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On a critical Choquard-type equation with vanishing potential in the Heisenberg group

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Abstract. In this article, we prove the existence of a weak solution to the sub-elliptic problem

$$-\Delta_{\mathbb{H}} u + V(\xi)u = \left(\int_{\mathbb{H}^N} \frac{|u(\eta)|^{Q_\alpha^*} + K(\eta)|u(\eta)|^p}{|\eta^{-1}\circ\xi|^\alpha} d\eta\right)\left(Q_\alpha^*|u|^{Q_\alpha^*-2}u + pK(\xi)|u|^{p-2}u\right)$$

in the Heisenberg group $\mathbb{H}^N$, where $0<\alpha<4$, $Q=2N+2$, $Q_\alpha^* = \frac{2Q-\alpha}{Q-2}$, $p \in (2, Q_\alpha^*)$, V is a non-negative continuous function that can vanish at infinity and K is continuous non-negative bounded function. We establish the existence of a weak solution by employing the penalization method in conjunction with the mountain pass theorem.

Keywords: vanishing potential, critical Choquard type non-linearity, Heisenberg group, penalization method.

2020 Mathematics Subject Classification: 35R03, 35A15, 35H20.

1 Introduction

In this paper, we study the existence of a weak solution to the problem:

$$-\Delta_{\mathbb{H}} u + V(\xi)u = \left(\int_{\mathbb{H}^N} \frac{|u(\eta)|^{Q_\alpha^*} + K(\eta)|u(\eta)|^p}{|\eta^{-1}\circ\xi|^\alpha} d\eta\right)\left(Q_\alpha^*|u|^{Q_\alpha^*-2}u + pK(\xi)|u|^{p-2}u\right) \tag{1.1}$$

in the Heisenberg group $\mathbb{H}^N$, where $0<\alpha<4$, $Q=2N+2$, $Q_\alpha^* = \frac{2Q-\alpha}{Q-2}$, $p \in (2, Q_\alpha^*)$, V is a non-negative continuous function that can vanish at infinity and K is continuous non-negative bounded function.

The Heisenberg group has found significant attraction due to its rich geometry, leading to various applications in the field of partial differential equations. In one of the pioneering works, Garofalo and Lanconelli [20] obtained the existence of a solution for a semilinear

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subelliptic PDE in the Heisenberg group via a variational technique. Since then, a wide range of results have been obtained, see [7,8,13,21,25,36,39] and references therein.

The equation (1.1) is analogous to the well-known steady state Schrödinger equation

$$\Delta\Psi + V(x)\Psi = (Z(x) * |\Psi|^q)|\Psi|^{q-2}\Psi \quad \text{in } \mathbb{R}^N, \tag{1.2}$$

where V is the potential, and Z is the response function describing the mutual interaction between the bosons. When $Z(x) = |x|^{-1}$, (1.2) reduces to the Choquard equation. A typical model, corresponding to $q = 2$, is

$$-\Delta\Psi + \Psi = \left(\frac{1}{|x|} * |\Psi|^2\right)\Psi \quad \text{in } \mathbb{R}^N,$$

which was originally introduced in the context of polaron models [19,33]. For a comprehensive overview of Choquard-type problems in $\mathbb{R}^N$, see [31].

On setting the response function $Z(x)$ to be the Dirac-delta function in (1.2), we obtain the Schrödinger equation in the following form:

$$-\Delta u + V(x)u = f(x,u) \quad \text{in } \mathbb{R}^N, \tag{1.3}$$

where $u > 0, V$ is a continuous function and f is a non-linear function. The function V is commonly referred to as the potential function. Equation (1.3) has been extensively studied under various assumptions on V and f. For bounded, coercive potentials V, see [15,30,34,36]. For vanishing potentials ($V(x) \to 0$ as $|x| \to \infty$), notable contributions include [3–5, 24, 29]. Other situations such as constant, periodic, asymptotically constant, asymptotically periodic, and radial potentials have also been addressed (see [1,2,10,32,40]). For the vanishing potential case Alves–Souto [4], Alves–Figueiredo-Yang [3] employed a penalization technique inspired by Del Pino–Felmer [16].

In recent years, considerable progress has been made on critical Choquard-type problems in the Heisenberg group. Goel and Sreenadh [21] established a Brezis–Nirenberg type result for Choquard equations in $\Omega \subset \mathbb{H}^N$. Sun et al. [36] developed a concentration-compactness lemma for the Choquard equation and applied it to obtain weak solutions of a Kirchhoff–Choquard problem with the critical Hardy–Littlewood–Sobolev exponent. Subsequent works extended and refined these results: Bai et al. [7] proved existence of weak solutions to Kirchhoff–Choquard equations with the same critical exponent in bounded domains, recovering compactness through suitable conditions on the Kirchhoff term, while Sun et al. [36] investigated Kirchhoff-–Choquard systems in bounded domains of $\mathbb{H}^N$. Yang et al. [38] studied (p,q)-type Choquard equations in $\mathbb{H}^N$, focusing on cases where the Choquard term is subcritical. Liang et al. [26] extended the concentration-compactness approach of [36] to the p-sub-Laplacian framework. Further contributions include Bai et al. [6], who considered critical Choquard-type problems with lower-order perturbation terms that may be sublinear or superlinear. Bai et al. [9], who established results for fractional p-sub-Laplacian equations with critical Choquard non-linearities. Most recently, Liang et al. [27] proved the existence of normalized solutions for critical Choquard-type equations with logarithmic perturbations in bounded domains of $\mathbb{H}^N$.

From the above survey, we conclude, to the best of our knowledge, that there are no existing works addressing problems involving vanishing potentials in the Heisenberg group. Moreover, we are not aware of any research in the Heisenberg group that employs the penalization technique. Although Li et al. [24] studied a problem similar to (1.1) in $\mathbb{R}^N$ for the

Laplacian, their approach did not involve the penalization method. Furthermore, to the best of our knowledge, Choquard-type problems have not been investigated via the penalization technique either in $\mathbb{R}^N$ or in $\mathbb{H}^N$. In this article, we obtain an existence result for (1.1) by employing the penalization method, inspired by the works of Alves et al. [3,4].

Before stating our main result, we specify our assumptions on V and K :

(Σ_1) $V : \mathbb{H}^N \to \mathbb{R}$, is continuous, non-negative, $V \in L^\infty(\mathbb{H}^N) \cap L^{\frac{Q}{2}}(\mathbb{H}^N)$ and there exists $\mu > 0, \Lambda > 0$ and $R > 0$ such that

$$\inf_{|\xi| \geq R} |\xi|^\mu V(\xi) \geq \Lambda.$$

(Σ_2) $K : \mathbb{H}^N \to \mathbb{R}$ is continuous, non-negative, bounded and $K \in L^{\frac{Q^*}{Q^*_\alpha - p}}$, where $Q^* = \frac{2Q}{Q-2}$.

From (Σ_1), we have $V \in L^{\frac{2Q}{4-\alpha}}(\mathbb{H}^N)$ by interpolation of the Lebesgue spaces $L^\infty(\mathbb{H}^N)$ and $L^{\frac{Q}{2}}(\mathbb{H}^N)$.

Next, we present and example of V and K satisfying (Σ_1) and (Σ_2), respectively.

Example 1.1. Let $R = 1$. Define

$$V_1(\xi) = \begin{cases} \dfrac{1}{|\xi|^a}, & \xi \in \mathbb{H}^N \setminus B_1(0), \ a > 2, \\ 1, & \xi \in B_1(0). \end{cases}$$

Then V_1 satisfies (Σ_1) for any $\mu > a$ with $\Lambda = 1$.

Examples for K.

(i) Define

$$K_1(\xi) = \begin{cases} \dfrac{1}{|\xi|^a}, & \xi \in \mathbb{H}^N \setminus B_1(0), \ a > \dfrac{Q(Q^*_\alpha - p)}{Q^*}, \\ 1, & \xi \in B_1(0), \end{cases}$$

where

$$p \in \left(2, \frac{2Q - \alpha}{Q - 2}\right).$$

Then, K_1 satisfies (Σ_2).

(ii) As a different example, consider

$$K_2(\xi) = e^{-|\xi|^4}.$$

Then K_2 also satisfies (Σ_2).

Now, we state the main result of this paper:

Theorem 1.2. *Let the conditions (Σ_1), (Σ_2) hold. Then, there exists a constant γ such that (1.1) admits a positive weak solution $u \in E$ (see Section 2 for the definition of E) for any $\Lambda \geq \gamma R^\mu$.*

We have organized the paper as follows. Section 2 gives the preliminaries for the Heisenberg group and some primary results on the Heisenberg group. Further, Section 3 converts the main problem into an auxiliary problem and describes its variational framework. In the last section, we prove Theorem 1.2.

2 Preliminaries

We recall some terminologies and definitions related to the Heisenberg group $\mathbb{H}^N = (\mathbb{R}^{2N+1}, \circ)$, where '$\circ$' denotes the group operation defined as

$$\xi \circ \xi' = (x + x', y + y', t + t' + 2(x' \cdot y - y' \cdot x)), \quad \text{for every } \xi = (x, y, t), \xi' = (x', y', t') \in \mathbb{H}^N,$$

where $x, y, x', y' \in \mathbb{R}^N, t, t' \in \mathbb{R}$. $\xi^{-1} = -\xi$ is the inverse, and therefore $(\xi')^{-1} \circ \xi^{-1} = (\xi \circ \xi')^{-1}$.

The natural group of dilations on $\mathbb{H}^N$ is defined as $\delta_s(\xi) = (sx, sy, s^2 t)$, for every $s > 0$. Hence, $\delta_s(\xi' \circ \xi) = \delta_s(\xi') \circ \delta_s(\xi)$ and $\delta_s(\delta_{s'}(\xi)) = \delta_{ss'}(\xi)$, for $s, s' > 0$. It can be easily proved that the Jacobian determinant of dilations $\delta_s : \mathbb{H}^N \to \mathbb{H}^N$ is constant and equal to s^Q, for every $\xi = (x, y, t) \in \mathbb{H}^N$. The natural number $Q = 2N + 2$ is called the homogeneous dimension of $\mathbb{H}^N$. The homogeneous norm on $\mathbb{H}^N$ is defined as follows

$$|\xi| = |\xi|_{\mathbb{H}} = \left[(|x|^2 + |y|^2)^2 + t^2 \right]^{\frac{1}{4}}, \quad \text{for every } \xi \in \mathbb{H}^N.$$

By definition, the homogeneous degree of the norm is 1, in terms of dilations. The following vector fields

$$T = \frac{\partial}{\partial t}, \quad X_j = \frac{\partial}{\partial x_j} + 2y_j \frac{\partial}{\partial t}, \quad Y_j = \frac{\partial}{\partial y_j} - 2x_j \frac{\partial}{\partial t}, \quad j = 1, \ldots, N,$$

generate the real Lie algebra of left-invariant vector fields. The vector fields, T, X_j, Y_j satisfy the following relations:

$$[X_j, Y_k] = -4\delta_{jk} T, \qquad [Y_j, Y_k] = [X_j, X_k] = [Y_j, T] = [X_j, T] = 0.$$

The Heisenberg gradient on $\mathbb{H}^N$ is given by

$$\nabla_{\mathbb{H}} = (X_1, X_2, \cdots, X_N, Y_1, Y_2, \cdots, Y_N),$$

and the Kohn-Laplacian on $\mathbb{H}^N$ is given by

$$\Delta_{\mathbb{H}} = \sum_{j=1}^{N} X_j^2 + Y_j^2 = \sum_{j=1}^{N} \frac{\partial^2}{\partial x_j^2} + \frac{\partial^2}{\partial y_j^2} + 4y_j \frac{\partial^2}{\partial x_j \partial t} - 4x_j \frac{\partial^2}{\partial x_j \partial t} + 4(x_j^2 + y_j^2) \frac{\partial^2}{\partial t^2}.$$

The Haar measure on $\mathbb{H}^N$ coincides with the Lebesgue measure and is Q-homogeneous with respect to dilations. More precisely, it is consistent with the $(2N + 1)$-dimensional Lebesgue measure. Consequently, the topological dimension of $\mathbb{H}^N$ is $2N + 1$, which is strictly smaller than its Hausdorff dimension $Q = 2N + 2$.

For a measurable set $\Omega \subseteq \mathbb{H}^N$, we denote by $|\Omega|$ its $(2N + 1)$-dimensional Lebesgue measure. Then, for every $s > 0$, we have

$$|\delta_s(\Omega)| = s^Q |\Omega|, \quad d(\delta_s(\xi)) = s^Q \, d\xi, \quad \text{and} \quad |B_r(\xi)| = \alpha_Q r^Q,$$

where $\alpha_Q = |B_1(0)|$. Here, $B_r(\xi)$ denotes the ball in $\mathbb{H}^N$ centered at ξ with radius r.

The horizontal Sobolev space is defined as

$$HW^{1,2}(\mathbb{H}^N) = \left\{ u \in L^2(\mathbb{H}^N) \ : \ X_j u, \, Y_j u \in L^2(\mathbb{H}^N), \ j = 1, \ldots, N \right\}.$$

It is a Hilbert space equipped with the inner product

$$\langle u, v \rangle = \int_{\mathbb{H}^N} \nabla_{\mathbb{H}} u(\xi) \cdot \nabla_{\mathbb{H}} v(\xi) \, d\xi + \int_{\mathbb{H}^N} u(\xi) \, v(\xi) \, d\xi,$$

which induces the norm

$$\|u\|_{HW^{1,2}(\mathbb{H}^N)} = \left(\int_{\mathbb{H}^N} \left|\nabla_{\mathbb{H}} u(\xi)\right|^2 d\xi + \int_{\mathbb{H}^N} |u(\xi)|^2 \, d\xi \right)^{\frac{1}{2}}.$$

We define

$$E = \left\{ u \in S^{1,2}(\mathbb{H}^N) : \int_{\mathbb{H}^N} V(\xi) \, |u(\xi)|^2 \, d\xi < \infty \right\},$$

where $S^{1,2}(\mathbb{H}^N)$ denotes the completion of $C_0^\infty(\mathbb{H}^N)$ with respect to the norm

$$\|u\|_{S^{1,2}(\mathbb{H}^N)} = \left(\int_{\mathbb{H}^N} |\nabla_{\mathbb{H}} u(\xi)|^2 \, d\xi \right)^{\frac{1}{2}}.$$

The space E is a Hilbert space endowed with the inner product

$$\langle u, v \rangle = \int_{\mathbb{H}^N} \left(\nabla_{\mathbb{H}} u(\xi) \cdot \nabla_{\mathbb{H}} v(\xi) + V(\xi) \, u(\xi) \, v(\xi) \right) d\xi,$$

which induces the norm

$$\|u\| = \left(\int_{\mathbb{H}^N} \left(|\nabla_{\mathbb{H}} u(\xi)|^2 + V(\xi) |u(\xi)|^2 \right) d\xi \right)^{1/2}.$$

The continuous embedding

$$E \subset S^{1,2}(\mathbb{H}^N) \hookrightarrow L^{Q^*}(\mathbb{H}^N)$$

holds (see [22]); that is,

$$\|u\|_{L^{Q^*}(\mathbb{H}^N)} \leq C\|u\|, \quad u \in E.$$

Since $S^{1,2}(\mathbb{H}^N) \hookrightarrow L^{Q^*}(\mathbb{H}^N)$, we have

$$S \, \|u\|^2_{L^{Q^*}(\mathbb{H}^N)} \leq \int_{\mathbb{H}^N} |\nabla_{\mathbb{H}} u(\xi)|^2 \, d\xi,$$

where S is the best Sobolev constant, defined by

$$S = \inf_{S^{1,2}(\mathbb{H}^N) \setminus \{0\}} \frac{\int_{\mathbb{H}^N} |\nabla_{\mathbb{H}} u(\xi)|^2 \, d\xi}{\left(\int_{\mathbb{H}^N} |u|^{Q^*} \, d\xi \right)^{\frac{2}{Q^*}}}.$$

It was proved by Jerison and Lee [23] that S is attained by the function

$$U(\xi) = U(x, y, t) = \frac{C}{\left(t^2 + \left(1 + |x|^2 + |y|^2\right)^2 \right)^{\frac{Q-2}{4}}}, \tag{2.1}$$

(up to dilation and translation), where $C > 0$ is a suitable constant.

Now, we state the Hardy–Littlewood–Sobolev inequality.

Lemma 2.1. *Let $r, s > 1$ and $0 < \alpha < Q$ satisfy*

$$\frac{1}{r} + \frac{\alpha}{Q} + \frac{1}{s} = 2.$$

If $f \in L^r(\mathbb{H}^N)$ and $h \in L^s(\mathbb{H}^N)$, then there exists a sharp constant $C(r,s,\alpha,Q) > 0$, independent of f and h, such that

$$\int_{\mathbb{H}^N}\int_{\mathbb{H}^N} \frac{f(\xi)\,h(\eta)}{|\eta^{-1}\circ\xi|^{\alpha}}\,d\eta\,d\xi \;\le\; C(r,s,\alpha,Q)\,\|f\|_{L^r(\mathbb{H}^N)}\,\|h\|_{L^s(\mathbb{H}^N)}. \tag{2.2}$$

In particular, if $r = s = \frac{2Q}{2Q-\alpha}$, then the sharp constant is given by

$$C(r,s,\alpha,Q) = C(Q,\alpha) = \left(\frac{\pi^{N+1}}{2^{N-1}N!}\right)^{\frac{\alpha}{Q}} \frac{N!\,\Gamma\left(\frac{Q-\alpha}{2}\right)}{\Gamma^2\left(\frac{2Q-\alpha}{2}\right)}, \tag{2.3}$$

where Γ denotes the Gamma function.

Frank and Lieb [18] proved that the equality holds in (2.2) if and only if $f,h \in L^{\frac{2Q}{2Q-\alpha}}(\mathbb{H}^N)$ defined as $f(\xi) = cW\left(\delta_\theta\left(\eta^{-1}\circ\xi\right)\right)$, $h(\xi) = c'W\left(\delta_\theta\left(\eta^{-1}\circ\xi\right)\right)$, where $c \in \mathbb{C}, \theta > 0$, and $\eta \in \mathbb{H}^N$(unless $f \equiv 0$ or $g \equiv 0$) and

$$W(\xi) = W(x,y,t) = \left(t^2 + \left(1+|x|^2+|y|^2\right)^2\right)^{-\frac{2Q-\alpha}{4}}, \quad \text{for all } \xi = (x,y,t) \in \mathbb{H}^N.$$

S_{HG} is defined as

$$S_{HG} = \inf_{S^{1,2}(\mathbb{H}^N)\setminus\{0\}} \frac{\int_{\mathbb{H}^N} |\nabla_{\mathbb{H}} u(\xi)|^2 d\xi}{\left(\int_{\mathbb{H}^N}\int_{\mathbb{H}^N} \frac{|u(\eta)|^{Q^*_\alpha}|u(\xi)|^{Q^*_\alpha}}{|\eta^{-1}\circ\xi|^{\alpha}} d\eta d\xi\right)^{\frac{1}{Q^*_\alpha}}}.$$

Goel and Sreenadh [21] proved that S_{HG} is attained by the function U (defined in (2.1)), up to the translation and dilation.(cf [21, Lemma 2.1]) and

$$S_{HG} = S\left(C(Q,\alpha)^{-\frac{1}{Q^*_\alpha}}\right),$$

where $C(Q,\alpha)$ is defined in (2.3).

To deal with the lack of compactness caused by the term with critical exponent, we use the Concentration-Compactness Principle, which was given by Lions [28] in the Euclidean framework and by Ivanov et al. [22] in the Carnot groups setting. We state the lemma in the Heisenberg group setting.

Lemma 2.2 ([22, Lemma 1.4.5])**.** *Let $\{u_n\}$ be a bounded sequence in $S^{1,2}(\mathbb{H}^N)$ converging weakly and a.e. to some $u \in S^{1,2}(\mathbb{H}^N)$. $|\nabla_{\mathbb{H}} u_n|^2 \rightharpoonup \omega$, $|u_n|^{Q^*} \rightharpoonup \zeta$ weakly in the sense of measures where ω and ζ are bounded non-negative measures on $\mathbb{H}^N$. Then we have:*

(1) there exists some at most countable set I, a family $\{z_i : i \in I\}$ of distinct points in $\mathbb{H}^N$, and a family $\{\zeta_i : i \in I\}$ of positive numbers such that

$$\zeta = |u|^{Q^*} + \sum_{i\in I} \zeta_i \delta_{z_i},$$

where δ_ξ is the Dirac-mass of mass 1 concentrated at $\xi \in \mathbb{H}^N$.

(2) In addition, we have

$$\omega \geq |\nabla_{\mathbb{H}} u|^2 + \sum_{i \in I} \omega_i \delta_{z_i}$$

for some family $\{\omega_i : i \in I\}$, $\omega_i > 0$ *satisfying*

$$S\zeta_i^{\frac{2}{Q^*}} \leq \omega_i, \quad \text{for all } i \in I.$$

In particular, $\sum_{i \in I} \zeta_i^{\frac{2}{Q^*}} < \infty$.

We next recall the Concentration–Compactness Principle to handle Choquard-type problems in the Heisenberg group:

Lemma 2.3 ([36, Theorem 3.1]). *Let* $\{u_n\}$ *be a bounded sequence in* $S^{1,2}(\mathbb{H}^N)$ *converging weakly and a.e. to some* u *and* ω, ζ *be the bounded nonnegative measures in Lemma 2.2. Assume that*

$$\int_{\mathbb{H}^N} \left(\frac{|u_n(\eta)|^{Q^*_\alpha}}{|\eta^{-1} \circ \xi|^\alpha} d\eta \right) |u_n(\xi)|^{Q^*_\alpha} \rightharpoonup \nu$$

weakly in the sense of measure where ν *is a bounded positive measure on* $\mathbb{H}^N$. *Then, there exists a countable sequence of points* $\{z_i\}_{i \in I} \subset \mathbb{H}^N$ *and families of positive numbers* $\{\nu_i : i \in I\}$, $\{\zeta_i : i \in I\}$ *and* $\{\omega_i : i \in I\}$ *such that*

$$\nu = \left(\int_{\mathbb{R}^N} \frac{|u(\eta)|^{Q^*_\alpha}}{|\eta^{-1} \circ \xi|^\alpha} d\eta \right) |u(\xi)|^{Q^*_\alpha} + \sum_{i \in I} \nu_i \delta_{z_i}, \qquad \sum_{i \in I} \nu_i^{\frac{1}{Q^*_\alpha}} < \infty,$$

$$\omega \geq |\nabla_{\mathbb{H}} u|^2 + \sum_{i \in I} \omega_i \delta_{z_i},$$

$$\zeta \geq |u|^{Q^*} + \sum_{i \in I} \zeta_i \delta_{z_i},$$

and

$$S_{HG} \nu_i^{\frac{1}{Q^*_\alpha}} \leq \omega_i, \quad \nu_i^{\frac{Q}{2Q-\alpha}} \leq C(Q,\alpha)^{\frac{Q}{2Q-\alpha}} \zeta_i,$$

where δ_ξ *is the Dirac-mass of mass 1 concentrated at* $\xi \in \mathbb{H}^N$.

Lemma 2.4 ([21, Lemma 2.5]). *Let* $0 < \alpha < Q$. *If* $\{u_n\}$ *is a bounded sequence in* $L^{\frac{2Q}{Q-2}}(\mathbb{H}^N)$ *such that* $u_n \to u$ *a.e. in* $\mathbb{H}^N$ *as* $n \to \infty$. *Then as* $n \to \infty$, *the following holds:*

$$\int_{\mathbb{H}^N} \left(|\xi|^{-\alpha} * |u_n|^{Q^*_\alpha} \right) |u_n|^{Q^*_\alpha} d\xi - \int_{\mathbb{H}^N} \left(|\xi|^{-\alpha} * |u_n - u|^{Q^*_\alpha} \right) |u_n - u|^{Q^*_\alpha} d\xi \to \int_{\mathbb{H}^N} \left(|\xi|^{-\alpha} * |u|^{Q^*_\alpha} \right) |u|^{Q^*_\alpha} d\xi.$$

By following the proof of [11, Proposition 2.2], we get the following embedding result:

Lemma 2.5. *Let* $g \in L^m(\mathbb{H}^N)$, *where* $m = \frac{Q^*}{Q^*-q}, q \in [1, Q^*)$. *Then* $S^{1,2}(\mathbb{H}^N) \hookrightarrow L^q(\mathbb{H}^N, |g|)$ *compactly, where* $L^q(\mathbb{H}^N, |g|) = \{u \text{ is a measurable function} \mid \int_{\mathbb{H}^N} g|u(\xi)|^q d\xi < \infty\}$.

3 Auxiliary problem and variational framework

In this section, adapting the arguments in [3, 29], we study the existence of a weak solution to an auxiliary problem associated with (1.1). Throughout this and the following sections, we use C to denote a generic constant, whose value may change from line to line.

Let $f(\xi,u) = Q_\alpha^*|u(\xi)|^{Q_\alpha^*-2}u(\xi) + pK(\xi)|u(\xi)|^{p-2}u(\xi)$. We penalize the left hand side of (1.1)

Define, for $\ell > 1$ and $R > 1$,

$$\tilde{f}(\xi,t) = \begin{cases} f(\xi,t) & \text{if } \ell f(\xi,t) \le V(\xi)t, \\ \frac{V(\xi)}{\ell}t & \text{if } \ell f(\xi,t) > V(\xi)t, \end{cases} \tag{3.1}$$

and

$$g(\xi,t) = \begin{cases} f(\xi,t) & \text{if } |\xi| \le R, \\ \tilde{f}(\xi,t) & \text{if } |\xi| > R. \end{cases} \tag{3.2}$$

From (3.1) and (3.2), it follows that

$$\begin{aligned} \tilde{f}(\xi,t) &\le f(\xi,t), \quad && \forall\, \xi \in \mathbb{H}^N, \\ g(\xi,t) &\le \frac{V(\xi)}{\ell}t, && |\xi| \ge R, \\ G(\xi,t) &= F(\xi,t), && |\xi| \le R, \\ G(\xi,t) &\le \frac{V(\xi)}{2\ell}t^2, && |\xi| \ge R, \end{aligned} \tag{3.3}$$

where $F(\xi,t) = \int_0^t f(\xi,s)ds$, and $G(\xi,t) = \int_0^t g(\xi,s)ds$.

We consider the following auxiliary problem associated with (1.1):

$$-\Delta_{\mathbb{H}}u(\xi) + V(\xi)u(\xi) = \left(\int_{\mathbb{H}^N} \frac{G(\eta,u(\eta))}{|\eta^{-1}\circ\xi|^\alpha}d\eta\right) g(\xi,u(\xi)) \quad \text{in } \mathbb{H}^N. \tag{3.4}$$

The energy functional $J : E \to \mathbb{R}$ associated with (3.4) is given by

$$J(u) = \frac{1}{2}\|u\|^2 - \frac{1}{2}\int_{\mathbb{H}^N}\left(\int_{\mathbb{H}^N} \frac{G(\eta,u(\eta))}{|\eta^{-1}\circ\xi|^\alpha}d\eta\right) G(\xi,u(\xi))d\xi. \tag{3.5}$$

One can verify that J is a C^1-functional and its derivative is given by

$$\begin{aligned} \langle J'(u),v\rangle = &\int_{\mathbb{H}^N} \left(\nabla_{\mathbb{H}}u(\xi)\cdot\nabla_{\mathbb{H}}v(\xi) + V(\xi)u(\xi)v(\xi)\right)d\xi \\ &- \int_{\mathbb{H}^N}\left(\int_{\mathbb{H}^N} \frac{G(\eta,u(\eta))}{|\eta^{-1}\circ\xi|^\alpha}d\eta\right) g(\xi,u(\xi))v(\xi)d\xi. \end{aligned}$$

We now present the result that constitutes the core of this paper.

Theorem 3.1. *Let the conditions* (Σ_1), (Σ_2) *hold. Then,* (3.4) *admits a positive weak solution* $u \in E$.

To prove Theorem 3.1, we first prove a series of lemmas.

Lemma 3.2. *The functional* J *satisfies the following properties:*

(i) there exist $\beta, r > 0$ such that $J(u) \geq \beta$ whenever $\|u\| = r$;

(ii) there exists $e \in E$ with $\|e\| > \rho$ (for some $\rho > 0$) such that $J(e) < \beta$.

Proof. Consider,

$$\begin{aligned}
J(u) &= \frac{1}{2}\|u\|^2 - \frac{1}{2}\int_{\mathbb{H}^N}\left(\int_{\mathbb{H}^N}\frac{G(\eta,u(\eta))}{|\eta^{-1}\circ\xi|^{\alpha}}d\eta\right)G(\xi,u(\xi))d\xi \\
&\geq \frac{1}{2}\|u\|^2 - \frac{1}{2}\int_{\mathbb{H}^N}\left(\int_{\mathbb{H}^N}\frac{F(\eta,u)}{|\eta^{-1}\circ\xi|^{\alpha}}d\eta\right)F(\xi,u)d\xi \\
&\geq \frac{1}{2}\|u\|^2 - \frac{1}{2}\int_{\mathbb{H}^N}\left(\int_{\mathbb{H}^N}\frac{|u(\eta)|^{Q_\alpha^*}+K(\eta)|u(\eta)|^p}{|\eta^{-1}\circ\xi|^{\alpha}}d\eta\right)\left(|u(\xi)|^{Q_\alpha^*}+K(\xi)|u(\xi)|^p\right)d\xi \\
&= \frac{1}{2}\|u\|^2 - \frac{1}{2}\int_{\mathbb{H}^N}\int_{\mathbb{H}^N}\frac{|u(\eta)|^{Q_\alpha^*}}{|\eta^{-1}\circ\xi|^{\alpha}}|u(\xi)|^{Q_\alpha^*}d\eta d\xi - \frac{1}{2}\int_{\mathbb{H}^N}\int_{\mathbb{H}^N}\frac{|u(\eta)|^{Q_\alpha^*}}{|\eta^{-1}\circ\xi|^{\alpha}}K(\xi)|u(\xi)|^p d\eta d\xi \\
&\quad - \frac{1}{2}\int_{\mathbb{H}^N}\int_{\mathbb{H}^N}\frac{K(\eta)|u(\eta)|^p}{|\eta^{-1}\circ\xi|^{\alpha}}K(\xi)|u(\xi)|^p d\eta d\xi - \frac{1}{2}\int_{\mathbb{H}^N}\int_{\mathbb{H}^N}\frac{K(\eta)|u(\eta)|^p}{|\eta^{-1}\circ\xi|^{\alpha}}|u(\xi)|^{Q_\alpha^*}d\eta d\xi.
\end{aligned}$$

Using (2.2), we get

$$\int_{\mathbb{H}^N}\int_{\mathbb{H}^N}\frac{K(\eta)|u(\eta)|^p}{|\eta^{-1}\circ\xi|^{\alpha}}K(\xi)|u(\xi)|^p d\eta d\xi \leq C\left(\int_{\mathbb{H}^N}|K(\xi)|^{\frac{2Q}{2Q-\alpha}}|u(\xi)|^{\frac{2Qp}{2Q-\alpha}}d\xi\right)^{\frac{2Q-\alpha}{Q}}. \tag{3.6}$$

By using Hölder's inequality and (Σ_2), we get

$$\begin{aligned}
\int_{\mathbb{H}^N}|K(\xi)|^{\frac{2Q}{2Q-\alpha}}|u(\xi)|^{\frac{2Qp}{2Q-\alpha}}d\xi &\leq \left(\int_{\mathbb{H}^N}|K(\xi)|^{\frac{Q^*}{Q_\alpha^*-p}}d\xi\right)^{\frac{Q_\alpha^*-p}{Q_\alpha^*}}\left(\int_{\mathbb{H}^N}|u(\xi)|^{Q^*}d\xi\right)^{\frac{p}{Q_\alpha^*}} \\
&\leq C\left(\int_{\mathbb{H}^N}|u(\xi)|^{Q^*}d\xi\right)^{\frac{p}{Q_\alpha^*}}.
\end{aligned}$$

On using this in (3.6), we get

$$\begin{aligned}
\int_{\mathbb{H}^N}\int_{\mathbb{H}^N}\frac{K(\eta)|u(\eta)|^p}{|\eta^{-1}\circ\xi|^{\alpha}}K(\xi)|u(\xi)|^p d\eta d\xi &\leq C\left(\int_{\mathbb{H}^N}|u(\xi)|^{Q^*}d\xi\right)^{\frac{p}{Q^*}}\left(\int_{\mathbb{H}^N}|u(\xi)|^{Q^*}d\xi\right)^{\frac{p}{Q^*}} \\
&= C\|u\|^{2p}_{L^{Q^*}}(\mathbb{H}^N).
\end{aligned}$$

Similarly, we have

$$\int_{\mathbb{H}^N}\int_{\mathbb{H}^N}\frac{|u(\eta)|^{Q_\alpha^*}}{|\eta^{-1}\circ\xi|^{\alpha}}K(\xi)|u(\xi)|^p d\eta d\xi \leq C\|u\|^{p+Q_\alpha^*}_{L^{Q^*}}$$

and

$$\int_{\mathbb{H}^N}\int_{\mathbb{H}^N}\frac{|u(\eta)|^{Q_\alpha^*}}{|\eta^{-1}\circ\xi|^{\alpha}}|u(\xi)|^{Q_\alpha^*}d\eta d\xi \leq C\|u\|^{2Q_\alpha^*}_{L^{Q^*}}.$$

Hence,

$$\begin{aligned}
J(u) &\geq \frac{1}{2}\|u\|^2 - \frac{1}{2}C\left(\|u\|^{p+Q_\alpha^*}_{L^{Q^*}} + \|u\|^{2Q_\alpha^*}_{L^{Q^*}} + \|u\|^{2p}_{L^{Q^*}}\right) \\
&\geq \frac{1}{2}\|u\|^2 - \frac{1}{2}C\left(\|u\|^{p+Q_\alpha^*} + \|u\|^{2Q_\alpha^*} + \|u\|^{2p}\right).
\end{aligned}$$

Thus for sufficiently small $\|u\| = r$, $J(u) \geq \beta$.

(ii) For $u \in C_0^\infty(B_R \setminus \{0\})$,

$$
\begin{aligned}
J(tu) \leq & \frac{t^2}{2} \int_{|\xi| \leq R} |\nabla_{\mathbb{H}} u|^2 + V(\xi) u^2 d\xi \\
& - \frac{t^{2p}}{2} \int_{|\eta| \leq R} \left(\int_{|\xi| \leq R} \frac{|u(\eta)|^{Q_\alpha^*} + K(\eta)|u(\eta)|^p}{|\eta^{-1} \circ \xi|^\alpha} d\eta \right) \left(|u(\xi)|^{Q_\alpha^*} + K(\xi)|u(\xi)|^p d\xi \right).
\end{aligned}
$$

Thus, we conclude that $J(tu) \to -\infty$ as $t \to \infty$. Hence for some $e \in E$ with $\|e\| > \rho$ such that $J(e) < \beta$. This completes the proof.

□

Lemma 3.3. *Let $\{u_n\}$ be a $(PS)_c$ sequence associated to the functional J in $E \subset S^{1,2}(\mathbb{H}^N)$. Then $\{u_n\}$ is bounded.*

Proof. Let $\{u_n\}$ be a $(PS)_c$ sequence associated to the functional J in $E \subset S^{1,2}(\mathbb{H}^N)$. Then

$$
J(u_n) \to c, \; J'(u_n) \to 0. \tag{3.7}
$$

Let $\Psi(u) = \frac{1}{2} \int_{\mathbb{H}^N} \left(\int_{\mathbb{H}^N} \frac{G(\eta, u(\eta))}{|\eta^{-1} \circ \xi|^\alpha} d\eta \right) G(\xi, u(\xi)) d\xi$.

We prove $\langle \Psi'(u), u \rangle \geq 2\theta \Psi(u) > 0$, for some $\theta \in (1, 2)$.

Consider,

$$
\begin{aligned}
\frac{1}{2\theta} \langle \Psi'(u), u \rangle - \Psi(u) &= \int_{\mathbb{H}^N} \left(\int_{\mathbb{H}^N} \frac{G(\eta, u(\eta))}{|\eta^{-1} \circ \xi|^\alpha} d\eta \right) \left(\frac{1}{2\theta} g(\xi, u(\xi)) u - \frac{1}{2} G(\xi, u(\xi)) \right) d\xi \\
&= \int_{|\xi| \leq R} \left(\int_{\mathbb{H}^N} \frac{G(\eta, u(\eta))}{|\eta^{-1} \circ \xi|^\alpha} d\eta \right) \left(\frac{1}{2\theta} f(\xi, u) u - \frac{1}{2} F(\xi, u) \right) d\xi \\
&\quad + \int_{|\xi| > R} \left(\int_{\mathbb{H}^N} \frac{G(\eta, u(\eta))}{|\eta^{-1} \circ \xi|^\alpha} d\eta \right) \left(\frac{1}{2\theta} g(\xi, u(\xi)) u - \frac{1}{2} G(\xi, u(\xi)) \right) d\xi
\end{aligned}
$$

Let,

$$
\begin{aligned}
A &= \{\sigma \in \mathbb{H}^N : |\sigma| > R \text{ and } \ell f(\sigma, u) \leq V(\sigma) u(\sigma)\}, \\
B &= \{\sigma \in \mathbb{H}^N : |\sigma| > R \text{ and } \ell f(\sigma, u) > V(\sigma) u(\sigma)\}.
\end{aligned} \tag{3.8}
$$

We have, $u f(\xi, u) \geq \theta F(\xi, u)$. Now for $|\xi| > R$,

$$
\begin{aligned}
\frac{1}{2\theta} \langle \Psi'(u), u \rangle - \Psi(u) &= \int_{|\xi| > R} \left(\int_{\mathbb{H}^N} \frac{G(\eta, u(\eta))}{|\eta^{-1} \circ \xi|^\alpha} d\eta \right) \left(\frac{1}{2\theta} g(\xi, u(\xi)) u - \frac{1}{2} G(\xi, u(\xi)) \right) d\xi \\
&\geq \int_A \left(\int_{\mathbb{H}^N} \frac{G(\eta, u(\eta))}{|\eta^{-1} \circ \xi|^\alpha} d\eta \right) \left(\frac{1}{2\theta} g(\xi, u(\xi)) u - \frac{1}{2} G(\xi, u(\xi)) \right) d\xi \\
&\quad + \int_B \left(\int_{\mathbb{H}^N} \frac{G(\eta, u(\eta))}{|\eta^{-1} \circ \xi|^\alpha} d\eta \right) \left(\frac{1}{2\theta} g(\xi, u(\xi)) u - \frac{1}{2} G(\xi, u(\xi)) \right) d\xi \\
&\geq \int_A \left(\int_{\mathbb{H}^N} \frac{G(\eta, u(\eta))}{|\eta^{-1} \circ \xi|^\alpha} d\eta \right) \left(\frac{1}{2\theta} f(\xi, u) u - \frac{1}{2} F(\xi, u) \right) d\xi \\
&\quad + \int_B \left(\int_{\mathbb{H}^N} \frac{G(\eta, u(\eta))}{|\eta^{-1} \circ \xi|^\alpha} d\eta \right) \left(\frac{1}{2\theta \ell} - \frac{1}{4\ell} \right) V(\xi) u^2 d\xi \\
&\geq 0.
\end{aligned}
$$

Consider,

$$\begin{aligned} c + \|u_n\| o(1) &= J(u_n) - \frac{1}{2\theta}\langle J'(u_n), u_n\rangle \\ &= \left(\frac{1}{2} - \frac{1}{2\theta}\right)\|u_n\|^2 + \frac{1}{2\theta}\langle \Psi'(u_n), u_n\rangle - \Psi(u_n) \\ &\geq \left(\frac{1}{2} - \frac{1}{2\theta}\right)\|u_n\|^2. \end{aligned}$$

Thus, $\{u_n\}$ is bounded. □

Lemma 3.4. *Let V and K satisfy (Σ_1), (Σ_2) and let $\{u_n\}$ be as $(PS)_c$ sequence of J. Then, there exists a positive number $c_1 > 0$ such that for $c < c_1$,*

$$\lim_{n\to\infty}\int_{B_R}\int_{\mathbb{H}^N}\frac{|(u_n-u)(\xi)|^{Q_\alpha^*}|(u_n-u)(\eta)|^{Q_\alpha^*}}{|\eta^{-1}\circ\xi|^\alpha}\,d\eta\,d\xi = 0$$

Proof. Let $\{u_n\}$ be a $(PS)_c$ sequence. By Lemma 3.3, the sequence $\{u_n\}$ is bounded in E. Since E is reflexive, there exists $u \in E$ such that, up to a subsequence, $u_n \rightharpoonup u$ weakly in E. Consequently, $u_n \to u$ strongly in $L^q_{\text{loc}}(\mathbb{H}^N)$ for every $q \in [2, Q^*)$, and $u_n(\xi) \to u(\xi)$ a.e. in $\mathbb{H}^N$. Moreover, $u_n \rightharpoonup u$ in $S^{1,2}(\mathbb{H}^N)$, and hence, by Sobolev embedding, $u_n \rightharpoonup u$ in $L^{Q^*}(\mathbb{H}^N)$. Therefore, there exist bounded nonnegative measures ω, ζ, and ν such that, as $n \to \infty$,

$$|\nabla_{\mathbb{H}} u_n|^2 \rightharpoonup \omega, \qquad |u_n|^{Q^*} \rightharpoonup \zeta, \qquad \int_{\mathbb{H}^N}\frac{|u_n(\eta)|^{Q_\alpha^*}}{|\eta^{-1}\circ\xi|^\alpha}\,d\eta\,|u_n(\xi)|^{Q_\alpha^*} \rightharpoonup \nu.$$

Therefore by Lemma 2.3, there exists a countable sequence of points $\{z_i\}_{i\in I} \subset \mathbb{H}^N$ and families of positive numbers $\{\nu_i : i \in I\}$, $\{\zeta_i : i \in I\}$ and $\{\omega_i : i \in I\}$ such that

$$\begin{gathered} \nu = \left(\int_{\mathbb{R}^N}\frac{|u(\eta)|^{Q_\alpha^*}}{|\eta^{-1}\circ\xi|^\alpha}dy\right)|u(x)|^{Q_\alpha^*} + \Sigma_{i\in I}\nu_i\delta_{z_i}, \qquad \Sigma_{i\in I}\nu_i^{\frac{1}{Q_\alpha^*}} < \infty, \\ \omega \geq |\nabla_{\mathbb{H}} u|^2 + \sum_{i\in I}\omega_i\delta_{z_i}, \\ \zeta \geq |u|^{Q^*} + \sum_{i\in I}\zeta_i\delta_{z_i}, \end{gathered}$$

and

$$S_{HG}\nu_i^{\frac{1}{Q_\alpha^*}} \leq \omega_i, \qquad \nu_i^{\frac{Q}{2Q-\alpha}} \leq C(Q,\alpha)^{\frac{Q}{2Q-\alpha}}\zeta_i,$$

where δ_ξ is the Dirac-mass of mass 1 concentrated at $\xi \in \mathbb{H}^N$.

Let $\phi \in C_0^\infty(\mathbb{H}^n)$ be such that $0 \leq \phi \leq 1$,

$$\phi(\xi) = \begin{cases} 1 & \text{if } \xi \in B_1(0), \\ 0 & \text{if } \xi \in \mathbb{H}^N \setminus B_2(0). \end{cases} \tag{3.9}$$

Let $\varepsilon > 0$. For fixed $i \in I$, define

$$\phi_{\varepsilon,i}(\xi) = \phi\left(\delta_{\frac{2}{\varepsilon}}(z_i^{-1}\circ\xi)\right).$$

From (3.7), we have $J'(u_n) \to 0$ as $n \to \infty$. Therefore, $\langle J'(u_n), \phi_{\varepsilon,i}u_n\rangle \to 0$ as $n \to \infty$. Hence,

$$\begin{aligned} \lim_{n\to\infty}\int_{\mathbb{H}^N} &\left(\nabla_{\mathbb{H}} u_n \cdot \nabla_{\mathbb{H}}\left(u_n \cdot \phi_{\varepsilon,i}(\xi)u_n\right. + V(\xi)|u_n|^2\phi_{\varepsilon,i}\right)d\xi \\ &- \int_{\mathbb{H}^N}\left(\int_{\mathbb{H}^N}\frac{G(\eta,u_n)}{|\eta^{-1}\circ\xi|^\alpha}d\eta\right)g(\xi,u_n)u_n(\xi)\phi_{\varepsilon,i}(\xi)d\xi = 0. \end{aligned} \tag{3.10}$$

Next, we claim that

$$\lim_{\varepsilon\to 0}\lim_{n\to\infty}\int_{\mathbb{H}^N} u_n \nabla_{\mathbb{H}} u_n \nabla_{\mathbb{H}} \phi_{\varepsilon,i} = 0. \tag{3.11}$$

Indeed, the boundedness of $\{u_n\}$, combined with Hölder's inequality and the Sobolev embedding theorem, yields

$$\begin{aligned}
&\lim_{\varepsilon\to 0}\lim_{n\to\infty}\left|\int_{\mathbb{H}^N} u_n(\xi)\nabla_{\mathbb{H}} u_n(\xi)\cdot\nabla_{\mathbb{H}}\phi_{\varepsilon,i}(\xi)\,d\xi\right| \\
&\quad\le \lim_{\varepsilon\to 0}\lim_{n\to\infty}\left(\int_{\mathbb{H}^N}|\nabla_{\mathbb{H}} u_n(\xi)|^2\,d\xi\right)^{\frac{1}{2}}\left(\int_{\mathbb{H}^N}|u_n(\xi)\nabla_{\mathbb{H}}\phi_{\varepsilon,i}(\xi)|^2\,d\xi\right)^{\frac{1}{2}} \\
&\quad\le \lim_{\varepsilon\to 0}\lim_{n\to\infty} C\left(\int_{B_\varepsilon(z_i)}|u_n(\xi)|^2|\nabla_{\mathbb{H}}\phi_{\varepsilon,i}(\xi)|^2\,d\xi\right)^{\frac{1}{2}} \\
&\quad\le \lim_{\varepsilon\to 0} C\left(\int_{B_\varepsilon(z_i)}|u(\xi)|^2|\nabla_{\mathbb{H}}\phi_{\varepsilon,i}(\xi)|^2\,d\xi\right)^{\frac{1}{2}} \\
&\quad\le \lim_{\varepsilon\to 0} C\left(\int_{B_\varepsilon(z_i)}|\nabla_{\mathbb{H}}\phi_{\varepsilon,i}(\xi)|^Q\,d\xi\right)^{\frac{1}{Q}}\left(\int_{B_\varepsilon(z_i)}|u(\xi)|^{Q^*}\,d\xi\right)^{\frac{1}{Q^*}}.
\end{aligned}$$

One can see that using change of variables, $\int_{B_\varepsilon(z_i)}|\nabla_{\mathbb{H}}\phi_{\varepsilon,i}(\xi)|^Q\,d\xi$ is bounded as $\varepsilon\to 0$. Thus, (3.11) holds. Further, by using (Σ_2) and Lemma 2.5, with $m=2$, we have

$$\lim_{n\to\infty}\int_{\mathbb{H}^N} V(\xi)|u_n(\xi)|^2 d\xi = \lim_{n\to\infty}\int_{\mathbb{H}^N} V(\xi)|u(\xi)|^2 d\xi.$$

So by Hölder's inequality,

$$\begin{aligned}
\lim_{\varepsilon\to 0}\lim_{n\to\infty}\int_{\mathbb{H}^N} V(\xi)|u_n(\xi)|^2\phi_{\varepsilon,i}(\xi)d\xi &= \lim_{\varepsilon\to 0}\int_{\mathbb{H}^N} V(\xi)|u(\xi)|^2\phi_{\varepsilon,i}(\xi)d\xi \\
&\le \lim_{\varepsilon\to 0}\left(\int_{B_\varepsilon(z_i)}|V(\xi)|^{\frac{Q}{2}}\right)\left(\int_{B_\varepsilon(z_i)}|u(\xi)|^{Q^*}d\xi\right)^{\frac{1}{Q^*}}.
\end{aligned}$$

Hence,

$$\lim_{\varepsilon\to 0}\lim_{n\to\infty}\int_{\mathbb{H}^N} V(\xi)|u_n(\xi)|^2\phi_{\varepsilon,i}(\xi)d\xi = 0. \tag{3.12}$$

Therefore, using Lemma 2.3, (3.11) and (3.12), we get

$$\omega_i \le \lim_{\varepsilon\to 0}\lim_{n\to\infty}\int_{\mathbb{H}^N}|\nabla_{\mathbb{H}} u_n|^2\phi_{\varepsilon,i}(\xi)d\xi = \lim_{\varepsilon\to 0}\lim_{n\to\infty}\int_{\mathbb{H}^N}\nabla_{\mathbb{H}} u_n\cdot\nabla_{\mathbb{H}}\left(\phi_{\varepsilon,i}(\xi)u_n(\xi)\right)d\xi. \tag{3.13}$$

Suppose that there exists $i\in I$ such that $z_i\in\overline{B_R(0)}$. Hence for small $\varepsilon>0$,

$$\begin{aligned}
&\int_{\mathbb{H}^N}\left(\int_{\mathbb{H}^N}\frac{G(\eta,u_n)}{|\eta^{-1}\circ\xi|^\alpha}d\eta\right)g(\xi,u_n)u_n(\xi)\phi_{\varepsilon,i}(\xi)d\xi \\
&\quad= \int_{|\xi|\le R}\left(\int_{|\eta|\le R}\frac{|u_n(\eta)|^{Q^*_\alpha}+K(\eta)|u_n(\eta)|^p}{|\eta^{-1}\circ\xi|^\alpha}d\eta\right)(Q^*_\alpha|u_n(\xi)|^{Q^*_\alpha}+pK(\xi)|u_n(\xi)|^p)\phi_{\varepsilon,i}(\xi)d\xi \\
&\qquad+ \int_{|\xi|\le R}\left(\int_{A}\frac{|u_n(\eta)|^{Q^*_\alpha}+K(\eta)|u_n(\eta)|^p}{|\eta^{-1}\circ\xi|^\alpha}d\eta\right)(Q^*_\alpha|u_n(\xi)|^{Q^*_\alpha}+pK(\xi)|u_n(\xi)|^p)\phi_{\varepsilon,i}(\xi)d\xi \\
&\qquad+ \int_{|\xi|\le R}\left(\int_{B}\frac{V(\eta)|u_n(\eta)|^2}{2\ell|\eta^{-1}\circ\xi|^\alpha}d\eta\right)(Q^*_\alpha|u_n(\xi)|^{Q^*_\alpha}+pK(\xi)|u_n(\xi)|^p)\phi_{\varepsilon,i}(\xi)d\xi.
\end{aligned} \tag{3.14}$$

We now claim

$$\lim_{n\to\infty}\int_{|\xi|\leq R}\int_{|\eta|\geq R}\frac{V(\eta)|u_n(\eta)|^2|u_n(\xi)|^p}{|\eta^{-1}\circ\xi|^\alpha}\phi_{\varepsilon,i}(\xi)d\eta d\xi\to 0\quad\text{as }\varepsilon\to 0. \tag{3.15}$$

By the Hardy–Littlewood–Sobolev inequality,

$$\int_{|\xi|\leq R}\int_{|\eta|\geq R}\frac{V(\eta)|u_n(\eta)|^2|u_n(\xi)|^p}{|\eta^{-1}\circ\xi|^\alpha}\phi_{\varepsilon,i}(\xi)d\eta d\xi\leq C\|V|u_n|^2\|_{L^{\frac{2Q}{2Q-\alpha}}(\mathbb{H}^N)}\|u_n\|^p_{L^{\frac{2Qp}{2Q-\alpha}}(B_\varepsilon(z_i))}.$$

On using Hölder's inequality, Sobolev embedding and (Σ_1), we get

$$\int_{|\xi|\leq R}\int_{|\eta|\geq R}\frac{V(\eta)|u_n(\eta)|^2|u_n(\xi)|^p}{|\eta^{-1}\circ\xi|^\alpha}\phi_{\varepsilon,i}(\xi)d\eta d\xi\leq C\|u\|^p_{L^{\frac{2Qp}{2Q-\alpha}}(B_\varepsilon(z_i))}.$$

Letting $\varepsilon\to 0$, we prove the claim. Similarly, by the Hardy–Littlewood–Sobolev inequality and (Σ_2), we have

$$\lim_{n\to\infty}\int_{|\xi|\leq R}\int_{|\eta|\leq R}\frac{K(\eta)|u_n(\eta)|^pK(\xi)|u_n(\xi)|^p}{|\eta^{-1}\circ\xi|^\alpha}\phi_{\varepsilon,i}(\xi)d\eta d\xi\to 0\quad\text{as }\varepsilon\to 0, \tag{3.16}$$

$$\lim_{n\to\infty}\int_{|\xi|\leq R}\int_{|\eta|\leq R}\frac{|u_n(\eta)|^{Q^*_\alpha}K(\xi)|u_n(\xi)|^p}{|\eta^{-1}\circ\xi|^\alpha}\phi_{\varepsilon,i}(\xi)d\eta d\xi\to 0,\quad\text{as }\varepsilon\to 0. \tag{3.17}$$

Next, we claim that

$$\lim_{n\to\infty}\int_{|\xi|\leq R}\int_{A}\frac{K(\eta)|u_n(\eta)|^p|u_n(\xi)|^{Q^*_\alpha}}{|\eta^{-1}\circ\xi|^\alpha}\phi_{\varepsilon,i}(\xi)d\eta d\xi\to 0\quad\text{as }\varepsilon\to 0.$$

We have $p\in\left(2,\frac{2Q-\alpha}{Q-2}\right)$, $K\in L^{\frac{Q^*_\alpha}{Q^*_\alpha-p}}(\mathbb{H}^N)$. Hence, $K^{\frac{2Q}{2Q-\alpha}}\in L^{\frac{Q^*}{Q^*-\frac{2Qp}{2Q-\alpha}}}(\mathbb{H}^N)$. By Lemma 2.5,

$$\int_A\left(K(\eta)|u_n(\eta)-u(\eta)|^p\right)^{\frac{2Q}{2Q-\alpha}}\to 0$$

as $n\to\infty$, i.e $K|u_n|^p\to K|u|^p$ in $L^{\frac{2Q}{2Q-\alpha}}(\mathbb{H}^N)$. By the Hardy–Littlewood–Sobolev inequality, we have

$$\int_A\frac{K(\eta)|u_n(\eta)|^p}{|\eta^{-1}\circ\xi|^\alpha}d\eta\to\int_A\frac{K(\eta)|u(\eta)|^p}{|\eta^{-1}\circ\xi|^\alpha}d\eta$$

in $L^{\frac{2Q}{\alpha}}(\mathbb{H}^N)$. Now since u_n is a bounded sequence in E, we have $\int_{|\xi|\leq R}(|u_n(\xi)|^{Q^*_\alpha}\phi_{\varepsilon,i})^{\frac{2Q}{2Q-\alpha}}d\xi\leq C$. Since $u_n\rightharpoonup u$ in E, as $n\to\infty$, $|u_n(\xi)|^{Q^*_\alpha}\phi_{\varepsilon,i}(\xi)\rightharpoonup|u(\xi)|^{Q^*_\alpha}\phi_{\varepsilon,i}(\xi)$ in $L^{\frac{2Q}{2Q-\alpha}}(|\xi|\leq R)$. Thus,

$$\int_{|\xi|\leq R}\int_A\frac{K(\eta)|u_n(\eta)|^p}{|\eta^{-1}\circ\xi|^\alpha}d\eta|u_n(\xi)|^{Q^*_\alpha}\phi_{\varepsilon,i}(\xi)d\xi\to\int_{|\xi|\leq R}\int_A\frac{K(\eta)|u(\eta)|^p}{|\eta^{-1}\circ\xi|^\alpha}d\eta|u(\xi)|^{Q^*_\alpha}\phi_{\varepsilon,i}(\xi)d\xi,$$

as $n\to\infty$. Now using the Hardy–Littlewood–Sobolev inequality,

$$\int_{|\xi|<R}\int_A\frac{K(\eta)|u(\eta)|^p}{|\eta^{-1}\circ\xi|^\alpha}d\eta|u(\xi)|^{Q^*_\alpha}\phi_{\varepsilon,i}(\xi)d\xi\leq C\|K|u|^p\|_{L^{\frac{2Q}{2Q-\alpha}}(|\xi|\leq R)}\|u\|^{Q^*_\alpha}_{L^{Q^*}(B_\varepsilon(z_i))}$$

and taking limit as $\varepsilon\to 0$, we prove the claim. Thus, we have

$$\lim_{n\to\infty}\int_{|\xi|}\int_A\frac{K(\eta)|u_n(\eta)|^p|u_n(\xi)|^{Q^*_\alpha}}{|\eta^{-1}\circ\xi|^\alpha}\phi_{\varepsilon,i}(\xi)d\eta d\xi\to 0\quad\text{as }\varepsilon\to 0.$$

Similarly, we can prove

$$\lim_{n\to\infty}\int_{|\xi|\le R}\int_B \frac{V(\eta)|u_n(\eta)|^2|u_n(\xi)|^{Q^*_\alpha}}{|\eta^{-1}\circ\xi|^\alpha}\phi_{\varepsilon,i}(\xi)d\eta d\xi \to 0 \quad \text{as } \varepsilon\to 0.$$

Hence, taking limit as $\varepsilon \to 0$ and $n\to\infty$ in (3.14) and using Lemma 2.3 we get

$$\begin{aligned}
&\lim_{\varepsilon\to\infty}\lim_{n\to\infty}\int_{\mathbb{H}^N}\left(\int_{\mathbb{H}^N}\frac{G(\eta,u_n)}{|\eta^{-1}\circ\xi|^\alpha}d\eta\right)g(\xi,u_n)u_n(\xi)\phi_{\varepsilon,i}(\xi)d\xi\\
&\le \lim_{\varepsilon\to\infty}\lim_{n\to\infty} Q^*_\alpha\int_{|\xi|\le R}\left(\int_{|\eta|\le R}\frac{|u_n(\eta)|^{Q^*_\alpha}}{|\eta^{-1}\circ\xi|^\alpha}d\eta\right)|u_n(\xi)|^{Q^*_\alpha}\phi_{\varepsilon,i}(\xi)d\xi\\
&\quad + Q^*_\alpha\int_{|\xi|\le R}\left(\int_A\frac{|u_n(\eta)|^{Q^*_\alpha}}{|\eta^{-1}\circ\xi|^\alpha}d\eta\right)|u_n(\xi)|^{Q^*_\alpha}\phi_{\varepsilon,i}(\xi)d\xi\\
&\le Q^*_\alpha\nu_i
\end{aligned}$$

where we have used (2.2) and Hölder's inequality. Hence, from (3.10) and (3.12)

$$\begin{aligned}
\omega_i &\le Q^*_\alpha\nu_i\,,\\
S_{HG}\nu_i^{\frac{Q-2}{2Q-\alpha}} &\le Q^*_\alpha\nu_i\,,\\
S_{HG} &\le Q^*_\alpha\nu_i^{\frac{Q-\alpha+2}{2Q-\alpha}}.
\end{aligned}$$

Hence, we conclude that $\nu_i > \delta > 0$, for some fixed δ. Now by Lemma 2.3, we have $\sum_{i\in I}\nu_i^{\frac{2}{Q^*_\alpha}} < \infty$. Hence, I is finite.

Consider,

$$\begin{aligned}
c+o(1) &= J(u_n) - \frac{1}{2}\langle J'(u_n),u_n\rangle\\
&\ge \frac12\int_{\mathbb{H}^N}\left(\int_{\mathbb{H}^N}\frac{G(\eta,u_n)}{|\eta^{-1}\circ\xi|^\alpha}d\eta\right)(g(\xi,u_n)u_n - 2G(\xi,u_n))d\xi\\
&= \frac12\int_{\mathbb{H}^N}\left(\int_{|\eta|\le R}\frac{|u_n|^{Q^*_\alpha}+K(\eta)|u_n|^p}{|\eta^{-1}\circ\xi|^\alpha}d\eta\right)((Q^*_\alpha-2)|u_n|^{Q^*_\alpha}+(p-2)K(\xi)|u_n|^p)d\xi\\
&\quad+\frac12\int_{\mathbb{H}^N}\left(\int_A\frac{|u_n|^{Q^*_\alpha}+K(\eta)|u_n|^p}{|\eta^{-1}\circ\xi|^\alpha}d\eta\right)((Q^*_\alpha-2)|u_n|^{Q^*_\alpha}+(p-2)K(\xi)|u_n|^p)d\xi\\
&\quad+\frac12\int_{\mathbb{H}^N}\left(\int_B\frac{V(\eta)|u_n|^2}{2\ell|\eta^{-1}\circ\xi|^\alpha}d\eta\right)((Q^*_\alpha-2)|u_n|^{Q^*_\alpha}+(p-2)K(\xi)|u_n|^p)d\xi\\
&\ge\left(\frac{Q^*_\alpha-2}{2}\right)\int_{\mathbb{H}^N}\left(\int_{|\eta|\le R}\frac{|u_n(\eta)|^{Q^*_\alpha}|u_n(\xi)|^{Q^*_\alpha}}{|\eta^{-1}\circ\xi|^\alpha}d\eta\right)\phi_{\varepsilon,i}(\xi)d\xi\\
&\quad+\left(\frac{Q^*_\alpha-2}{2}\right)\int_{\mathbb{H}^N}\left(\int_A\frac{|u_n(\eta)|^{Q^*_\alpha}|u_n(\xi)|^{Q^*_\alpha}}{|\eta^{-1}\circ\xi|^\alpha}d\eta\right)\phi_{\varepsilon,i}(\xi)d\xi\\
&\ge\left(\frac{Q^*_\alpha-2}{2Q^*_\alpha}\right)\omega_i\\
&\ge\left(\frac{Q^*_\alpha-2}{2Q^*_\alpha}\right)S_{HG}\nu_i^{\frac{Q-2}{2Q-\alpha}}\\
&\ge\left(\frac{Q^*_\alpha-2}{2}\right)\left(\frac{S_{HG}}{Q^*_\alpha}\right)^{\frac{2Q-\alpha}{Q-\alpha+2}}
\end{aligned}$$

as $\varepsilon \to 0$. Therefore,

$$c \geq \left(\frac{Q_\alpha^* - 2}{2}\right)\left(\frac{S_{HG}}{Q_\alpha^*}\right)^{\frac{2Q-\alpha}{Q-2+\alpha}}.$$

Hence for level less than $c_1 = \left(\frac{Q_\alpha^*-2}{2}\right)\left(\frac{S_{HG}}{Q_\alpha^*}\right)^{\frac{2Q-\alpha}{Q-2+\alpha}}$, I is empty or $\nu_i = 0$ for all $z_i \in B_R(0)$ and using Lemma 2.4 we get the desired result. □

Lemma 3.5. *Let V and K satisfy (Σ_1), (Σ_2) and $\{u_n\}$ be a $(PS)_c$ sequence for J. Then, at least for a subsequence and for any $\varepsilon > 0$ there exists $r \geq R$ such that, for all r*

$$\limsup_{n\to\infty} \int_{|\xi|\geq 2r} (|\nabla_{\mathbb{H}} u_n(\xi)|^2 + V(\xi)u_n(\xi)^2)d\xi < \varepsilon.$$

Proof. For $r > R$, let $\psi_r \in C^\infty(\mathbb{H}^n)$ be such that $0 \leq \psi_r \leq 1$,

$$\psi_r(\xi) = \begin{cases} 0 & \text{if } \xi \in B_r(0), \\ 1 & \text{if } \xi \in \mathbb{H}^N \setminus B_{2r}(0), \end{cases} \tag{3.18}$$

and $|\nabla_{\mathbb{H}}\psi_r(\xi)| \leq \frac{2}{r}$. Since $J'(u_n) \to 0$ as $n \to \infty$, $\langle J'(u_n), \psi_r u_n\rangle \to 0$ as $n \to \infty$. Thus,

$$\begin{aligned}
&\lim_{n\to\infty}\int_{\mathbb{H}^N} \nabla_{\mathbb{H}} u_n(\xi)\cdot\nabla_{\mathbb{H}}(u_n(\xi)\psi_r(\xi))d\xi + \lim_{n\to\infty}\int_{\mathbb{H}^N} V(\xi)|u_n(\xi)|^2\psi_r(\xi)d\xi \\
&\quad = \lim_{n\to\infty}\int_{\mathbb{H}^N}\left(\int_{\mathbb{H}^N}\frac{G(\eta,u_n)}{|\eta^{-1}\circ\xi|^\alpha}d\eta\right)g(\xi,u_n)u_n\psi_r(\xi)d\xi,
\end{aligned}$$

$$\begin{aligned}
&\lim_{n\to\infty}\int_{\mathbb{H}^N} |\nabla_{\mathbb{H}} u_n(\xi)|^2\psi_r(\xi)d\xi + \lim_{n\to\infty}\int_{\mathbb{H}^N} V(\xi)|u_n(\xi)|^2\psi_r(\xi)d\xi \\
&\quad + \lim_{n\to\infty}\int_{\mathbb{H}^N} u_n(\xi)\nabla_{\mathbb{H}} u_n(\xi)\cdot\nabla_{\mathbb{H}}\psi_r(\xi)d\xi \\
&\quad = \lim_{n\to\infty}\int_{\mathbb{H}^N}\left(\int_{\mathbb{H}^N}\frac{G(\eta,u_n)}{|\eta^{-1}\circ\xi|^\alpha}d\eta\right)g(\xi,u_n)u_n\psi_r(\xi)d\xi.
\end{aligned}$$

Next, we claim that

$$\lim_{r\to\infty}\lim_{n\to\infty}\int_{\mathbb{H}^N} u_n(\xi)\nabla_{\mathbb{H}} u_n(\xi)\cdot\nabla_{\mathbb{H}}\psi_r(\xi)d\xi = 0. \tag{3.19}$$

Since $\nabla_{\mathbb{H}}\psi_r(\xi) = 0$ for $|\xi| \leq r$ and $|\xi| \geq 2r$, it follows from Hölder's inequality, boundedness of $\{u_n\}$ and compact embedding $S^{1,2}(\mathbb{H}^N) \hookrightarrow L^2_{\text{loc}}(\mathbb{H}^N)$ that

$$\begin{aligned}
\int_{\mathbb{H}^N} u_n(\xi)\nabla_{\mathbb{H}} u_n(\xi)\cdot\nabla_{\mathbb{H}}\psi_r(\xi)d\xi &\leq \frac{2}{r}\int_{r\leq|\xi|\leq 2r}|u_n(\xi)||\nabla_{\mathbb{H}} u_n(\xi)|d\xi \\
&\leq \frac{2}{r}\left(\int_{r\leq|\xi|\leq 2r}|u_n|^2d\xi\right)^{\frac{1}{2}}\left(\int_{r\leq|\xi|\leq 2r}|\nabla u_n|^2d\xi\right)^{\frac{1}{2}} \\
&\leq \frac{C}{r}\left(\int_{r\leq|\xi|\leq 2r}|u_n|^2d\xi\right)^{\frac{1}{2}}. \\
&\to \frac{C}{r}\left(\int_{r\leq|\xi|\leq 2r}|u(\xi)|^2d\xi\right)^{\frac{1}{2}} \quad \text{as } n\to\infty.
\end{aligned}$$

Thus, we have

$$\begin{aligned}\lim_{n\to\infty}\int_{\mathbb{H}^N} u_n(\xi)\nabla_{\mathbb{H}}u_n(\xi)\cdot\nabla_{\mathbb{H}}\psi_r(\xi)d\xi &\le \frac{C}{r}\left(\int_{r\le|\xi|\le 2r}|u(\xi)|^2d\xi\right)^{\frac{1}{2}}\\ &\le \frac{C}{r}\left(\int_{r\le|\xi|\le 2r}d\xi\right)^{\frac{1}{Q}}\left(\int_{r\le|\xi|\le 2r}|u(\xi)|^{Q^*}d\xi\right)^{\frac{1}{Q^*}}\\ &\le C\left(\int_{r\le|\xi|}|u(\xi)|^{Q^*}d\xi\right)^{\frac{1}{Q^*}},\end{aligned}$$

which shows that

$$\lim_{r\to\infty}\lim_{n\to\infty}\int_{\mathbb{H}^N} u_n(\xi)\nabla_{\mathbb{H}}u_n(\xi)\cdot\nabla_{\mathbb{H}}\psi_r(\xi)d\xi = 0.$$

This proves (3.19). Next, we see that

$$\int_{\mathbb{H}^N}\left(\int_{\mathbb{H}^N}\frac{G(\eta,u_n)}{|\eta^{-1}\circ\xi|^\alpha}d\eta\right)g(\xi,u_n)u_n\psi_r(\xi)d\xi \le \int_{|\xi|\ge r}\left(\int_{\mathbb{H}^N}\frac{G(\eta,u_n)}{|\eta^{-1}\circ\xi|^\alpha}d\eta\right)g(\xi,u_n)u_n\psi_r(\xi)d\xi.$$

Since $r>R$, using (3.3), boundedness of $\{u_n\}$, (Σ_1), and (Σ_2), we have

$$\begin{aligned}&\int_{\mathbb{H}^N}\left(\int_{\mathbb{H}^N}\frac{G(\eta,u_n)}{|\eta^{-1}\circ\xi|^\alpha}d\eta\right)g(\xi,u_n)u_n\psi_r(\xi)d\xi\\ &\quad\le \int_{|\xi|\ge r}\left(\int_{\mathbb{H}^N}\frac{G(\eta,u_n)}{|\eta^{-1}\circ\xi|^\alpha}d\eta\right)\frac{V(\xi)|u_n(\xi)|^2}{\ell}d\xi\\ &\quad\le \int_{|\xi|\ge r}\left(\int_{\mathbb{H}^N}\frac{|u_n(\eta)|^{Q^*_\alpha}+K(\eta)|u_n(\eta)|^p}{|\eta^{-1}\circ\xi|^\alpha}d\eta\right)\frac{V(\xi)|u_n(\xi)|^2}{\ell}d\xi\\ &\qquad+\int_{|\xi|\ge r}\left(\int_{\mathbb{H}^N}\frac{V(\eta)|u_n(\eta)|^2}{2\ell|\eta^{-1}\circ\xi|^\alpha}d\eta\right)\frac{V(\xi)|u_n(\xi)|^2}{\ell}d\xi\\ &\quad\le C\left(\int_{|\xi|\ge r}|V(\xi)|^{\frac{2Q}{2Q-\alpha}}|u_n(\xi)|^{\frac{4Q}{2Q-\alpha}}d\xi\right)^{\frac{2Q-\alpha}{2Q}}\left(\int_{\mathbb{H}^N}\left(|u_n(\eta)|^{Q^*}+|K(\eta)|^{\frac{2Q}{2Q-\alpha}}|u_n(\eta)|^{\frac{2Qp}{2Q-\alpha}}\right)d\eta\right)^{\frac{2Q-\alpha}{2Q}}\\ &\qquad+C\left(\int_{|\xi|\ge r}|V(\xi)|^{\frac{2Q}{2Q-\alpha}}|u_n(\xi)|^{\frac{4Q}{2Q-\alpha}}d\xi\right)^{\frac{2Q-\alpha}{2Q}}\left(\int_{\mathbb{H}^N}|V(\eta)|^{\frac{2Q}{2Q-\alpha}}|u_n(\eta)|^{\frac{4Q}{2Q-\alpha}}d\eta\right)^{\frac{2Q-\alpha}{2Q}}\\ &\quad\le C\left(\int_{|\xi|\ge r}|V(\xi)|^{\frac{2Q}{2Q-\alpha}}|u_n(\xi)|^{\frac{4Q}{2Q-\alpha}}d\xi\right)^{\frac{2Q-\alpha}{2Q}}\left(\int_{\mathbb{H}^N}\left(|u_n(\eta)|^{Q^*}+|K(\eta)|^{\frac{2Q}{2Q-\alpha}}|u_n(\eta)|^{\frac{2Qp}{2Q-\alpha}}\right)d\eta\right)^{\frac{2Q-\alpha}{2Q}}\\ &\qquad+C\left(\int_{|\xi|\ge r}|V(\xi)|^{\frac{2Q}{2Q-\alpha}}|u_n(\xi)|^{\frac{4Q}{2Q-\alpha}}d\xi\right)^{\frac{2Q-\alpha}{2Q}}\left(\int_{\mathbb{H}^N}|V(\eta)|^{\frac{2Q}{4-\alpha}}d\eta\right)^{\frac{4-\alpha}{2Q}}\left(\int_{\mathbb{H}^N}|u_n(\eta)|^{Q^*}d\eta\right)^{\frac{2Q-4}{2Q}}\\ &\quad\le C\left(\int_{|\xi|\ge r}|V(\xi)|^{\frac{2Q}{2Q-\alpha}}|u_n(\xi)|^{\frac{4Q}{2Q-\alpha}}d\xi\right)^{\frac{2Q-\alpha}{2Q}}.\end{aligned}\tag{3.20}$$

Since $\alpha\in(0,4)$, it follows that $\frac{4Q}{2Q-\alpha}\in(2,Q^*)$. Moreover, as $V\in L^{\frac{2Q}{4-\alpha}}(\mathbb{H}^N)$, we have

$$V^{\frac{2Q}{2Q-\alpha}}\in L^{\frac{Q^*}{Q^*-\frac{4Q}{2Q-\alpha}}}(\mathbb{H}^N).$$

Therefore, by applying Lemma 2.5, we deduce that

$$\lim_{n\to\infty}\int_{|\xi|\ge r}|V(\xi)|^{\frac{2Q}{2Q-\alpha}}|u_n(\xi)|^{\frac{4Q}{2Q-\alpha}}d\xi = \int_{|\xi|\ge r}|V(\xi)|^{\frac{2Q}{2Q-\alpha}}|u(\xi)|^{\frac{4Q}{2Q-\alpha}}d\xi.$$

Hence,

$$\lim_{r\to\infty}\lim_{n\to\infty}\int_{|\xi|\geq r}|V(\xi)|^{\frac{2Q}{2Q-\alpha}}|u_n(\xi)|^{\frac{4Q}{2Q-\alpha}}d\xi=0.$$

Substituting this into (3.20), we obtain

$$\lim_{r\to\infty}\lim_{n\to\infty}\int_{\mathbb{H}^N}\left(\int_{\mathbb{H}^N}\frac{G(\eta,u_n)}{|\eta^{-1}\circ\xi|^{\alpha}}d\eta\right)g(\xi,u_n)u_n\psi_r(\xi)d\xi=0.$$

This completes the proof. □

We now find the upper bound for the functional J. Before that, we state some asymptotic estimates which can be proved similar to [21, Lemma 3.1] and [14, Lemma 1.1]. The minimizer for the Sobolev inequality

$$S\,\|u\|^2_{L^{Q^*}(\mathbb{H}^N)}\leq\int_{\mathbb{H}^N}|\nabla_{\mathbb{H}}u(\xi)|^2\,d\xi,$$

is attained by

$$U(\xi)=U(x,y,t)=\frac{C}{((t^2+(1+|x|^2+|y^2|)^2))^{\frac{Q-2}{4}}},$$

where C is a suitable positive constant. Let,

$$U_\varepsilon(\xi)=\frac{C}{((t^2+(\varepsilon+|x|^2+|y^2|)^2))^{\frac{Q-2}{4}}},\tag{3.21}$$

where C is a suitable constant. Hence

$$U_\varepsilon(\xi)=\varepsilon^{-\frac{Q-2}{2}}U(\delta_{\frac{1}{\sqrt{\varepsilon}}}(\xi)).$$

Define,

$$\phi(\xi)=\begin{cases}1 & \text{if } \xi\in B_{\frac{R}{2}}(0),\\ 0 & \text{if } \xi\in B^c_R(0),\end{cases}$$

$$u_\varepsilon(\xi)=U_\varepsilon(\xi)\phi(\xi),$$

$$v_\varepsilon(\xi)=\frac{u_\varepsilon(\xi)}{\left(\int_{\mathbb{H}^N}\int_{\mathbb{H}^N}\frac{|u_\varepsilon(\eta)|^{Q^*_\alpha}|u_\varepsilon(\xi)|^{Q^*_\alpha}}{|\eta^{-1}\circ\xi|^{\alpha}}d\eta d\xi\right)^{\frac{1}{2Q^*_\alpha}}}.\tag{3.22}$$

Since $S^{1,2}(\mathbb{H}^N)\hookrightarrow L^2_{\mathrm{loc}}(\mathbb{H}^N)$ and $\phi\in C^\infty_c(\mathbb{H}^N)$, we have $u_\varepsilon\in L^2_{\mathrm{loc}}(\mathbb{H}^N)$. Consequently, $v_\varepsilon\in E$. Further, we have

$$\int_{\mathbb{H}^N}|\nabla_{\mathbb{H}}u_\varepsilon(\xi)|^2d\xi=\varepsilon^{-\frac{Q-2}{2}}\int_{\mathbb{H}^N}|\nabla_{\mathbb{H}}U(\xi)|^2d\xi+O(1).\tag{3.23}$$

$$\int_{\mathbb{H}^N}|u_\varepsilon(\xi)|^{Q^*}d\xi=\varepsilon^{-\frac{Q}{2}}\int_{\mathbb{H}^N}|U(\xi)|^{Q^*}d\xi+O(1).\tag{3.24}$$

Using the Hardy–Littlewood–Sobolev inequality, we get

$$\int_{\mathbb{H}^N}\int_{\mathbb{H}^N}\frac{|u_\varepsilon(\eta)|^{Q^*_\alpha}|u_\varepsilon(\xi)|^{Q^*_\alpha}}{|\eta^{-1}\circ\xi|^{\alpha}}d\eta d\xi\leq C(Q,\alpha)\varepsilon^{-\frac{2Q-\alpha}{2}}\left(\int_{\mathbb{H}^N}|U(\xi)|^{Q^*}d\xi\right)^{\frac{2Q-\alpha}{Q}}+O(1)\tag{3.25}$$

Following the calculations in [21, Lemma 3.1], we obtain

$$\int_{\mathbb{H}^N}\int_{\mathbb{H}^N}\frac{|u_\varepsilon(\eta)|^{Q^*_\alpha}|u_\varepsilon(\xi)|^{Q^*_\alpha}}{|\eta^{-1}\circ\xi|^{\alpha}}d\eta d\xi\geq\varepsilon^{-\frac{2Q-\alpha}{2}}\int_{\mathbb{H}^N}\int_{\mathbb{H}^N}\frac{|U(\eta)|^{Q^*_\alpha}|U(\xi)|^{Q^*_\alpha}}{|\eta^{-1}\circ\xi|^{\alpha}}d\eta d\xi-O(1).\tag{3.26}$$

Lemma 3.6.

$$\int_{\mathbb{H}^N} |u_\varepsilon(\xi)|^q d\xi \leq \begin{cases} C\varepsilon^{\frac{Q}{2}-\frac{Q-2}{2}q} + O(1), & q \in (2, Q^*), \\ 1 + \log \frac{R}{\sqrt{\varepsilon}}, & q = 2, Q = 4, \\ C\varepsilon^{-\frac{Q-4}{2}} + O(1), & q = 2, Q > 4. \end{cases}$$

Proof. If $q \in (2, Q^*)$, then

$$\begin{aligned} \int_{\mathbb{H}^N} |u_\varepsilon(\xi)|^q d\xi &\leq \int_{|\xi|\leq R} |U_\varepsilon(\xi)|^q d\xi \\ &\leq \varepsilon^{\frac{Q}{2}-\frac{Q-2}{2}q} \int_{|\xi|\leq \frac{R}{\sqrt{\varepsilon}}} |U(\xi)|^q d\xi \\ &= C\varepsilon^{\frac{Q}{2}-\frac{Q-2}{2}q} \int_{|\xi|\leq \frac{R}{\sqrt{\varepsilon}}} \frac{1}{(t^2 + (1 + |x|^2 + |y|^2)^2)^{\frac{Q-2}{4}q}} d\xi \\ &\leq C\varepsilon^{\frac{Q}{2}-\frac{Q-2}{2}q} \int_{|\xi|\leq \frac{R}{\sqrt{\varepsilon}}} \frac{1}{(1 + |\xi|^4)^{\frac{Q-2}{4}q}} d\xi \\ &\leq C\varepsilon^{\frac{Q}{2}-\frac{Q-2}{2}q} \left(\int_{1\leq|\xi|\leq \frac{R}{\sqrt{\varepsilon}}} \frac{1}{|\xi|^{(Q-2)q}} d\xi + \int_{|\xi|\leq 1} 1 d\xi \right) \\ &\leq C\varepsilon^{\frac{Q}{2}-\frac{Q-2}{2}q} \left(1 + \int_1^{\frac{R}{\sqrt{\varepsilon}}} \frac{1}{s^{(Q-2)q-Q+1}} ds \right) \\ &\leq C\varepsilon^{\frac{Q}{2}-\frac{Q-2}{2}q} \left(1 + \frac{1}{\varepsilon^{\frac{Q}{2}-\frac{Q-2}{2}q}} \right), \end{aligned}$$

that is,

$$\int_{\mathbb{H}^N} |u_\varepsilon(\xi)|^q d\xi \leq C\varepsilon^{\frac{Q}{2}-\frac{Q-2}{2}q} + O(1).$$

For $q = 2$, $Q = 4$, we have

$$\begin{aligned} \int_{\mathbb{H}^N} |u_\varepsilon(\xi)|^2 d\xi &\leq \int_{|\xi|\leq R} |U_\varepsilon(\xi)|^2 d\xi \\ &\leq C \int_{|\xi|\leq \frac{R}{\sqrt{\varepsilon}}} \frac{C}{1 + |\xi|^4} d\xi \\ &\leq C \left(1 + \int_1^{\frac{R}{\sqrt{\varepsilon}}} \frac{1}{s} ds \right) \\ &\leq 1 + \log \frac{R}{\sqrt{\varepsilon}}. \end{aligned}$$

For $q = 2$, $Q > 4$, we have

$$\begin{aligned} \int_{\mathbb{H}^N} |u_\varepsilon(\xi)|^2 d\xi &\leq \varepsilon^{-\frac{Q-4}{2}} \int_{|\xi|\leq R} |U_\varepsilon(\xi)|^2 d\xi \\ &\leq C\varepsilon^{-\frac{Q-4}{2}} \left(1 + \int_1^{\frac{R}{\sqrt{\varepsilon}}} \frac{1}{s^{Q-3}} ds \right) \\ &\leq C\varepsilon^{-\frac{Q-4}{2}} + O(1). \end{aligned}$$

□

Using these estimates and following the calculations as in [14, Lemma 1.1] we obtain the following: Using (3.23), (3.25) and (3.26) we get,

$$\int_{\mathbb{H}^N} |\nabla_{\mathbb{H}} v_\varepsilon(\xi)|^2 d\xi = S_{HG} + O(\varepsilon^{\frac{Q-2}{2}}), \tag{3.27}$$

Using Lemma 3.6, (3.26) and the Hardy–Littlewood–Sobolev inequality,

$$\int_{\mathbb{H}^N}\int_{\mathbb{H}^N} \frac{|v_\varepsilon(\eta)|^p |v_\varepsilon(\xi)|^p}{|\eta^{-1}\circ\xi|^\alpha} d\eta d\xi \le O\left(\varepsilon^{\frac{2Q-\alpha}{2}-\frac{Q-2}{2}p}\right) + O\left(\varepsilon^{\frac{Q-2}{2}p}\right). \tag{3.28}$$

Using Lemma 3.6 and (3.26),

$$\int_{\mathbb{H}^N} |v_\varepsilon(\xi)|^2 d\xi \le \begin{cases} C\varepsilon\left(1+\log\left(\frac{R}{\sqrt{\varepsilon}}\right)\right), \text{ if } Q=4, \\ O(\varepsilon^2), \text{ if } Q>4. \end{cases} \tag{3.29}$$

Thus, from (3.28) and (3.29), we conclude

$$\lim_{\varepsilon\to 0}\int_{\mathbb{H}^N}\int_{\mathbb{H}^N} \frac{|v_\varepsilon(\eta)|^p |v_\varepsilon(\xi)|^p}{|\eta^{-1}\circ\xi|^\alpha} d\eta d\xi = 0$$

and

$$\lim_{\varepsilon\to 0}\int_{\mathbb{H}^N} |v_\varepsilon(\xi)|^2 d\xi = 0.$$

Now, we prove a bound for the energy functional J.

Lemma 3.7. *There exists $v \in E, v > 0$, such that* $\sup_{t>0} J(tv) < \frac{1}{2}\left(\frac{1}{Q_\alpha^*}\right)^{\frac{1}{Q_\alpha^*-1}}\left(1-\left(\frac{1}{Q_\alpha^*}\right)^{Q_\alpha^*}\right) S_{HG}^{\frac{Q_\alpha^*}{Q_\alpha^*-1}}$.

Proof. Let $v_\varepsilon \in E$ be defined as in (3.22). Define

$$h(t) = \frac{t^2}{2}\|v_\varepsilon\|^2 - \frac{t^{2p}}{2}\int_{|\xi|\le R}\int_{|\eta|\le R} \frac{K(\eta)|v_\varepsilon(\eta)|^p K(\xi)|v_\varepsilon(\xi)|^p}{|\eta^{-1}\circ\xi|^\alpha} d\eta d\xi - \frac{t^{2Q_\alpha^*}}{2}. \tag{3.30}$$

We observe that $h(t) \ge J(tv_\varepsilon)$, with $h(t) \to 0$ as $t \to 0$ and $h(t) \to -\infty$ as $t \to \infty$. Since $2 < 2p < 2Q_\alpha^*$, we have $h(t) > 0$ for small $t > 0$. Therefore, h attains a global maximum. Let t_ε denote a point at which h achieves its global maximum. Then

$$\|v_\varepsilon\|^2 = pt_\varepsilon^{2(p-1)}\int_{|\xi|\le R}\int_{|\eta|\le R} \frac{K(\eta)|v_\varepsilon(\eta)|^p K(\xi)|v_\varepsilon(\xi)|^p}{|\eta^{-1}\circ\xi|^\alpha} d\eta d\xi + Q_\alpha^* t_\varepsilon^{2(Q_\alpha^*-1)}. \tag{3.31}$$

Hence,

$$Q_\alpha^* t_\varepsilon^{2(Q_\alpha^*-1)} \le \|v_\varepsilon\|^2,$$

$$t_\varepsilon \le \left(\frac{1}{Q_\alpha^*}\|v_\varepsilon\|^2\right)^{\frac{1}{2(Q_\alpha^*-1)}}.$$

Therefore, using this preceding inequality in (3.30),

$$\int_{|\xi|\le R} |\nabla_{\mathbb{H}} v_\varepsilon(\xi)|^2 d\xi \le p\left(\frac{1}{Q_\alpha^*}\|v_\varepsilon\|^2\right)^{\frac{2(p-1)}{2(Q_\alpha^*-1)}}\int_{|\xi|\le R}\int_{|\eta|\le R} \frac{K(\eta)|v_\varepsilon(\eta)|^p K(\xi)|v_\varepsilon(\xi)|^p}{|\eta^{-1}\circ\xi|^\alpha} d\eta d\xi$$
$$+ Q_\alpha^* t_\varepsilon^{2(Q_\alpha^*-1)}.$$

From (3.27) for sufficiently small ε,

$$Q_\alpha^* t_\varepsilon^{2(Q_\alpha^*-1)} \geq \frac{S_{HG}}{2}.$$

Hence,

$$\begin{aligned} h(t_\varepsilon) \leq \frac{t_\varepsilon^2}{2} \int_{|\xi|\leq R} |\nabla_{\mathbb{H}} v_\varepsilon(\xi)|^2 d\xi + C \int_{|\xi|\leq R} |v_\varepsilon(\xi)|^2 d\xi \\ - C \int_{|\xi|\leq R} \int_{|\eta|\leq R} \frac{K(\eta)|v_\varepsilon(\eta)|^p K(\xi)|v_\varepsilon(\xi)|^p}{|\eta^{-1}\circ\xi|^\alpha} d\eta d\xi - \frac{t_\varepsilon^{2Q_\alpha^*}}{2}. \end{aligned}$$

Let $\rho(\tau) = \frac{\tau^2}{2}\|v_\varepsilon\|^2 - \frac{\tau^{2Q_\alpha^*}}{2}$. By an argument similar to that used for h, the function ρ attains a global maximum. Let $\tau' = \left(\frac{1}{Q_\alpha^*}\right)^{\frac{1}{Q_\alpha^*-1}} \|v_\varepsilon\|^{\frac{1}{Q_\alpha^*-1}}$ be the point at which ρ attains its maximum. Hence,

$$\begin{aligned} \rho(\tau') &= \frac{1}{2}\left(\frac{1}{Q_\alpha^*}\right)^{\frac{1}{Q_\alpha^*-1}} \|v_\varepsilon\|^{\frac{2Q_\alpha^*}{Q_\alpha^*-1}} - \frac{1}{2}\left(\frac{1}{Q_\alpha^*}\right)^{\frac{Q_\alpha^*}{Q_\alpha^*-1}} \|v_\varepsilon\|^{\frac{2Q_\alpha^*}{Q_\alpha^*-1}} \\ &= \frac{1}{2}\|v_\varepsilon\|^{\frac{2Q_\alpha^*}{Q_\alpha^*-1}} \left(\left(\frac{1}{Q_\alpha^*}\right)^{\frac{1}{Q_\alpha^*-1}} - \left(\frac{1}{Q_\alpha^*}\right)^{\frac{Q_\alpha^*}{Q_\alpha^*-1}} \right). \end{aligned}$$

Therefore using (3.27), (3.28) and (3.29),

$$\begin{aligned} h(t) &\leq \frac{1}{2}\left(\frac{1}{Q_\alpha^*}\right)^{\frac{1}{Q_\alpha^*-1}} \|v_\varepsilon\|^{\frac{2Q_\alpha^*}{Q_\alpha^*-1}} \left(1 - \left(\frac{1}{Q_\alpha^*}\right)^{Q_\alpha^*}\right) + C\int_{|\xi|\leq R} |v_\varepsilon(\xi)|^2 d\xi \\ &\quad - C \int_{|\xi|\leq R} \int_{|\eta|\leq R} \frac{K(\eta)|v_\varepsilon(\eta)|^p K(\xi)|v_\varepsilon(\xi)|^p}{|\eta^{-1}\circ\xi|^\alpha} d\eta d\xi \\ &\leq \frac{1}{2}\left(\left(\frac{1}{Q_\alpha^*}\right)^{\frac{1}{Q_\alpha^*-1}} - \left(\frac{1}{Q_\alpha^*}\right)^{\frac{Q_\alpha^*}{Q_\alpha^*-1}} \right) S_{HG}^{\frac{Q_\alpha^*}{Q_\alpha^*-1}}. \end{aligned}$$

□

Proof of Theorem 3.1. Let,

$$c_0 = \min\left\{ \left(\frac{Q_\alpha^*-2}{2}\right)\left(\frac{S_{HG}}{Q_\alpha^*}\right)^{\frac{2Q-\alpha}{Q-2+\alpha}}, \frac{1}{2}\left(\left(\frac{1}{Q_\alpha^*}\right)^{\frac{1}{Q_\alpha^*-1}} - \left(\frac{1}{Q_\alpha^*}\right)^{\frac{Q_\alpha^*}{Q_\alpha^*-1}} \right) S_{HG}^{\frac{Q_\alpha^*}{Q_\alpha^*-1}} \right\}.$$

$c_0 > 0$. We conclude that from Lemma 3.4, Lemma 3.5 and Lemma 3.6, if $c \leq c_0$, then $I = \emptyset$. To conclude the result, it is enough to prove that the $(PS)_c-$sequence $\{u_n\}$ is convergent in E. From (3.7) we have $\langle J'(u_n), u_n - u\rangle \to 0$ and so $\langle J'(u_n - u), u_n - u\rangle \to 0$. Hence,

$$\begin{aligned} \int_{\mathbb{H}^N} |\nabla_{\mathbb{H}}(u_n-u)(\xi)|^2 + V(\xi)|(u_n-u)(\xi)|^2 d\xi \\ - \int_{\mathbb{H}^N}\left(\int_{\mathbb{H}^N} \frac{G(\eta, u_n-u)}{|\eta^{-1}\circ\xi|^\alpha} d\eta\right) g(\xi, u_n-u)(u_n-u) d\xi \to 0. \end{aligned}$$

We claim that $\int_{\mathbb{H}^N}\left(\int_{\mathbb{H}^N} \frac{G(\eta,u_n-u)}{|\eta^{-1}\circ\xi|^\alpha} d\eta\right) g(\xi, u_n-u)(u_n-u) \to 0$ as $n \to \infty$.

Using Lemma 3.5, for given $\varepsilon > 0$, if $r > R$ is sufficiently large then for all $n \geq n_0$,

$$\int_{|\xi|\geq 2r}\left(\int_{\mathbb{H}^N} \frac{G(\eta, u_n-u)}{|\eta^{-1}\circ\xi|^\alpha} d\eta\right) g(\xi, u_n-u)(u_n-u) d\xi \leq \frac{\varepsilon}{2}.$$

Consider

$$\begin{aligned}
&\int_{|\xi|\le R}\left(\int_{\mathbb{H}^N}\frac{G(\eta,u_n-u)}{|\eta^{-1}\circ\xi|^{\alpha}}d\eta\right)g(\xi,u_n-u)(u_n-u)d\xi\\
&=\int_{|\xi|\le R}\left(\int_{|\eta|\le R}\frac{|(u_n-u)(\eta)|^{Q_\alpha^*}+K(\eta)|(u_n-u)(\eta)|^{p}}{|\eta^{-1}\circ\xi|^{\alpha}}d\eta\right)(Q_\alpha^*|(u_n-u)(\xi)|^{Q_\alpha^*}+pK(\xi)|(u_n-u)(\xi)|^{p})d\xi\\
&+\int_{|\xi|\le R}\left(\int_{A}\frac{|(u_n-u)(\eta)|^{Q_\alpha^*}+K(\eta)|(u_n-u)(\eta)|^{p}}{|\eta^{-1}\circ\xi|^{\alpha}}d\eta\right)(Q_\alpha^*|(u_n-u)(\xi)|^{Q_\alpha^*}+pK(\xi)|(u_n-u)(\xi)|^{p})d\xi\\
&+\int_{|\xi|\le R}\left(\int_{B}\frac{V(\eta)|(u_n-u)(\eta)|^{2}}{|\eta^{-1}\circ\xi|^{\alpha}}d\eta\right)(Q_\alpha^*|(u_n-u)(\xi)|^{Q_\alpha^*}+pK(\xi)|(u_n-u)(\xi)|^{p})d\xi.
\end{aligned}\tag{3.32}$$

Now by virtue of Lemma 3.4, Lemma 2.5 and compact embedding in bounded domain,

$$\int_{|\xi|\le 2r}\left(\int_{\mathbb{H}^N}\frac{G(\eta,u_n-u)}{|\eta^{-1}\circ\xi|^{\alpha}}d\eta\right)g(\xi,u_n-u)(u_n-u)d\xi\le\varepsilon.$$

Hence combining the results, we get $\|u_n-u\|\to 0$ in E, as $n\to\infty$. Therefore, by the Mountain Pass Theorem [37, Theorem 1.17], equation (3.4) admits a weak solution. Moreover, the positivity of the solution follows from the strong maximum principle [12]. □

4 Proof of Theorem 1.2

In this section, we prove the main result of this paper. The following theorem provides the L^∞-regularity of the weak solution to (3.4). The proof is based on an argument of Brezis–Kato type, as carried out in $\mathbb{R}^N$ in [4, Lemma 2.10]. For the adaptation of the Brezis–Kato argument to the Heisenberg group, we refer to [20, Theorem 4.2]. We omit the details.

Theorem 4.1. *Let u be the solution obtained for the equation (3.4). Then, there exists a constant M_0, such that*

$$\|u\|_{L^\infty(\mathbb{H}^N)}\le M_0.$$

Before presenting the proof of Theorem 1.2, we prove the following lemma:

Lemma 4.2. *Let u be a solution of (3.4). Then, there exists $\ell_0>0$ such that*

$$\sup_{u\in E}\frac{H(u)(\xi)}{\ell_0}\le\frac{1}{2},$$

where $H(u)(\xi)=\int_{\mathbb{H}^N}\frac{G(\eta,u(\eta))}{|\eta^{-1}\circ\xi|^{\alpha}}d\eta$

Proof. Consider,

$$-\Delta_{\mathbb{H}}u+V(\xi)u=\left(\int_{\mathbb{H}^N}\frac{G(\eta,u(\eta))}{|\eta^{-1}\circ\xi|^{\alpha}}d\eta\right)g(\xi,u(\xi))$$

and

$$-\Delta_{\mathbb{H}}u=\left(-V(\xi)+\left(\int_{\mathbb{H}^N}\frac{G(\eta,u(\eta))}{|\eta^{-1}\circ\xi|^{\alpha}}d\eta\right)\frac{g(\xi,u(\xi))}{u}\right)u.$$

One can verify that $\int_{\mathbb{H}^N}\left(\frac{G(\eta,u(\eta))}{|\eta^{-1}\circ\xi|^\alpha}d\eta\right)\frac{g(\xi,u(\xi))}{u} \in L^{\frac{Q}{2}}(\mathbb{H}^N)$. Hence by Brezis–Kato argument [35, Lemma B.3], $u \in L^t_{\text{loc}}(\mathbb{H}^N)$ for all $t < \infty$. Consider,

$$\begin{aligned}
|H(u)(\xi)| &= \left|\int_{\mathbb{H}^N} \frac{G(\eta,u(\eta))}{|\eta^{-1}\circ\xi|^\alpha}d\eta\right| \\
&\le \int_{\mathbb{H}^N} \frac{|u(\eta)|^{Q^*_\alpha}+K(\eta)|u(\eta)|^p}{|\eta^{-1}\circ\xi|^\alpha}d\eta \\
&\le \int_{|\eta^{-1}\circ\xi|^\alpha\le 1} \frac{|u(\eta)|^{Q^*_\alpha}+K(\eta)|u(\eta)|^p}{|\eta^{-1}\circ\xi|^\alpha}d\eta + \int_{|\eta^{-1}\circ\xi|^\alpha> 1} \frac{|u(\eta)|^{Q^*_\alpha}+K(\eta)|u(\eta)|^p}{|\eta^{-1}\circ\xi|^\alpha}d\eta \\
&\le \int_{|\eta^{-1}\circ\xi|^\alpha\le 1} \frac{|u(\eta)|^{Q^*_\alpha}+K(\eta)|u(\eta)|^p}{|\eta^{-1}\circ\xi|^\alpha}d\eta + C \\
&\le \int_{|\eta^{-1}\circ\xi|^\alpha\le 1} \frac{|u(\eta)|^{Q^*_\alpha}}{|\eta^{-1}\circ\xi|^\alpha}d\eta + C\int_{|\eta^{-1}\circ\xi|^\alpha\le 1} \frac{|u(\eta)|^{p}}{|\eta^{-1}\circ\xi|^\alpha}d\eta + C \\
&\le \left(\int_{|\eta^{-1}\circ\xi|^\alpha\le 1} |u(\eta)|^{Q^*_\alpha t}d\eta\right)^{\frac{1}{t}}\left(\int_{|\eta^{-1}\circ\xi|^\alpha\le 1} \frac{1}{|\eta^{-1}\circ\xi|^{\alpha t'}}d\eta\right)^{\frac{1}{t'}} \\
&\quad + C\left(\int_{|\eta^{-1}\circ\xi|^\alpha\le 1} |u(\eta)|^{ps}d\eta\right)^{\frac{1}{s}}\left(\int_{|\eta^{-1}\circ\xi|^\alpha\le 1} \frac{1}{|\eta^{-1}\circ\xi|^{\alpha s'}}d\eta\right)^{\frac{1}{s'}} + C
\end{aligned}$$

where $s', t' \in \left[1, \frac{Q}{\alpha}\right)$ and $\frac{1}{s'}+\frac{1}{s}=1$, $\frac{1}{t'}+\frac{1}{t}=1$. Hence, by using the Hardy–Littlewood–Sobolev inequality, we have

$$|H(u)(\xi)| \le C.$$

Thus, we may choose $\ell_0 > 0$ such that

$$\sup_{u\in E}\frac{H(u)(\xi)}{\ell_0} \le \frac{1}{2}. \qquad \square$$

We take $\ell \ge \ell_0 > 0$ and consider the penalized problem.

In order to prove the existence of solution for (1.1), we show that there exists $R > 1$ such that u satisfies the inequality

$$f(\xi,u) \le \frac{V(\xi)}{\ell}u$$

for $|\xi| \ge R$.

Proof of Theorem 1.2. Let

$$v(\xi) = \frac{R^{Q-2}\|u\|_{L^\infty(\mathbb{H}^N)}}{|\xi|^{Q-2}}.$$

One can verify that $v \in C^\infty(\mathbb{H}^N)\setminus\{0\}$. Observe that $u \le v$ for $|\xi| = R$. Define

$$w = \begin{cases} (u-v)_+ & \text{if } |\xi| \ge R, \\ 0 & \text{if } |\xi| < R. \end{cases}$$

We see that $w = 0$ on $\partial B_R(0)$, $w \ge 0$ and $\Delta_{\mathbb{H}} v = 0$ in $\mathbb{H}^N \setminus B_R(0)$, since $\frac{C}{|\xi|^{Q-2}}$ is a fundamental solution where C is suitable constant[17, Theorem 2.1]. Therefore,

$$\begin{aligned}\int_{\mathbb{H}^N} |\nabla_{\mathbb{H}} w(\xi)|^2 d\xi &= \int_{\mathbb{H}^N} \nabla_{\mathbb{H}}(u-v)(\xi)\cdot\nabla_{\mathbb{H}} w(\xi)d\xi \\ &= \int_{|\xi|\geq R} H(\xi)g(\xi,u(\xi))w(\xi) - V(\xi)u(\xi)w(\xi)d\xi \\ &\leq \int_{|\xi|\geq R}\left(\frac{H(\xi)}{\ell_0}-1\right)V(\xi)u(\xi)w(\xi)d\xi \\ &\leq 0.\end{aligned}$$

Thus $w \equiv 0$ and hence, $u \leq v \leq \frac{R^{Q-2}M}{|\xi|^{Q-2}}$ if $|\xi| \geq R$. Consider,

$$\begin{aligned}\frac{f(\xi,u)}{u}p &\leq K(\xi)|u|^{p-2} + Q_\alpha^*|u|^{Q_\alpha^*-2} \\ &\leq pM^{p-2}K(\xi)\left(\frac{R}{|\xi|}\right)^{(p-2)(Q-2)} + Q_\alpha^* M^{Q_\alpha^*-2}\left(\frac{R}{|\xi|}\right)^{(Q_\alpha^*-2)(Q-2)} \\ &\leq C\left(\frac{R}{|\xi|}\right)^{(p-2)(Q-2)} \\ &\leq \frac{C\ell R^{(p-2)(Q-2)}}{\ell|\xi|^{(p-2)(Q-2)}}.\end{aligned}$$

Let $\gamma = C\ell, \Lambda \geq \gamma R^\mu$ and $\mu = (p-2)(Q-2)$. Then, from (Σ_1), we have

$$\frac{f(\xi,u)}{u} \leq \frac{1}{\ell}\frac{\Lambda}{|\xi|^\mu} \leq \frac{V(\xi)}{\ell}.$$

This completes the proof. □

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