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Existence of normalized solutions for the nonlinear Schrödinger–Poisson–Boltzmann system

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Abstract. In this paper, we consider the following nonlinear Schrödinger–Poisson–Boltzmann (SPB) system under an L^2-norm constraint

$$\begin{cases} -\Delta u + \lambda u + \phi u = |u|^{p-2} u & \text{in } \mathbb{R}^3, \\ -\Delta \phi + \kappa^2 \phi = 4\pi u^2 & \text{in } \mathbb{R}^3, \\ \|u\|_2 = a, \end{cases}$$

where $p \in \left(2, \frac{10}{3}\right) \setminus \{3\}$, $\lambda \in \mathbb{R}$, $a > 0$ is a prescribed constant and $\kappa > 0$ is a parameter. We prove that the above system admits a positive ground state normalized solution with the Lagrange multiplier $\lambda > 0$ when either $2 < p < 3$ and $a > 0$ is sufficiently small, or $3 < p < \frac{10}{3}$ and $a > 0$ is large enough. Moreover, we prove that when $2 < p < 3$, $\kappa > 0$ and $a > 0$ is small enough, the ground state normalized solutions are radially symmetric up to translation, and as $\kappa \to 0$, they converge to a radially symmetric ground state normalized solution of the Schrödinger–Poisson–Slater system under the same L^2-norm constraint.

Keywords: Schrödinger–Poisson–Boltzmann system, ground state, constrained minimization, radial symmetry, limit behavior.

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1 Introduction

1.1 Physical background

This paper concerns the study of ground state normalized solutions for the following nonlinear Schrödinger–Poisson–Boltzmann (SPB) system under the L^2-norm constraint $\|u\|_2 = a$

$$\begin{cases} -\Delta u + \lambda u + \phi u = |u|^{p-2} u & \text{in } \mathbb{R}^3, \\ -\Delta \phi + \kappa^2 \phi = 4\pi u^2 & \text{in } \mathbb{R}^3, \end{cases} \tag{1.1}$$

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where $a > 0$ is a prescribed constant, $\kappa > 0$ is a parameter, $\lambda \in \mathbb{R}$ and $p \in \left(2, \frac{10}{3}\right) \setminus \{3\}$. System (1.1) arises from seeking the standing wave solutions $\Psi(x,t) = e^{i\lambda t}u(x)$ to the following system

$$\begin{cases} i\partial_t \Psi + \Delta_x \Psi - \phi\Psi + |\Psi|^{p-2}\Psi = 0 & \text{in } \mathbb{R}^3, \\ -\Delta\phi + \kappa^2\phi = 4\pi |\Psi|^2 & \text{in } \mathbb{R}^3, \end{cases} \tag{1.2}$$

where $\Psi : \mathbb{R}^3 \times [0,\infty) \to \mathbb{C}$ represents the wave function and $\phi : \mathbb{R}^3 \to \mathbb{R}$ denotes the electrostatic potential in the Poisson–Boltzmann equation. Therefore, system (1.2) describes a self-consistent quantum-electrostatic system, in which the wave function Ψ and the electrostatic potential ϕ mutually influence each other through nonlinear term and source term, collectively determining the dynamical behavior of this system.

The Poisson–Boltzmann (PB) equation describes an electric double layer model developed independently by Gouy [16] and Chapman [10], which explains the interaction between the ions in the solution and the charged layer. Subsequently, Andelman [1] proposed the Poisson–Boltzmann theory, in which the Poisson–Boltzmann equation is employed to depict the distribution of electrostatic potential within electrolyte solutions. Blossey [4] made further improvements to this theory. Since the electrolyte solutions formed after the dissolution of 1:1 salts (such as NaCl) play a crucial role in many biochemical reactions and physical processes, according to the Boltzmann distribution and Gauss's law, we obtain the following Poisson–Boltzmann equation for 1:1 salts

$$\Delta\phi = \kappa^2 \sinh\phi,$$

where ϕ denotes the electrostatic potential, Δ is the Laplace operator, κ is the inverse of Debye-Hückel screening length λ_D which satisfies

$$\kappa^2 = \frac{2n_0 e^2}{\varepsilon k_B T},$$

where ε stands for the dielectric constant, k_B denotes the Boltzmann constant, T represents the absolute temperature, e is the elementary charge, and n_0 corresponds to the ionic concentration in a neutral electrolyte solution of (1:1) salt (see [4, Section 1.2]). In many physical situations, when the value of the electrostatic potential ϕ is relatively small, the above equation can be linearized into the following equation

$$\Delta\phi = \kappa^2\phi,$$

which not only demonstrates its convenience in the process of mathematical solving, but also has extensive and significant applications in fields such as biophysics, electrochemistry, and colloid chemistry. Moreover, the above linearized Poisson–Boltzmann equation is a classic example of the homogeneous modified Helmholtz equation, where the Laplacian of a function is directly proportional to the function itself (see [4, Section 1.3]). For more physical details, one can refer to the papers [1,4,9,21] and the references therein.

1.2 Problems and main results

In order to explore the existence of solutions to the SPB system (1.1), we would like to recall two distinct approaches in terms of the frequency λ. The first one is to fix $\lambda \in \mathbb{R}$ and search for the coupled solution (u,ϕ) of system (1.1), which is referred to the fixed frequency problem.

In this situation, any solution to system (1.1) can be identified as a critical point of the action functional $F_\lambda(u,\phi)$ on $H^1(\mathbb{R}^3)$, where

$$F_\lambda(u,\phi) := \frac{1}{2}\|\nabla u\|_2^2 + \frac{\lambda}{2}\|u\|_2^2 + \frac{1}{2}\int_{\mathbb{R}^3}\phi u^2 \mathrm{d}x - \frac{1}{16\pi}\|\nabla\phi\|_2^2 - \frac{\kappa^2}{16\pi}\|\phi\|_2^2 - \frac{1}{p}\|u\|_p^p,$$

and the potential ϕ is given by

$$\phi(x) := \left(\frac{e^{-\kappa|\cdot|}}{|\cdot|} * u^2\right)(x) = \int_{\mathbb{R}^3}\frac{e^{-\kappa|x-y|}}{|x-y|}u^2(y)\mathrm{d}y.$$

Due to the presence of ϕu, system (1.1) is classified as a nonlocal problem. A typical example of such nonlocal problems is the Schrödinger–Poisson–Slater system, and a substantial body of literature has investigated various properties of solutions to this system over the past decades. For further related results, we refer to [31,32,38] and the references therein.

Moreover, Han, Huh and Seok [18] studied the following Schrödinger equation coupled with a neutral scalar field N

$$\begin{cases} -\frac{1}{2m}\Delta u + \frac{q}{4m^2}|u|^2 u + (1+\frac{\kappa q}{2m})Nu + \omega u = 0 & \text{in } \mathbb{R}^2, \\ \left(-\Delta + \kappa^2 q^2\right)N + q(1+\frac{\kappa q}{2m})u^2 = 0 & \text{in } \mathbb{R}^2, \end{cases} \tag{1.3}$$

where m, κ, $q > 0$ respectively represent the mass of the particle, the Chern–Simons coupling constant and the Maxwell coupling constant. Since $N := -q\left(1+\frac{\kappa q}{2m}\right)\left(G^{\kappa q} * u^2\right)$, where $G^{\kappa q}$ is the Yukawa potential in two dimension (see [26, Theorem 6.23]), (1.3) is also a nonlocal problem. They proved the existence of standing wave solutions for system (1.3) in the radially symmetric space $H_r^1(\mathbb{R}^2)$ by applying Mountain-Pass theorem. Inspired by [18], Kang, Liu and Tang [23] considered the problem (1.3) in non-radial symmetric space $H^1(\mathbb{R}^2)$. By combining the Nehari manifold, Moser iteration and some analytical skills, they got a positive ground state solution to (1.3), which is classical and spherically symmetric.

D'Avenia and Siciliano [11] explored the following nonlinear Schrödinger–Bopp–Podolsky system

$$\begin{cases} -\Delta u + \omega u + q^2\tilde{\phi}u = |u|^{p-2}u & \text{in } \mathbb{R}^3, \\ -\Delta\tilde{\phi} + a^2\Delta^2\tilde{\phi} = 4\pi u^2 & \text{in } \mathbb{R}^3, \end{cases}$$

where u, $\tilde{\phi} : \mathbb{R}^3 \to \mathbb{R}$, a, $\omega > 0$, $p > 0$ and $q \neq 0$. Depending on the two parameters q and p, the authors established both existence and nonexistence results using variational methods. Moreover, they demonstrated that these solutions converge to those of the classical Schrödinger–Poisson–Slater system as $a \to 0$ in the radial case. However, as far as we know, there is scarcely any literature dedicated to studying the existence of solutions for the SPB system (1.1).

Alternatively, it is of great interest to study solutions to system (1.1) having prescribed L^2-norm. Namely, for any given mass $a > 0$, we consider solutions to system (1.1) under the L^2-norm constraint

$$S_a := \left\{u \in H^1(\mathbb{R}^3) : \|u\|_2 = a\right\}, \tag{1.4}$$

which is called the fixed mass problem. Such solutions are often referred to as the normalized solutions for the following system which is given by

$$\begin{cases} -\Delta u + \lambda u + \phi u = |u|^{p-2}u & \text{in } \mathbb{R}^3, \\ -\Delta\phi + \kappa^2\phi = 4\pi u^2 & \text{in } \mathbb{R}^3, \\ \|u\|_2 = a, \end{cases} \tag{1.5}$$

and the frequency $\lambda \in \mathbb{R}$ cannot be fixed any longer and appears as a Lagrange multiplier.

For the sake of simplicity, by applying the reduction argument introduced in [11, 23], we simplify the system (1.5) to the following nonlocal equation with the constraint S_a

$$-\Delta u + \lambda u + \phi_u u = |u|^{p-2} u \quad \text{in } \mathbb{R}^3, \tag{1.6}$$

where $\phi_u := \phi \in H^1(\mathbb{R}^3)$ uniquely solves

$$-\Delta \phi + \kappa^2 \phi = 4\pi u^2 \quad \text{in } \mathbb{R}^3$$

(see Section 2.3). Therefore, for given $a > 0$, to search for the coupled solution (u, ϕ, λ) of system (1.5) is equivalent to find the solution pair (u, λ) of (1.6) under the constraint S_a, where $\|u\|_2 = a$ and $\phi_u = \phi$ in (1.6). In this case, the normalized solution u of (1.6) can be obtained as a critical point of the constrained functional $I|_{S_a}$, which is given by

$$\begin{aligned} I(u) &:= \frac{1}{2} \|\nabla u\|_2^2 + \frac{1}{4} \int_{\mathbb{R}^3} \phi_u u^2 \mathrm{d}x - \frac{1}{p} \|u\|_p^p \\ &= \frac{1}{2} \|\nabla u\|_2^2 + \frac{1}{4} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{e^{-\kappa|x-y|}}{|x-y|} u^2(x) u^2(y) \mathrm{d}x\mathrm{d}y - \frac{1}{p} \|u\|_p^p, \end{aligned} \tag{1.7}$$

and λ is the corresponding Lagrange multiplier.

We deduce that $\bar{p} := \frac{10}{3}$ is the mass critical exponent and $2^* = 6$ is the Sobolev critical exponent for $I|_{S_a}$ by scaling argument and the Gagliardo–Nirenberg inequality (see Lemma 2.1). For further clarification, we agree that the mass subcritical case, mass critical case and mass supercritical case mean that $2 < p < \frac{10}{3}$, $p = \bar{p} = \frac{10}{3}$ and $\frac{10}{3} < p < 6$, respectively.

When $2 < p < \frac{10}{3}$, namely, the mass subcritical case, it holds that the constrained functional $I|_{S_a}$ is coercive and bounded from below (see Lemma 2.10). This leads to the following global minimization problem

$$I_{a^2} := \inf_{u \in S_a} I(u). \tag{1.8}$$

In particular, if $(u, \phi, \lambda) \in S_a \times H^1(\mathbb{R}^3) \times \mathbb{R}$ is a coupled solution to system (1.5) such that $I(u) = I_{a^2}$, then $u \in S_a$ is the ground state normalized solution of (1.5).

The first study on normalized solutions to the Schrödinger–Poisson–Slater system was conducted by Sánchez and Soler [33], who established the foundational framework for this problem. Specifically, they studied the following system in $\mathbb{R}^3$

$$\begin{cases} -\frac{1}{2}\Delta u + Vu - C_S |u|^{\frac{2}{3}} u = \beta u & \lim_{|x| \to \infty} u(x) = 0, \\ -\Delta V = u^2 & \lim_{|x| \to \infty} V(x) = 0, \\ \|u\|_2 = M > 0, \end{cases}$$

where $V := \frac{1}{4\pi|x|} * u^2 \in \mathcal{D}^{1,2}(\mathbb{R}^3)$ and C_S denotes the Slater constant. The corresponding energy functional is

$$E(u) := \frac{1}{2} \|\nabla u\|_2^2 + \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{1}{8\pi |x-y|} u^2(x) u^2(y) \mathrm{d}x\mathrm{d}y - \frac{3C_S}{4} \|u\|_{\frac{8}{3}}^{\frac{8}{3}}.$$

By applying the concentration-compactness argument in [28, 29] and nonzero weak convergence after translations in [26], they proved that when $M > 0$ is sufficiently small, all the minimizing sequences for I_M are compact, where

$$I_M := \inf \left\{ E(u) : u \in H^1(\mathbb{R}^3), \|u\|_2 = M \right\}.$$

Kikuchi [24] proved that if C_S is sufficiently large, then the infimum of the minimization problem $I(\mu) := \inf\left\{\mathcal{E}(v) : v \in H^1(\mathbb{R}^3), \|u\|_2^2 = \mu\right\}$ is achieved for any $\mu > 0$, where

$$\mathcal{E}(v) := \frac{1}{2}\|\nabla v\|_2^2 + \frac{1}{4}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\frac{1}{|x-y|}v^2(x)v^2(y)\mathrm{d}x\mathrm{d}y - \frac{C_S}{3}\|v\|_3^3.$$

More generally, Bellazzini and Siciliano [6, 7] explored the following Schrödinger–Poisson equation

$$-\Delta u + \varphi u - |u|^{p-2}u = \omega u \quad \text{in } \mathbb{R}^3, \tag{1.9}$$

where $\omega \in \mathbb{R}$, $p \in \left(2, \frac{10}{3}\right) \setminus \{3\}$ and $\varphi := \frac{1}{|x|} * u^2$ satisfying $-\Delta\varphi = 4\pi u^2$. They considered the following minimization problem

$$\mathcal{I}_\rho := \inf_{u \in B_\rho} \mathcal{I}(u), \tag{1.10}$$

where

$$\mathcal{I}(u) := \frac{1}{2}\|\nabla u\|_2^2 + \frac{1}{4}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\frac{1}{|x-y|}u^2(x)u^2(y)\mathrm{d}x\mathrm{d}y - \frac{1}{p}\|u\|_p^p$$

and

$$B_\rho := \left\{u \in H^1(\mathbb{R}^3) : \|u\|_2 = \rho\right\},$$

and they proved that $\mathcal{I}_\rho$ admits a minimizer when either $3 < p < \frac{10}{3}$ and $\rho > 0$ is large enough, or $2 < p < 3$ and $\rho > 0$ is sufficiently small. Subsequently, Georgiev, Prinari and Visciglia [13] studied the radial symmetry of the minimizers of problem (1.10) up to translation provided that $2 < p < 3$ and $\rho > 0$ is sufficiently small. When $p \in [3, \frac{10}{3}]$, Jeanjean and Luo [22] specified a threshold value of $\rho > 0$ that separates the existence and nonexistence of minimizers for the minimization problem (1.10).

Based on the above research results, by applying minimizing method, He, Li and Chen [17] considered the following nonlinear Schrödinger–Bopp–Podolsky system with prescribed mass

$$\begin{cases} -\Delta u + \omega u + \tilde{\phi} u = |u|^{p-2}u & \text{in } \mathbb{R}^3, \\ -\Delta\tilde{\phi} + a^2\Delta^2\tilde{\phi} = 4\pi u^2 & \text{in } \mathbb{R}^3, \\ \|u\|_2 = \rho > 0, \end{cases} \tag{1.11}$$

where $\omega > 0$, $a > 0$, $p \in \left(2, \frac{10}{3}\right)$ and $\tilde{\phi} := \frac{1-e^{-|x|/a}}{|x|} * u^2$. When $a = 1$, they respectively proved the existence of positive normalized solutions of system (1.11) when $2 < p < 3$ and ρ is sufficiently small, as well as when $3 < p < \frac{10}{3}$ and $\rho > 0$ is sufficiently large. Applying critical point theory, minimization method and some ideas borrowed from [6, 7, 13], de Paula Ramos and Siciliano [12] proved that when $p \in \left(2, \frac{10}{3}\right)\setminus\{3\}$ and $a > 0$ is a parameter, system (1.11) has a ground state normalized solution. Meanwhile, under the assumptions that $p \in \left(2, \frac{14}{5}\right)$ and $\rho > 0$ is sufficiently small, they demonstrated that the ground state normalized solution is radially symmetric up to translation, and these solutions converge to a ground state normalized solution of the Schrödinger–Poisson–Slater system as $a \to 0$.

In the light of [11] and [12], Hernandez and Siciliano [20] studied the existence and multiplicity of solutions for the following Schrödinger–Bopp–Podolsky system under an L^2-norm constraint

$$\begin{cases} -\Delta u + \tilde{\phi} u = \omega u & \text{in } \Omega, \\ -\Delta\tilde{\phi} + a^2\Delta^2\tilde{\phi} = u^2 & \text{in } \Omega, \\ u = \tilde{\phi} = \Delta\tilde{\phi} = 0 & \text{on } \partial\Omega, \\ \int_\Omega u^2 \mathrm{d}x = 1, \end{cases}$$

where Ω is an open bounded and smooth domain in $\mathbb{R}^3$, and $a > 0$ is the Bopp–Podolsky parameter. The unknowns are $u, \tilde{\phi} : \Omega \to \mathbb{R}$ and $\omega \in \mathbb{R}$. By applying variational methods and topological invariants from Ljusternik–Schnirelmann theory in the critical point theory, they established that for any $a > 0$, there exist infinitely many nontrivial solutions with diverging energy and norm-divergent behavior. Moreover, they also showed that the ground state solutions of the above system converge to a ground state solution of the corresponding Schrödinger–Poisson system under the same L^2-norm constraint as $a \to 0$.

Motivated by [17] and [12], Li and Zhang [25] proved the existence, asymptotic behavior and the multiplicity of normalized solutions for the following Schrödinger–Bopp–Podolsky system with a critical term

$$\begin{cases} -\Delta u + \tilde{\phi} u = \lambda u + \mu |u|^{p-2} u + u^5 & \text{in } \mathbb{R}^3, \\ -\Delta \tilde{\phi} + \Delta^2 \tilde{\phi} = 4\pi u^2 & \text{in } \mathbb{R}^3, \\ \|u\|_2^2 = m^2 > 0, \end{cases} \tag{1.12}$$

where $\lambda \in \mathbb{R}$, $2 < p < 6$ and $\mu > 0$ is a parameter. For $p \in (\frac{10}{3}, 6)$, by applying Lagrange multipliers argument and Mountain-Pass theorem, they obtained the existence and asymptotic behavior of positive normalized ground state solutions for (1.12). For $p \in (2, \frac{10}{3}]$, they proved the existence of a normalized ground state solution to (1.12) by combining Mountain-Pass theorem with Lebesgue dominated convergence theorem. Moreover, they also verified the multiplicity of normalized solutions to (1.12).

When $\frac{10}{3} < p < 6$, that is, the mass supercritical case, and the minimizing method becomes inapplicable since the constrained functional $I|_{S_a}$ is unbounded from below, that is, $I_{a^2} = -\infty$. Therefore, by applying a Mountain-Pass argument, Bellazzini, Jeanjean and Luo [8] proved that for any $\rho > 0$ sufficiently small, there exist critical points of the constrained functional $\mathcal{I}|_{B_\rho}$ of the Schrödinger–Poisson equation (1.9).

However, when $p = \frac{10}{3}$, that is, the mass critical case, it becomes delicate to say that $I_{a^2} > -\infty$ or $I_{a^2} = -\infty$ since it depends on the range of value of $a > 0$.

Therefore, the purpose of this paper is to consider the existence and limit behavior of the ground state normalized solutions for system (1.5). As far as we know, no relevant results can be found in the existing literature. Next, we state our main results as follows.

Theorem 1.1. *Suppose that $p \in (2, 3)$ and $a, \kappa > 0$. Then there exists $a_0 > 0$, such that for any $a \in (0, a_0)$, system (1.5) admits a ground state normalized solution $(u, \phi_u, \lambda) \in H^1(\mathbb{R}^3) \times H^1(\mathbb{R}^3) \times (0, \infty)$ with $u > 0$ in $\mathbb{R}^3$.*

Theorem 1.2. *Suppose that $p \in \left(3, \frac{10}{3}\right)$ and $a, \kappa > 0$. Then there exists $a_1 > 0$, such that for any $a \in (a_1, \infty)$, system (1.5) admits a ground state normalized solution $(u, \phi_u, \lambda) \in H^1(\mathbb{R}^3) \times H^1(\mathbb{R}^3) \times (0, \infty)$ with $u > 0$ in $\mathbb{R}^3$.*

Let $S_a^r := S_a \cap H_r^1(\mathbb{R}^3)$. Then, the following theorem reveals the radial symmetry of the ground state normalized solutions for system (1.5) when $p \in (2, 3)$ and $a > 0$ is sufficiently small.

Theorem 1.3. *Suppose that $p \in (2, 3)$, $\kappa > 0$ and $a_0 > 0$ is obtained in Theorem 1.1. Let $(u, \phi_u, \lambda) \in S_a \times H^1(\mathbb{R}^3) \times (0, \infty)$ be a ground state normalized solution to system (1.5) with $u > 0$. Then there exists $a_2 \in (0, a_0]$ such that up to translation, $(u, \phi_u, \lambda) \in S_a^r \times H_r^1(\mathbb{R}^3) \times (0, \infty)$ for any $a \in (0, a_2)$.*

Our final result concerns the limit behavior of the ground state normalized solutions $(u_\kappa, \phi_\kappa, \lambda_\kappa)$ to system (1.5) as $\kappa \to 0$. For this purpose, we introduce the following Schrödinger–Poisson–Slater system

$$\begin{cases} -\Delta u + \lambda u + \varphi u = |u|^{p-2} u & \text{in } \mathbb{R}^3, \\ -\Delta \varphi = 4\pi u^2 & \text{in } \mathbb{R}^3, \\ \|u\|_2 = a, \end{cases} \tag{1.13}$$

where $\varphi := \frac{1}{|x|} * u^2 \in \mathcal{D}^{1,2}(\mathbb{R}^3)$ and the corresponding energy functional is

$$\mathcal{J}_0(u) := \frac{1}{2} \|\nabla u\|_2^2 + \frac{1}{4} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{1}{|x-y|} u^2(x) u^2(y) \mathrm{d}x\mathrm{d}y - \frac{1}{p} \|u\|_p^p. \tag{1.14}$$

Theorem 1.4. *Suppose that $p \in (2,3)$, $\kappa > 0$ and $a_2 > 0$ is obtained in Theorem 1.3. Let $\{(u_\kappa, \phi_\kappa, \lambda_\kappa)\} \subset S_a \times H^1(\mathbb{R}^3) \times (0,\infty)$ be a set of ground state normalized solutions to system (1.5) with $u_\kappa > 0$. Then when $a \in (0, a_2)$ is small enough, the Schrödinger–Poisson–Slater system (1.13) admits a ground state normalized solution $(u_0, \varphi_0, \lambda_0) \in S_a^r \times \mathcal{D}_r^{1,2}(\mathbb{R}^3) \times (0,\infty)$ such that, up to a subsequence and up to translation, the following holds*

$$(u_\kappa, \phi_\kappa, \lambda_\kappa) \to (u_0, \varphi_0, \lambda_0) \quad \text{in } S_a^r \times \mathcal{D}_r^{1,2}(\mathbb{R}^3) \times (0,\infty) \quad \text{as } \kappa \to 0. \tag{1.15}$$

1.3 Comments on the Theorems 1.1–1.4

- The proofs of Theorems 1.1 and 1.2 adopt the following approach: if we could prove that the minimizing sequences for the constrained functional $I|_{S_a}$ are precompact, then we conclude that the infimum $I_{a^2} := \inf_{u \in S_a} I(u)$ is achieved. For this end, we apply the abstract framework for constrained minimization problems developed by Bellazzini and Siciliano [6,7]. Furthermore, since it is known that $\varphi := \frac{1}{|x|} * u^2 \in \mathcal{D}^{1,2}(\mathbb{R}^3)$ uniquely solves $-\Delta \varphi = 4\pi u^2$ and $\tilde{\phi} := \frac{1-e^{-|x|/a}}{|x|} * u^2 \in \{\phi \in \mathcal{D}^{1,2}(\mathbb{R}^3) : \Delta\phi \in L^2(\mathbb{R}^3)\}$ uniquely solves $-\Delta\tilde{\phi} + a^2\Delta^2\tilde{\phi} = 4\pi u^2$ for any $u \in H^1(\mathbb{R}^3)$, through a detailed study of operator $-\Delta + \kappa^2$ in the second equation of the SPB system (1.5), we conclude that for any $u \in H^1(\mathbb{R}^3)$, there exists a unique $\phi_u := \frac{e^{-\kappa|x|}}{|x|} * u^2 \in H^1(\mathbb{R}^3)$ satisfying $-\Delta\phi + \kappa^2\phi = 4\pi u^2$. Notably, according to Sections 2.4 and 2.5, by applying the regularity of solutions and the Pohožaev identity of system (1.5) which is obviously different from that of the Schrödinger–Poisson–Slater system (1.13), we further deduce that the ground state normalized solution u and the corresponding Lagrange multiplier λ of system (1.5) are positive, which is a novelty compared with [6,7,12].

- The proof of Theorem 1.3 follows from making small adjustments to the implicit function argument proposed by Georgiev, Prinari and Visciglia [13]. We remark that the argument rests on the study of the behavior of ground state normalized solutions under an appropriate scaling (see Section 3.3). To some extent, this scaling permits one to explore the asymptotic behavior of ground state normalized solutions to Schrödinger–Poisson–Slater system (1.13) as $a \to 0$ by means of solutions to the following semilinear partial differential equation

$$-\Delta u + \omega_0 u = |u|^{p-2} u \quad \text{in } \mathbb{R}^3,$$

where $\omega_0 > 0$ is a certain constant. Through further calculations in this paper, we demonstrate that the range of p can be extended from $(2, \frac{14}{5})$ to $(2,3)$. Therefore, when

$2 < p < 3$ and $a > 0$ is small enough, by applying the above argument, we obtain the radial symmetry of the ground state normalized solutions to system (1.5), which extends the result in [12, Theorem C].

- The proof of Theorem 1.4 is inspired by the results presented in [11, Theorem 1.3], [12, Theorem D] and [20, Theorem 1.3]. Based on Theorem 1.3 and the Sobolev embedding $H_r^1(\mathbb{R}^3) \hookrightarrow \mathcal{D}_r^{1,2}(\mathbb{R}^3)$, we conclude that the set of ground state normalized solutions $\{(u_\kappa, \phi_\kappa, \lambda_\kappa)\}$ is a subset of $S_a^r \times \mathcal{D}_r^{1,2}(\mathbb{R}^3) \times (0, \infty)$. Moreover, by applying Lemmas 3.6, 3.7, Proposition 3.8 and the Sobolev embedding $H_r^1(\mathbb{R}^3) \hookrightarrow\hookrightarrow L^p(\mathbb{R}^3)$ with $p \in (2, 6)$, we get the limit behavior of radially symmetric ground state normalized solutions to system (1.5) as $\kappa \to 0$. In conclusion, our findings supplement and extend the results presented in [6, 7, 12, 20].

This paper is organized as follows. In Section 1, we state the main results and make some comments on the proofs of Theorems 1.1–1.4. In section 2, we introduce some preliminary results, which are crucial to our subsequent arguments. Specifically, in Section 2.2, we establish some properties of operator $-\Delta + \kappa^2$, which play a crucial role in our subsequent analysis. Sections 2.3, 2.4 and 2.5 aim at exploring the energy functional, the regularity of solutions, and the Pohožaev identity respectively. Moreover, in Section 2.6, we present some lemmas and propositions for the constrained minimization problem (1.8) developed by Bellazzini and Siciliano [6, 7]. Finally, Sections 3.1–3.4 are dedicated to proving Theorems 1.1–1.4 respectively.

We now introduce some notations which will be used in the sequel. Let $H^1(\mathbb{R}^3)$ be the usual Sobolev space endowed with the standard inner product and norm $\|\cdot\|_{H^1}$. $H_r^1(\mathbb{R}^3)$ denotes the usual radially symmetric space. $L^p(\mathbb{R}^3)$ denotes the usual Lebesgue space with the norm $\|\cdot\|_p$ for $1 \le p \le \infty$. $\mathcal{D}^{1,2}(\mathbb{R}^3)$ denotes the Sobolev space defined as the completion of $C_c^\infty(\mathbb{R}^3)$ with respect to the norm $\|\cdot\|_{\mathcal{D}^{1,2}} := \|\nabla \cdot\|_2$. For any $\rho > 0$ and $x \in \mathbb{R}^3$, $B_\rho(x)$ denotes the ball of radius ρ centered at x. As usual, C and C_i $(i = 1, 2, 3 \ldots)$ denote various positive constants that may change from line to line but are not essential for the analysis of the problem.

2 Preliminaries

2.1 Some important inequalities

Lemma 2.1 ([35])**.** *Let $2^* := \frac{2N}{N-2}$, for any $N \ge 3$ and $p \in [2, 2^*)$, there exists a constant $C_{N,p}$ depending on N, p such that the following well-known Gagliardo–Nirenberg inequality holds*

$$\|u\|_{L^p(\mathbb{R}^N)}^p \le C_{N,p}^p \|\nabla u\|_{L^2(\mathbb{R}^N)}^{p\gamma_p} \|u\|_{L^2(\mathbb{R}^N)}^{p-p\gamma_p} \quad \forall u \in H^1(\mathbb{R}^N),$$

where $\gamma_p := \frac{N(p-2)}{2p}$. In particular, when $p = 2^*$, the following Sobolev inequality holds

$$\mathcal{S} \|u\|_{L^{2^*}(\mathbb{R}^N)}^2 \le \|\nabla u\|_{L^2(\mathbb{R}^N)}^2 \quad \forall u \in \mathcal{D}^{1,2}(\mathbb{R}^N),\ N \ge 3,$$

where the constant $\mathcal{S}$ is called the best Sobolev constant.

Lemma 2.2 (Hardy–Littlewood–Sobolev inequality [26])**.** *Let p, $r > 1$ and $0 < \lambda < N$ with $\frac{1}{p} + \frac{\lambda}{N} + \frac{1}{r} = 2$, and for any $f \in L^p(\mathbb{R}^N)$ and $g \in L^r(\mathbb{R}^N)$, there exists a sharp constant $C_{N,\lambda,p,r}$ such that*

$$\left| \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{f(x)g(y)}{|x-y|^\lambda} \mathrm{d}x\mathrm{d}y \right| \le C_{N,\lambda,p,r} \|f\|_p \|g\|_r .$$

Lemma 2.3 ([19,37]). *Let $\alpha \in (0, N)$, the Riesz potential operator J_α of order α (also referred to as the fractional integral operator) is defined by*

$$J_\alpha(f)(x) := \int_{\mathbb{R}^N} \frac{f(y)}{|x-y|^{N-\alpha}} \mathrm{d}y, \qquad x \in \mathbb{R}^N.$$

When $1 < p < \frac{N}{\alpha}$ and $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{N}$, there exists a constant $C_{N,\alpha}$ such that $\|J_\alpha(f)\|_q \leq C_{N,\alpha} \|f\|_p$ for any $f \in L^p(\mathbb{R}^N)$.

2.2 The operator $-\Delta + \kappa^2$

In the field of electrostatics, by the Gauss's law (or Poisson equation), for a given charge-density distribution ρ in $\mathbb{R}^3$, the electrostatic potential ϕ satisfies $-\Delta\phi = \rho$. If $\rho = 4\pi\delta_y$ with $y \in \mathbb{R}^3$, where δ_y denotes Dirac's delta measure at y, then it is known that the fundamental solution of $-\Delta\phi = 4\pi\delta_y$ (or Green's function for $-\Delta$) is $G(x-y)$, where $G(x) := \frac{1}{|x|}$ (see [26, Theorem 6.20]). Particularly, for the following Poisson–Boltzmann equation

$$-\Delta\phi + \kappa^2 \sinh(\phi) = \rho \quad \text{in } \mathbb{R}^3, \tag{2.1}$$

where ϕ, κ and ρ are consistent with those in the previous text. According to Section 1.1, when the surface potential ϕ is small enough, (2.1) can be linearized by the following equation

$$-\Delta\phi + \kappa^2\phi = \rho \quad \text{in } \mathbb{R}^3.$$

In particular, when $\kappa > 0$ and $\rho = 4\pi\delta_y$, we find that $G^\kappa(x-y)$ is the fundamental solution of the following equation

$$-\Delta\phi + \kappa^2\phi = 4\pi\delta_y \quad \text{in } \mathbb{R}^3,$$

where

$$G^\kappa(x) := \frac{e^{-\kappa|x|}}{|x|}. \tag{2.2}$$

Namely, G^κ is the Green's function for $-\Delta + \kappa^2$ (see [2, Table 10.1]), which is also referred to as the Yukawa potential in three dimension (see [26, Theorem 6.23]).

Lemma 2.4. *For any $\kappa > 0$, we have $G^\kappa \in L^p(\mathbb{R}^3)$ with $1 \leq p < 3$, and $\nabla G^\kappa \in L^1(\mathbb{R}^3)$. Moreover, in the sense of distributions, there holds*

$$-\Delta G^\kappa(x-y) + \kappa^2 G^\kappa(x-y) = 4\pi\delta_y, \tag{2.3}$$

where δ_y denotes Dirac's delta measure at y (often written as $\delta(x-y)$).

Proof. By applying spherical coordinate transformation and variable substitution, we deduce from (2.2) that

$$\begin{aligned} \|G^\kappa\|_p^p &= \int_{\mathbb{R}^3} \frac{e^{-\kappa p|x|}}{|x|^p} \mathrm{d}x = 4\pi \int_0^\infty r^{2-p} e^{-\kappa p r} \mathrm{d}r = \frac{4\pi}{(\kappa p)^{3-p}} \int_0^\infty y^{(3-p)-1} e^{-y} \mathrm{d}y \\ &= \frac{4\pi}{(\kappa p)^{3-p}} \Gamma(3-p), \end{aligned}$$

where $\Gamma(s)$ represents the Gamma function and thus we obtain that $1 \leq p < 3$. Since

$$\frac{\partial G^\kappa}{\partial x_i} = -\left(\frac{\kappa x_i}{|x|^2} + \frac{x_i}{|x|^3} \right) e^{-\kappa|x|},\ i = 1,2,3 \quad \text{and} \quad \nabla G^\kappa = -\left(\frac{\kappa x}{|x|^2} + \frac{x}{|x|^3} \right) e^{-\kappa|x|}, \tag{2.4}$$

it follows from spherical coordinate transformation and integration by parts that

$$\int_{\mathbb{R}^3} |\nabla G^\kappa|\, \mathrm{d}x = \int_{\mathbb{R}^3} \left| -\left(\frac{\kappa x}{|x|^2} + \frac{x}{|x|^3} \right) \right| e^{-\kappa|x|} \mathrm{d}x \le \int_{\mathbb{R}^3} \frac{\kappa e^{-\kappa|x|}}{|x|} \mathrm{d}x + \int_{\mathbb{R}^3} \frac{e^{-\kappa|x|}}{|x|^2} \mathrm{d}x = \frac{8\pi}{\kappa} < \infty.$$

To sum up, for any $\kappa > 0$, $G^\kappa \in L^p(\mathbb{R}^3)$ with $1 \le p < 3$ and $\nabla G^\kappa \in L^1(\mathbb{R}^3)$.

Next, from [26, Theorem 6.20] or [11, Lemma 3.3], in order to prove (2.3), let $y = 0$ and then we need to show that for any $\psi \in C_c^\infty(\mathbb{R}^3)$, it holds that

$$-\int_{\mathbb{R}^3} G^\kappa(x-0)\Delta\psi \mathrm{d}x + \kappa^2 \int_{\mathbb{R}^3} G^\kappa(x-0)\psi \mathrm{d}x = 4\pi\psi(0). \tag{2.5}$$

Since $G^\kappa \in L^1_{\text{loc}}(\mathbb{R}^3)$, it suffices to show that

$$\lim_{r\to 0^+} I(r) = 4\pi\psi(0), \tag{2.6}$$

where

$$I(r) := -\int_{|x|>r} G^\kappa(x-0)\Delta\psi\, \mathrm{d}x + \kappa^2 \int_{|x|>r} G^\kappa(x-0)\psi\, \mathrm{d}x.$$

Since ψ has compact support, we consider the annulus $A := \{x \in \mathbb{R}^3 : r < |x| < R\}$ for R large enough. Since $-\Delta G^\kappa + \kappa^2 G^\kappa = 0$ in A, by applying a standard integration by parts, we have

$$\begin{aligned}
I(r) &= -\int_A G^\kappa \Delta\psi\, \mathrm{d}x + \kappa^2 \int_A G^\kappa \psi\, \mathrm{d}x \\
&= \int_A \nabla G^\kappa \nabla\psi\, \mathrm{d}x - \int_{|x|=r} G^\kappa \nabla\psi \cdot \nu\, \mathrm{d}\mathbf{S} + \kappa^2 \int_A G^\kappa \psi\, \mathrm{d}x \\
&= -\int_A \psi \Delta G^\kappa\, \mathrm{d}x + \int_{|x|=r} \psi \nabla G^\kappa \cdot \nu\, \mathrm{d}\mathbf{S} - \int_{|x|=r} G^\kappa \nabla\psi \cdot \nu\, \mathrm{d}\mathbf{S} + \kappa^2 \int_A G^\kappa \psi\, \mathrm{d}x \\
&= \int_A \psi \left(-\Delta G^\kappa + \kappa^2 G^\kappa\right) \mathrm{d}x + \int_{|x|=r} \psi \nabla G^\kappa \cdot \nu\, \mathrm{d}\mathbf{S} - \int_{|x|=r} G^\kappa \nabla\psi \cdot \nu\, \mathrm{d}\mathbf{S} \\
&= \int_{|x|=r} \psi \nabla G^\kappa \cdot \nu\, \mathrm{d}\mathbf{S} - \int_{|x|=r} G^\kappa \nabla\psi \cdot \nu\, \mathrm{d}\mathbf{S} := I_1(r) - I_2(r),
\end{aligned}$$

where ν is the unit outward normal to A and $\mathbf{S}$ is the unit sphere in $\mathbb{R}^3$. When considering $I_1(r)$ on the sphere $|x| = r$, it follows from (2.4) that $\nabla G^\kappa \cdot \nu = \frac{\kappa r+1}{r^2} e^{-\kappa r}$. Since ψ is continuous, we get

$$I_1(r) = \int_{|x|=r} \psi \nabla G^\kappa \cdot \nu\, \mathrm{d}\mathbf{S} = \int_{\mathbf{S}_1} \psi(r\sigma)(\kappa r + 1) e^{-\kappa r}\, \mathrm{d}\sigma \to 4\pi\psi(0) \quad \text{as } r \to 0^+,$$

where σ represents a unit vector on the unit sphere surface $\mathbf{S}_1$.

As for $I_2(r)$, since ψ has compact support, there exists some constant $C_1 > 0$ such that $|\nabla\psi \cdot \nu| \le C_1$. Moreover, $\left| |x|^2 G^\kappa \right| < |x|^{\frac{1}{2}}$ when $|x|$ is small enough. Thus,

$$|I_2(r)| = \left| \int_{|x|=r} G^\kappa \nabla\psi \cdot \nu\, \mathrm{d}\mathbf{S} \right| \le \int_{|x|=r} |G^\kappa|\, |\nabla\psi \cdot \nu|\, \mathrm{d}\mathbf{S} \le 4\pi C_1 r^{\frac{1}{2}} \to 0 \quad \text{as } r \to 0^+.$$

Namely, (2.6) has been verified and then (2.3) holds. □

Lemma 2.5. *For any $\kappa > 0$, if $G^\kappa, \nabla G^\kappa \in L^1(\mathbb{R}^3)$ and $f \in L^p(\mathbb{R}^3)$ with $1 \le p \le \infty$, the following conclusions hold:*

(i) The function $y \mapsto G^\kappa(x-y)f(y)$ *is integrable on* $\mathbb{R}^3$*. Moreover, let*

$$\phi(x) := (G^\kappa * f)(x) = \int_{\mathbb{R}^3} \frac{e^{-\kappa|x-y|}}{|x-y|} f(y)\,\mathrm{d}y. \tag{2.7}$$

Then for any $1 \le p \le \infty$*, there holds*

$$\phi \in L^p(\mathbb{R}^3) \quad \text{and} \quad \|\phi\|_p = \|G^\kappa * f\|_p \le \|G^\kappa\|_1 \|f\|_p\,.$$

(ii) In the sense of distributions, ϕ *solves*

$$-\Delta\phi + \kappa^2\phi = 4\pi f \qquad \text{in } \mathcal{D}'(\mathbb{R}^3), \tag{2.8}$$

where $\mathcal{D}'(\mathbb{R}^3)$ *denotes the dual space of* $C_c^\infty(\mathbb{R}^3)$*. Moreover,* ϕ *has a distributional derivative, which is given by*

$$\partial_i\phi(x) = \int_{\mathbb{R}^3} \frac{\partial G^\kappa(x-y)}{\partial x_i} f(y)\,\mathrm{d}y, \quad i = 1,2,3, \tag{2.9}$$

and $\nabla\phi = \int_{\mathbb{R}^3} \nabla G^\kappa(x-y)f(y)\,\mathrm{d}y \in L^p(\mathbb{R}^3)$ *with* $1 \le p \le \infty$*.*

Proof. (i) For any $\kappa > 0$, it follows from Lemma 2.4 that $G^\kappa \in L^1(\mathbb{R}^3)$. Therefore, from [3, Theorem 4.15], it follows that (2.7) is well defined for almost everywhere $x \in \mathbb{R}^3$. Moreover, $\phi \in L^p(\mathbb{R}^3)$ and $\|\phi\|_p = \|G^\kappa * f\|_p \le \|G^\kappa\|_1 \|f\|_p$ with $1 \le p \le \infty$.

(ii) To verify (2.8), we need to show that for any $\psi \in C_c^\infty(\mathbb{R}^3)$, it holds that

$$-\int_{\mathbb{R}^3} \phi\Delta\psi\,\mathrm{d}x + \kappa^2 \int_{\mathbb{R}^3} \phi\psi\,\mathrm{d}x = 4\pi \int_{\mathbb{R}^3} f\psi\,\mathrm{d}x. \tag{2.10}$$

First, let $F(x,y) := G^\kappa(x-y)f(y)\Delta\psi(x)$. Since $G^\kappa \in L^1(\mathbb{R}^3)$, $f \in L^p(\mathbb{R}^3)$ with $1 \le p \le \infty$, and $\psi \in C_c^\infty(\mathbb{R}^3)$, we deduce that $F \in L^1(\mathbb{R}^3 \times \mathbb{R}^3)$. Next, by substituting (2.7) into the first integral of (2.10) and applying Fubini's theorem, we deduce that

$$-\int_{\mathbb{R}^3} \phi\Delta\psi\,\mathrm{d}x = \int_{\mathbb{R}^3} f(y)\left(-\int_{\mathbb{R}^3} G^\kappa(x-y)\Delta\psi(x)\,\mathrm{d}x\right)\mathrm{d}y. \tag{2.11}$$

It follows from (2.5) that

$$-\int_{\mathbb{R}^3} G^\kappa(x-y)\Delta\psi(x)\,\mathrm{d}x = -\kappa^2 \int_{\mathbb{R}^3} G^\kappa(x-y)\psi(x)\,\mathrm{d}x + 4\pi\psi(y). \tag{2.12}$$

By substituting (2.12) into (2.11), it follows that

$$-\int_{\mathbb{R}^3} \phi\Delta\psi\,\mathrm{d}x = -\kappa^2 \int_{\mathbb{R}^3} f(y)\left(\int_{\mathbb{R}^3} G^\kappa(x-y)\psi(x)\,\mathrm{d}x\right)\mathrm{d}y + 4\pi \int_{\mathbb{R}^3} f(y)\psi(y)\,\mathrm{d}y. \tag{2.13}$$

Next, substituting (2.7) into the second integral of (2.10) and applying Fubini's theorem once again, we have

$$\kappa^2 \int_{\mathbb{R}^3} \phi\psi\,\mathrm{d}x = \kappa^2 \int_{\mathbb{R}^3} f(y)\left(\int_{\mathbb{R}^3} G^\kappa(x-y)\psi(x)\,\mathrm{d}x\right)\mathrm{d}y. \tag{2.14}$$

Combining (2.13) and (2.14), we have

$$-\int_{\mathbb{R}^3} \phi\Delta\psi\,\mathrm{d}x + \kappa^2 \int_{\mathbb{R}^3} \phi\psi\,\mathrm{d}x = 4\pi \int_{\mathbb{R}^3} f(y)\psi(y)\,\mathrm{d}y = 4\pi \int_{\mathbb{R}^3} f\psi\,\mathrm{d}x,$$

thus, (2.10) holds, namely, (2.8) holds true.

To prove (2.9), we start from the definition of distributional derivative. For any test function $\psi \in C_c^\infty(\mathbb{R}^3)$, we have that the distributional derivative $\partial_i \phi(x)$ satisfies

$$\int_{\mathbb{R}^3} \partial_i \phi(x)\psi(x)\,\mathrm{d}x = -\int_{\mathbb{R}^3} \phi(x)\partial_i\psi(x)\,\mathrm{d}x, \qquad i = 1,2,3. \tag{2.15}$$

Since $G^\kappa \in L^1(\mathbb{R}^3)$ and $f \in L^p(\mathbb{R}^3)$ with $1 \le p \le \infty$, by substituting (2.7) into the right side of (2.15) and applying Fubini's theorem, it follows that

$$-\int_{\mathbb{R}^3} \phi(x)\partial_i\psi(x)\,\mathrm{d}x = -\int_{\mathbb{R}^3} f(y)\left(\int_{\mathbb{R}^3} G^\kappa(x-y)\partial_i\psi(x)\,\mathrm{d}x\right)\mathrm{d}y. \tag{2.16}$$

Thus, according to (2.16), by applying integration by parts to $\int_{\mathbb{R}^3} G^\kappa(x-y)\partial_i\psi(x)\,\mathrm{d}x$ and the fact that $G^\kappa(x-y)\psi(x)$ vanishes as x goes to infinity due to the compact support of ψ, we infer that

$$\int_{\mathbb{R}^3} G^\kappa(x-y)\partial_i\psi(x)\,\mathrm{d}x = -\int_{\mathbb{R}^3} \psi(x)\frac{\partial G^\kappa(x-y)}{\partial x_i}\,\mathrm{d}x,$$

and

$$-\int_{\mathbb{R}^3} \phi(x)\partial_i\psi(x)\,\mathrm{d}x = \int_{\mathbb{R}^3} f(y)\left(\int_{\mathbb{R}^3} \psi(x)\frac{\partial G^\kappa(x-y)}{\partial x_i}\,\mathrm{d}x\right)\mathrm{d}y.$$

Since $\nabla G^\kappa \in L^1(\mathbb{R}^3)$, we have $\frac{\partial G^\kappa(x-y)}{\partial x_i} \in L^1(\mathbb{R}^3)$. Then, it follows from Fubini's theorem and (2.15) that

$$\int_{\mathbb{R}^3} \partial_i\phi(x)\psi(x)\,\mathrm{d}x = -\int_{\mathbb{R}^3} \phi(x)\partial_i\psi(x)\,\mathrm{d}x = \int_{\mathbb{R}^3}\left(\int_{\mathbb{R}^3} \frac{\partial G^\kappa(x-y)}{\partial x_i} f(y)\,\mathrm{d}y\right)\psi(x)\mathrm{d}x,$$

which shows that

$$\partial_i\phi(x) = \int_{\mathbb{R}^3} \frac{\partial G^\kappa(x-y)}{\partial x_i} f(y)\,\mathrm{d}y,$$

namely, (2.9) holds true and this completes the proof. □

Lemma 2.6. *For any $u \in H^1(\mathbb{R}^3)$ and $\kappa > 0$, the following conclusions hold:*

(i) There exists a unique $\phi_u \in H^1(\mathbb{R}^3)$ solving the following equation

$$-\Delta\phi + \kappa^2\phi = 4\pi u^2 \tag{2.17}$$

in the weak sense. Moreover, there exists some constant $C > 0$ such that $\|\phi_u\|_{H^1} \le C\|u\|_{H^1}^2$.

(ii) Let $\Phi : H^1(\mathbb{R}^3) \to H^1(\mathbb{R}^3)$ be the mapping defined by $\Phi(u) := \phi_u$. Then Φ maps bounded sets into bounded sets. Moreover, Φ is continuous.

(iii) When $1 \le p \le 3$, it holds that

$$\phi_u := \left(G^\kappa * u^2\right)(x) = \int_{\mathbb{R}^3} \frac{e^{-\kappa|x-y|}}{|x-y|}u^2(y)\mathrm{d}y \in L^p(\mathbb{R}^3)$$

and

$$\nabla\phi_u := \left(\nabla G^\kappa * u^2\right)(x) = \int_{\mathbb{R}^3} \nabla G^\kappa(x-y)u^2(y)\mathrm{d}y \in L^p(\mathbb{R}^3).$$

Moreover, $\|\phi_u\|_2 \le C\|u\|_4^2$ and $\int_{\mathbb{R}^3}\phi_u u^2\mathrm{d}x \le C\|u\|_4^4$.

(iv) *If* $u \in H_r^1(\mathbb{R}^3)$, *then* $\phi_u \in H_r^1(\mathbb{R}^3)$. *For any* $\theta > 0$, $\phi_{\theta u} = \theta^2 \phi_u$ *and* $\phi_{u(\cdot+z)}(x) = \phi_u(x+z)$.

(v) *If* $\{u_n\}$ *is bounded in* $H^1(\mathbb{R}^3)$ *and* $u_n \rightharpoonup u$ *in* $H^1(\mathbb{R}^3)$ *as* $n \to \infty$, *then, there exists a unique* $\phi_u \in H^1(\mathbb{R}^3)$ *solving* (2.17) *in the weak sense such that, up to a subsequence,* $\phi_{u_n} \rightharpoonup \phi_u$ *in* $H^1(\mathbb{R}^3)$,

$$\|\phi_{u_n}\|_2^2 = \|\phi_{u_n} - \phi_u\|_2^2 + \|\phi_u\|_2^2 + o_n(1)$$

and

$$\|\nabla\phi_{u_n}\|_2^2 = \|\nabla\phi_{u_n} - \nabla\phi_u\|_2^2 + \|\nabla\phi_u\|_2^2 + o_n(1).$$

Furthermore, for any $\psi \in H^1(\mathbb{R}^3)$, *the following holds*

$$\lim_{n\to\infty} \int_{\mathbb{R}^3} \phi_{u_n} u_n \psi \, dx = \int_{\mathbb{R}^3} \phi_u u \psi \, dx. \tag{2.18}$$

Proof. (i) For any given $u \in H^1(\mathbb{R}^3)$, define a linear functional $L_u(v) : H^1(\mathbb{R}^3) \to \mathbb{R}$ by

$$L_u(v) := 4\pi \int_{\mathbb{R}^3} u^2 v \, dx.$$

Then, from the continuous Sobolev embedding $H^1(\mathbb{R}^3) \hookrightarrow L^4(\mathbb{R}^3)$, we have

$$|L_u(v)| = 4\pi \left| \int_{\mathbb{R}^3} u^2 v \, dx \right| \le C \|u\|_4^2 \|v\|_2 \le C \|u\|_{H^1}^2 \|v\|_{H^1} < \infty.$$

Hence, $L_u(v)$ is continuous in $H^1(\mathbb{R}^3)$. Now we define the following inner product $\langle u, v\rangle$ on $H^1(\mathbb{R}^3)$

$$\langle u, v\rangle := \int_{\mathbb{R}^3} \nabla u \nabla v \, dx + \kappa^2 \int_{\mathbb{R}^3} uv \, dx.$$

Then, from Lemma 2.5 and the Riesz representation theorem, for every $v \in H^1(\mathbb{R}^3)$, there exists a unique $\phi_u \in H^1(\mathbb{R}^3)$ satisfying

$$\int_{\mathbb{R}^3} \nabla\phi_u \nabla v \, dx + \kappa^2 \int_{\mathbb{R}^3} \phi_u v \, dx = 4\pi \int_{\mathbb{R}^3} u^2 v \, dx,$$

which exactly means that ϕ_u solves (2.17) in the weak sense. In particular, let $v = \phi_u$ in the above equation, from Hölder's inequality, Lemma 2.5-(i) and Sobolev embedding theorem, we have that

$$\|\nabla\phi_u\|_2^2 + \kappa^2 \|\phi_u\|_2^2 = 4\pi \int_{\mathbb{R}^3} \phi_u u^2 \, dx \le 4\pi \|\phi_u\|_2 \|u^2\|_2 \le 4\pi \|G^\kappa\|_1 \|u\|_4^4 \le C \|u\|_{H^1}^4,$$

which implies that there exists some constant $C > 0$ such that $\|\phi_u\|_{H^1} \le C \|u\|_{H^1}^2$.

(ii) Let $\{u_n\}$ be any bounded sequence in $H^1(\mathbb{R}^3)$. Then there exists $M > 0$ such that for all $n \in \mathbb{N}^+$, it holds that $\|u_n\|_{H^1} \le M$. Thus, according to (i), we have $\|\Phi(u_n)\|_{H^1} = \|\phi_{u_n}\|_{H^1} \le CM^2$, namely, $\Phi : H^1(\mathbb{R}^3) \to H^1(\mathbb{R}^3)$ maps bounded sets into bounded sets.

Next, we claim that if $u_n \to u$ in $H^1(\mathbb{R}^3)$ as $n \to \infty$, then $\Phi(u_n) \to \Phi(u)$ in $H^1(\mathbb{R}^3)$ as $n \to \infty$. Indeed, according to (i), we conclude that both $\Phi(u_n) = \phi_{u_n} \in H^1(\mathbb{R}^3)$ and $\Phi(u) = \phi_u \in H^1(\mathbb{R}^3)$ solve (2.17) in the weak sense. Therefore, for any $v \in H^1(\mathbb{R}^3)$, it holds that

$$\int_{\mathbb{R}^3} \nabla\Phi(u_n) \nabla v \, dx + \kappa^2 \int_{\mathbb{R}^3} \Phi(u_n) v \, dx = 4\pi \int_{\mathbb{R}^3} u_n^2 v \, dx$$

and

$$\int_{\mathbb{R}^3} \nabla\Phi(u) \nabla v \, dx + \kappa^2 \int_{\mathbb{R}^3} \Phi(u) v \, dx = 4\pi \int_{\mathbb{R}^3} u^2 v \, dx.$$

Subtracting the above two equations, we deduce that

$$\int_{\mathbb{R}^3} (\nabla\Phi(u_n) - \nabla\Phi(u))\, \nabla v \, dx + \kappa^2 \int_{\mathbb{R}^3} (\Phi(u_n) - \Phi(u))\, v \, dx = 4\pi \int_{\mathbb{R}^3} \left(u_n^2 - u^2\right) v \, dx.$$

Let $v = \Phi(u_n) - \Phi(u)$. Then we have

$$\begin{aligned} C\, \|\Phi(u_n) - \Phi(u)\|_{H^1}^2 &\le \|\nabla (\Phi(u_n) - \Phi(u))\|_2^2 + \kappa^2 \|\Phi(u_n) - \Phi(u)\|_2^2 \\ &= 4\pi \int_{\mathbb{R}^3} \left(u_n^2 - u^2\right) (\Phi(u_n) - \Phi(u))\, dx. \end{aligned}$$

Since $u_n \to u$ in $H^1(\mathbb{R}^3)$ as $n \to \infty$ and $\Phi(u_n)$, $\Phi(u) \in H^1(\mathbb{R}^3)$, from Hölder's inequality and Sobolev's embedding theorem, we infer that

$$\begin{aligned} &\left| \int_{\mathbb{R}^3} \left(u_n^2 - u^2\right) (\Phi(u_n) - \Phi(u))\, dx \right| \\ &\le \int_{\mathbb{R}^3} |u_n + u|\, |u_n - u|\, |\Phi(u_n)|\, dx + \int_{\mathbb{R}^3} |u_n + u|\, |u_n - u|\, |\Phi(u)|\, dx \\ &\le C\, \|u_n + u\|_4\, \|u_n - u\|_4\, \|\Phi(u_n)\|_2 + C\, \|u_n + u\|_4\, \|u_n - u\|_4\, \|\Phi(u)\|_2 \\ &\le C\, \|u_n + u\|_4\, \|u_n - u\|_{H^1}\, \|\Phi(u_n)\|_2 + C\, \|u_n + u\|_4\, \|u_n - u\|_{H^1}\, \|\Phi(u)\|_2 \to 0 \end{aligned}$$

as $n \to \infty$. Then $\|\Phi(u_n) - \Phi(u)\|_{H^1}^2 \to 0$, namely, $\Phi(u_n) \to \Phi(u)$ in $H^1(\mathbb{R}^3)$ as $n \to \infty$.

(iii) Fix $u \in H^1(\mathbb{R}^3)$, from Lemma 2.5 and conclusion (i), we have that $\phi_u, \nabla\phi_u \in L^p(\mathbb{R}^3)$ with $1 \le p \le 3$. Moreover, from Lemma 2.5-(i) and the fact that $\|G^\kappa\|_1 = \frac{4\pi}{\kappa^2}$, we deduce that

$$\|\phi_u\|_2 \le \|G^\kappa\|_1 \left\|u^2\right\|_2 = \frac{4\pi}{\kappa^2} \|u\|_4^2 \le C\, \|u\|_4^2.$$

Then, from Hölder's inequality, we have

$$\int_{\mathbb{R}^3} \phi_u u^2 dx \le \|\phi_u\|_2 \left\|u^2\right\|_2 \le \frac{4\pi}{\kappa^2} \|u\|_4^4 \le C\, \|u\|_4^4.$$

(iv) It is only necessary to check that if $u \in H_r^1(\mathbb{R}^3)$, then $\phi_u \in H_r^1(\mathbb{R}^3)$. Let $u \in H_r^1(\mathbb{R}^3)$, and let $O(3)$ denote the three-dimensional orthogonal group. Then for any $A \in O(3)$ and x, $y \in \mathbb{R}^3$, it holds that $|Ax| = |x|$ and $|Ax - Ay| = |x - y|$. Thus, we get that $\phi_u(Ax) = \phi_u(x)$ by applying a change of variable in the integrals.

(v) Since $\{u_n\}$ is bounded in $H^1(\mathbb{R}^3)$ and $u_n \rightharpoonup u$ in $H^1(\mathbb{R}^3)$, then, up to a subsequence, $u_n \to u$ in $L_{\text{loc}}^r(\mathbb{R}^3)$ with $r \in [1, 6)$ and $u_n(x) \to u(x)$ a.e. in $\mathbb{R}^3$ as $n \to \infty$. Since $\{\phi_{u_n}\}$ is bounded in $H^1(\mathbb{R}^3)$ from (ii), there exists $\bar{\phi}_u \in H^1(\mathbb{R}^3)$ such that, up to a subsequence, $\phi_{u_n} \rightharpoonup \bar{\phi}_u$ in $H^1(\mathbb{R}^3)$, $\phi_{u_n} \to \bar{\phi}_u$ in $L_{\text{loc}}^r(\mathbb{R}^3)$ with $r \in [1, 6)$ and $\phi_{u_n} \to \bar{\phi}_u$ a.e. in $\mathbb{R}^3$. Next, we claim that $\bar{\phi}_u = \phi_u$. Indeed, for any $\psi \in C_c^\infty(\mathbb{R}^3)$, it holds that

$$\int_{\mathbb{R}^3} \nabla\phi_{u_n} \nabla\psi \, dx + \kappa^2 \int_{\mathbb{R}^3} \phi_{u_n} \psi \, dx = 4\pi \int_{\mathbb{R}^3} u_n^2 \psi \, dx.$$

Since $\nabla\phi_{u_n} \rightharpoonup \nabla\bar{\phi}_u$ in $L^2(\mathbb{R}^3)$ and $\nabla\psi \in L^2(\mathbb{R}^3)$, we infer that

$$\int_{\mathbb{R}^3} \nabla\phi_{u_n} \nabla\psi \, dx \to \int_{\mathbb{R}^3} \nabla\bar{\phi}_u \nabla\psi \, dx.$$

From $\phi_{u_n} \to \bar{\phi}_u$, $u_n \to u$ in $L_{\text{loc}}^r(\mathbb{R}^3)$ with $r \in [1, 6)$, $\psi \in C_c^\infty(\mathbb{R}^3)$ and Hölder's inequality, we have

$$\left| \int_{\mathbb{R}^3} (\phi_{u_n} - \bar{\phi}_u)\, \psi \, dx \right| \le \int_{\text{supp}\,\psi} |(\phi_{u_n} - \bar{\phi}_u)\, \psi| \, dx \le \|\phi_{u_n} - \bar{\phi}_u\|_{L^2(\text{supp}\,\psi)} \|\psi\|_2 \to 0$$

and

$$\left| \int_{\mathbb{R}^3} (u_n^2 - u^2)\, \psi \, dx \right| \le \int_{\mathrm{supp}\, \psi} \left| (u_n^2 - u^2)\, \psi \right| \, dx$$
$$\le \|u_n + u\|_{L^4(\mathrm{supp}\, \psi)} \|u_n - u\|_{L^4(\mathrm{supp}\, \psi)} \|\psi\|_2 \to 0$$

as $n \to \infty$. Therefore, it holds that

$$\int_{\mathbb{R}^3} \nabla \bar{\phi}_u \nabla \psi \, dx + \kappa^2 \int_{\mathbb{R}^3} \bar{\phi}_u \psi \, dx = 4\pi \int_{\mathbb{R}^3} u^2 \psi \, dx,$$

which implies that $\bar{\phi}_u$ solves (2.17) in the weak sense. By the uniqueness of the solution to (2.17) according to (i), we have $\bar{\phi}_u = \phi_u$. Thus, it follows from $\phi_u \in H^1(\mathbb{R}^3)$ that, up to a subsequence, $\phi_{u_n} \rightharpoonup \phi_u$ in $H^1(\mathbb{R}^3)$, $\phi_{u_n} \to \phi_u$ in $L^r_{\mathrm{loc}}(\mathbb{R}^3)$ with $r \in [1, 6)$ and $\phi_{u_n} \to \phi_u$ a.e. in $\mathbb{R}^3$.

We claim that up to a subsequence, $\nabla \phi_{u_n} \to \nabla \phi_u$ a.e. in $\mathbb{R}^3$ as $n \to \infty$. Indeed, it follows from (iii) that

$$\nabla \phi_u = \int_{\mathbb{R}^3} \nabla G^\kappa (x - y) u^2(y) \mathrm{d}y.$$

Thus, from (2.4), we infer that

$$\begin{aligned}
|\nabla \phi_{u_n} - \nabla \phi_u| &= \left| \int_{\mathbb{R}^3} \left(u_n^2(y) - u^2(y) \right) \nabla G^\kappa (x - y) \mathrm{d}y \right| \\
&\le \int_{\mathbb{R}^3} \left| u_n^2(y) - u^2(y) \right| \left| \kappa \frac{x - y}{|x - y|^2} + \frac{x - y}{|x - y|^3} \right| e^{-\kappa |x - y|} \mathrm{d}y \\
&\le \int_{\mathbb{R}^3} \left| u_n^2(y) - u^2(y) \right| \frac{\kappa e^{-\kappa |x - y|}}{|x - y|} \mathrm{d}y + \int_{\mathbb{R}^3} \left| u_n^2(y) - u^2(y) \right| \frac{e^{-\kappa |x - y|}}{|x - y|^2} \mathrm{d}y \\
&=: J_n^1 + J_n^2.
\end{aligned}$$

Let $R \ge 1$. By Hölder's inequality and the facts that $e^{-t} \le 1$ and $e^t \ge \frac{t^2}{2}$ for $t \ge 0$, we deduce that

$$\begin{aligned}
J_n^1 \le\ & \|u_n + u\|_{L^4(B_R(x))} \|u_n - u\|_{L^4(B_R(x))} \left(\int_{|y - x| \le R} \frac{\kappa^2 e^{-2\kappa |x - y|}}{|x - y|^2} \mathrm{d}y \right)^{\frac{1}{2}} \\
& + \|u_n + u\|_{L^4(B_R^c(x))} \|u_n - u\|_{L^4(B_R^c(x))} \left(\int_{|y - x| \ge R} \frac{\kappa^2}{|x - y|^2 \, e^{2\kappa |x - y|}} \mathrm{d}y \right)^{\frac{1}{2}} \\
\le\ & \|u_n + u\|_{L^4(\mathbb{R}^3)} \|u_n - u\|_{L^4(B_R(x))} \left(\int_{|y - x| \le R} \frac{\kappa^2}{|x - y|^2} \mathrm{d}y \right)^{\frac{1}{2}} \\
& + \|u_n + u\|_{L^4(\mathbb{R}^3)} \|u_n - u\|_{L^4(B_R^c(x))} \left(\int_{|y - x| \ge R} \frac{1}{2 \, |x - y|^4} \mathrm{d}y \right)^{\frac{1}{2}}
\end{aligned}$$

and

$$
\begin{aligned}
J_n^2 &= \int_{|y-x|\le R} \left|u_n^2(y)-u^2(y)\right| \frac{e^{-\kappa|x-y|}}{|x-y|^2}\mathrm{d}y + \int_{|y-x|\ge R} \left|u_n^2(y)-u^2(y)\right| \frac{e^{-\kappa|x-y|}}{|x-y|^2}\mathrm{d}y \\
&\le \int_{|y-x|\le R} \frac{|u_n(y)+u(y)|}{|x-y|}\frac{|u_n(y)-u(y)|}{|x-y|}\mathrm{d}y \\
&\quad + \int_{|y-x|\ge R} |u_n(y)+u(y)|\,|u_n(y)-u(y)| \frac{e^{-\kappa|x-y|}}{|x-y|}\mathrm{d}y \\
&\le \left(\int_{|y-x|\le R} (|u_n(y)|+|u(y)|)^2 \frac{1}{|x-y|^2}\mathrm{d}y\right)^{\frac12} \left(\int_{|y-x|\le R} |u_n(y)-u(y)|^2 \frac{1}{|x-y|^2}\mathrm{d}y\right)^{\frac12} \\
&\quad + \|u_n+u\|_{L^4(B_R^c(x))} \|u_n-u\|_{L^4(B_R^c(x))} \left(\int_{|y-x|\ge R} \frac{1}{|x-y|^2 e^{2\kappa|x-y|}}\mathrm{d}y\right)^{\frac12} \\
&\le C(R)\left(\|u_n\|_{L^2(\mathbb{R}^3)}+\|u\|_{L^2(\mathbb{R}^3)}\right)\|u_n-u\|_{L^2(B_R(x))} \\
&\quad + \|u_n+u\|_{L^4(\mathbb{R}^3)} \|u_n-u\|_{L^4(B_R^c(x))} \left(\int_{|y-x|\ge R} \frac{1}{2\kappa^2|x-y|^4}\mathrm{d}y\right)^{\frac12}.
\end{aligned}
$$

Fix R, then let $n\to\infty$, and subsequently let $R\to\infty$, since $u_n\to u$ in $L^r_{\mathrm{loc}}(\mathbb{R}^3)$ with $r\in[1,6)$, we conclude that both $J_n^1\to 0$ and $J_n^2\to 0$. Thus, $\nabla\phi_{u_n}\to\nabla\phi_u$ a.e. in $\mathbb{R}^3$ as $n\to\infty$. Since $\{\phi_{u_n}\}$ is bounded in $H^1(\mathbb{R}^3)$, $\phi_u\in H^1(\mathbb{R}^3)$ and $\phi_{u_n}\to\phi_u$ a.e. in $\mathbb{R}^3$ as $n\to\infty$, from Brezis–Lieb lemma ([36, Lemma 1.32]), we deduce that

$$
\|\phi_{u_n}\|_2^2 = \|\phi_{u_n}-\phi_u\|_2^2 + \|\phi_u\|_2^2 + o_n(1) \text{ and } \|\nabla\phi_{u_n}\|_2^2 = \|\nabla\phi_{u_n}-\nabla\phi_u\|_2^2 + \|\nabla\phi_u\|_2^2 + o_n(1).
$$

Finally, for any $\psi\in C_c^\infty(\mathbb{R}^3)$, we infer that

$$
\begin{aligned}
\left|\int_{\mathbb{R}^3}(\phi_{u_n}u_n\psi-\phi_u u\psi)\,\mathrm{d}x\right| &\le \int_{\mathbb{R}^3}|(\phi_{u_n}-\phi_u)\,u_n\psi|\,\mathrm{d}x + \int_{\mathbb{R}^3}|\phi_u\,(u_n-u)\,\psi|\,\mathrm{d}x \\
&\le \|\phi_{u_n}-\phi_u\|_{L^4(\operatorname{supp}\psi)}\|u_n\|_4\|\psi\|_2 + \|\phi_u\|_2\|u_n-u\|_{L^4(\operatorname{supp}\psi)}\|\psi\|_4 \\
&\to 0
\end{aligned}
$$

as $n\to\infty$, namely, (2.18) holds since $C_c^\infty(\mathbb{R}^3)$ is dense in $H^1(\mathbb{R}^3)$. □

2.3 The functional setting

It is easy to see that the critical points (u,ϕ) of the C^1 functional

$$
F(u,\phi) := \frac12\|\nabla u\|_2^2 + \frac12\int_{\mathbb{R}^3}\phi u^2\mathrm{d}x - \frac{1}{16\pi}\|\nabla\phi\|_2^2 - \frac{\kappa^2}{16\pi}\|\phi\|_2^2 - \frac1p\|u\|_p^p
$$

on $H^1(\mathbb{R}^3)\times H^1(\mathbb{R}^3)$ restricted to the sphere S_a are the weak solutions of system (1.5). Since the constrained functional $F|_{S_a}$ is strongly unbounded from below and above, the usual techniques of the critical point theory cannot be used directly. To deal with this issue, we shall reduce the functional F to another functional I depending on the single variable u, following a procedure introduced by Benci and Fortunato in [5] for this kind of problem.

From Lemma 2.6, we find that system (1.5) can be written as the following nonlocal equation on the constraint S_a

$$
-\Delta u + \lambda u + \phi_u u = |u|^{p-2}u \quad \text{in } \mathbb{R}^3,
$$

where $\phi_u = \frac{e^{-\kappa|x|}}{|x|} * u^2 \in H^1(\mathbb{R}^3)$. By multiplying the second equation of (1.5) by $\frac{1}{16\pi}\phi_u$ and integrating in $\mathbb{R}^3$, we obtain that

$$\frac{1}{16\pi}\|\nabla\phi_u\|_2^2 + \frac{\kappa^2}{16\pi}\|\phi_u\|_2^2 = \frac{1}{4}\int_{\mathbb{R}^3}\phi_u u^2 \mathrm{d}x.$$

Then, by substituting the above equation into $F(u,\phi)$, we obtain the energy functional

$$I(u) := \frac{1}{2}\|\nabla u\|_2^2 + \frac{1}{4}\int_{\mathbb{R}^3}\phi_u u^2 \mathrm{d}x - \frac{1}{p}\|u\|_p^p$$

with the constraint S_a. Obviously, I is well defined and is of class C^1 in $H^1(\mathbb{R}^3)$. Then, for any $\varphi \in H^1(\mathbb{R}^3)$, we have

$$\langle I'(u), \varphi\rangle = \int_{\mathbb{R}^3}\nabla u \nabla\varphi \,\mathrm{d}x + \int_{\mathbb{R}^3}\phi_u u\varphi \,\mathrm{d}x - \int_{\mathbb{R}^3}|u|^{p-2}u\varphi \,\mathrm{d}x.$$

Hence, the critical point (u,ϕ) of $F|_{S_a}$ is equivalent to the critical point u of $I|_{S_a}$.

2.4 The regularity of solutions

We remark here that the weak solutions are classical solutions. Indeed, let $(u,\phi_u) \in H^1(\mathbb{R}^3) \times H^1(\mathbb{R}^3)$ be a weak solution of system (1.5), namely, u and ϕ_u satisfy (1.6). From (1.6) and the fact that $\phi_u \in L^p(\mathbb{R}^3)$ with $1 \le p \le 3$, it follows that $u \in W^{2,2}_{\mathrm{loc}}(\mathbb{R}^3)$ by applying Sobolev embedding theorem and L^p estimates (see [14, Theorem 9.9]). By reapplying Sobolev embedding theorem to u, we conclude that $u \in C^{0,\alpha}_{\mathrm{loc}}(\mathbb{R}^3)$ with $0 < \alpha \le \frac{1}{2}$. Similarly, we have $\phi_u \in W^{2,3}_{\mathrm{loc}}(\mathbb{R}^3)$, and thus $\phi_u \in C^{0,\alpha}_{\mathrm{loc}}(\mathbb{R}^3)$ with $0 < \alpha \le 1 - \frac{3}{p}$, $p > 3$. Therefore, from (1.6) and Schauder estimates (see [34, Theorem B.1]), we get that u, $\phi_u \in C^{2,\alpha}_{\mathrm{loc}}(\mathbb{R}^3)$.

2.5 The Pohožaev identity

The following lemma presents two forms of the Pohožaev identity of system (1.5), which are widely used in the study of the fixed mass problem.

Lemma 2.7. *Let $2 < p < 6$ and $a, \kappa > 0$. Then, for any nontrivial solution $(u,\phi_u) \in H^1(\mathbb{R}^3) \times H^1(\mathbb{R}^3)$ of system (1.5), the following two Pohožaev identities*

$$\frac{1}{2}\|\nabla u\|_2^2 + \frac{3}{2}\lambda\|u\|_2^2 + \frac{5}{4}\int_{\mathbb{R}^3}\phi_u u^2 \mathrm{d}x - \frac{\kappa}{4}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} e^{-\kappa|x-y|}u^2(x)u^2(y)\mathrm{d}x\mathrm{d}y - \frac{3}{p}\|u\|_p^p = 0 \quad (2.19)$$

and

$$P(u) := \|\nabla u\|_2^2 + \frac{1}{4}\int_{\mathbb{R}^3}\phi_u u^2 \mathrm{d}x + \frac{\kappa}{4}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} e^{-\kappa|x-y|}u^2(x)u^2(y)\mathrm{d}x\mathrm{d}y - \frac{3(p-2)}{2p}\|u\|_p^p = 0 \quad (2.20)$$

hold, where ϕ_u is defined in Lemma 2.6-(iii).

Proof. For any nontrivial solution $(u,\phi_u) \in H^1(\mathbb{R}^3) \times H^1(\mathbb{R}^3)$ of system (1.5), we have

$$\|\nabla u\|_2^2 + \lambda\|u\|_2^2 + \int_{\mathbb{R}^3}\phi_u u^2 \mathrm{d}x - \|u\|_p^p = 0 \quad (2.21)$$

and

$$\|\nabla\phi_u\|_2^2 + \kappa^2\|\phi_u\|_2^2 = 4\pi\int_{\mathbb{R}^3}\phi_u u^2 \mathrm{d}x. \quad (2.22)$$

We claim that u satisfies (2.19). Indeed, if (u, ϕ_u) solves system (1.5), from Section 2.4, we have $u, \phi_u \in C^2(B_R)$, where B_R is an arbitrary ball in $\mathbb{R}^3$ centered at the origin with radius R. Thus, from [11, Appendix A.3], we deduce that

$$\int_{B_R} -\Delta u\,(x\cdot\nabla u)\,\mathrm{d}x = -\frac{1}{2}\int_{B_R}|\nabla u|^2\,\mathrm{d}x - \frac{1}{R}\int_{\partial B_R}|x\cdot\nabla u|^2\,\mathrm{d}\sigma + \frac{R}{2}\int_{\partial B_R}|\nabla u|^2\,\mathrm{d}\sigma,$$

$$\int_{B_R}\phi_u u\,(x\cdot\nabla u)\,\mathrm{d}x = -\frac{1}{2}\int_{B_R}u^2\,(x\cdot\nabla\phi_u)\,\mathrm{d}x - \frac{3}{2}\int_{B_R}\phi_u u^2\mathrm{d}x + \frac{R}{2}\int_{\partial B_R}\phi_u u^2\mathrm{d}\sigma,$$

$$\int_{B_R}u\,(x\cdot\nabla u)\,\mathrm{d}x = -\frac{3}{2}\int_{B_R}u^2\mathrm{d}x + \frac{R}{2}\int_{\partial B_R}u^2\mathrm{d}\sigma,$$

and

$$\int_{B_R}|u|^{p-2}u\,(x\cdot\nabla u)\,\mathrm{d}x = -\frac{3}{p}\int_{B_R}|u|^p\,\mathrm{d}x + \frac{R}{p}\int_{\partial B_R}|u|^p\,\mathrm{d}\sigma,$$

where $\mathrm{d}\sigma$ represents the surface area element on the sphere ∂B_R.

By multiplying the first equation of system (1.5) by $x\cdot\nabla u$, multiplying the second equation by $x\cdot\nabla\phi_u$, and integrating over B_R, and according to the above four equations, we get

$$\begin{aligned}
&-\frac{1}{2}\int_{B_R}|\nabla u|^2\,\mathrm{d}x - \frac{1}{R}\int_{\partial B_R}|x\cdot\nabla u|^2\,\mathrm{d}\sigma + \frac{R}{2}\int_{\partial B_R}|\nabla u|^2\,\mathrm{d}\sigma - \frac{3}{2}\lambda\int_{B_R}u^2\mathrm{d}x + \frac{R}{2}\lambda\int_{\partial B_R}u^2\mathrm{d}\sigma\\
&\quad -\frac{1}{2}\int_{B_R}u^2\,(x\cdot\nabla\phi_u)\,\mathrm{d}x - \frac{3}{2}\int_{B_R}\phi_u u^2\mathrm{d}x + \frac{R}{2}\int_{\partial B_R}\phi_u u^2\mathrm{d}\sigma\\
&= -\frac{3}{p}\int_{B_R}|u|^p\,\mathrm{d}x + \frac{R}{p}\int_{\partial B_R}|u|^p\,\mathrm{d}\sigma
\end{aligned}\tag{2.23}$$

and

$$\begin{aligned}
4\pi\int_{B_R}u^2\,(x\cdot\nabla\phi_u)\,\mathrm{d}x &= \int_{B_R}-\Delta\phi_u\,(x\cdot\nabla\phi_u)\,\mathrm{d}x + \kappa^2\int_{B_R}\phi_u\,(x\cdot\nabla\phi_u)\,\mathrm{d}x\\
&= -\frac{1}{2}\int_{B_R}|\nabla\phi_u|^2\,\mathrm{d}x - \frac{1}{R}\int_{\partial B_R}|x\cdot\nabla\phi_u|^2\,\mathrm{d}\sigma + \frac{R}{2}\int_{\partial B_R}|\nabla\phi_u|^2\,\mathrm{d}\sigma\\
&\quad -\frac{3}{2}\kappa^2\int_{B_R}\phi_u^2\,\mathrm{d}x + \frac{R}{2}\kappa^2\int_{\partial B_R}\phi_u^2\,\mathrm{d}\sigma.
\end{aligned}\tag{2.24}$$

Substituting (2.24) into (2.23), we obtain that

$$\begin{aligned}
&-\frac{1}{2}\int_{B_R}|\nabla u|^2\,\mathrm{d}x + \frac{1}{16\pi}\int_{B_R}|\nabla\phi_u|^2\,\mathrm{d}x + \frac{3\kappa^2}{16\pi}\int_{B_R}\phi_u^2\,\mathrm{d}x\\
&\quad -\frac{3}{2}\lambda\int_{B_R}u^2\mathrm{d}x - \frac{3}{2}\int_{B_R}\phi_u u^2\mathrm{d}x + \frac{3}{p}\int_{B_R}|u|^p\,\mathrm{d}x\\
&= \frac{1}{R}\int_{\partial B_R}\left(|x\cdot\nabla u|^2 - \frac{1}{8\pi}|x\cdot\nabla\phi_u|^2\right)\mathrm{d}\sigma - \frac{R}{2}\int_{\partial B_R}\left(|\nabla u|^2 + \lambda u^2 + \phi_u u^2 - \frac{1}{8\pi}|\nabla\phi_u|^2\right)\mathrm{d}\sigma\\
&\quad + \frac{R}{16\pi}\kappa^2\int_{\partial B_R}\phi_u^2\mathrm{d}\sigma + \frac{R}{p}\int_{\partial B_R}|u|^p\,\mathrm{d}\sigma.
\end{aligned}$$

Now, let $R = R_n$ with $R_n \to \infty$ as $n \to \infty$. Then, we have that $B_{R_n} \to \mathbb{R}^3$ and the right-hand side of the above equation tends to zero as $n \to \infty$. Thus, we get that

$$\begin{aligned}
&\frac{1}{2}\int_{\mathbb{R}^3}|\nabla u|^2\,\mathrm{d}x - \frac{1}{16\pi}\int_{\mathbb{R}^3}|\nabla\phi_u|^2\,\mathrm{d}x - \frac{3\kappa^2}{16\pi}\int_{\mathbb{R}^3}\phi_u^2\mathrm{d}x\\
&\qquad + \frac{3}{2}\lambda\int_{\mathbb{R}^3}u^2\mathrm{d}x + \frac{3}{2}\int_{\mathbb{R}^3}\phi_u u^2\mathrm{d}x - \frac{3}{p}\int_{\mathbb{R}^3}|u|^p\,\mathrm{d}x = 0.
\end{aligned}$$

Then, by inserting (2.22) into the above equation, we infer that

$$\frac{1}{2}\|\nabla u\|_2^2+\frac{3}{2}\lambda\|u\|_2^2+\frac{5}{4}\int_{\mathbb{R}^3}\phi_u u^2\mathrm{d}x-\frac{\kappa^2}{8\pi}\|\phi_u\|_2^2-\frac{3}{p}\|u\|_p^p=0. \tag{2.25}$$

For the convenience of subsequent calculations, we will adopt the method in [11, Appendix A.2] to transform $\|\phi_u\|_2^2$ in (2.25) into another form. For any $f\in L^1(\mathbb{R}^3)$, we define the following Fourier transform

$$\mathcal{F}[f](x):=\frac{1}{(2\pi)^{\frac{3}{2}}}\int_{\mathbb{R}^3}e^{-ix\cdot y}f(y)\,\mathrm{d}y.$$

For any $f,g\in L^2(\mathbb{R}^3)$, it holds that

$$\mathcal{F}[f*g]=(2\pi)^{\frac{3}{2}}\mathcal{F}[f]\,\mathcal{F}[g]\quad\text{and}\quad\int_{\mathbb{R}^3}\mathcal{F}[f]\,\mathcal{F}[g]\,\mathrm{d}x=\int_{\mathbb{R}^3}fg\,\mathrm{d}x.$$

Moreover, when $f\in L^1(\mathbb{R}^3)\cap L^2(\mathbb{R}^3)$, it follows from [26, Theorem 5.3] that $\mathcal{F}[f]\in L^2(\mathbb{R}^3)$ and $\|\mathcal{F}[f]\|_2^2=\|f\|_2^2$. Since

$$\mathcal{F}\left[e^{-\kappa|\cdot|}\right](x)=\sqrt{\frac{2}{\pi}}\frac{2\kappa}{\left(\kappa^2+|x|^2\right)^2}\quad\text{and}\quad\mathcal{F}[G^\kappa](x)=\mathcal{F}\left[\frac{e^{-\kappa|\cdot|}}{|\cdot|}\right](x)=\sqrt{\frac{2}{\pi}}\frac{1}{\kappa^2+|x|^2},$$

we have

$$\mathcal{F}[G^\kappa*G^\kappa]=(2\pi)^{\frac{3}{2}}\mathcal{F}[G^\kappa]\,\mathcal{F}[G^\kappa]=\frac{4\sqrt{2\pi}}{\left(\kappa^2+|x|^2\right)^2}\quad\text{and}\quad(G^\kappa*G^\kappa)(x)=\frac{2\pi}{\kappa}e^{-\kappa|x|}.$$

Since $\phi_u\in L^p(\mathbb{R}^3)$ with $1\le p\le 3$, we conclude that

$$\begin{aligned}\frac{\kappa^2}{8\pi}\|\phi_u\|_2^2&=\frac{\kappa^2}{8\pi}\int_{\mathbb{R}^3}|\mathcal{F}[\phi_u]|^2\,\mathrm{d}x=\frac{\kappa^2}{8\pi}(2\pi)^{\frac{3}{2}}\int_{\mathbb{R}^3}\mathcal{F}\left[u^2\right]\mathcal{F}[G^\kappa]\,\mathcal{F}[\phi_u]\,\mathrm{d}x\\&=\frac{\kappa^2}{8\pi}\int_{\mathbb{R}^3}\mathcal{F}\left[u^2\right]\mathcal{F}[G^\kappa*\phi_u]\,\mathrm{d}x=\frac{\kappa^2}{8\pi}\int_{\mathbb{R}^3}\left((G^\kappa*G^\kappa)*u^2\right)u^2\,\mathrm{d}x\\&=\frac{\kappa}{4}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}e^{-\kappa|x-y|}u^2(x)u^2(y)\mathrm{d}x\mathrm{d}y.\end{aligned}$$

Then by substituting the above result into (2.25), we obtain (2.19), and thus (2.20) follows from (2.19) and (2.21). This completes the proof of Lemma 2.7. □

2.6 The abstract minimization problem

In [6] and [7], Bellazzini and Siciliano developed an abstract framework to prove the existence of solutions to constrained minimization problems. We shall follow this framework in our proofs of Theorems 1.1 and 1.2. Next, we define the following functionals which will be frequently used in subsequent results:

$$A(u):=\|\nabla u\|_2^2,\quad B(u):=\int_{\mathbb{R}^3}\phi_u u^2\mathrm{d}x,\quad C(u):=\|u\|_p^p,\quad T(u):=\frac{1}{4}B(u)-\frac{1}{p}C(u).$$

For any $u\in H^1(\mathbb{R}^3)$, $\beta\in\mathbb{R}$ and $\theta>0$, we define $u^\theta\in H^1(\mathbb{R}^3)$ as the scaling

$$u^\theta(x):=\theta^{1-\frac{3}{2}\beta}u\left(\frac{x}{\theta^\beta}\right)\quad\text{for a.e. }x\in\mathbb{R}^3 \tag{2.26}$$

such that $\left\|u^\theta\right\|_2=\theta\|u\|_2$. Then, the following proposition is a corollary of [6, Theorem 2.1].

Proposition 2.8. *If $2 < p < 3$, let T be a C^1 functional on $H^1(\mathbb{R}^3)$ and let $\{u_n\} \subset S_a$ be a minimizing sequence for I_{a^2} such that, up to a subsequence, $u_n \rightharpoonup \bar{u}$ in $H^1(\mathbb{R}^3)$ as $n \to \infty$ with $\bar{u} \in H^1(\mathbb{R}^3)\setminus\{0\}$. Assume $a, \kappa > 0$ and the following conditions are satisfied.*

(i) If $0 < \mu < a$, then $I_{a^2} \le I_{\mu^2} + I_{a^2-\mu^2}$.

(ii) For all $a > 0$, $-\infty < I_{a^2} < 0$ with $I(0) = 0$.

(iii) The function $a \mapsto I_{a^2}$ is continuous and $I_{a^2}/a^2 \to 0$ as $a \to 0^+$.

(iv) Let $\alpha_n := \left(a^2 - \|\bar{u}\|_2^2\right) \|u_n - \bar{u}\|_2^{-2}$, then the functional T satisfies

$$T(u_n - \bar{u}) + T(\bar{u}) = T(u_n) + o_n(1) \tag{2.27}$$

and

$$T\left(\alpha_n(u_n - \bar{u})\right) - T(u_n - \bar{u}) = o_n(1). \tag{2.28}$$

Then, for every $a > 0$, the set $M(a)$ is nonempty, where

$$M(a) := \bigcup_{\mu \in (0,a]} \left\{ u \in S_\mu : I(u) = I_{\mu^2} = \inf_{u \in S_\mu} I(u) \right\}.$$

If we suppose further that for any $u \in M(a)$, there exists $\beta \in \mathbb{R}$ such that for any $\theta > 0$,

$$h_\beta^u(\theta) := I(u^\theta) - \theta^2 I(u)$$

is differentiable and $\left(h_\beta^u\right)'(1) \neq 0$, then, for any $\mu \in (0, a)$, the function $\mu \mapsto I_{\mu^2}/\mu^2$ is monotone decreasing. Thus, for any $\mu \in (0, a)$, the following strong subadditivity inequality holds

$$I_{a^2} = \frac{\mu^2}{a^2} I_{a^2} + \frac{a^2 - \mu^2}{a^2} I_{a^2} < I_{\mu^2} + I_{a^2-\mu^2}. \tag{2.29}$$

Therefore, it follows that $\bar{u} \in S_a$ and $I(\bar{u}) = I_{a^2} = \inf_{u \in S_a} I(u) < 0$.

If we suppose further that

$$\langle T'(u_n), u_n \rangle = O_n(1) \tag{2.30}$$

and

$$\langle T'(u_n) - T'(u_m), u_n - u_m \rangle = o_n(1) \quad \text{as } n, m \to \infty, \tag{2.31}$$

then, $\|u_n - \bar{u}\|_{H^1} \to 0$ as $n \to \infty$.

Proposition 2.9 ([6, Lemma 2.1]). *Let $3 < p < \frac{10}{3}$, a, $\kappa > 0$ and $\mu := \|\bar{u}\|_2 \in (0, a]$. Suppose that $\{u_n\} \subset S_a$ is a minimizing sequence for I_{a^2} such that, up to a subsequence, $u_n \rightharpoonup \bar{u}$ in $H^1(\mathbb{R}^3)$ as $n \to \infty$ with $\bar{u} \in H^1(\mathbb{R}^3)\setminus\{0\}$, and let $T : H^1(\mathbb{R}^3) \to \mathbb{R}$ be a C^1 functional satisfying (2.27)–(2.29), then $\bar{u} \in S_a$ and $I(\bar{u}) = I_{a^2}$. If we further assume that (2.30) and (2.31) also hold, then, $\|u_n - \bar{u}\|_{H^1} \to 0$ as $n \to \infty$.*

After introducing the above abstract framework, we now state some results showing how this framework applies to the minimization problem (1.8).

Lemma 2.10. *Assume that $2 < p < \frac{10}{3}$ and a, $\kappa > 0$, then the functional I is coercive and bounded from below on S_a. Moreover, it holds that $I_{a^2} = \inf_{u \in S_a} I(u) \in (-\infty, 0]$.*

Proof. For any $u \in S_a$, it follows from $\frac{3(p-2)}{2} < 2$ and Lemma 2.1 that

$$I(u) = \frac{1}{2}\|\nabla u\|_2^2 + \frac{1}{4}\int_{\mathbb{R}^3} \phi_u u^2 \mathrm{d}x - \frac{1}{p}\|u\|_p^p \geq \frac{1}{2}\|\nabla u\|_2^2 - \frac{C_{3,p}^p a^{\frac{6-p}{2}}}{p}\|\nabla u\|_2^{\frac{3(p-2)}{2}},$$

which implies that I is coercive on S_a and then I is bounded from below on S_a, namely, $I_{a^2} := \inf_{S_a} I(u) > -\infty$. Moreover, for any $u \in S_a$ and $t > 0$, let $u_t(x) := t^{\frac{3}{2}}u(tx)$ for a.e. $x \in \mathbb{R}^3$. Then it holds that $u_t \in S_a$ and

$$\begin{aligned} I_{a^2} \leq I(u_t) &= \frac{t^2}{2}\|\nabla u\|_2^2 + \frac{t}{4}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{e^{-\frac{\kappa}{t}|x-y|}}{|x-y|}u^2(x)u^2(y)\mathrm{d}x\mathrm{d}y - \frac{t^{\frac{3(p-2)}{2}}}{p}\|u\|_p^p \\ &\leq \frac{t^2}{2}\|\nabla u\|_2^2 + \frac{t}{4}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{1}{|x-y|}u^2(x)u^2(y)\mathrm{d}x\mathrm{d}y - \frac{t^{\frac{3(p-2)}{2}}}{p}\|u\|_p^p \to 0 \quad \text{as } t \to 0. \end{aligned}$$

Hence, when $2 < p < \frac{10}{3}$, it holds that $I_{a^2} \in (-\infty, 0]$ for any $a > 0$. □

Lemma 2.11. *If $2 < p < \frac{10}{3}$ and $\kappa > 0$, then I satisfies the following weak subadditivity inequality*

$$I_{a^2} \leq I_{\mu^2} + I_{a^2-\mu^2} \quad \text{for any } 0 < \mu < a.$$

Proof. Now argue as in the [15, Lemma 3.1], and we complete the proof. □

Lemma 2.12. *For any $\kappa > 0$, we have*

(i) If $2 < p < 3$, then there exists $a_0 > 0$ such that $I_{a^2} < 0$ for any $a \in (0, a_0)$. Moreover, it holds that $I_{a^2} < 0$ for any $a > 0$.

(ii) If $3 < p < \frac{10}{3}$, then there exists $a_1 > 0$ such that $I_{a^2} < 0$ for any $a \in (a_1, \infty)$.

Proof. From (2.26), we have $u^\theta(x) := \theta^{1-\frac{3}{2}\beta}u\left(\frac{x}{\theta^\beta}\right)$ for a.e. $x \in \mathbb{R}^3$, where $u \in H^1(\mathbb{R}^3)$, $\beta \in \mathbb{R}$ and $\theta > 0$. Then, from direct calculations, we get that $\|u^\theta\|_2 = \theta\|u\|_2$ and

$$\begin{aligned} &A(u^\theta) = \left\|\nabla u^\theta\right\|_2^2 = \theta^{2-2\beta}A(u), \quad C(u^\theta) = \left\|u^\theta\right\|_p^p = \theta^{(1-\frac{3}{2}\beta)p+3\beta}C(u), \\ &B(u^\theta) = \theta^{4-\beta}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\frac{e^{-\kappa\theta^\beta|x-y|}}{|x-y|}u^2(x)u^2(y)\mathrm{d}x\mathrm{d}y < \theta^{4-\beta}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\frac{1}{|x-y|}u^2(x)u^2(y)\mathrm{d}x\mathrm{d}y. \end{aligned}$$

(i) When $2 < p < 3$, fix $u \in S_1$ and $\beta = -2$ such that $\left(1 - \frac{3}{2}\beta\right)p + 3\beta = 4p - 6 < 2 - 2\beta = 4 - \beta = 6$, then $u^a \in S_a$. Thus, as $a \to 0^+$, we have

$$I(u^a) < \frac{a^6}{2}A(u) + \frac{a^6}{4}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\frac{1}{|x-y|}u^2(x)u^2(y)\mathrm{d}x\mathrm{d}y - \frac{a^{4p-6}}{p}C(u) = \mathcal{J}_0(u_a) \to 0^-,$$

where $\mathcal{J}_0$ is the energy functional of the Schrödinger–Poisson–Slater system (1.14) in Section 1.2. Then, there exists $a_0 > 0$ small enough such that

$$I_{a^2} < 0 \quad \text{for any } a \in (0, a_0].$$

Let $\bar{a} \in (a_0, \sqrt{2}a_0]$. For every $a \in (a_0, \bar{a}]$, we have $a^2 - a_0^2 \leq \bar{a}^2 - a_0^2 \leq 2a_0^2 - a_0^2 = a_0^2$ and then we deduce from Lemma 2.11 that

$$I_{a^2} \leq I_{a_0^2} + I_{a^2-a_0^2} < 0,$$

which implies that $I_{a^2} < 0$ for any $a \in (0, \bar{a}]$. Thus, for given $\kappa > 0$, by iterating the above procedure, it follows that $I_{a^2} < 0$ for every $a > 0$ and (i) holds.

(ii) When $3 < p < \frac{10}{3}$, fix $u \in S_1$ and $\beta = -2$ such that $\left(1 - \frac{3}{2}\beta\right) p + 3\beta = 4p - 6 > 2 - 2\beta = 4 - \beta = 6$, then $u^a \in S_a$ and as $a \to \infty$, we have

$$I(u^a) < \frac{a^6}{2} A(u) + \frac{a^6}{4} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{1}{|x-y|} u^2(x) u^2(y) \mathrm{d}x\mathrm{d}y - \frac{a^{4p-6}}{p} C(u) = \mathcal{J}_0(u_a) \to -\infty.$$

Thus, for given $\kappa > 0$, it follows that there exists $a_1 > 0$ such that if $a > a_1$, then

$$I_{a^2} \le I(u^a) < \mathcal{J}_0(u^a) < 0,$$

and this completes the proof of Lemma 2.12. □

Lemma 2.13. *If $a, \kappa > 0$ and $\{u_n\} \subset S_a$ is a minimizing sequence for I_{a^2} such that, up to a subsequence, $u_n \rightharpoonup \bar{u}$ in $H^1(\mathbb{R}^3)$ as $n \to \infty$, then, up to translation, the following statements hold:*

(i) For any $a > 0$ and $2 < p < 3$, $\bar{u} \neq 0$.

(ii) For any $a \in (a_1, \infty)$ and $3 < p < \frac{10}{3}$, $\bar{u} \neq 0$, where a_1 is obtained in Lemma 2.12.

Proof. It follows from Lemma 2.10 that $\{u_n\}$ is bounded in $H^1(\mathbb{R}^3)$. Let

$$\delta := \lim_{n\to\infty} \left(\sup_{y\in\mathbb{R}^3} \int_{B(y,1)} |u_n|^2 \,\mathrm{d}x \right).$$

If $\delta = 0$, from [29, Lemma I.1] or [36, Lemma 1.21], we conclude that $u_n \to 0$ in $L^p(\mathbb{R}^3)$ as $n \to \infty$ with $p \in (2, 6)$. Hence, from Lemma 2.12-(i), for any $a > 0$ and $2 < p < 3$, it holds that

$$0 \le \lim_{n\to\infty} I(u_n) = \lim_{n\to\infty} \left(\frac{1}{2} A(u_n) + \frac{1}{4} B(u_n) \right) = I_{a^2} < 0,$$

which leads to a contradiction. Similarly, for any $a \in (a_1, \infty)$ and $3 < p < \frac{10}{3}$, we also obtain a contradiction. Thus, we deduce that $\delta > 0$ in both scenarios (i) and (ii) and then there exists a sequence $\{y_n\} \subset \mathbb{R}^3$ such that

$$\int_{B(y_n,1)} |u_n|^2 \,\mathrm{d}x \ge \frac{\delta}{2} > 0.$$

Let $\tilde{u}_n(\cdot) := u_n(\cdot + y_n)$. Then $\{\tilde{u}_n\} \subset S_a$ is also a bounded minimizing sequence for I_{a^2} and

$$\int_{B(0,1)} |\tilde{u}_n|^2 \,\mathrm{d}x \ge \frac{\delta}{2} > 0 \quad \text{for } n \in \mathbb{N}^+ \text{ large enough.}$$

Hence, it follows that conclusions (i) and (ii) hold. □

Lemma 2.14. *Let $p \in (2, \frac{10}{3})$ and $\{u_n\} \subset S_a$ be a minimizing sequence for I_{a^2} such that, up to a subsequence, $u_n \rightharpoonup \bar{u}$ in $H^1(\mathbb{R}^3)$ as $n \to \infty$ with $\bar{u} \in H^1(\mathbb{R}^3) \setminus \{0\}$. Then the C^1 functional T satisfies (2.27), (2.28) and (2.30). If the strong subadditivity inequality (2.29) holds, then T also satisfies (2.31).*

Proof. From Lemma 2.10 and Lemma 2.6-(i), we know that $\{u_n\}$ is bounded in $H^1(\mathbb{R}^3)$ and $\|\phi_{u_n}\|_{H^1} \leq C\|u_n\|^2_{H^1}$. Therefore, there exists $\bar{u} \in H^1(\mathbb{R}^3)$ such that, up to a subsequence, $u_n \rightharpoonup \bar{u}$ in $H^1(\mathbb{R}^3)$, $u_n \to \bar{u}$ in $L^r_{\text{loc}}(\mathbb{R}^3)$ with $r \in [1,6)$ and $u_n \to \bar{u}$ a.e. in $\mathbb{R}^3$ as $n \to \infty$. From the conclusions (ii) and (v) of Lemma 2.6, we deduce that $\{\phi_{u_n}\}$ is bounded in $H^1(\mathbb{R}^3)$ and there exists a unique $\phi_{\bar{u}} \in H^1(\mathbb{R}^3)$ solving (2.17) in the weak sense such that, up to a subsequence, $\phi_{u_n} \rightharpoonup \phi_{\bar{u}}$ in $H^1(\mathbb{R}^3)$.

We now introduce the following notations:

$$G^\kappa(x,y) := G^\kappa(x-y) = \frac{e^{-\kappa|x-y|}}{|x-y|}, \qquad A := \int_{\mathbb{R}^3} \phi_{\bar{u}}\bar{u}^2 \mathrm{d}x,$$
$$I_n^{(1)} := \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} G^\kappa(x,y)u_n^2(y)\bar{u}^2(x)\mathrm{d}x\mathrm{d}y = \int_{\mathbb{R}^3} \phi_{u_n}\bar{u}^2\mathrm{d}x,$$
$$I_n^{(2)} := \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} G^\kappa(x,y)u_n(y)\bar{u}(y)u_n(x)\bar{u}(x)\mathrm{d}x\mathrm{d}y,$$
$$I_n^{(3)} := \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} G^\kappa(x,y)u_n^2(y)u_n(x)\bar{u}(x)\mathrm{d}x\mathrm{d}y = \int_{\mathbb{R}^3} \phi_{u_n}u_n\bar{u}\mathrm{d}x,$$
$$I_n^{(4)} := \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} G^\kappa(x,y)u_n(y)\bar{u}(y)\bar{u}^2(x)\mathrm{d}x\mathrm{d}y.$$

From direct calculation we deduce that

$$B(u_n - \bar{u}) - B(u_n) + B(\bar{u}) = 2I_n^{(1)} + 4I_n^{(2)} - 4I_n^{(3)} - 4I_n^{(4)} + 2A.$$

Next, we will show that $\lim_{n\to\infty} I_n^{(i)} = A$, $i = 1,2,3,4$. Since $\phi_{u_n} \rightharpoonup \phi_{\bar{u}}$ in $H^1(\mathbb{R}^3)$, we infer that $\phi_{u_n} \rightharpoonup \phi_{\bar{u}}$ in $L^2(\mathbb{R}^3)$. It follows from $\bar{u}^2 \in L^2(\mathbb{R}^3)$ that $\lim_{n\to\infty} I_n^{(1)} = A$. To prove $\lim_{n\to\infty} I_n^{(2)} = A$, let

$$\hat{v}_n(x) := \int_{\mathbb{R}^3} \frac{e^{-\kappa|x-y|}}{|x-y|}u_n(y)\bar{u}(y)\mathrm{d}y.$$

Then we claim that $\hat{v}_n \to \phi_{\bar{u}}$ a.e. in $\mathbb{R}^3$ as $n \to \infty$. Indeed, for any $R \geq 1$, by Hölder's inequality, $e^{-t} \leq 1$ and $e^t \geq \frac{t^2}{2}$ for $t \geq 0$, we deduce that

$$\begin{aligned}
|\hat{v}_n(x) - \phi_{\bar{u}}(x)| &\leq \|u_n - \bar{u}\|_{L^3(B_R(x))}\|\bar{u}\|_{L^6(B_R(x))}\left(\int_{|y-x|\leq R}\frac{e^{-2\kappa|x-y|}}{|x-y|^2}\mathrm{d}y\right)^{\frac{1}{2}} \\
&\quad + \|u_n - \bar{u}\|_{L^3(B_R^c(x))}\|\bar{u}\|_{L^6(B_R^c(x))}\left(\int_{|y-x|\geq R}\frac{1}{|x-y|^2 e^{2\kappa|x-y|}}\mathrm{d}y\right)^{\frac{1}{2}} \\
&\leq \|u_n - \bar{u}\|_{L^3(B_R(x))}\|\bar{u}\|_{L^6(\mathbb{R}^3)}\left(\int_{|y-x|\leq R}\frac{1}{|x-y|^2}\mathrm{d}y\right)^{\frac{1}{2}} \\
&\quad + \|u_n - \bar{u}\|_{L^3(B_R^c(x))}\|\bar{u}\|_{L^6(\mathbb{R}^3)}\left(\int_{|y-x|\geq R}\frac{1}{2\kappa^2|x-y|^4}\mathrm{d}y\right)^{\frac{1}{2}}.
\end{aligned}$$

which implies that $\hat{v}_n \to \phi_{\bar{u}}$ a.e. in $\mathbb{R}^3$ by first letting $n \to \infty$ (for fixed R) and then letting $R \to \infty$. Since $\{u_n\} \subset S_a$ and $\bar{u} \in H^1(\mathbb{R}^3)$, from Hölder's inequality and Sobolev embedding theorem, we have

$$\|u_n\bar{u}\|_{\frac{6}{5}} \leq C\|u_n\|_{\frac{12}{5}}\|\bar{u}\|_{\frac{12}{5}} \leq C\|u_n\|_{H^1}\|\bar{u}\|_{H^1} < \infty,$$

namely, $u_n\bar{u} \in L^{\frac{6}{5}}(\mathbb{R}^3)$. Let $\alpha = 2$, $N = 3$, $f = |u_n\bar{u}|$, $q = 6$ and $p = \frac{6}{5}$ in Lemma 2.3. Then, we have

$$|\hat{v}_n| \le \int_{\mathbb{R}^3} \frac{e^{-\kappa|x-y|}}{|x-y|} |u_n(y)\bar{u}(y)|\,\mathrm{d}y \le \int_{\mathbb{R}^3} \frac{|u_n(y)\bar{u}(y)|}{|x-y|}\mathrm{d}y := J_2\left(|u_n\bar{u}|\right)(x),$$

and thus it holds that

$$\|\hat{v}_n\|_6 \le \|J_2\left(|u_n\bar{u}|\right)\|_6 \le C\,\|u_n\bar{u}\|_{\frac{6}{5}} \le C\,\|u_n\|_{H^1}\,\|\bar{u}\|_{H^1} < \infty,$$

namely, $\hat{v}_n \in L^6(\mathbb{R}^3)$. It follows from Hölder's inequality and Sobolev embedding theorem that

$$\|\hat{v}_n u_n\|_2 \le \|\hat{v}_n\|_6\,\|u_n\|_3 \le C\,\|u_n\|_{H^1}^2\,\|\bar{u}\|_{H^1} < \infty.$$

Since $\hat{v}_n \to \phi_{\bar{u}}$, $u_n \to \bar{u}$ a.e. in $\mathbb{R}^3$, we have $\hat{v}_n u_n \to \phi_{\bar{u}}\bar{u}$ a.e. in $\mathbb{R}^3$ and $\hat{v}_n u_n \rightharpoonup \phi_{\bar{u}}\bar{u}$ in $L^2(\mathbb{R}^3)$. Due to $\bar{u} \in L^2(\mathbb{R}^3)$, we get $\lim_{n\to\infty} I_n^{(2)} = A$. Let $\psi = \bar{u}$ in Lemma 2.6-(v), and thus it holds that $\lim_{n\to\infty} I_n^{(3)} = A$. Since $\hat{v}_n \to \phi_{\bar{u}}$ a.e. in $\mathbb{R}^3$ and $\hat{v}_n \in L^6(\mathbb{R}^3)$, we get $\hat{v}_n \rightharpoonup \phi_{\bar{u}}$ in $L^6(\mathbb{R}^3)$ as $n \to \infty$. By the fact that $\bar{u}^2 \in L^{\frac{6}{5}}(\mathbb{R}^3)$, we deduce that $\lim_{n\to\infty} I_n^{(4)} = A$. In summary, we have

$$B(u_n - \bar{u}) + B(\bar{u}) = B(u_n) + o_n(1).$$

Moreover, it follows from the Brezis–Lieb lemma that

$$C(u_n - \bar{u}) + C(\bar{u}) = C(u_n) + o_n(1),$$

and thus (2.27) holds.

From Lemma 2.6-(iii) and Hölder's inequality, we infer that

$$B(u_n) = \int_{\mathbb{R}^3} \phi_{u_n} u_n^2 \mathrm{d}x \le C\,\|u_n\|_4^4 \le C\,\|u_n\|_{H^1}^4 < \infty.$$

It follows from Brezis-Lieb lemma that $\|u_n - \bar{u}\|_2^2 + \|\bar{u}\|_2^2 = \|u_n\|_2^2 + o_n(1) = a^2 + o_n(1)$. Therefore, from Proposition 2.8-(iv), we have $\alpha_n = \left(a^2 - \|\bar{u}\|_2^2\right)\|u_n - \bar{u}\|_2^{-2} \to 1$ as $n \to \infty$, and thus we obtain that

$$\begin{aligned} B\left(\alpha_n(u_n - \bar{u})\right) - B(u_n - \bar{u}) &= \left(\alpha_n^4 - 1\right) B(u_n - \bar{u}) = o_n(1), \\ C\left(\alpha_n(u_n - \bar{u})\right) - C(u_n - \bar{u}) &= \left(\alpha_n^p - 1\right) C(u_n - \bar{u}) = o_n(1), \end{aligned}$$

thus, (2.28) holds. Since $\{u_n\}$ is bounded in $H^1(\mathbb{R}^3)$, from Lemma 2.6-(iii), we conclude that

$$\left\langle T'(u_n), u_n\right\rangle = \int_{\mathbb{R}^3} \phi_{u_n} u_n^2 \mathrm{d}x - \|u_n\|_p^p \le \int_{\mathbb{R}^3} \phi_{u_n} u_n^2 \mathrm{d}x \le C\,\|u_n\|_4^4 \le C\,\|u_n\|_{H^1}^4 < \infty,$$

namely, (2.30) holds.

Since (2.27) and (2.28) hold, if the strong subadditivity inequality (2.29) also holds, from Proposition 2.8, we conclude that $\bar{u} \in S_a$, namely, $\|\bar{u}\|_2^2 = a^2$. Thus, $\|u_n - \bar{u}\|_2 = o_n(1)$ and $\|u_n - u_m\|_2 = o_n(1)$ as $n,\ m \to \infty$. Notice that $2 < p < \frac{10}{3} < 6$, from Lemma 2.1 and interpolation inequality, we have that

$$\|u_n - u_m\|_p \le \|u_n - u_m\|_2^{\alpha}\,\|u_n - u_m\|_6^{1-\alpha} \le \mathcal{S}^{-\frac{1-\alpha}{2}}\,\|u_n - u_m\|_2^{\alpha}\,\|\nabla u_n - \nabla u_m\|_2^{1-\alpha} = o_n(1),$$

where $\mathcal{S}$ denotes the sharp constant and α satisfies $\frac{\alpha}{2} + \frac{1-\alpha}{6} = \frac{1}{p}$, therefore, $\|u_n - u_m\|_p = o_n(1)$. From Hölder's inequality and Sobolev's embedding theorem, we have

$$\int_{\mathbb{R}^3} |u_n|^{p-1} |u_n - u_m| \, dx \le \left(\int_{\mathbb{R}^3} \left| |u_n|^{p-1} \right|^{\frac{p}{p-1}} dx \right)^{\frac{p-1}{p}} \|u_n - u_m\|_p$$
$$= \|u_n\|_p^{p-1} \|u_n - u_m\|_p = o_n(1),$$

and thus

$$\left| \int_{\mathbb{R}^3} \left(|u_n|^{p-2} u_n - |u_m|^{p-2} u_m \right) (u_n - u_m) \, dx \right|$$
$$\le \int_{\mathbb{R}^3} |u_n|^{p-1} |u_n - u_m| \, dx + \int_{\mathbb{R}^3} |u_m|^{p-1} |u_n - u_m| \, dx$$
$$\le \|u_n\|_p^{p-1} \|u_n - u_m\|_p + \|u_m\|_p^{p-1} \|u_n - u_m\|_p = o_n(1).$$

Hence, we have $\langle C'(u_n) - C'(u_m), u_n - u_m \rangle = o_n(1)$ as n, $m \to \infty$. From $\phi_{u_n} \in H^1(\mathbb{R}^3)$, Lemma 2.6-$(i)$ and Sobolev's embedding theorem, we infer that $\phi_{u_n} \in L^6(\mathbb{R}^3)$ and $\|\phi_{u_n}\|_6 \le C \|\phi_{u_n}\|_{H^1} \le C \|u_n\|_{H^1}^2$. By Hölder's inequality and $\|u_n - u_m\|_p = o_n(1)$ with $p \in (2, \frac{10}{3})$, we have

$$\int_{\mathbb{R}^3} \phi_{u_n} u_n |u_n - u_m| \, dx \le C \|\phi_{u_n}\|_6 \|u_n\|_2 \|u_n - u_m\|_3 \le C \|u_n\|_{H^1}^2 \|u_n\|_2 \|u_n - u_m\|_3 = o_n(1).$$

Therefore, as n, $m \to \infty$, we have $\langle B'(u_n) - B'(u_m), u_n - u_m \rangle = o_n(1)$, and thus

$$\left\langle T'(u_n) - T'(u_m), u_n - u_m \right\rangle = o_n(1),$$

namely, (2.31) holds. □

3 Proofs of main results

Based on the preparatory lemmas presented in Section 2, we now proceed to prove the main results.

3.1 Proof of Theorem 1.1

To prove Theorem 1.1, we need to demonstrate the following two crucial lemmas.

Lemma 3.1. *Assume $2 < p < \frac{10}{3}$ and $a, \kappa > 0$, let $\{u_n\} \subset S_a$ be a minimizing sequence for I_{a^2}. Then the function $a \mapsto I_{a^2}$ is continuous and $I_{a^2}/a^2 \to 0$ as $a \to 0^+$.*

Proof. When $2 < p < \frac{10}{3}$, we have $\frac{3(p-2)}{2} < 2$. Now arguing as in Step 4 in the proof of Theorem 4.1 in [7], we can complete the proof. □

Lemma 3.2. *If $a, \kappa > 0$, $2 < p < 3$ and $M(a)$ is defined in Proposition 2.8, then for any $u \in M(a)$, there exists $\beta \in \mathbb{R}$ such that for any $\theta > 0$,*

$$h_\beta^u(\theta) := I(u^\theta) - \theta^2 I(u)$$

is differentiable and $(h_u)'(1) \neq 0$, where $u^\theta(x) := \theta^{1-\frac{3}{2}\beta} u\left(\frac{x}{\theta^\beta}\right)$.

Proof. For any $u \in M(a)$ and $\beta \in \mathbb{R}$, it follows from (1.7) and the definition of u^θ that

$$\begin{aligned} h_\beta^u(\theta) &= I(u^\theta) - \theta^2 I(u) \\ &= \frac{1}{2}\left(\theta^{(2-2\beta)} - \theta^2\right)\|\nabla u\|_2^2 + \frac{1}{4}\left(\theta^{4-\beta}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\frac{e^{-\kappa\theta^\beta|x-y|}}{|x-y|}u^2(x)u^2(y)\mathrm{d}x\mathrm{d}y - \theta^2\int_{\mathbb{R}^3}\phi_u u^2\mathrm{d}x\right) \\ &\quad - \frac{1}{p}\left(\theta^{(1-\frac{3}{2}\beta)p+3\beta} - \theta^2\right)\|u\|_p^p. \end{aligned}$$

Thus, $h_\beta^u(\theta)$ is differentiable and

$$\begin{aligned} \left(h_\beta^u\right)'(1) = -\beta\|\nabla u\|_2^2 &+ \frac{1}{4}\left((2-\beta)\int_{\mathbb{R}^3}\phi_u u^2\mathrm{d}x - \kappa\beta\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}e^{-\kappa|x-y|}u^2(x)u^2(y)\mathrm{d}x\mathrm{d}y\right) \\ &- \frac{1}{p}\left((1-\frac{3}{2}\beta)p + 3\beta - 2\right)\|u\|_p^p. \end{aligned}$$

To prove this lemma, we argue by contradiction. Assume that there exists a sequence $\{u_n\} \subset M(a)$ with $\|u_n\|_2 := \mu_n \le a$ and $\mu_n \to 0$ as $n \to \infty$ such that $\left(h_\beta^{u_n}\right)'(1) = 0$ for all $\beta \in \mathbb{R}$, then we conclude that $I(u_n) = I_{\mu_n^2} \to 0$ as $n \to \infty$ according to Lemma 3.1. Moreover, since (u_n, ϕ_{u_n}) solves system (1.1) under the constraint S_{μ_n}, from Lemma 2.7, we infer that (u_n, ϕ_{u_n}) satisfies the following Pohožaev identity

$$\|\nabla u_n\|_2^2 + \frac{1}{4}\int_{\mathbb{R}^3}\phi_{u_n}u_n^2\mathrm{d}x + \frac{\kappa}{4}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}e^{-\kappa|x-y|}u_n^2(x)u_n^2(y)\mathrm{d}x\mathrm{d}y - \frac{3(p-2)}{2p}\|u_n\|_p^p = 0. \tag{3.1}$$

Let $\beta = \frac{2}{3}$, from $\left(h_{\frac{2}{3}}^{u_n}\right)'(1) = 0$, we deduce that

$$\frac{1}{2}\|\nabla u_n\|_2^2 = \frac{1}{4}\int_{\mathbb{R}^3}\phi_{u_n}u_n^2\mathrm{d}x - \frac{\kappa}{8}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}e^{-\kappa|x-y|}u_n^2(x)u_n^2(y)\mathrm{d}x\mathrm{d}y. \tag{3.2}$$

It follows from (3.2), Lemma 2.6 and Lemma 2.1 that

$$\|\nabla u_n\|_2^2 \le \frac{1}{2}\int_{\mathbb{R}^3}\phi_{u_n}u_n^2\mathrm{d}x \le C\|u_n\|_4^4 \le C\|\nabla u_n\|_2^3\|u_n\|_2. \tag{3.3}$$

Let $\beta = 0$, from $(h_0^{u_n})'(1) = 0$, we have

$$\|u_n\|_p^p = \frac{p}{2(p-2)}\int_{\mathbb{R}^3}\phi_{u_n}u_n^2\mathrm{d}x \le C(p)\|\nabla u_n\|_2^3\|u_n\|_2. \tag{3.4}$$

By (3.3), (3.4), $\|u_n\|_2 = \mu_n \to 0$ and $I(u_n) = I_{\mu_n^2} \to 0$ as $n \to \infty$, we infer that

$$\|\nabla u_n\|_2^2,\ \int_{\mathbb{R}^3}\phi_{u_n}u_n^2\mathrm{d}x,\ \|u_n\|_p^p \to 0 \quad \text{as } n \to \infty. \tag{3.5}$$

It follows from Lemma 2.2 that

$$\begin{aligned} \int_{\mathbb{R}^3}\phi_{u_n}u_n^2\mathrm{d}x &= \int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\frac{e^{-\kappa|x-y|}}{|x-y|}u_n^2(x)u_n^2(y)\mathrm{d}x\mathrm{d}y \\ &\le \int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\frac{1}{|x-y|}u_n^2(x)u_n^2(y)\mathrm{d}x\mathrm{d}y \le C\|u_n\|_{\frac{12}{5}}^4. \end{aligned} \tag{3.6}$$

Similar to the Cases (a)–(d) of Step 5 in the proof of Theorem 4.1 in [6], when $2 < p \le \frac{8}{3}$, by applying (3.3)–(3.6), interpolation inequality and Lemma 2.1, we obtain a contradiction in all four cases. Then, we discuss the case when $\frac{8}{3} < p < 3$. Substituting (3.2) into Pohožaev identity (3.1), we obtain that

$$\frac{1}{4}\int_{\mathbb{R}^3}\phi_{u_n}u_n^2\mathrm{d}x = \frac{p-2}{2p}\|u_n\|_p^p, \tag{3.7}$$

and thus we have

$$I(u_n) = \frac{1}{2}\|\nabla u_n\|_2^2 + \frac{1}{4}\int_{\mathbb{R}^3}\phi_{u_n}u_n^2\mathrm{d}x - \frac{1}{p}\|u_n\|_p^p = \frac{1}{2}\|\nabla u_n\|_2^2 - \frac{4-p}{2p}\|u_n\|_p^p. \tag{3.8}$$

Let $\{w_n\}, \{v_n\} \subset M(a)$ such that $w_n := v_n^{\mu_n} = \mu_n^{1-\frac{3}{2}\beta}v_n\left(\frac{x}{\mu_n^\beta}\right)$, where $\beta \in \mathbb{R}$, $\|v_n\|_2 = 1$ and $\|w_n\|_2 = \mu_n \to 0$ as $n \to \infty$. Moreover, let $C(p) > 0$ such that when $\|\nabla v_n\|_2^2$ is small enough, it holds that $\frac{1}{2}\|\nabla v_n\|_2^2 - \frac{4-p}{2p}\|v_n\|_p^p < -C(p)$. Now, let $\beta = \frac{2(2-p)}{10-3p}$ such that $-2\beta = (1-\frac{3}{2}\beta)p + 3\beta - 2 = \frac{4(p-2)}{10-3p}$. From (3.8), we have

$$\begin{aligned}\frac{I_{\mu_n^2}}{\mu_n^2} \le \frac{I(w_n)}{\mu_n^2} &= \frac{\mu_n^{-2\beta}}{2}\|\nabla v_n\|_2^2 - \frac{4-p}{2p}\mu_n^{(1-\frac{3}{2}\beta)p+3\beta-2}\|v_n\|_p^p \\ &= \mu_n^{\frac{4(p-2)}{10-3p}}\left(\frac{1}{2}\|\nabla v_n\|_2^2 - \frac{4-p}{2p}\|v_n\|_p^p\right) \le -C(p)\mu_n^{\frac{4(p-2)}{10-3p}}.\end{aligned} \tag{3.9}$$

From (3.6), (3.8) and interpolation inequality, we infer that

$$\|u_n\|_p^p = \frac{p}{2(p-2)}\int_{\mathbb{R}^3}\phi_{u_n}u_n^2\mathrm{d}x \le C(p)\|u_n\|_{\frac{12}{5}}^4 \le C(p)\|u_n\|_2^{4\alpha}\|u_n\|_p^{4(1-\alpha)} = C(p)\mu_n^{4\alpha}\|u_n\|_p^{4(1-\alpha)},$$

where $\alpha := \frac{5p-12}{6(p-2)}$. Since $\frac{8}{3} < p < 3$, namely, $p > 4(1-\alpha)$, we cannot get a contradiction with $\|u_n\|_p \to 0$ as $n \to \infty$. But we deduce from the above inequality that

$$\|u_n\|_p^p \le C(p)\mu_n^{\frac{2(5p-12)}{3p-8}} \quad \text{and} \quad \frac{I(u_n)}{\mu_n^2} \ge -\frac{1}{p}\frac{\|u_n\|_p^p}{\mu_n^2} \ge -C(p)\mu_n^{\frac{4(p-2)}{3p-8}}. \tag{3.10}$$

Combining (3.9) and (3.10), we infer that

$$-C(p)\mu_n^{\frac{4(p-2)}{3p-8}} \le \frac{I_{\mu_n^2}}{\mu_n^2} \le -C(p)\mu_n^{\frac{4(p-2)}{10-3p}},$$

which contradicts $\mu_n \to 0$ as $n \to \infty$ since $\frac{4(p-2)}{3p-8} > \frac{4(p-2)}{10-3p}$ for any $\frac{8}{3} < p < 3$. In summary, for all the cases where $2 < p < 3$, we have derived contradictions. □

Proof of Theorem 1.1. Let $a_0 > 0$ be given by Lemma 2.12 and fix $a \in (0, a_0)$. Let $\{u_n\} \subset S_a$ be a minimizing sequence for $I_{a^2} < 0$ and then from Ekeland's variational principle ([36, Theorem 2.4]), we can know that $\{u_n\} \subset S_a$ is also a Palais–Smale sequence for I. From Lemma 2.10, we know that $\{u_n\}$ is bounded in $H^1(\mathbb{R}^3)$. It follows from Lemma 2.13 that there exists $\bar{u} \in H^1(\mathbb{R}^3)\setminus\{0\}$ such that $\|\bar{u}\|_2 \in (0, a]$, then, up to a subsequence, $u_n \rightharpoonup \bar{u}$ as $n \to \infty$. From Lemma 2.14, we know that (2.27), (2.28), (2.30) and (2.31) are satisfied. Further, Lemmas 2.11, 3.1, 3.2 ensure that the assumptions of Proposition 2.8 hold. Then, it follows from Proposition 2.8 that, up to a subsequence, $u_n \to \bar{u}$ in $H^1(\mathbb{R}^3)$ as $n \to \infty$. Moreover,

$\bar{u} \in S_a$ and $I(\bar{u}) = I_{a^2} < 0$. Therefore, $\bar{u}$ is the critical point of the constrained functional $I|_{S_a}$, that is, $\bar{u}$ is the normalized solution to system (1.5). From Section 2.3 and Lemma 2.7, we know that $\bar{u}$ satisfies the Pohožaev identity (2.20). Since $P(\bar{u}) = P(|\bar{u}|) = 0$, it follows that $\bar{u}$ is nonnegative. From Section 2.4 and the strong maximum principle, we conclude that $\bar{u}$ is a classical solution and $\bar{u} > 0$ in $\mathbb{R}^3$. Moreover, there exists a Lagrange multiplier $\lambda \in \mathbb{R}$ such that (2.21) holds.

When $2 < p < 3$, since $I_{a^2} < 0$ for any $a \in (0, a_0)$ from Lemma 2.12, we claim that there exists some constant $\sigma > 0$ such that $\|\nabla \bar{u}\|_2 > \sigma$. Indeed, if $\|\nabla \bar{u}\|_2 = 0$, from the fact that $\int_{\mathbb{R}^3} \phi_{\bar{u}} \bar{u}^2 dx \leq C \|\bar{u}\|_4^4$ and Lemma 2.1, we obtain that $\int_{\mathbb{R}^3} \phi_{\bar{u}} \bar{u}^2 dx = \|\bar{u}\|_p^p = 0$ and thus $I_{a^2} = 0$, which contradicts $I(\bar{u}) = I_{a^2} < 0$. When $2 < p \leq \frac{12}{5}$, from (2.21), Pohožaev identity (2.20) and $\bar{u} > 0$ in $\mathbb{R}^3$, we infer that

$$\begin{aligned}
\lambda \|\bar{u}\|_2^2 &= -\|\nabla \bar{u}\|_2^2 - \int_{\mathbb{R}^3} \phi_{\bar{u}} \bar{u}^2 dx + \|\bar{u}\|_p^p \\
&= 3\|\nabla \bar{u}\|_2^2 + \kappa \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} e^{-\kappa|x-y|} \bar{u}^2(x)\bar{u}^2(y) dx dy + \frac{12-5p}{p}\|\bar{u}\|_p^p > 3\sigma^2 > 0.
\end{aligned}$$

When $\frac{12}{5} < p < 3$, let $\sigma := \left(\frac{5p-12}{3p} C_{3,p}^p a_0^{\frac{6-p}{2}}\right)^{\frac{2}{10-3p}} > 0$ and we have

$$\begin{aligned}
\lambda \|\bar{u}\|_2^2 &= 3\|\nabla \bar{u}\|_2^2 + \kappa \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} e^{-\kappa|x-y|} \bar{u}^2(x)\bar{u}^2(y) dx dy - \frac{5p-12}{p}\|\bar{u}\|_p^p \\
&\geq \|\nabla \bar{u}\|_2^{\frac{3(p-2)}{2}} \left(3\|\nabla \bar{u}\|_2^{\frac{10-3p}{2}} - \frac{5p-12}{p} C_{3,p}^p a_0^{\frac{6-p}{2}}\right) > 0.
\end{aligned}$$

Thus, it holds that $\lambda > 0$ when $2 < p < 3$ and we have completed the proof. □

3.2 Proof of Theorem 1.2

Lemma 3.3. *Let $a, \kappa > 0$, $3 < p < \frac{10}{3}$ and a_1 be obtained in Lemma 2.12. Moreover, let $\{u_n\} \subset S_a$ be a minimizing sequence for I_{a^2}. Then, for any $0 < a_1 < \mu < a$, the following strong subadditivity inequality holds*

$$I_{a^2} < I_{\mu^2} + I_{a^2-\mu^2}.$$

Proof. Since $\{u_n\} \subset S_a$ is bounded, we claim that there exist positive constants k_1, k_2 such that

$$0 < k_1 \leq A(u_n) = \|\nabla u_n\|_2^2 \leq k_2. \tag{3.11}$$

Indeed, if $A(u_n) = o_n(1)$, from Lemma 2.6-(i) and Lemma 2.1, we infer that $B(u_n) = C(u_n) = o_n(1)$ and $I_{a^2} = 0$, which contradicts $I_{a^2} < 0$ from Lemma 2.12-(ii).

Let $u_n^\theta(x) := \theta^{1-\frac{3}{2}\beta} u_n\left(\frac{x}{\theta^\beta}\right)$ for a.e. $x \in \mathbb{R}^3$, then $\|u_n^\theta\|_2 = \theta\|u_n\|_2 = \theta a$. Fix $\beta = -2$ and it holds that

$$\begin{aligned}
I\left(u_n^\theta\right) &= \frac{\theta^6}{2}A(u_n) + \frac{\theta^6}{4}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{e^{-\kappa\theta^{-2}|x-y|}}{|x-y|} u_n^2(x)u_n^2(y) dx dy - \frac{\theta^{4p-6}}{p}C(u_n) \\
&\leq \frac{\theta^6}{2}A(u_n) + \frac{\theta^6}{4}N(u_n) - \frac{\theta^{4p-6}}{p}C(u_n) = \mathcal{J}_0\left(u_n^\theta\right) \\
&= \theta^2\left(I(u_n) + \frac{1}{2}\left(\theta^4 - 1\right)A(u_n) + \frac{1}{4}\left(\theta^4 N(u_n) - B(u_n)\right) - \frac{1}{p}\left(\theta^{4p-8} - 1\right)C(u_n)\right) \\
&=: \theta^2\left(I(u_n) + f(\theta, u_n)\right),
\end{aligned} \tag{3.12}$$

where $N(u_n) := \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{1}{|x-y|} u_n^2(x) u_n^2(y) \mathrm{d}x\mathrm{d}y$ and $\mathcal{J}_0$ is defined in (1.14). When $a > a_1$, it follows from [7, Lemma 3.2] that

$$\mathcal{J}_0(u_n) = \frac{1}{2}A(u_n) + \frac{1}{4}N(u_n) - \frac{1}{p}C(u_n) < 0. \tag{3.13}$$

Therefore, when $3 < p < \frac{10}{3}$, it follows from (3.11) and (3.13) that

$$f'(\theta, u_n)|_{\theta=1} = 2A(u_n) + N(u_n) - \frac{4p-8}{p}C(u_n) < (6-2p)\,A(u_n) + (3-p)\,N(u_n) < 0. \tag{3.14}$$

Since $(4p-8)(4p-9) > 12$ when $3 < p < \frac{10}{3}$, from (3.11) and (3.13), we derive that

$$\begin{aligned} f''(\theta, u_n)|_{\theta=1} &= 6A(u_n) + 3N(u_n) - \frac{(4p-8)(4p-9)}{p}C(u_n) \\ &< \left(6 - \frac{(4p-8)(4p-9)}{2}\right) A(u_n) + \left(3 - \frac{(4p-8)(4p-9)}{4}\right) N(u_n) < 0. \end{aligned} \tag{3.15}$$

Since $f(1, u_n) = \frac{1}{4}(N(u_n) - B(u_n)) > 0$ and $f(\theta, u_n) \to -\infty$ as $\theta \to \infty$, we conclude from (3.14) and (3.15) that there exists $\theta_1 > 1$ such that $f(\theta_1, u_n) = 0$. Hence, for any $\theta > \theta_1 > 1$, we get $f(\theta, u_n) < 0$ and from (3.12), we have

$$I_{\theta^2 a^2} \le I\left(u_n^\theta\right) < \theta^2 I(u_n) \to \theta^2 I_{a^2} \quad \text{as } n \to \infty.$$

Therefore, for any $0 < a_1 < \mu < a$, it holds that

$$I_{a^2} = \frac{\mu^2}{a^2} I_{\frac{a^2}{\mu^2}\mu^2} + \frac{a^2-\mu^2}{a^2} I_{\frac{a^2}{a^2-\mu^2}(a^2-\mu^2)} < I_{\mu^2} + I_{a^2-\mu^2}.$$

This completes the proof of Lemma 3.3. □

Proof of Theorem 1.2. Let $a_1 > 0$ be given by Lemma 2.12 and fix $a > a_1$. Let $\{u_n\} \subset S_a$ be a minimizing sequence for $I_{a^2} < 0$ and then from Ekeland's variational principle ([36, Theorem 2.4]), we can know that $\{u_n\} \subset S_a$ is also a Palais–Smale sequence for I. From Lemma 2.10, we know that $\{u_n\}$ is bounded in $H^1(\mathbb{R}^3)$. It follows from Lemma 2.13 that there exists $\bar{u} \in H^1(\mathbb{R}^3)\setminus\{0\}$ such that $\|\bar{u}\|_2 \in (0, a]$, then, up to a subsequence, $u_n \rightharpoonup \bar{u}$ as $n \to \infty$. From Lemma 2.14, we know that (2.27), (2.28), (2.30) and (2.31) are satisfied. Lemma 3.3 guarantees that the strong subadditivity inequality (2.29) holds. Then, it follows from Proposition 2.9 that, up to a subsequence, $u_n \to \bar{u}$ in $H^1(\mathbb{R}^3)$ as $n \to \infty$. Moreover, $\bar{u} \in S_a$ and $I(\bar{u}) = I_{a^2} < 0$ for any $a \in (a_1, \infty)$ from Lemma 2.12. Therefore, $\bar{u}$ is the critical point of the constrained functional $I|_{S_a}$, namely, $\bar{u}$ is the normalized solution to system (1.5). Similar to the proof of Theorem 1.1, we conclude that $\bar{u}$ is a positive classical solution and the corresponding Lagrange multiplier $\lambda > 0$ when $3 < p < \frac{10}{3}$. This completes the proof of Theorem 1.2. □

3.3 Proof of Theorem 1.3

In Section 2.6, we have introduced the scaling $u^\theta(x) := \theta^{1-\frac{3}{2}\beta} u\left(\frac{x}{\theta^\beta}\right)$ for a.e. $x \in \mathbb{R}^3$ and $\|u^\theta\|_2 = \theta\|u\|_2$. Specifically, when $2 < p < 3$ and $u \in H^1(\mathbb{R}^3)$, let $\beta = \frac{4-2p}{10-3p}$ and $u^a(x) :=$

$a^{\frac{4}{10-3p}}u\left(a^{\frac{2p-4}{10-3p}}x\right)$. Then for any $u \in S_1$, we have $\|u^a\|_2 = a$ and

$$\begin{aligned} I(u^a) &= a^{\frac{12-2p}{10-3p}}\left(\frac{1}{2}\|\nabla u\|_2^2 + \frac{a^{\alpha(p)}}{4}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\frac{e^{-\kappa a^{\frac{4-2p}{10-3p}}|x-y|}}{|x-y|}u^2(x)u^2(y)\mathrm{d}x\mathrm{d}y - \frac{1}{p}\|u\|_p^p\right) \\ &=: a^{\frac{12-2p}{10-3p}}\mathcal{I}(u), \end{aligned}$$

where $\alpha(p) := \frac{8(3-p)}{10-3p} > 0$. Therefore, the minimization problem (1.8) is equivalent to the following one

$$\mathcal{I}_{a^2} := \inf_{u\in S_1}\mathcal{I}(u).$$

According to the arguments in [13], we introduce the following minimization problem

$$K_{a,p} := \inf_{u\in S_1}\mathcal{E}_{a,p}(u),$$

where

$$\mathcal{E}_{a,p}(u) := \frac{1}{2}\|\nabla u\|_2^2 + \frac{a^{\alpha(p)}}{4}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\frac{e^{-\kappa a^{\frac{4-2p}{10-3p}}|x-y|}}{|x-y|}u^2(x)u^2(y)\mathrm{d}x\mathrm{d}y - \frac{1}{p}\|u\|_p^p.$$

We denote by $\mathcal{N}_{a,p}$ the corresponding set of minimizers

$$\mathcal{N}_{a,p} := \left\{u \in S_1 : \mathcal{E}_{a,p}(u) = K_{a,p}\right\}.$$

Thus, Theorem 1.3 is equivalent to the following proposition.

Proposition 3.4. *Let $2 < p < 3$ and a_0 be obtained in Theorem 1.1. Then, there exists $a_2 = a_2(p) \in (0, a_0]$ such that for any $0 < a < a_2$, every function $u \in \mathcal{N}_{a,p}$ is radially symmetric. Consequently, $\phi_u \in H_r^1(\mathbb{R}^3)$ up to translation.*

Remark 3.5. It follows from Lemma 2.6-(iv) and the implicit function argument in [13] that Proposition 3.4 holds. Therefore, the detailed proof of Theorem 1.3 will not be presented here, and readers can refer to [13].

3.4 Proof of Theorem 1.4

In this section, we shall study the limit behavior of the ground state normalized solutions $(u_\kappa, \phi_\kappa, \lambda_\kappa)$ to system (1.5) as $\kappa \to 0$.

Lemma 3.6. *([12, Lemma 3.4]) Let $p \in (2, 3)$, $\kappa > 0$ and $a \in (0, a_2)$ with $a_2 > 0$ obtained in Theorem 1.3. Moreover, let $\mathcal{J}_0$ be defined in (1.14) and let*

$$I_\kappa(u_\kappa) := \frac{1}{2}\|\nabla u_\kappa\|_2^2 + \frac{1}{4}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\frac{e^{-\kappa|x-y|}}{|x-y|}u_\kappa^2(x)u_\kappa^2(y)\mathrm{d}x\mathrm{d}y - \frac{1}{p}\|u_\kappa\|_p^p.$$

If

$$\left\{u_\kappa \in S_a\backslash\{0\} : I_\kappa(u_\kappa) = I_{\kappa,a^2} := \inf_{u_\kappa\in S_a} I_\kappa(u_\kappa) < 0\right\}$$

is a family of minimizers for the functional I_κ, then it holds that $I_{\kappa,a^2} \to \mathcal{J}_{0,a^2} := \inf_{u\in S_a}\mathcal{J}_0(u) < 0$ as $\kappa \to 0$.

Proof. According to $\frac{e^{-\kappa|x|}}{|x|} \leq \frac{1}{|x|}$, we know that $I_{\kappa,a^2} \leq \mathcal{J}_{0,a^2}$ for any $\kappa \geq 0$. Therefore, it suffices to prove that $\mathcal{J}_{0,a^2} \leq \liminf_{\kappa\to 0} I_{\kappa,a^2}$. Given $\kappa > 0$, we have

$$\mathcal{J}_{0,a^2} \leq \mathcal{J}_0(u_\kappa) = I_\kappa(u_\kappa) + \frac{1}{4}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{1-e^{-\kappa|x-y|}}{|x-y|} u_\kappa^2(x)u_\kappa^2(y)\mathrm{d}x\mathrm{d}y =: I_\kappa(u_\kappa) + \frac{1}{4}g(\kappa).$$

In this situation, we only need to prove that $g(\kappa) \to 0$ as $\kappa \to 0$. Fix $\kappa \in (0,1)$. It follows from a change of variable that

$$\begin{aligned}|g(\kappa)| \leq &\int_{\left\{z\in\mathbb{R}^3:|z|\leq\frac{1}{\kappa}\right\}} \frac{1-e^{-\kappa|z|}}{|z|}\int_{\mathbb{R}^3} |u_\kappa(x)|^2 |u_\kappa(x-z)|^2 \mathrm{d}x\mathrm{d}z \\ &+ \int_{\left\{z\in\mathbb{R}^3:|z|\geq\frac{1}{\kappa}\right\}} \frac{1-e^{-\kappa|z|}}{|z|}\int_{\mathbb{R}^3} |u_\kappa(x)|^2 |u_\kappa(x-z)|^2 \mathrm{d}x\mathrm{d}z =: g_1(\kappa)+g_2(\kappa).\end{aligned}$$

For any $u_\kappa \in S_a$, since $1-e^{-t} \leq t$ for all $t \geq 0$, let $t = \kappa|z| > 0$, we have

$$\begin{aligned}g_1(\kappa) &\leq \int_{\left\{z\in\mathbb{R}^3:|z|\leq\frac{1}{\kappa}\right\}} \kappa\int_{\mathbb{R}^3} |u_\kappa(x)|^2 |u_\kappa(x-z)|^2 \mathrm{d}x\mathrm{d}z \\ &\leq \kappa\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} |u_\kappa(x)|^2 |u_\kappa(x-z)|^2 \mathrm{d}x\mathrm{d}z = \kappa a^4.\end{aligned}$$

For any $u_\kappa \in S_a$, it follows from $1-e^{-t} \leq 1$ for all $t \geq 0$ that

$$\begin{aligned}g_2(\kappa) &\leq \int_{\left\{z\in\mathbb{R}^3:|z|\geq\frac{1}{\kappa}\right\}} \frac{1}{|z|}\int_{\mathbb{R}^3} |u_\kappa(x)|^2 |u_\kappa(x-z)|^2 \mathrm{d}x\mathrm{d}z \\ &\leq \kappa\int_{\left\{z\in\mathbb{R}^3:|z|\geq\frac{1}{\kappa}\right\}}\int_{\mathbb{R}^3} |u_\kappa(x)|^2 |u_\kappa(x-z)|^2 \mathrm{d}x\mathrm{d}z \\ &\leq \kappa\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} |u_\kappa(x)|^2 |u_\kappa(x-z)|^2 \mathrm{d}x\mathrm{d}z = \kappa a^4.\end{aligned}$$

Therefore, it holds that $g_1(\kappa), g_2(\kappa) \to 0$ as $\kappa \to 0$, that is, $g(\kappa) \to 0$ as $\kappa \to 0$, which implies that $I_{\kappa,a^2} \to \mathcal{J}_{0,a^2}$ as $\kappa \to 0$. Moreover, from $p \in (2,3)$, $a \in (0,a_2)$, (1.14), [7] and Lemma 2.12-(i), we have $\mathcal{J}_{0,a^2} < 0$. □

Lemma 3.7 ([12, Lemma 3.5])**.** *Let $\{f_0\} \cup \{f_\kappa : 0 < \kappa < 1\} \subset L^{\frac{6}{5}}(\mathbb{R}^3)$. Let $\varphi_0 \in \mathcal{D}^{1,2}(\mathbb{R}^3)$ be the unique solution to the following equation*

$$-\Delta\varphi = f_0. \tag{3.16}$$

Given $\kappa \in (0,1)$, let $\phi_\kappa \in H^1(\mathbb{R}^3)$ be the unique solution to the following equation

$$-\Delta\phi + \kappa^2\phi = f_\kappa. \tag{3.17}$$

As $\kappa \to 0$, it holds that

(i) If $f_\kappa \rightharpoonup f_0$ in $L^{\frac{6}{5}}(\mathbb{R}^3)$, then $\phi_\kappa \rightharpoonup \varphi_0$ in $\mathcal{D}^{1,2}(\mathbb{R}^3)$.

(ii) If $f_\kappa \to f_0$ in $L^{\frac{6}{5}}(\mathbb{R}^3)$, then $\phi_\kappa \to \varphi_0$ in $\mathcal{D}^{1,2}(\mathbb{R}^3)$ and $\kappa^2\phi_\kappa \to 0$ in $L^2(\mathbb{R}^3)$.

Proof. (i) It follows from $\phi_\kappa \in H^1(\mathbb{R}^3)$ and Sobolev's embedding theorem that $\|\phi_\kappa\|_6 \leq C\|\phi_\kappa\|_{H^1}$, and thus $\phi_\kappa \in \mathcal{D}^{1,2}(\mathbb{R}^3)$. Fix $\kappa \in (0,1)$, since $\phi_\kappa \in \mathcal{D}^{1,2}(\mathbb{R}^3)$ is the unique solution to (3.17), from Hölder's inequality and Lemma 2.1, we deduce that

$$\|\nabla\phi_\kappa\|_2^2 \leq \|\nabla\phi_\kappa\|_2^2 + \kappa^2\|\phi_\kappa\|_2^2 = \int_{\mathbb{R}^3} f_\kappa\phi_\kappa\mathrm{d}x \leq C\|f_\kappa\|_{\frac{6}{5}}\|\phi_\kappa\|_6 \leq C\|f_\kappa\|_{\frac{6}{5}}\|\nabla\phi_\kappa\|_2, \tag{3.18}$$

thus, $\|\nabla\phi_\kappa\|_2 \le C\|f_\kappa\|_{\frac{6}{5}}$. Since $f_\kappa \rightharpoonup f_0$ in $L^{\frac{6}{5}}(\mathbb{R}^3)$ as $\kappa \to 0$, we deduce that $\{f_\kappa : 0 < \kappa < 1\}$ is a bounded subset of $L^{\frac{6}{5}}(\mathbb{R}^3)$. It follows from $\phi_\kappa \in \mathcal{D}^{1,2}(\mathbb{R}^3)$ that $\{\phi_\kappa : 0 < \kappa < 1\}$ is a bounded subset of $\mathcal{D}^{1,2}(\mathbb{R}^3)$. In particular, we conclude that for any $\{\kappa_n\}_{n\in\mathbb{N}} \subset (0,1)$ satisfying $\lim_{n\to\infty}\kappa_n = 0$, there exists $\phi_* \in \mathcal{D}^{1,2}(\mathbb{R}^3)$ such that, up to a subsequence, $\phi_{\kappa_n} \rightharpoonup \phi_*$ in $\mathcal{D}^{1,2}(\mathbb{R}^3)$ as $n \to \infty$. Next, it suffices to prove that if $\{\kappa_n\}_{n\in\mathbb{N}} \subset (0,1)$ satisfies $\kappa_n \to 0$ and $\phi_{\kappa_n} \rightharpoonup \phi_*$ in $\mathcal{D}^{1,2}(\mathbb{R}^3)$ as $n \to \infty$, then $\phi_* = \varphi_0$.

Fix $n \in \mathbb{N}^+$, since the function ϕ_{κ_n} is a solution to (3.17), for any $\psi \in C_c^\infty(\mathbb{R}^3)$, it holds that

$$\int_{\mathbb{R}^3} \nabla\phi_{\kappa_n}\nabla\psi \mathrm{d}x + \kappa_n^2\int_{\mathbb{R}^3}\phi_{\kappa_n}\psi \mathrm{d}x = \int_{\mathbb{R}^3} f_{\kappa_n}\psi \mathrm{d}x. \tag{3.19}$$

Since $\phi_{\kappa_n} \in L^6(\mathbb{R}^3)$, $\phi_{\kappa_n} \rightharpoonup \phi_*$ in $\mathcal{D}^{1,2}(\mathbb{R}^3)$ and $\kappa_n^2\int_{\mathbb{R}^3}\phi_{\kappa_n}\psi \mathrm{d}x \le \kappa_n^2 C\|\phi_{\kappa_n}\|_6\|\psi\|_{\frac{6}{5}} \to 0$ as $n \to \infty$, from (3.19), we take the limit as $n \to \infty$ to conclude that

$$\int_{\mathbb{R}^3}\nabla\phi_*\nabla\psi \mathrm{d}x = \int_{\mathbb{R}^3} f_0\psi \mathrm{d}x \quad \text{for any } \psi \in C_c^\infty(\mathbb{R}^3). \tag{3.20}$$

Therefore, we infer that $\phi_* = \varphi_0$ according to (3.20) and the uniqueness of the solution $\varphi_0 \in \mathcal{D}^{1,2}(\mathbb{R}^3)$ to (3.16).

(ii) Since $f_\kappa \to f_0$ in $L^{\frac{6}{5}}(\mathbb{R}^3)$, we deduce from (i) that $\phi_\kappa \rightharpoonup \varphi_0$ in $\mathcal{D}^{1,2}(\mathbb{R}^3)$ as $\kappa \to 0$. According to the weak lower semicontinuity of $\|\nabla\cdot\|_2^2$, we conclude that

$$\|\nabla\varphi_0\|_2^2 \le \liminf_{\kappa\to 0}\|\nabla\phi_\kappa\|_2^2. \tag{3.21}$$

Let $\{\psi_n\}_{n\in\mathbb{N}} \subset C_c^\infty(\mathbb{R}^3)$ be a sequence such that $\psi_n \to \varphi_0$ in $\mathcal{D}^{1,2}(\mathbb{R}^3)$ as $n \to \infty$. Fix $\kappa \in (0,1)$, let $E_\kappa : \mathcal{D}^{1,2}(\mathbb{R}^3) \to \mathbb{R}$ be the energy functional of (3.17), which is given by

$$E_\kappa(\phi) = \frac{1}{2}\|\nabla\phi\|_2^2 + \frac{\kappa^2}{2}\|\phi\|_2^2 - \int_{\mathbb{R}^3} f_\kappa\phi \, \mathrm{d}x.$$

By the facts that $\phi_\kappa \in H^1(\mathbb{R}^3)$ is the unique solution to (3.17) and $H^1(\mathbb{R}^3) \hookrightarrow \mathcal{D}^{1,2}(\mathbb{R}^3)$, we deduce from (i) that $E_\kappa(\phi_\kappa) = \inf_{\phi\in\mathcal{D}^{1,2}(\mathbb{R}^3)} E_\kappa(\phi)$. It follows from $\{\psi_n\}_{n\in\mathbb{N}} \subset C_c^\infty(\mathbb{R}^3)$ and $C_c^\infty(\mathbb{R}^3)$ is dense in $\mathcal{D}^{1,2}(\mathbb{R}^3)$ that

$$\begin{aligned}\frac{1}{2}\|\nabla\phi_\kappa\|_2^2 &= E_\kappa(\phi_\kappa) - \frac{\kappa^2}{2}\|\phi_\kappa\|_2^2 + \int_{\mathbb{R}^3} f_\kappa\phi_\kappa \, \mathrm{d}x \le E_\kappa(\psi_n) + \int_{\mathbb{R}^3} f_\kappa\phi_\kappa \, \mathrm{d}x \\ &= \frac{1}{2}\|\nabla\psi_n\|_2^2 + \frac{\kappa^2}{2}\|\psi_n\|_2^2 - \int_{\mathbb{R}^3} f_\kappa(\psi_n - \phi_\kappa) \, \mathrm{d}x.\end{aligned} \tag{3.22}$$

Since $f_\kappa \to f_0$ in $L^{\frac{6}{5}}(\mathbb{R}^3)$ and $\phi_\kappa \rightharpoonup \varphi_0$ in $\mathcal{D}^{1,2}(\mathbb{R}^3)$ as $\kappa \to 0$, we deduce that

$$\int_{\mathbb{R}^3} f_\kappa\psi_n \mathrm{d}x \to \int_{\mathbb{R}^3} f_0\psi_n \mathrm{d}x \quad \text{and} \quad \int_{\mathbb{R}^3} f_\kappa\phi_\kappa \mathrm{d}x \to \int_{\mathbb{R}^3} f_0\varphi_0 \mathrm{d}x. \tag{3.23}$$

Thus, from (3.22) and (3.23), we infer that

$$\limsup_{\kappa\to 0}\frac{1}{2}\|\nabla\phi_\kappa\|_2^2 \le \frac{1}{2}\|\nabla\psi_n\|_2^2 - \int_{\mathbb{R}^3} f_0(\psi_n - \varphi_0) \, \mathrm{d}x.$$

From $\psi_n \to \varphi_0$ in $\mathcal{D}^{1,2}(\mathbb{R}^3)$ as $n \to \infty$, we obtain that

$$\limsup_{\kappa\to 0}\|\nabla\phi_\kappa\|_2^2 \le \|\nabla\varphi_0\|_2^2 \quad \text{as } n \to \infty. \tag{3.24}$$

It follows from (3.21) and (3.24) that $\|\nabla\phi_\kappa\|_2^2 \to \|\nabla\varphi_0\|_2^2$ as $\kappa \to 0$, and thus $\phi_\kappa \to \varphi_0$ in $\mathcal{D}^{1,2}(\mathbb{R}^3)$ as $\kappa \to 0$.

Finally, since $\varphi_0 \in \mathcal{D}^{1,2}(\mathbb{R}^3)$ is the unique solution to equation (3.16), we conclude that

$$\kappa^2 \|\phi_\kappa\|_2^2 = \int_{\mathbb{R}^3} f_\kappa\phi_\kappa \mathrm{d}x - \|\nabla\phi_\kappa\|_2^2 \to \int_{\mathbb{R}^3} f_0\varphi_0 \mathrm{d}x - \|\nabla\varphi_0\|_2^2 = 0 \quad \text{as } \kappa \to 0,$$

namely, (ii) holds. □

The following proposition shows that if $p \in (2,3)$, then the Lagrange multipliers of solutions to the Schrödinger–Poisson–Slater system (1.13) tend to a certain $\tilde{\lambda} > 0$ as $a \to 0$.

Proposition 3.8 ([13, Proposition 1.3])**.** *Assume $p \in (2,3)$, $\kappa > 0$ and $\tilde{\lambda} > 0$, when $a \in (0, a_2)$, where a_2 is obtained in Theorem 1.3, for every $\varepsilon > 0$, there exists $a(\varepsilon) \in (0, a_2)$ such that*

$$\sup_{\lambda\in\mathcal{A}_a} |\lambda - \tilde{\lambda}| < \varepsilon \quad \forall\, 0 < a < a(\varepsilon),$$

where

$$\mathcal{A}_a := \{\lambda \in \mathbb{R} : \text{there exists } u \in H^1(\mathbb{R}^3) \text{ such that } (u,\varphi,\lambda) \in H^1(\mathbb{R}^3) \times \mathcal{D}^{1,2}(\mathbb{R}^3) \times \mathbb{R}$$
$$\text{is a ground state solution to the Schrödinger–Poisson–Slater system } (1.13)\}.$$

Proof of Theorem 1.4. Since $\kappa > 0$, $2 < p < 3$ and $0 < a < a_2$, according to Theorems 1.1 and 1.3, for any ground state normalized solution $(u_\kappa, \phi_\kappa, \lambda_\kappa)$ to system (1.5) with $u_\kappa \in S_a\backslash\{0\}$ and $I_\kappa(u_\kappa) = I_{\kappa,a^2} < 0$, we deduce that up to translation, $(u_\kappa, \phi_\kappa, \lambda_\kappa) \in S_a^r \times H_r^1(\mathbb{R}^3) \times (0,\infty)$, which implies that $(u_\kappa, \phi_\kappa, \lambda_\kappa) \in S_a^r \times \mathcal{D}_r^{1,2}(\mathbb{R}^3) \times (0,\infty)$.

It follows from Lemma 2.10 that the set $\{u_\kappa \in S_a : I_\kappa(u_\kappa) = I_{\kappa,a^2} < 0\}$ is bounded in $H_r^1(\mathbb{R}^3)$ for any $\kappa > 0$. Let $\{\kappa_n\}_{n\in\mathbb{N}^+} \subset (0,1)$ be a sequence such that $\kappa_n \to 0$ as $n \to \infty$. Then, for any $n \in \mathbb{N}^+$, both $\{u_{\kappa_n}\} \subset S_a^r$ and $\{\phi_{\kappa_n}\} \subset \mathcal{D}_r^{1,2}(\mathbb{R}^3)$ are bounded and thus there exists $u_0 \in H_r^1(\mathbb{R}^3)$ such that, up to a subsequence, $u_{\kappa_n} \rightharpoonup u_0$ in $H_r^1(\mathbb{R}^3)$ as $n \to \infty$, $u_{\kappa_n} \to u_0$ in $L^p(\mathbb{R}^3)$ with $p \in (2,6)$ and $u_{\kappa_n} \to u_0$ a.e. in $\mathbb{R}^3$. Since $u_{\kappa_n}^2 \to u_0^2$ in $L^{\frac{6}{5}}(\mathbb{R}^3)$, from Lemma 3.7, we deduce that as $n \to \infty$, up to a subsequence, $\phi_{\kappa_n} \to \varphi_0$ in $\mathcal{D}_r^{1,2}(\mathbb{R}^3)$, where φ_0 is the unique solution of $-\Delta\varphi = 4\pi u_0^2$ in $\mathbb{R}^3$. Since for any $n \in \mathbb{N}^+$, $(u_{\kappa_n}, \phi_{\kappa_n}, \lambda_{\kappa_n}) \in S_a^r \times \mathcal{D}_r^{1,2}(\mathbb{R}^3) \times (0,\infty)$ is a ground state normalized solution to (1.5), from (2.21), Lemmas 2.1 and 2.6-(iii), we infer that

$$\begin{aligned}|\lambda_{\kappa_n}| &= \frac{1}{a^2}\left|\|u_{\kappa_n}\|_p^p - \|\nabla u_{\kappa_n}\|_2^2 - \int_{\mathbb{R}^3} \phi_{\kappa_n} u_{\kappa_n}^2 \mathrm{d}x\right| \\ &\le \frac{1}{a^2}\left(C\|\nabla u_{\kappa_n}\|_2^{\frac{3(p-2)}{2}} a^{\frac{6-p}{2}} + \|\nabla u_{\kappa_n}\|_2^2 + C\|\nabla u_{\kappa_n}\|_2^3 a\right) < \infty,\end{aligned}$$

namely, $\{\lambda_{\kappa_n}\} \subset (0,\infty)$ is a bounded sequence, and then there exists $\lambda_0 \in [0,\infty)$ such that, up to a subsequence, $\lambda_{\kappa_n} \to \lambda_0$ as $n \to \infty$.

Moreover, since $\mathcal{J}_0$, defined in (1.14), is the functional of the Schrödinger–Poisson–Slater system (1.13), it follows from Lemma 3.6 that $\{u_{\kappa_n}\}$ is a minimizing sequence for $\mathcal{J}_0$ at level $\mathcal{J}_{0,a^2} < 0$. Hence, from $u_{\kappa_n} \rightharpoonup u_0$ in $H_r^1(\mathbb{R}^3)$, $\phi_{\kappa_n} \to \varphi_0$ in $\mathcal{D}_r^{1,2}(\mathbb{R}^3)$, $\lambda_{\kappa_n} \to \lambda_0$, Proposition 2.8, Lemma 2.13 and [7], we conclude that $u_0 \in S_a^r\backslash\{0\}$, $u_{\kappa_n} \to u_0$ in $H_r^1(\mathbb{R}^3)$ as $n \to \infty$ and $\mathcal{J}_0(u_0) = \mathcal{J}_{0,a^2} < 0$. Thus, $(u_0, \varphi_0, \lambda_0) \in S_a^r \times \mathcal{D}_r^{1,2}(\mathbb{R}^3) \times [0,\infty)$ is a ground state normalized solution to the Schrödinger–Poisson–Slater system (1.13) with

$$\lambda_0 := \frac{1}{a^2}\left(\|u_0\|_p^p - \|\nabla u_0\|_2^2 - \int_{\mathbb{R}^3} \varphi_0 u_0^2 \,\mathrm{d}x\right).$$

According to Proposition 3.8, for every $\varepsilon > 0$, there exists $a(\varepsilon) \in (0, a_2)$ such that $\lambda_0 > 0$ when $a \in (0, a_2)$ is sufficiently small. Therefore, we conclude that, up to a subsequence and up to translation, it holds that

$$(u_\kappa, \phi_\kappa, \lambda_\kappa) \to (u_0, \varphi_0, \lambda_0) \quad \text{in } S_a^r \times \mathcal{D}_r^{1,2}(\mathbb{R}^3) \times (0, \infty) \quad \text{as } \kappa \to 0.$$

That is, (1.15) holds. Thus, we have completed the proof of Theorem 1.4. □

Declarations

- **Data availability** Data sharing not applicable to this article as no data sets were generated or analyzed during the current study.
- **Conflict of interests** On behalf of all authors, the corresponding author states that there is no conflict of interest.

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References

[1] D. Andelman, Electrostatic properties of membranes: the Poisson–Boltzmann theory, in: *Handbook of biological physics*, Structure and Dynamics of Membranes – From Cells to Vesicles, Vol. 1, Elsevier (North-Holland), Amsterdam, 1995, 603–642. https://doi.org/10.1016/S1383-8121(06)80005-9.

[2] G. B. Arfken, H. J. Weber, F. E. Harris, *Mathematical methods for physicists: A comprehensive guide*, 7th ed., Academic press, Amsterdam, 2011. https://doi.org/10.1016/C2009-0-30629-7; Zbl 1239.00005

[3] H. Brezis, *Functional analysis, Sobolev spaces and partial differential equations*, Universitext, Vol. 2, Springer, New York, 2011. https://doi.org/10.1007/978-0-387-70914-7; Zbl 1220.46002

[4] R. Blossey, *The Poisson–Boltzmann equation: An introduction*, SpringerBriefs in Physics, Springer, Cham, 2023. https://doi.org/10.1007/978-3-031-24782-8; Zbl 1544.82002

[5] V. Benci, D. Fortunato, An eigenvalue problem for the Schrödinger–Maxwell equations, *Topol. Methods Nonlinear Anal.* **11**(1998), No. 2, 283–293. https://doi.org/10.12775/TMNA.1998.019; Zbl 0926.35125

[6] J. Bellazzini, G. Siciliano, Stable standing waves for a class of nonlinear Schrödinger–Poisson equations, *Z. Angew. Math. Phys.* **62**(2011). No. 2, 267–280. https://doi.org/10.1007/s00033-010-0092-1; Zbl 1339.35280

[7] J. Bellazzini, G. Siciliano, Scaling properties of functionals and existence of constrained minimizers, *J. Funct. Anal.* **261**(2011), No. 9, 2486–2507. https://doi.org/10.1016/j.jfa.2011.06.014; Zbl 1357.49053

[8] J. Bellazzini, L. Jeanjean, T. J. Luo, Existence and instability of standing waves with prescribed norm for a class of Schrödinger–Poisson equations, *Proc. London Math. Soc.* **107**(2013), No. 2, 303–339. https://doi.org/10.1112/plms/pds072; Zbl 1284.35391

[9] D. M. Chipman, Solution of the linearized Poisson–Boltzmann equation, *J. Chem. Phys.* **120**(2004) No. 12, 5566–5575. https://doi.org/10.1063/1.1648632.

[10] D. L. Chapman, A contribution to the theory of electrocapillarity, *Lond. Edinb. Dublin Philos. Mag. J. Sci.* **25**(1913) No. 148, 475–481. https://doi.org/10.1080/14786440408634187; JFM 44.0918.01

[11] P. d'Avenia, G. Siciliano, Nonlinear Schrödinger equation in the Bopp–Podolsky electrodynamics: Solutions in the electrostatic case, *J. Differential Equations* **267**(2019), No. 2, 1025–1065. https://doi.org/10.1016/j.jde.2019.02.001; Zbl 1432.35080

[12] G. de Paula Ramos, G. Siciliano, Existence and limit behavior of least energy solutions to constrained Schrödinger–Bopp–Podolsky systems in $\mathbb{R}^3$, *Z. Angew. Math. Phys.* **74**(2023), No. 2, 56. https://doi.org/10.1007/s00033-023-01950-w; Zbl 1514.35169

[13] V. Georgiev, F. Prinari, N. Visciglia, On the radiality of constrained minimizers to the Schrödinger–Poisson–Slater energy, *Ann. Inst. Henri Poincaré, Anal. Non Linéaire.* **29**(2012) No. 3, 369–376. https://doi.org/10.1016/j.anihpc.2011.12.001; Zbl 1260.35204

[14] D. Gilbarg, N. S. Trudinger, *Elliptic partial differential equations of second order*, Classics in Mathematics, Vol. 224, Springer, Berlin, Heidelberg, 1977. https://doi.org/10.1007/978-3-642-61798-0; Zbl 1042.35002

[15] T. X. Gou, Z. T. Zhang, Normalized solutions to the Chern–Simons–Schrödinger system, *J. Funct. Anal.* **280**(2021), No. 5, 108894, https://doi.org/10.1016/j.jfa.2020.108894; Zbl 1455.35080

[16] M. Gouy, Sur la constitution de la charge électrique à la surface d'un électrolyte, *J. Phys. Theor. Appl.* **9**(1910) No. 1, 457–468. JFM 41.0957.01

[17] C. M. He, L. Li, S. J. Chen, Normalized solutions for Schrödinger–Bopp–Podolsky system, (2022), arXiv: 2206.04008. https://doi.org/10.48550/arXiv.2206.04008

[18] J. Han, H. Huh, J. Seok, Chern–Simons limit of the standing wave solutions for the Schrödinger equations coupled with a neutral scalar field, *J. Funct. Anal.* **266**(2014), No. 1, 318–342. https://doi.org/10.1016/j.jfa.2013.09.019; Zbl 1304.35642

[19] L. I. Hedberg, On certain convolution inequalities, *Proc. Amer. Math. Soc.* **36**(1972), No. 2, 505–510. https://doi.org/10.2307/2039187; Zbl 0283.26003

[20] L. S. Hernandez, G. Siciliano, Existence and asymptotic behavior of solutions to eigenvalue problems for Schrödinger–Bopp–Podolsky equations, *Electron. J. Differential Equations* **2023**, No. 66, 1–18. https://doi.org/10.58997/ejde.2023.66; Zbl 1532.35184

[21] M. Itagaki, Higher order three-dimensional fundamental solutions to the Helmholtz and the modified Helmholtz equations, *Eng. Anal. Bound. Element.* **15**(1995), No. 3, 289–293. https://doi.org/10.1016/0955-7997(95)00032-J

[22] L. Jeanjean, T. J. Luo, Sharp nonexistence results of prescribed L^2-norm solutions for some class of Schrödinger–Poisson and quasi-linear equations, *Z. Angew. Math. Phys.* **64**(2013), No. 4, 937–954. https://doi.org/10.1007/s00033-012-0272-2; Zbl 1294.35140

[23] J. C. Kang, X. Q. Liu, C. L. Tang, Chern–Simons limit of ground state solutions for the Schrödinger equations coupled with a neutral scalar field, *J. Differential Equations* **343**(2023), 152–185. https://doi.org/10.1016/j.jde.2022.10.008; Zbl 1503.35211

[24] H. Kikuchi, Existence and stability of standing waves for Schrödinger–Poisson–Slater equation, *Adv. Nonlinear Stud.* **7**(2007), No. 3, 403–437. https://doi.org/10.1515/ans-2007-0305; Zbl 1133.35013

[25] Y. Q. Li, B. L. Zhang, Critical Schrödinger–Bopp–Podolsky system with prescribed mass, *J. Geom. Anal.* **33**(2023), No. 7, 220. https://doi.org/10.1007/s12220-023-01287-w; Zbl 1514.35170

[26] E. H. Lieb, M. Loss, *Analysis*, 2nd ed., Graduate Studies in Mathematics, Vol. 14, American Mathematical Society, Providence, RI, 2001. https://doi.org/10.1090/gsm/014; Zbl 0966.26002

[27] J. Lin, C. S. Chen, C. S. Liu, Fast solution of three-dimensional modified Helmholtz equations by the method of fundamental solutions, *Commun. Comput. Phys.* **20**(2016), No. 2, 512–533. https://doi.org/10.4208/cicp.060915.301215a; Zbl 1388.65180

[28] P. L. Lions, The concentration-compactness principle in the calculus of variations. The locally compact case, part I, *Ann. Inst. Henri Poincaré, Anal. Non Linéaire.* **1**(1984), No. 2, 109–145. https://doi.org/10.1016/S0294-1449(16)30428-0; Zbl 0541.49009

[29] P. L. Lions, The concentration-compactness principle in the calculus of variations. The locally compact case, part II, *Ann. Inst. Henri Poincaré, Anal. Non Linéaire* **1**(1984), No. 4, 223–283. https://doi.org/10.1016/S0294-1449(16)30422-X; Zbl 0704.49004

[30] R. S. Palais, The principle of symmetric criticality, *Commun. Math. Phys.* **69**(1979), No. 1, 19–30. https://doi.org/10.1007/BF01941322; Zbl 0417.58007

[31] D. Ruiz, Semiclassical states for coupled Schrödinger–Maxwell equations: concentration around a sphere, *Math. Models Methods Appl. Sci.* **15**(2005), No. 1, 141–164. https://doi.org/10.1142/S0218202505003939; Zbl 1074.81023

[32] D. Ruiz, The Schrödinger–Poisson equation under the effect of a nonlinear local term, *J. Funct. Anal.* **237**(2006), No. 2, 655–674. https://doi.org/10.1016/j.jfa.2006.04.005; Zbl 1136.35037

[33] O. Sánchez, J. Soler, Long-time dynamics of the Schrödinger–Poisson–Slater system, *J. Stat. Phys.* **114**(2004), No. 1-2, 179–204. https://doi.org/10.1023/B:JOSS.0000003109.97208.53; Zbl 1060.82039

[34] M. Struwe, *Variational methods: Applications to nonlinear partial differential equations and Hamiltonian systems*, 4th ed., Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge, Vol. 34, Springer, Berlin, Heidelberg, 2008. https://doi.org/10.1007/978-3-540-74013-1; Zbl 1284.49004

[35] M. I. Weinstein, Nonlinear Schrödinger equations and sharp interpolation estimates, *Commun. Math. Phys.* **87**(1983), 567–576. https://doi.org/10.1007/BF01208265; Zbl 0527.35023

[36] M. Willem, *Minimax theorems*, Progress in Nonlinear Differential Equations and Their Applications, Vol. 24, Birkhäuser, Boston, 1996. https://doi.org/10.1007/978-1-4612-4146-1; Zbl 0856.49001

[37] K. K. Yang, H. Wang, Homogeneous fractional integral operators on Lebesgue and Morrey spaces, Hardy–Littlewood–Sobolev and Olsen-type inequalities, (2022), arXiv: 2212.14774. https://doi.org/10.48550/arXiv.2212.14774

[38] L. G. Zhao, F. K. Zhao, On the existence of solutions for the Schrödinger–Poisson equations, *J. Math. Anal. Appl.* **346**(2008), No. 1, 155–169. https://doi.org/10.1016/j.jmaa.2008.04.053; Zbl 1159.35017