^0 = \emptyset$.*

Proof. Arguing by contradiction, let $(u,v) \in \mathcal{N}^0$. Then $I(u,v) = 0$, and

$$\left(\|(u,v)\|^p_{s_1,p} + \|(u,v)\|^t_{s_2,t}\right) - \lambda\int_\Omega H_1(x)|u|^\theta dx - \mu\int_\Omega H_2(x)|v|^\theta dx$$
$$= \int_\Omega (R_1+R_2)(x)|u|^q|v|^r dx \quad (3.6)$$

and

$$0 = (p - p^*_{s_1})\|(u,v)\|^p_{s_1,p} + (t - p^*_{s_1})\|(u,v)\|^t_{s_2,t} - (\theta - p^*_{s_1})\lambda\int_\Omega H_1(x)|u|^\theta \ln|u| dx$$
$$- (\theta - p^*_{s_1})\mu\int_\Omega H_2(x)|v|^\theta \ln|v| dx - \lambda\int_\Omega H_1(x)|u|^\theta dx - \mu\int_\Omega H_2(x)|v|^\theta dx. \quad (3.7)$$

From equations (3.6) and (3.7), and Lemma 2.10, we get

$$\begin{aligned}(\|(u,v)\|^{p}_{s_1,p}+\|(u,v)\|^{t}_{s_2,t})-\int_\Omega (R_1+R_2)(x)|u|^\theta dx \\ \leq (\lambda C_{H_1}+\mu C_{H_2})L\left(\|(u,v)\|^{p}_{s_1,p}+\|(u,v)\|^{t}_{s_2,t}\right) \\ +\ln\|(u,v)\|\left(\lambda\int_\Omega H_1(x)|u|^\theta dx+\mu\int_\Omega H_2(x)|v|^\theta dx\right).\end{aligned} \tag{3.8}$$

Then, again by (3.7)

$$\begin{aligned}&(1-(\lambda C_{H_1}+\mu C_{H_2})L)\left(\|(u,v)\|^{p}_{s_1,p}+\|(u,v)\|^{t}_{s_2,t}\right)\\ &\leq \ln\|(u,v)\|\left[\lambda\int_\Omega H_1(x)|u|^\theta dx+\mu\int_\Omega H_2(x)|v|^\theta dx\right]+\int_\Omega (R_1+R_2)(x)|v|^q|v|^r dx\\ &\leq \ln\|(u,v)\|\left[\lambda\int_\Omega H_1(x)|u|^\theta dx+\mu\int_\Omega H_2(x)|v|^\theta dx\right]\\ &\quad+\frac{1}{p-p^*_{s_1}}\left[-\lambda(p-\theta)\int_\Omega H_1(x)|u|^\theta\ln|u|dx\right.\\ &\quad\left.-\mu(p-\theta)\int_\Omega H_2(x)|v|^\theta\ln|v|dx+\lambda\int_\Omega H_1(x)|u|^\theta dx+\mu\int_\Omega H_2(x)|v|^\theta dx\right].\end{aligned}$$

That is,

$$\begin{aligned}&(1-(\lambda C_{H_1}+\mu C_{H_2})L)\left(\|(u,v)\|^{p}_{s_1,p}+\|(u,v)\|^{t}_{s_2,t}\right)\\ &\leq \ln\|(u,v)\|\left(\lambda\int_\Omega H_1(x)|u|^\theta dx+\mu\int_\Omega H_2(x)|v|^\theta dx\right)\\ &\quad+\frac{1}{p-p^*_{s_1}}\left[-\lambda(p-\theta)\int_\Omega H_1(x)|u|^\theta\ln|u|dx-\mu(p-\theta)\int_\Omega H_2(x)|v|^\theta\ln|v|dx\right]\\ &\quad+\frac{\lambda C_{H_1}+\mu C_{H_2}}{p^*_{s_1}-p}\left[|\Omega|^{p^*_s}S_{q,r}^{\frac{p^*_{s_1}}{p}}\right]\left(\|(u,v)\|^{p}_{s_1,p}+\|(u,v)\|^{t}_{s_2,t}\right),\end{aligned} \tag{3.9}$$

which means that

$$\begin{aligned}&\left[1-(\lambda C_{H_1}+\mu C_{H_2})L-\frac{\lambda C_{H_1}+\mu C_{H_2}}{p^*_{s_1}-p}\left[|\Omega|^{p^*_s}S_{q,r}^{\frac{p^*_{s_1}}{p}}\right]\right]\left(\|(u,v)\|^{p}_{s_1,p}+\|(u,v)\|^{t}_{s_2,t}\right)\\ &\leq \ln\|(u,v)\|\left[\lambda\int_\Omega H_1(x)|u|^\theta dx+\mu\int_\Omega H_2(x)|v|^\theta dx\right]\\ &\quad+\frac{1}{p-p^*_{s_1}}\left(-\lambda(p-\theta)\int_\Omega H_1(x)|u|^\theta\ln|u|dx-\mu(p-\theta)\int_\Omega H_2(x)|v|^\theta\ln|v|dx\right).\end{aligned}$$

Using again equation (3.7), in view of $p<p^*_{s_1}$, we have

$$\begin{aligned}0&>\ln\|(u,v)\|\left(\lambda\int_\Omega H_1(x)|u|^\theta dx+\mu\int_\Omega H_2(x)|v|^\theta dx\right)\\ &\quad+\frac{1}{p-p^*_{s_1}}\left(-\lambda(p-\theta)\int_\Omega H_1(x)|u|^\theta\ln|u|dx-\mu(p-\theta)\int_\Omega H_2(x)|v|^\theta\ln|v|dx\right).\end{aligned}$$

Thus, it follows from $\lambda C_{H_1}+\mu C_{H_2}<\frac{1}{L+M}$ that $\|(u,v)\|\leq 1$, where, here we consider M as in Remark 3.4.

Otherwise, using (3.9), and one more time (3.6) and (3.7),

$$\begin{aligned}
&\left[1-(\lambda C_{H_1}+\mu C_{H_2})L\right]\left(\|(u,v)\|_{s_1,p}^{p}+\|(u,v)\|_{s_2,t}^{t}\right)\\
&\quad\le \ln\|(u,v)\|\Big(\lambda\int_\Omega H_1(x)|u|^\theta dx+\mu\int_\Omega H_2(x)|v|^\theta dx\Big)+\int_\Omega (R_1+R_2)(x)|u|^q|v|^r dx\\
&\quad\le (p-p_{s_1}^*)\ln\|(u,v)\|\int_\Omega (R_1+R_2)(x)|u|^q|v|^r dx\\
&\qquad+(\lambda C_{H_1}+\mu C_{H_2})M\left(\|(u,v)\|_{s_1,p}^{p}+\|(u,v)\|_{s_2,t}^{t}\right). \qquad (3.10)
\end{aligned}$$

Now, we can note that

$$\int_\Omega (R_1+R_2)(x)|u|^q|v|^r dx\le C_B S_{q,r}^{\frac{p_{s_1}^*}{p}}\|(u,v)\|^{p_{s_1}^*}, \qquad (3.11)$$

thus

$$[1-(\lambda C_{H_1}+\mu C_{H_2})(L+M)-C_B S_{q,r}^{\frac{p_{s_1}^*}{p}}]\left(\|(u,v)\|_{s_1,p}^{p}+\|(u,v)\|_{s_2,t}^{t}\right)\le 0.$$

Since $(\lambda C_{H_1}+\mu C_{H_2})<\frac{1}{L+M}$, and

$$C_B<K:=\frac{1-(\lambda C_{H_1}+\mu C_{H_2})(L+M)}{S_{q,r}^{\frac{p_{s_1}^*}{p}}},$$

it follows that $\|(u,v)\|=0$. Then $(u,v)=(0,0)$, which is a contradiction. Therefore $\mathcal{N}^0=\emptyset$. □

For the next result, we consider

$$\Lambda_{\lambda,\mu}:=\min\left\{1,\left(\frac{1-(\lambda C_{H_1}+\mu C_{H_2})L-C_B S_{q,r}^{\frac{p_{s_1}^*}{p}}}{C_B S_{q,r}^{\frac{p_{s_1}^*}{p}}}\right)^{\frac{1}{p_{s_1}^*-p}}\right\}.$$

Lemma 3.6. *If $0<\lambda C_{H_1}+\mu C_{H_2}<\frac{1}{L+M}$ and $C_B<K$, then $\|(U,V)\|\ge\Lambda_{\lambda,\mu}$, for all $(U,V)\in\mathcal{N}^-$ and $\|(u,v)\|\le 1$, for all $(u,v)\in\mathcal{N}^+$.*

Proof. Let $(U,V)\in\mathcal{N}^-$. Then

$$\begin{aligned}
&\int_\Omega (R_1+R_2)(x)|U|^q|V|^r dx\\
&\quad=\left(\|(u,v)\|_{s_1,p}^{p}+\|(u,v)\|_{s_2,t}^{t}\right)-\lambda\int_\Omega H_1(x)|U|^\theta\ln|U|dx-\mu\int_\Omega H_2(x)|V|^\theta\ln|V|dx, \qquad (3.12)
\end{aligned}$$

and

$$\begin{aligned}
&(p-p_{s_1}^*)\|(U,V)\|_{s_1,p}^{p}+(t-p_{s_1}^*)\|(U,V)\|_{s_2,t}^{t}\\
&\quad<\lambda\int_\Omega H_1(x)|U|^\theta dx+\mu\int_\Omega H_2(x)|V|^\theta dx\\
&\qquad-(p_{s_1}^*-\theta)\Big(\lambda\int_\Omega H_1(x)|U|^\theta\ln|U|dx+\mu\int_\Omega H_2(x)|V|^\theta\ln|V|dx\Big). \qquad (3.13)
\end{aligned}$$

Similarly to equation (3.10) in Lemma 3.5, we have

$$\begin{aligned}
&[1-(\lambda C_{H_1}+\mu C_{H_2})L]\|(u,v)\left(\|_{s_1,p}^{p}+\|(u,v)\|_{s_2,t}^{t}\right)\\
&\quad\le\ln\|(U,V)\|\left(\lambda\int_\Omega H_1(x)|U|^\theta dx+\mu\int_\Omega H_2(x)|V|^\theta dx\right)+C_B S_{p_{s_1}^*}^{q}S_{p_{s_1}^*}^{r}\|(U,V)\|^{p_{s_1}^*}. \qquad (3.14)
\end{aligned}$$

We consider two cases:

i) If $\lambda \int_\Omega H_1(x)|U|^\theta dx + \mu \int_\Omega H_2(x)|V|^\theta dx > 0$, then $\|(U,V)\| \geq 1$, because otherwise, being $\|(U,V)\| < 1$, we have

$$[1-(\lambda C_{H_1}+\mu C_{H_2})L]\left(\|(u,v)\|_{s_1,p}^p + \|(u,v)\|_{s_2,t}^t\right) < C_B S_{p_{s_1}^*}^q S_{p_{s_1}^*}^r \|(U,V)\|^{p_{s_1}^*},$$

and together with $C_B < K$, we get

$$\|(U,V)\| > \left(\frac{1-(\lambda C_{H_1}+\mu C_{H_2})L}{C_B S_{p_{s_1}^*}^q S_{p_{s_1}^*}^r}\right)^{\frac{1}{p_{s_1}^*-p}} > 1, \tag{3.15}$$

which is a contradiction. Thus $\|(U,V)\| \geq 1$.

ii) If $\lambda \int_\Omega H_1(x)|U|^\theta dx + \mu \int_\Omega H_2(x)|V|^\theta dx < 0$, we have two more cases to analyse:

2.1) If $\|(U,V)\| > 1$, similar to case i) above, we get the same equation (3.15).

2.2) If $\|(U,V)\| < 1$, we have by equations (3.12) and (3.13),

$$\begin{aligned}
&[1-(\lambda C_{H_1}+\mu C_{H_2})L]\left(\|(u,v)\|_{s_1,p}^p + \|(u,v)\|_{s_2,t}^t\right) \\
&\quad < C_B S_{p_{s_1}^*}^q S_{p_{s_1}^*}^r \|(U,V)\|^p \\
&\qquad + (p-\theta)\left(\lambda \int_\Omega H_1(x)|U|^\theta \ln|U| dx + \mu \int_\Omega H_2(x)|V|^\theta \ln|V| dx\right) \\
&\qquad + C_B S_{p_{s_1}^*}^q S_{p_{s_1}^*}^r \|(U,V)\|^{p_{s_1}^*}.
\end{aligned}$$

So, because $p \leq \theta$,

$$\|(U,V)\| > \left(\frac{1-(\lambda C_{H_1}+\mu C_{H_2})L - C_B S_{p_{s_1}^*}^q S_{p_{s_1}^*}^r}{C_B S_{p_{s_1}^*}^q S_{p_{s_1}^*}^r}\right)^{\frac{1}{p_{s_1}^*-p}} > 1.$$

Thus, $\|(U,V)\| \geq \Lambda_{\lambda,\mu}$, for all $(U,V) \in \mathcal{N}^-$.

Now, if $(u,v) \in \mathcal{N}^+$, a similar discussion shows us that $\|(u,v)\| \leq 1$. □

Lemma 3.7. *If $0 < \lambda C_{H_1} + \mu C_{H_2} < \frac{1}{L+M}$ and $C_B < K$, then $\mathcal{N}^-$ is a closed subset o W.*

Proof. The proof follows directly from Lemmas 3.5 and 3.6. □

Lemma 3.8. *If $0 < \lambda C_{H_1} + \mu C_{H_2} < \frac{1}{L+M}$. Then the functional E is bounded from below on $\mathcal{N}$.*

Proof. Because of the relationship between the functionals E and I, for $(u,v) \in \mathcal{N}$, we have

$$\begin{aligned}
E(u,v) = &\left(\frac{1}{p}-\frac{1}{\theta}\right)\left(\|(u,v)\|_{s_1,p}^p + \|(u,v)\|_{s_2,t}^t\right) + \frac{1}{\theta^2}\left(\lambda \int_\Omega H_1(x)|u|^\theta dx + \mu \int_\Omega H_2(x)|v|^\theta dx\right) \\
&- \left(\frac{1}{p_{s_1}^*}-\frac{1}{\theta}\right)\int_\Omega (R_1+R_2)(x)|u|^q|v|^r dx.
\end{aligned}$$

Because $(u,v) \in \mathcal{N}$, we have also

$$\begin{aligned}
&\int_\Omega (R_1+R_2)(x)|u|^q|v|^r dx \\
&\quad = \left(\|(u,v)\|_{s_1,p}^p + \|(u,v)\|_{s_2,t}^t\right) - \lambda \int_\Omega H_1(x)|u|^\theta \ln|u| dx - \mu \int_\Omega H_2(x)|v|^\theta \ln|v| dx.
\end{aligned}$$

So

$$\begin{aligned} E(u,v) = &\left(\frac{1}{p}-\frac{1}{p^*_{s_1}}\right)\|(u,v)\|^p_{s_1,p}+\left(\frac{1}{t}-\frac{1}{p^*_{s_1}}\right)\|(u,v)\|^t_{s_2,t} \\ &+\frac{1}{\theta^2}\Big(\lambda\int_\Omega H_1(x)|u|^\theta dx+\mu\int_\Omega H_2(x)|v|^\theta dx\Big) \\ &+\left(\frac{1}{\theta}-\frac{1}{p^*_{s_1}}\right)\left(\lambda\int_\Omega H_1(x)|u|^\theta \ln|u|dx+\mu\int_\Omega H_2(x)|v|^\theta \ln|v|dx\right). \end{aligned}$$

Now, concerning to exposed above, we consider two cases:

i) If $\lambda\int_\Omega H_1(x)|u|^\theta dx+\mu\int_\Omega H_2(x)|v|^\theta dx>0$, because $0<t<p<\theta<p^*_{s_1}$, it follows that

$$E(u,v)>\left(\frac{1}{p}-\frac{1}{p^*_{s_1}}\right)\|(u,v)\|^p_{s_1,p}+\left(\frac{1}{t}-\frac{1}{p^*_{s_1}}\right)\|(u,v)\|^t_{s_2,t}\geq 0.$$

ii) If $\lambda\int_\Omega H_1(x)|u|^\theta dx+\mu\int_\Omega H_2(x)|v|^\theta dx\leq 0$, it follows from Lemma 2.10 that

$$\begin{aligned} E(u,v) = &\left(\frac{1}{p}-\frac{1}{p^*_{s_1}}\right)[1-(\lambda C_{H_1}+\mu C_{H_2})L]\|(u,v)\left(\|^p_{s_1,p}+\|(u,v)\|^t_{s_2,t}\right) \\ &+\left[\frac{1}{\theta^2}-\left(\frac{1}{\theta}-\frac{1}{p^*_{s_1}}\right)\ln\|(u,v)\|\right]\left(\lambda\int_\Omega H_1(x)|u|^\theta dx+\mu\int_\Omega H_2(x)|v|^\theta dx\right). \end{aligned}$$

Now, if $\frac{1}{\theta^2}-\left(\frac{1}{\theta}-\frac{1}{p^*_{s_1}}\right)\ln\|(u,v)\|\geq 0$, then $E(u,v)\geq 0$.

Otherwise, if $\frac{1}{\theta^2}-\left(\frac{1}{\theta}-\frac{1}{p^*_{s_1}}\right)\ln\|(u,v)\|<0$, there exists a constant $C>0$ such that

$$\begin{aligned} E(u,v) \geq &\left(\frac{1}{p}-\frac{1}{p^*_{s_1}}\right)[1-(\lambda C_{H_1}+\mu C_{H_2})L]\left(\|(u,v)\|^p_{s_1,p}+\|(u,v)\|^t_{s_2,t}\right) \\ &+\left[\frac{1}{\theta^2}-\left(\frac{1}{\theta}-\frac{1}{p^*_{s_1}}\right)\ln\|(u,v)\|\right]\Big(\lambda\int_\Omega H_1(x)|u|^\theta dx+\mu\int_\Omega H_2(x)|v|^\theta dx\Big) \\ \geq &\left(\frac{1}{p}-\frac{1}{p^*_{s_1}}\right)[1-(\lambda C_{H_1}+\mu C_{H_2})L]\left(\|(u,v)\|^p_{s_1,p}+\|(u,v)\|^t_{s_2,t}\right) \\ &-C\left[\frac{1}{\theta^2}-\left(\frac{1}{\theta}-\frac{1}{p^*_{s_1}}\right)\ln\|(u,v)\|\right]\geq -C. \end{aligned}$$

Thus, E is bounded from below on $\mathcal{N}$. □

4 Ground state solution for positive weight function

After the introductory part, in the two following sections we establish the existence of a ground state solution for problem (P). Specifically in this section, we carry on with the weights functions $H_1,H_2>0$, it is H_1,H_2 being non sign-changing functions. For this, we use the Mountain Pass theorem to show the existence of a level $c_*\in W$, where the functional E satisfy the $(PS)_{c_*}$ condition.

The version of Mountain Pass Theorem used in this paper is given in the below lemma.

Lemma 4.1. *Let $H_1,H_2\in C(\overline{\Omega})$ and $H_1(x),H_2(x)>0$ for all $x\in\Omega$. Then there exist $\eta,\zeta>0$, such that* *(i)* $E(u,v) \geq \eta > 0$ *for all* $\|(u,v)\| = \zeta$,

(ii) *there exists* $(u,v) \in W$ *such that* $E(u,v) < 0$ *if* $\|(u,v)\| > \zeta$.

Proof. Because of the definition of E and inequality (3.11) and $1 < q < p < p+r = p^*_{s_1}$, we have

$$\begin{aligned}E(u,v) \geq\ & \left[\frac{1}{p} - \frac{1}{\theta}(\lambda C_{H_1} + \mu C_{H_2})L\right] \|(u,v)\|^p_{s_1,p} + \left[\frac{1}{t} - \frac{1}{\theta}(\lambda C_{H_1} + \mu C_{H_2})L\right] \|(u,v)\|^t_{s_2,t} \\ & - \frac{1}{\theta}\Big(\ln\|(u,v)\| - \frac{1}{\theta}\Big)\left[\lambda \int_\Omega H_1(x)|u|^\theta dx + \mu \int_\Omega H_2(x)|v|^\theta dx\right] - C_B S_{q,r}^{\frac{p^*_s}{p}} \|(u,v)\|^{p^*_{s_1}} \\ \geq\ & \left[\frac{1}{p} - \frac{1}{\theta}(\lambda C_{H_1} + \mu C_{H_2})L\right] \|(u,v)\|^p_{s_1,p} \\ & - \frac{1}{\theta}\Big(\ln\|(u,v)\| - \frac{1}{\theta}\Big)\left[\lambda \int_\Omega H_1(x)|u|^\theta dx + \mu \int_\Omega H_2(x)|v|^\theta dx\right] - C_B S_{q,r}^{\frac{p^*_s}{p}} \|(u,v)\|^{p^*_{s_1}},\end{aligned}$$

which implies that, for all $(u,v) \in W$ with $0 < \|(u,v)\| \leq 1$, we have

$$E(u,v) \geq \left[\frac{1}{p} - \frac{1}{\theta}(\lambda C_{H_1} + \mu C_{H_2})L\right] \|(u,v)\|^p_{s_1,p} - C_B S_{q,r}^{\frac{p^*_{s_1}}{p}} \|(u,v)\|^{p^*_{s_1}}.$$

Choosing $\zeta \in (0,1]$ small enough, such that

$$\left(\frac{1}{p} - \frac{1}{\theta}(\lambda C_{H_1} + \mu C_{H_2})L\right) - C_B S_{q,r}^{\frac{p^*_{s_1}}{p}} \|(u,v)\|^{p^*_{s_1}} \zeta^{p^*_{s_1}} > 0,$$

we have

$$E(u,v) \geq \left[\left(\frac{1}{p} - \frac{1}{\theta}(\lambda C_{H_1} + \mu C_{H_2})L\right) - C_B S_{q,r}^{\frac{p^*_{s_1}}{p}} \|(u,v)\|^{p^*_{s_1}} \zeta^{p^*_{s_1}}\right] \zeta^p > 0,$$

for all $(u,v) \in W$, with $\|(u,v)\| = \zeta$. Thus (i) holds.

Otherwise, for all $(u,v) \in W \setminus \{0,0\}$ and $k > 0$, we have

$$\begin{aligned}E(ku,kv) = k^{p^*_{s_1}} \Bigg[& \frac{k^{p-p^*_{s_1}}}{p}\|(u,v)\|^p_{s_1,p} + \frac{k^{t-p^*_{s_1}}}{t}\|(u,v)\|^t_{s_2,t} - \frac{k^{\theta-p^*_{s_1}}\ln|k|}{\theta}\int_\Omega H_1(x)|u|^\theta dx \\ & - \frac{k^{\theta-p^*_{s_1}}}{\theta}\lambda\int_\Omega H_1(x)|u|^\theta \ln|u| dx + \frac{k^{\theta-p^*_{s_1}}}{\theta^2}\lambda\int_\Omega H_1(x)|u|^\theta dx \\ & - \frac{k^{\theta-p^*_{s_1}}\ln|k|}{\theta}\int_\Omega H_2(x)|v|^\theta dx - \frac{k^{\theta-p^*_{s_1}}}{\theta}\mu\int_\Omega H_2(x)|v|^\theta \ln|v| dx \\ & + \frac{k^{\theta-p^*_{s_1}}}{\theta^2}\mu\int_\Omega H_2(x)|v|^\theta dx - \frac{1}{p^*_{s_1}}\int_\Omega (R_1+R_2)(x)|u|^q|v|^r dx\Bigg],\end{aligned}$$

and because $1 < \theta < q < p < p^*_{s_1}$ it implies that there exists $k_0 > 0$ large enough such that $\|(k_0u,k_0v)\| > \zeta$ and $E(k_0u,k_0v) < 0$. So, taking $(u,v) = (k_0u,k_0v)$, item (ii) holds □

Lemma 4.2. *Let* (u_n,v_n) *be a* $(PS)_{c_*}$ *sequence of the functional* E*. Then, a sequence* (u_n,v_n) *is bounded in* W.

Proof. For $c_* \in \mathbb{R}$, we assume that $\{(u_n,v_n)\}_n \subset W$ with $\|(u_n,v_n)\| > 1$ is a $(PS)_{c_*}$ sequence with, it is $E(u_n,v_n) \to c_*$ and $E'(u_n,v_n)(u_n,v_n) \to 0$, as $n \to \infty$. On the other hand, since that (u_n,v_n) is a $(PS)_{c_*}$ sequence for functional E, we have

$$\begin{aligned} c_* + o_n(1) &\geq E(u_n,v_n) - \frac{1}{p}E'(u_n,v_n)(u_n,v_n) \\ &= \left(\frac{1}{t} - \frac{1}{p}\right) \|(u,v)\|^t_{s_2,p} \\ &\quad + \left(\frac{1}{p} - \frac{1}{\theta}\right) \left[\lambda \int_\Omega H_1(x)|u|^\theta \ln|u|dx + \mu \int_\Omega H_2(x)|v|^\theta \ln|v|dx\right] \\ &\quad + \frac{1}{\theta^2}\left[\lambda \int_\Omega H_1(x)|u|^\theta dx + \mu \int_\Omega H_2(x)|v|^\theta dx\right] \\ &\geq \frac{1}{\theta^2}\left[\lambda \int_\Omega H_1(x)|u_n|^\theta dx + \mu \int_\Omega H_2(x)|v_n|^\theta dx\right]. \end{aligned} \tag{4.1}$$

We have also

$$\begin{aligned} c_* + o_n(1) &\geq E(u_n,v_n) - \frac{1}{p^*_{s_1}}E'(u_n,v_n)(u_n,v_n) \\ &= \left(\frac{1}{p} - \frac{1}{p^*_{s_1}}\right) \|(u_n,v_n)\|^p_{s_1,p} + \left(\frac{1}{t} - \frac{1}{p^*_{s_1}}\right) \|(u_n,v_n)\|^t_{s_2,t} \\ &\quad + \frac{1}{\theta^2}\left[\lambda \int_\Omega H_1(x)|u_n|^\theta dx + \mu \int_\Omega H_2(x)|v_n|^\theta dx\right] \\ &\quad - \left(\frac{1}{p} - \frac{1}{p^*_{s_1}}\right)\left[\lambda \int_\Omega H_1(x)|u_n|^\theta \ln u_n dx + \mu \int_\Omega H_2(x)|v_n|^\theta \ln v_n dx\right] \\ &\geq \left(\frac{1}{p} - \frac{1}{p^*_{s_1}}\right)\|(u_n,v_n)\|^p_{s_1,p} \\ &\quad - \left(\frac{1}{p} - \frac{1}{p^*_{s_1}}\right)\left[\lambda \int_\Omega H_1(x)|u_n|^\theta \ln u_n + \mu \int_\Omega H_2(x)|v_n|^\theta \ln v_n\right]. \end{aligned} \tag{4.2}$$

Combining equations (4.1) and (4.2), Lemma 2.6 and Lemma 2.10, we have

$$\begin{aligned} \left[\left(\frac{1}{p} - \frac{1}{p^*_{s_1}}\right)[1 - (\lambda C_{H_1} + \mu C_{H_2})L]\right] \|(u,v)\| &\leq \left(\frac{1}{p} - \frac{1}{p^*_{s_1}}\right) \frac{\theta^2[(u_n,v_n)]^\sigma}{e\sigma}[c_* + \mathrm{o}(1)] \\ &\leq C\left(1 + \|(u_n,v_n)\|^{1+\sigma}\right) + \mathrm{o}(1), \end{aligned}$$

where $\sigma \in (0, p-1)$. Thus $\{(u_n,v_n)\}_n$ is bounded in W. □

Lemma 4.3. *Let (u_n,v_n) be a $(PS)_{c_*}$ sequence of the functional E. Then, functional E satisfies the $(PS)_{c_*}$ condition at any level c_*.*

Proof. By Lemmas 2.8 and 2.9, we get

$$\lim_{n\to\infty} \int_\Omega H_1(x)u|u_n|^{\theta-1} \ln|u|dx = \int_\Omega H_1(x)|u|^\theta \ln|u|dx, \tag{4.3}$$

$$\lim_{n\to\infty} \int_\Omega H_2(x)v|v_n|^{\theta-1} \ln|v|dx = \int_\Omega H_2(x)|v|^\theta \ln|v|dx, \tag{4.4}$$

$$\lim_{n\to\infty}\int_\Omega (R_1+R_2)(x)|u_n|^{q-2}u_n|v_n|^r(u_n-u)dx=0, \tag{4.5}$$

$$\lim_{n\to\infty}\int_\Omega (R_1+R_2)(x)|v_n|^{r-2}v_n|u_n|^\theta(v_n-v)dx=0. \tag{4.6}$$

Note that

$$\begin{aligned}\|(u_n-u,v_n-v)\|^p &= (E'(u_n,v_n)-E'(u,v))(u_n-u,v_n-v)\\ &+\lambda\int_\Omega H_1(x)\left(|u_n|^{\theta-1}\ln|u_n|-|u|^{\theta-1}\ln|u|\right)(u_n-u)dx\\ &+\mu\int_\Omega H_2(x)\left(|v_n|^{\theta-1}\ln|v_n|-|v|^{\theta-1}\ln|v|\right)(v_n-v)dx\\ &+\frac{q}{p^*_{s_1}}\int_\Omega (R_1+R_2)(x)\left(|v_n|^r u_n^{q-2}u_n-|v|^r|u|^{q-2}u\right)(u_n-u)dx\\ &+\frac{r}{p^*_{s_1}}\int_\Omega (R_1+R_2)(x)\left(|u_n|^q v_n^{r-2}v_n-|u|^q|v|^{r-2}v^{r-2}v\right)(v_n-v)dx.\end{aligned}$$

Since $(E'(u_n,v_n)-E'(u,v))(u_n-u,v_n-v)\to 0$ as $n\to\infty$, it follow from equations (4.3), (4.4), (4.5), (4.6) and Lemma 2.5, that $(u_n,v_n)\to(u,v)$ in W. This yields the proof. □

Now we can prove the first of the main results in the paper. More precisely, in what follows, we prove Theorem 1.1.

4.1 Proof of Theorem 1.1

Define $\mathfrak{N}:=\{(u,v)\in W\setminus\{0,0\}\mid E'(u,v)(u,v)=0\}$. By the previous lemmas, $\mathfrak{N}$ is nonempty. Let $(u,v)\in\mathfrak{N}$, from Lemma 2.10, we have

$$\begin{aligned}0 &= \|(u,v)\|^p_{s_1,p}+\|(u,v)\|^t_{s_2,t}-\left(\lambda\int_\Omega H_1(x)|u|^\theta\ln|u|dx+\mu\int_\Omega H_2(x)|v|^\theta\ln|v|dx\right)\\ &\quad-\int_\Omega (R_1+R_2)(x)|u|^q|v|^r dx\\ &\geq (1-(\lambda C_{H_1}+\mu C_{H_2})L)\left(\|(u,v)\|^p_{s_1,p}+\|(u,v)\|^t_{s_2,t}\right)\\ &\quad-\ln\|(u,v)\|\left(\lambda\int_\Omega H_1(x)|u|^\theta\ln|u|dx+\mu\int_\Omega H_2(x)|v|^\theta\ln|v|dx\right)\\ &\quad-\int_\Omega (R_1+R_2)(x)|u|^q|v|^r dx.\end{aligned} \tag{4.7}$$

If $\|(u,v)\|\leq 1$, it follows from the above equation (4.7) that

$$[1-(\lambda C_{H_1}+\mu C_{H_2})]\left(\|(u,v)\|^p_{s_1,p}+\|(u,v)\|^t_{s_2,t}\right)\leq\int_\Omega (R_1+R_2)(x)|u|^q|v|^r dx\leq C_B S_{q,r}^{\frac{p^*_{s_1}}{p}}.$$

Then,

$$\|(u,v)\|\geq\left[\frac{1-(\lambda C_{H_1}+\mu C_{H_2})L}{C_B S_{q,r}^{\frac{p^*_{s_1}}{p}}}\right]^{\frac{1}{p^*_{s_1}-p}},$$

which implies that

$$\int_\Omega (R_1+R_2)(x)|u|^q|v|^r dx \geq [1-(\lambda C_{H_1}+\mu C_{H_2})L]\left(\|(u,v)\|^p_{s_1,p}+\|(u,v)\|^t_{s_2,t}\right)$$

$$\geq \left[\frac{[1-(\lambda C_{H_1}+\mu C_{H_2})L]^{p^*_{s_1}}}{\left(C_B S_{q,r}^{\frac{p^*_{s_1}}{p}}\right)^p}\right]^{\frac{1}{p^*_{s_1}-p}}. \tag{4.8}$$

Define

$$\tilde{C} := \left[\frac{[1-(\lambda C_{H_1}+\mu C_{H_2})L]^{p^*_{s_1}}}{\left(C_B S_{q,r}^{\frac{p^*_{s_1}}{p}}\right)^p}\right]^{\frac{1}{p^*_{s_1}-p}},$$

and

$$\mathfrak{C} := \inf\{E(u,v) \mid (\tilde{U_n},\tilde{V_n}) \in \mathfrak{N}\},$$

then $\mathfrak{C}>0$, because otherwise, there exists $\{(\tilde{U_n},\tilde{V_n})\}_n \subset \mathfrak{N}$ such that $E'(\tilde{U_n},\tilde{V_n}) \to 0$.

It follows from

$$\begin{aligned} E(u,v) = & \left(\frac{1}{p}-\frac{1}{\theta}\right)\|(u,v)\|^p_{s_1,p} + \left(\frac{1}{t}-\frac{1}{\theta}\right)\|(u,v)\|^t_{s_2,t} + \frac{1}{\theta}I(u,v) \\ & + \frac{1}{\theta^2}\left(\lambda\int_\Omega H_1(x)|u|^\theta dx + \mu\int_\Omega H_2(x)|v|^\theta dx\right) \\ & - \left(\frac{1}{p^*_{s_1}}-\frac{1}{\theta}\right)\int_\Omega (R_1+R_2)(x)|u|^q|v|^r dx, \end{aligned}$$

that

$$\frac{1}{\theta^2}\left(\lambda\int_\Omega H_1(x)|\tilde{U_n}|^\theta dx + \mu\int_\Omega H_2(x)|\tilde{V_n}|^\theta dx\right) - \left(\frac{1}{p^*_{s_1}}-\frac{1}{p}\right)\int_\Omega (R_1+R_2)(x)|\tilde{U_n}|^q|\tilde{V_n}|^r dx \to 0,$$

when $n\to\infty$, and by Lemma 4.2. But, from equation (4.8) it follows that

$$\int_\Omega (R_1+R_2)(x)|\tilde{U_n}|^q|\tilde{V_n}|^r dx \geq \tilde{C} > 0,$$

and it implies that $0 \geq \tilde{C} > 0$, which is a contradiction. Thus, $\mathfrak{C}>0$.

Finally, let $\{(u_n,v_n)\}_n \subset \mathfrak{N}$ be a minimizing sequence. Then $E'(u_n,v_n)(u_n,v_n)=0$ and $\lim_{n\to\infty} E(u_n,v_n)=\mathfrak{C}>0$. Again by Lemma 4.2, there exists $(u_0,v_0)\in W\setminus\{0,0\}$ such that

$$(u_n,v_n)\to(u_0,v_0)$$

in W. Hence $E(u_0,v_0)=\mathfrak{C}$ and $E'(u_0,v_0)=0$, and it means that (u_0,v_0) is a nontrivial ground state solution of problem (P).

5 Nontrivial solutions for sign-changing weight functions

To conclude this paper, we now proceed with the proof of the existence of solutions to the problem (P), but now for the cases in which the weight functions H_1, H_2 are sign-changing functions.

Lemma 5.1. *E has a nontrivial and nonnegative minimizer on $\mathcal{N}^+$.*

Proof. The proof of this lemma will be shown in two steps.

Step 1. The strong convergence of minimizing sequence. By Lemma 3.8, we get

$$c^+ = \inf_{(u,v)\in\mathcal{N}^+} E(u,v).$$

Claim: $c^+ < 0$. Indeed, for each $(u,v) \in \mathcal{N}^+$, we get

$$\begin{aligned}\|(u,v)\|^p_{s_1,p} + \|(u,v)\|^t_{s_2,t} &= \lambda \int_\Omega H_1(x)|u|^\theta \ln|u|dx + \mu \int_\Omega H_2(x)|v|^\theta \ln|v|dx \\ &\quad + \int_\Omega (R_1(x)+R_2(x))|u|^q|v|^r dx \end{aligned} \tag{5.1}$$

and there holds

$$\begin{aligned} E(u,v) &= \frac{1}{p}\|(u,v)\|^p_{s_1,p} + \frac{1}{t}\|(u,v)\|^t_{s_2,t} - \frac{\lambda}{\theta}\int_\Omega H_1(x)|u|^\theta \ln|u|dx + \frac{\lambda}{\theta^2}\int_\Omega H_1(x)|u|^\theta dx \\ &\quad - \frac{\mu}{\theta}\int_\Omega H_2(x)|v|^\theta \ln|v|dx + \frac{\mu}{\theta^2}\int_\Omega H_2(x)|v|^\theta dx - \frac{1}{p^*_{s_1}}\int_\Omega (R_1(x)+R_2(x))|u|^q|v|^r dx \\ &= \frac{1}{p}\left[\lambda \int_\Omega H_1(x)|u|^\theta \ln|u|dx + \mu\int_\Omega H_2(x)|v|^\theta \ln|v|dx + \int_\Omega (R_1(x)+R_2(x))|u|^q|v|^r dx\right] \\ &\quad - \frac{\lambda}{\theta}\int_\Omega H_1(x)|u|^\theta \ln|u|dx + \frac{\lambda}{\theta^2}\int_\Omega H_1(x)|u|^\theta dx - \frac{\mu}{\theta}\int_\Omega H_2(x)|v|^\theta \ln|v|dx \\ &\quad + \frac{\mu}{\theta^2}\int_\Omega H_2(x)|v|^\theta dx - \frac{1}{p^*_{s_1}}\int_\Omega (R_1(x)+R_2(x))|u|^q|v|^r dx \\ &= \left(\frac{1}{p} - \frac{1}{\theta}\right)\left[\lambda\int_\Omega H_1(x)|u|^\theta \ln|u|dx + \mu\int_\Omega H_2(x)|v|^\theta \ln|v|dx\right] \\ &\quad + \frac{1}{\theta^2}\left[\lambda\int_\Omega H_1(x)|u|^\theta dx + \mu\int_\Omega H_2(x)|v|^\theta dx\right] \\ &\quad + \left(\frac{1}{\theta} - \frac{1}{p^*_{s_1}}\right)\int_\Omega (R_1(x)+R_2(x))|u|^q|v|^r dx. \end{aligned}$$

Using the following inequality

$$\begin{aligned} &(p-\theta)\left[\lambda\int_\Omega H_1(x)|u_n|^\theta \ln|u|dx + \mu\int_\Omega H_2(x)|v_n|^\theta \ln|v|dx\right] \\ &\quad > (p^*_{s_1} - p)\int_\Omega (R_1(x)+R_2(x))|u|^q|v|^r dx + \lambda\int_\Omega H_1(x)|u|^\theta dx + \mu\int_\Omega H_2(x)|v|^\theta dx \end{aligned}$$

and $p < \theta$ we have

$$\begin{aligned} &\lambda\int_\Omega H_1(x)|u_n|^\theta \ln|u|dx + \mu\int_\Omega H_2(x)|v_n|^\theta \ln|v|dx \\ &\quad < -\frac{p^*_{s_1} - p}{\theta - p}\int_\Omega (R_1(x)+R_2(x))|u|^q|v|^r dx \\ &\quad\quad - \frac{1}{\theta - p}\left[\lambda\int_\Omega H_1(x)|u|^\theta dx + \mu\int_\Omega H_2(x)|v|^\theta dx\right]. \end{aligned} \tag{5.2}$$

Using (5.2) implies that

$$\begin{aligned} E(u,v) &= \left(\frac{1}{p}-\frac{1}{\theta}\right)\left[\lambda\int_\Omega H_1(x)|u|^\theta \ln|u|dx + \mu\int_\Omega H_2(x)|v|^\theta \ln|v|dx\right] \\ &\quad + \frac{1}{\theta^2}\left[\lambda\int_\Omega H_1(x)|u|^\theta dx + \mu\int_\Omega H_2(x)|v|^\theta dx\right] \\ &\quad + \left(\frac{1}{p}-\frac{1}{p^*_{s_1}}\right)\int_\Omega (R_1(x)+R_2(x))|u|^q|v|^r dx \\ &< (p-p^*_{s_1})\left(\frac{1}{\theta-p}-\frac{1}{p(p^*_{s_1})}\right)\int_\Omega (R_1(x)+R_2(x))|u|^q|v|^r dx \\ &\quad + \left(\frac{1}{\theta^2}-\frac{1}{\theta-p}\right)\left[\lambda\int_\Omega H_1(x)|u|^\theta dx + \mu\int_\Omega H_2(x)|v|^\theta dx\right] \\ &< 0. \end{aligned}$$

Let $\{(u_n,v_n)\}\subset\mathcal{N}^+$ be a minimizing sequence. Then,

$$\begin{aligned} \|(u,v)\|^p_{s_1,p}+\|(u,v)\|^t_{s_2,t} &= \lambda\int_\Omega H_1(x)|u_n|^\theta \ln|u|dx + \mu\int_\Omega H_2(x)|v_n|^\theta \ln|v|dx \\ &\quad + \int_\Omega (R_1(x)+R_2(x))|u|^q|v|^r dx, \end{aligned} \tag{5.3}$$

and using $\Phi''_{(u_n,v_n)}(1)>0$,

$$\begin{aligned} \Phi''_{(u_n,v_n)}(1) &= \|(u,v)\|^p_{s_1,p}+\|(u,v)\|^t_{s_2,t} - \lambda\int_\Omega H_1(x)|u_n|^\theta \ln|u_n|dx - \mu\int_\Omega H_2(x)|v|^\theta \ln|v|dx \\ &\quad - \int_\Omega (R_1+R_2)(x)|u_n|^q|v_n|^r dx - \lambda\int_\Omega H_1(x)|u_n|^\theta dx \\ &\quad - \mu\int_\Omega H_2(x)|v_n|^\theta dx + (p-p^*_{s_1})\int_\Omega (R_1(x)+R_2(x))|u|^q|v|^r dx \\ &> 0. \end{aligned} \tag{5.4}$$

This implies

$$-\lambda\int_\Omega H_1(x)|u_n|^\theta dx - \mu\int_\Omega H_2(x)|v_n|^\theta dx + (p-p^*_{s_1})\int_\Omega (R_1(x)+R_2(x))|u|^q|v|^r dx > 0. \tag{5.5}$$

Since $\mathcal{N}^+$ is bounded by Lemma 3.7, up to subsequence we assume that

$$\begin{cases} (u_n,v_n)\rightharpoonup(u^+,v^+) & \text{in } W; \\ u_n\to u^+,\ v_n\to v^+ & \text{strongly in } L^t(\Omega),\ \text{for } 1\le t<p^*_s \\ u_n(x)\to u^+(x),\ v_n(x)\to v^+(x) & \text{a.e in } \Omega. \end{cases} \tag{5.6}$$

Similarly to Lemma 4.3, we get

$$\lim_{n\to\infty}\int_\Omega H_1(x)u_n|u_n|^\theta \ln|u_n|dx = \int_\Omega H_1(x)|u^+|^\theta \ln|u^+|dx, \tag{5.7}$$

$$\lim_{n\to\infty}\int_\Omega H_2(x)v_n|v_n|^\theta \ln|v_n|dx = \int_\Omega H_2(x)|v^+|^\theta \ln|v^+|dx, \tag{5.8}$$

$$\lim_{n\to\infty}\int_\Omega R_1(x)|u_n|^q|v_n|^r dx = \int_\Omega R_2(x)|u^+|^q|v^+|^r dx, \tag{5.9}$$

$$\lim_{n\to\infty}\int_\Omega R_2(x)|u_n|^q|v_n|^r dx = \int_\Omega R_2(x)|u^+|^q|v^+|^r dx. \tag{5.10}$$

If $(u_n, v_n) \nrightarrow (u^+, v^+)$ in W, then

$$\|(u^+, v^+)\|^2 \leq \liminf_{n\to+\infty} \|(u_n, v_n)\|^2. \tag{5.11}$$

This implies that

$$\begin{aligned}
&\|(u^+, v^+)\|^p_{s_1,p} + \|(u^+, v^+)\|^t_{s_2,t} - \lambda \int_\Omega H_1(x)|u^+|^\theta \ln|u^+|dx - \mu \int_\Omega H_1(x)|v^+|^\theta \ln|v^+|dx \\
&\quad - \int_\Omega R_1(x)|u^+|^q|v^+|^r dx - \int_\Omega R_2(x)|u^+|^q|v^+|^r dx \\
&< \liminf_{n\to+\infty} \Big[\|(u_n, v_n)\|^2 - \lambda \int_\Omega H_1(x)|u_n|^\theta \ln|u_n|dx - \mu \int_\Omega H_1(x)|v_n|^\theta \ln|v_n|dx \\
&\quad - \int_\Omega R_1(x)|u_n|^q|v_n|^r dx - \int_\Omega R_2(x)|u_n|^q|v_n|^r dx\Big] = 0.
\end{aligned} \tag{5.12}$$

Now we prove that for (u^+, v^+) there exists $0 < t_{(u^+,v^+)} \neq 1$ such that

$$t_{(u^+,v^+)}(u^+, v^+) \in \mathcal{N}^+.$$

Since $c^+ < 0$, we can show that $(u^+, v^+) \neq (0, 0)$. By (5.5), we deduce that

$$\lambda \int_\Omega R_1(x)|u^+|^q dx + \mu \int_\Omega R_2(x)|v^+|^r dx < 0,$$

then by Lemma 3.3 there exists $t_{(u^+,v^+)} > 0$ such that

$$t_{(u^+,v^+)}(u^+, v^+) \in \mathcal{N}^+ \quad \text{and} \quad \Psi'_{(u^+,v^+)}(t_{(u^+,v^+)}) = 0$$

By (5.12), $\Psi'_{(u^+,v^+)}(1)) < 0$. Thus, $t_{(u^+,v^+)} \neq 1$. Note that $t_{(u^+,v^+)}(u^+, v^+)$ is minimizer of $g(t) = E(\tau(u^+, \tau v^+)$. Thus,

$$E(t_{(u^+,v^+)}u^+, t_{(u^+,v^+)}v^+) < E(u^+, v^+)) \leq \lim_{n\to+\infty} E(u_n, v_n) = \inf_{(u,v)\in\mathcal{N}^+} E(u, v),$$

which is absurd. Therefore, we obtain $(u_n, v_n) \to (u^+, v^+)$ in W.

Step 2. Existence of nonnegative minimizers. From $(u_n, v_n) \to (u^+, v^+)$ in W and Lemma 3.8, we get

$$-\lambda \int_\Omega H_1(x)|u^+|^\theta dx - \mu \int_\Omega H_2(x)|v^+|^\theta dx + (p - p^*_{s_1}) \int_\Omega (R_1 + R_2)(x)|u^+|^q|v^+|^r dx > 0. \tag{5.13}$$

Thus, we obtain that $(u^+, v^+) \in \mathcal{N}^+$. This gives that (u^+, v^+) is a minimizer of E on $\mathcal{N}^+$. We can prove that $(|u^+|, |v^+|)$ is also a minimizer in $\mathcal{N}^+$. Since

$$E(|u^+|, |v^+|) \leq E(u^+, v^+)$$

and because (5.13) holds for $(|u^+|, |v^+|)$, one can show that

$$\begin{aligned}
&\|(u^+, v^+)\|^p_{s_1,p} + \|(u^+, v^+)\|^t_{s_2,t} - \lambda \int_\Omega H_1(x)|u^+|^\theta \ln|u^+|dx - \mu \int_\Omega H_1(x)|v^+|^\theta \ln|v^+|dx \\
&= \int_\Omega R_1(x)|u^+|^q|v^+|^r dx + \int_\Omega R_2(x)|u^+|^q|v^+|^r dx.
\end{aligned} \tag{5.14}$$

Using $[|u^+|]_s^p \le [u^+]_s^p$, we obtain

$$\begin{aligned}&\|(u^+, v^+)\|_{s_1,p}^p + \|(u^+, v^+)\|_{s_2,t}^t - \lambda \int_\Omega H_1(x)|u^+|^\theta \ln|u^+|dx - \mu \int_\Omega H_1(x)|v^+|^\theta \ln|v^+|dx \\ &\quad \le \int_\Omega R_1(x)|u^+|^q|v^+|^r dx + \int_\Omega R_2(x)|u^+|^q|v^+|^r dx. \end{aligned} \tag{5.15}$$

If

$$\begin{aligned}&\|(u^+, v^+)\|_{s_1,p}^p + \|(u^+, v^+)\|_{s_2,t}^t - \lambda \int_\Omega H_1(x)|u^+|^\theta \ln|u^+|dx - \mu \int_\Omega H_1(x)|v^+|^\theta \ln|v^+|dx \\ &\quad < \int_\Omega R_1(x)|u^+|^q|v^+|^r dx + \int_\Omega R_2(x)|u^+|^q|v^+|^r dx, \end{aligned} \tag{5.16}$$

then $\Psi'_{(|u^+|,|v^+|)}(1)) < 0$. For $(|u^+|, |v^+|)$, by Lemma 3.3 there exist $t_{(|u^+|,|v^+|)} \in \mathcal{N}^+$ and $\Psi'_{(|u^+|,|v^+|)} t_{(|u^+|,|v^+|)} = 0$. Thus, $t_{(|u^+|,|v^+|)} \neq 1$. For other hand $t_{(|u^+|,|v^+|)}$ is a minimizer of

$$\xi(t) := E(u^+, v^+).$$

Thus,

$$E(t_{(|u^+|,|v^+|)}|u^+|, t_{(|u^+|,|v^+|)}|v^+| \le E(|u^+|, |v^+|) \le E(u^+, v^+) = \inf_{(u,v)\in\mathcal{N}^+} E(u, v),$$

which is absurd. Thus, $(|u^+|, |v^+|) \in \mathcal{N}^+$ and

$$E(|u^+|, |v^+|) = \inf_{(u,v)\in\mathcal{N}^+} E(u, v).$$

In conclusion, we get nonnegative minimizer E on $\mathcal{N}^+$. □

Lemma 5.2. *E has a nontrivial and nonnegative minimizer on $\mathcal{N}^-$.*

Proof. By Lemma 3.8, we know $c^- := \inf_{(u,v)\in\mathcal{N}^-} E(u, v)$ is attained. Let $\{(u_n, v_n)\}_n \subset \mathcal{N}^-$ be a minimizing sequence such that $E(u_n, v_n) \to c^-$. Then

$$\begin{aligned}&\|(u_n, v_n)\|_{s_1,p}^p + \|(u_n, v_n)\|_{s_2,t}^t - \lambda \int_\Omega H_1(x)|u_n|^\theta dx - \mu \int_\Omega H_2(x)|v_n|^\theta dx \\ &\qquad = \int_\Omega (R_1 + R_2)(x)|u_n|^q|v_n|^r dx, \end{aligned} \tag{5.17}$$

and,

$$\begin{aligned}0 > {} & \lambda(p-\theta) \int_\Omega H_1(x)|u_n|^\theta \ln|u_n|dx + \mu(p-\theta) \int_\Omega H_2(x)|v_n|^\theta \ln|v_n|dx \\ & - \lambda(p-\theta) \int_\Omega H_1(x)|u_n|^\theta dx - \mu(p-\theta) \int_\Omega H_2(x)|v_n|^\theta dx \\ & - (p_{s_1}^* - p) \int_\Omega (R_1 + R_2)(x)|u_n|^q|v_n|^r dx. \end{aligned} \tag{5.18}$$

We claim that $\{(u_n, v_n)\}_n$ is bounded in $\mathcal{N}^-$.

Without loss of generality, we assume that $\|(u_n, v_n)\| \ge 1$. Then

$$\begin{aligned}c^- + o_n(1) &= E(u_n, v_n) - \frac{1}{p} E'(u_n, v_n)(u_n, v_n) \\ &= \frac{1}{\theta^2}\left(\lambda \int_\Omega H_1(x)|u_n|^\theta dx + \mu \int_\Omega H_2(x)|v_n|^\theta dx\right) \\ &\quad + \left(\frac{1}{p} - \frac{1}{p_{s_1}^*}\right) \int_\Omega (R_1 + R_2)(x)|u_n|^q|v_n|^r dx. \end{aligned} \tag{5.19}$$

So

$$c^- + o_n(1) \geq \frac{1}{\theta^2}\left(\lambda \int_\Omega H_1(x)|u_n|^\theta dx + \mu \int_\Omega H_2(x)|v_n|^\theta dx\right). \tag{5.20}$$

If $\lambda \int_\Omega H_1(x)|u_n|^\theta dx + \mu \int_\Omega H_2(x)|v_n|^\theta dx \geq 0$, then from equations (5.19), (5.20), we have

$$\|(u,v)\| \leq \left(\frac{1}{\frac{1}{p} - \frac{1}{\theta}(\lambda C_{H_1} + \mu C_{H_2})L}\right)^{\frac{1}{p-2}}.$$

If, $\lambda \int_\Omega H_1(x)|u_n|^\theta dx + \mu \int_\Omega H_2(x)|v_n|^\theta dx < 0$, from equation (5.20), we have

$$\begin{aligned} c^- + o_n(1) &= E(u_n, v_n) \\ &\geq \left[\frac{1}{p} - \frac{1}{\theta}(\lambda C_{H_1} + \mu C_{H_2})L\right]\left(\|(u_n,v_n)\|^p_{s_1,p} + \|(u_n,v_n)\|^t_{s_2,t}\right) \\ &\quad + \left[\frac{1}{\theta^2} - \left(\frac{1}{\theta} - \frac{1}{p^*_{s_1}}\right)\ln\|(u_n,v_n)\|\right]\left(\lambda \int_\Omega H_1(x)|u_n|^\theta dx + \mu \int_\Omega H_2(x)|v_n|^\theta dx\right). \end{aligned}$$

We have two cases to analyze:

i) If $\frac{1}{\theta^2} - (\frac{1}{\theta} - \frac{1}{p^*_{s_1}})\ln\|(u_n,v_n)\| \leq 0$, we have $c^- + o_n(1) = (\frac{1}{p} - \frac{1}{p^*_{s_1}})[1 - (\lambda C_{H_1} + \mu C_{H_2})L]$ $(\|(u_n,v_n)\|^p_{s_1,p} + \|(u_n,v_n)\|^t_{s_2,t})$, which implies that

$$\|(u_n,v_n)\| \leq \left[\frac{p(p^*_{s_1})(c^- + o_n(1))}{(p^*_{s_1} - p)[1 - (\lambda C_{H_1} + \mu C_{H_2})L]}\right]^{\frac{1}{p}}.$$

ii) If $\frac{1}{\theta^2} - (\frac{1}{\theta} - \frac{1}{p^*_{s_1}})\ln\|(u_n,v_n)\| > 0$, then $\|(u_n,v_n)\| < \exp\left(\frac{p^*_{s_1} - \theta}{\theta^3(p^*_{s_1})}\right)$. So there exists $C > 0$ such that $\|(u_n,v_n)\| < C$.

Thus, every minimizer sequence of E on $\mathcal{N}^-$ is bounded.
Now, since $\{(u_n,v_n)\}$ is bounded in $\mathcal{N}^-$, up to a subsequence, we may assume that

$$\begin{aligned} (u_n,v_n) &\rightharpoonup (u_-,v_-) \quad \text{weakly in } W; \\ (u_n,v_n) &\to (u_-,v_-) \quad \text{strongly in } L^\nu(\Omega) \times L^\tau(\Omega); \\ (u_n,v_n) &\to (u_+,v_+) \quad \text{a.e in } \Omega, \end{aligned}$$

where $1 < \nu, \tau < \min\{p^*_{s_1}, p^*_{s_1}\}$. It implies that

$$\begin{aligned} \lim_{n\to\infty} \int_\Omega H_1(x)|u_n|^\theta \ln|u_n| dx &= \int_\Omega H_1(x)|u_-|^\theta \ln|u_-| dx; \\ \lim_{n\to\infty} \int_\Omega H_2(x)|v_n|^\theta \ln|v_n| dx &= \int_\Omega H_2(x)|v_-|^\theta \ln|v_-| dx; \\ \lim_{n\to\infty} \int_\Omega H_1(x)|u_n|^\theta dx &= \int_\Omega H_1(x)|u_-|^\theta dx; \\ \lim_{n\to\infty} \int_\Omega H_1(x)|v_n|^\theta dx &= \int_\Omega H_2(x)|v_-|^\theta dx. \end{aligned}$$

An easy computation, combined to equation (5.18), shows that

$$I(u_-,v_-) \leq \liminf_{n\to\infty} I(u_n,v_n) = 0.$$

It means that $(u_-,v_-) \in \mathcal{N}^-$. Now, by Lemma 3.3, item $[i]$, there exists $k_{(u,v)} > 0$ such that $I'_{(u_-,v_-)} k_{(u_-,v_-)} = 0$ and $k_{(u_-,v_-)} \neq 1$. Since $(u_n,v_n) \not\to (u_-,v_-)$, we get

$$k_{(u_-,v_-)}(u_n,v_n) \not\to k_{(u_-,v_-)}(u_-,v_-)$$

in W. Another easy computation shows that

$$E(k_{(u_-,v_-)}u_-, k_{(u_-,v_-)}v_-) \leq E(k_{(u_-,v_-)}u_n, k_{(u_-,v_-)}v_n).$$

Now, observe that the function $z(k) := E(ku_n,kv_n)$ attains its maximum at $k = k_{(u_-,v_-)}$. So

$$\begin{aligned} E(k_{(u_-,v_-)}u_-, k_{(u_-,v_-)}v_-) &< \liminf_{n\to\infty} E(k_{(u_-,v_-)}u_n, k_{(u_-,v_-)}v_n) \\ &\leq \lim_{n\to\infty} E(u_n,v_n) = \inf_{(u,v)\in\mathcal{N}^-} E(u,v), \end{aligned}$$

which is a contradiction. Thus $(u_n,v_n) \to (u_-,v_-)$ in W.

Because $(u_-,v_-) \in \mathcal{N}^-$, then (u_-,v_-) is a minimizer of E on $\mathcal{N}^-$. Moreover, a similar discussion as Theorem 5.1 – *Step* 2 – one can show that $(|u_-|,|v_-|)$ is a minimizer of E on $\mathcal{N}^-$. This yields the proof. □

Now is possible to proof Theorem 1.2.

5.1 Proof of Theorem 1.2

From Lemmas 5.1 and 5.2, E has two non-negative minimizers $(u_+,v_+) \in \mathcal{N}^+$ and $(u_-,v_-) \in \mathcal{N}^-$. Then, from Lemma 3.2, E has two non-negative critical points on W, which is non-trivial and non-negative local least energy solution of problem (P). This two solution are distinct, because is obviously that $\mathcal{N}^- \cap \mathcal{N}^+ = \emptyset$.

We claim that (u_+,v_+) and (u_-,v_-) are not semi-trivial solution.

Supposing, otherwise, $v_+ = 0$ in (u_+,v_+), we get that u_+ is a non-trivial solution of the problem

$$\begin{cases} (-\Delta)_p^{s_1} u + (-\Delta)_t^{s_2} u = \lambda H_1(x)|u|^{\theta-2}u \ln|u| & \text{in } \Omega, \\ (u,v) \in W_0^{s_1,p}(\Omega) \times W_0^{s_2,t}(\Omega). \end{cases} \tag{P'}$$

Then $[u_+]_{s_1,p}^p + [u_+]_{s_2,t}^t = \lambda \int_\Omega H_1(x)|u_+|^\theta \ln|u_+| dx$, and because $(u_+,0) \in \mathcal{N}^+$ and $\Phi''_{(u,v)}(1) > 0$, we get

$$\lambda(p-\theta)\int_\Omega H_1(x)|u_+|^\theta dx < \lambda \int_\Omega H_1(x)|u_+|^\theta dx.$$

Because $p < \theta$, we have $\int_\Omega H_1(x)|u_+|^\theta dx < 0$.

Now we choose $w \in W_0^{t,p}(\Omega) \setminus \{0\}$ such that $\int_\Omega H_1(x)|w|^\theta dx < 0$.

For (u_+,w), by Lemma 3.3, there exists a unique $k_1 > 0$ such that $k_1(u_+,w) \in \mathcal{N}^+$. Moreover, we have

$$k_{\max} = \exp\left[\frac{\|u_+\|_{s_1,p}^p + \|u_+\|_{s_2,t}^t - \left[\int_\Omega \lambda H_1(x)|u_+|^\theta \ln|u_+|^\theta + \mu H_2(x)|w|^\theta \ln|w|^\theta\right]dx}{\lambda\int_\Omega H_1(x)^\theta dx + \mu\int_\Omega H_2(x)|w|^\theta dx} - \frac{1}{p-p_{s_1}^*}\right],$$

and

$$E(k_1u_+,k_1w) = \inf_{0<k<k_{\max}} E(ku_+,kw).$$

It follows that

$$c^+ \leq E(k_1 u_+, k_1 w) < E(u_+, w) < E(u_+, 0) = c^+,$$

which is a contradiction. Thus (u_+, v_+) is not a semi-trivial solution for problem (P).

Otherwise, (u_-, v_-) is not a semi-trivial solution for problem (P), by using the same above argument, but this time assuming $v_- = 0$. In this case $(u_-, 0)$ is a nontrivial solution for problem (P') and $\int_\Omega H_1(x)|u_-|^\theta dx > 0$ and $w \in W_0^{t,p}(\Omega) \setminus \{0\}$ is taking such that $\lambda \int_\Omega H_1(x)|u_-|^\theta dx + \mu \int_\Omega H_2(x)|w|^\theta dx < 0$.

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