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A study on the oscillation of second-order neutral differential equations in noncanonical form

This article is dedicated to the memory of my father, Müslim Tunç (1932–1978).

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Abstract. This paper analyzes the oscillation of the second-order neutral differential equation in noncanonical form

$$\left[r(t)\left(\left(x(t)+p(t)x(\pi(t))\right)'\right)^{\alpha}\right]'+k(t)x^{\beta}(\eta(t))=0.$$

Using a combination of the linearization technique and the monotonicity properties of the neutral term, we derive new conditions for the studied equation to be oscillatory. Our findings provide new results applicable to linear, sublinear, superlinear, half-linear and other nonlinear forms of the studied equation, and moreover, for all these cases, the oscillation of the studied equation is attained via only one condition. Further, we illustrate the significance of our results with various examples, highlighting their superiority over known criteria in the existing literature.

Keywords: noncanonical, second-order, oscillation, neutral.

2020 Mathematics Subject Classification: 34C10, 34K11, 34K40.

1 Introduction

The purpose of the present paper is to investigate the oscillatory behavior of solutions to the second-order noncanonical differential equation with bounded and unbounded neutral coefficients

$$\left[r(t)\left(\left(x(t)+p(t)x(\pi(t))\right)'\right)^{\alpha}\right]'+k(t)x^{\beta}(\eta(t))=0,\quad t\in\mathcal{I}_0:=[t_0,\infty),\tag{1.1}$$

where $t_0\in\mathbb{R}_+:=(0,\infty)$, α and β are the ratios of odd positive integers with $\alpha\geq 1$. We also assume that:

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(C_1) $r \in C(\mathcal{I}_0, \mathbb{R}_+)$ satisfies

$$\int_{t_0}^{\infty} \frac{1}{r^{1/\alpha}(t)} dt < \infty; \tag{1.2}$$

(C_2) $p, k \in C(\mathcal{I}_0, \mathbb{R}_0)$, $p(t) \geq \ell > 1$, and k does not vanish eventually;

(C_3) $\pi, \eta \in C(\mathcal{I}_0, \mathbb{R})$, $\pi(t) \leq t$, π is strictly increasing, η is nondecreasing, $\eta(t) \leq \pi(t)$, and $\lim_{t\to\infty} \pi(t) = \lim_{t\to\infty} \eta(t) = \infty$.

For simplicity, we put $z(t) := x(t) + p(t)x(\pi(t))$. Under a *solution* of (1.1), we mean a function $x \in C(\mathcal{I}_x, \mathbb{R})$, $\mathcal{I}_x := [t_x, \infty)$ for some $t_x \geq t_0$ such that $z \in C^1(\mathcal{I}_x, \mathbb{R})$, $r(z')^\alpha \in C^1(\mathcal{I}_x, \mathbb{R})$ and x satisfies (1.1) on $\mathcal{I}_x$. Since we are only interested in the oscillatory behavior of solutions, every solution $x(t)$ of (1.1) considered here is assumed to be continuable and nontrivial, i.e., $x(t)$ exists on some half-line $\mathcal{I}_x$ and $\sup_{t\geq T}\{|x(t)|\} > 0$ for any $T \geq t_x$. We say that a solution of (1.1) is *oscillatory* if it has an unbounded set of zeros on $\mathcal{I}_x$; it is called *nonoscillatory*, otherwise. The equation itself is called oscillatory if all its solutions oscillate.

Neutral differential equations are a type of functional differential equations in which the highest-order derivative of the unknown function appears both with and without deviating arguments. Equations of this type have attracted the interest of researchers not only from a theoretical point of view, but also because of their numerous applications in different fields. Those interested in the applications of such equations can refer to [17, 18] for some classical applications and to [7, 12] for more recent applications.

It is clearly observed from the literature review that the oscillation theory is one of the continuously developing important branches of the qualitative theory of differential equations. Its foundations are based on the pioneering work of Sturm [23] on well-known results concerning the zeros of solutions of second-order self-adjoint differential equations. Since then, oscillation criteria have been established for various classes of differential equations by researchers using different techniques and/or methods and the interesting results obtained have contributed significantly to the growth and development of this theory. There are different techniques and/or methods to establish oscillation criteria; among them, Riccati technique, integral averaging technique, comparison theorems and linearization techniques are some of the most influential; see, e.g., the books by Agarwal et al. [2,3], the papers [1,5,6,9–11,13,15,16,20–22,24] and the references quoted there.

We would like to point out that Eq. (1.1) is said to be in *noncanonical* form if (1.2) is satisfied, and Eq. (1.1) is said to be in *canonical* form if

$$\int_{t_0}^{\infty} \frac{1}{r^{1/\alpha}(t)} dt = \infty. \tag{1.3}$$

In particular, for the delay differential equation in canonical form

$$x''(t) + k(t)x(\eta(t)) = 0, \tag{1.4}$$

Koplatadze et al. [19] presented the following interesting oscillation criterion of lim sup-type

$$\limsup_{t\to\infty} \left\{ \eta(t) \int_t^{\infty} k(s)ds + \int_{\eta(t)}^{t} \eta(s)k(s)ds + \frac{1}{\eta(t)} \int_{t_0}^{\eta(t)} s\eta(s)k(s)ds \right\} > 1. \tag{1.5}$$

Later, Baculikova [4] extended the technique developed by Koplatadze et al.[19] to the second-order differential equation with deviating argument in noncanonical form

$$(r(t)x'(t))' + k(t)f(x(\eta(t))) = 0, \tag{1.6}$$

where f is nondecreasing and covers sublinear and linear cases ($f(u) = u^\beta$ with $\beta \in (0,1]$) and provided new oscillation criteria for equation (1.6).

Grace et al. [14] applied the same technique to the second-order neutral differential equation in noncanonical form

$$\left[r(t)\left(x(t)+p(t)x(\pi(t))\right)'\right]' + k(t)x^\beta(\eta(t)) = 0, \tag{1.7}$$

where $0 \le p(t) \le d < 1$ and $\beta \in (0,1]$, and derived new oscillation results for (1.7).

Tunç et al. [25] implemented the same technique to the equation with distributed deviating arguments in noncanonical form

$$\left[r(t)\left(x(t)+p(t)x(\pi(t))\right)'\right]' + \int_{c_1}^{c_2} k(t,\varrho)x^\beta(\phi(t,\varrho))d\varrho = 0, \tag{1.8}$$

where $0 < c_1 < c_2 < \infty$, $p(t) \ge \ell > 1$ and $\beta \in (0,1]$, and established novel criteria for the oscillation of (1.8).

From the above observations, it is clear that the results in [4, 14, 25] only cover the linear and sublinear cases, i.e., the results in [4, 14, 25] are only applicable to the cases ($\alpha = \beta = 1$) and ($\alpha = 1$ and $\beta < 1$); and so they provide no information for the superlinear case ($\alpha = 1$ and $\beta > 1$), half-linear case ($\beta = \alpha \neq 1$) and the other nonlinear cases such as ($1 \neq \alpha \neq \beta$) and ($\alpha > 1$ and $\beta = 1$).

Motivated by the above mentioned researches; using a combination of the linearization technique and the monotonicity properties of the neutral term, our goal here is to establish new oscillation criteria for Equation (1.1) that are applicable not only to the linear and sublinear cases, but also to the superlinear, half-linear and the other nonlinear cases the mentioned above. Furthermore, in contrast to the results reported in [14, 25], a key feature of our results is that the oscillation of the equation considered here is ensured through only one condition. It should also be noted that, as can be seen from the details in the proofs, the proofs for all these cases of α and β are not straightforward and require considerable effort. On the other hand, since we focus on the cases $1 < p(t) < \infty$ and $p(t) \to \infty$ as $t \to \infty$, our current results cannot be applied to the case $0 \le p(t) < 1$. However, with some standard calculations, the criteria here can be adjusted for the case $0 \le p(t) < 1$ as well. Additionally, a series of carefully selected examples are provided to illustrate the significance of our results and their superiority over known criteria in the existing literature.

We wish also point out that if x is a solution of (1.1), under our assumptions, $-x$ is also a solution of (1.1). Therefore, when considering nonoscillatory solutions, it is sufficient to consider only positive ones since the proofs for negative solutions are similar due to the form of (1.1).

2 Oscillation results for (1.1) in the case where $\beta \le \alpha$

In this section, we present the oscillation results for (1.1) in the case where $\beta \le \alpha$. For notational purposes, it would be convenient to set:

$$\kappa = (\ell-1)^\beta \ell^{-\beta},\ b(t) = \pi^{-1}(\eta(t)),\ A(t) = \int_{t_*}^{t} \frac{1}{r^{1/\alpha}(s)}ds,\ t_* \in \mathcal{I}_0,\ \lambda(t) = \int_{t}^{\infty} \frac{1}{r^{1/\alpha}(s)}ds.$$

Lemma 2.1. *Let x be an eventually positive solution of (1.1) on $\mathcal{I}_0$. If*

$$\int_{t_0}^{\infty} \lambda^\alpha(t)k(t)p^{-\beta}(b(t))dt = \infty, \tag{2.1}$$

then the corresponding function $z(t) = x(t) + p(t)x(\pi(t))$ belongs to the class $\mathcal{N}_0$ eventually, where

$$z(t) \in \mathcal{N}_0 \Leftrightarrow z(t) > 0, \quad r^{1/\alpha}(t)z'(t) < 0, \quad \left(r^{1/\alpha}z'\right)'(t) \leq 0.$$

Proof. Since x is an eventually positive solution of (1.1) on $\mathcal{I}_0$, there exists a $t_1 \in \mathcal{I}_0$ such that $x(t) > 0$ for $t \geq t_1$. In view of $\lim_{t\to\infty} \pi(t) = \lim_{t\to\infty} \eta(t) = \infty$, we can choose $t_2 \geq t_1$ such that $x(\pi(t)) > 0$ and $x(\eta(t)) > 0$ for $t \geq t_2$. Now, for $t \geq t_2$, $z(t) > 0$ and

$$\left(r(z')^\alpha\right)'(t) = -k(t)x^\beta(\eta(t)) \leq 0,$$

which implies that z' is of one sign eventually, i.e., either $z'(t) > 0$ or $z'(t) < 0$ eventually. We claim that $z'(t) < 0$ eventually. If this is not the case, then there exists a $t_3 \geq t_2$ such that $z'(t) > 0$ for $t \geq t_3$. Applying the chain-rule to $(r(z')^\alpha)'(t)$ yields

$$\left(r(z')^\alpha\right)'(t) = \alpha \left(r^{1/\alpha}(t)z'(t)\right)^{\alpha-1} \left(r^{1/\alpha}(t)z'(t)\right)'. \tag{2.2}$$

From (2.2) and the fact that α is the ratio of odd positive integers, we deduce that

$$\operatorname{sgn}\left(r^{1/\alpha}(t)z'(t)\right)' = \operatorname{sgn}\left(r(t)(z'(t))^\alpha\right)'.$$

Hence

$$z(t) > 0, \quad r^{1/\alpha}(t)z'(t) > 0, \quad \left(r^{1/\alpha}z'\right)'(t) \leq 0 \quad \text{for } t \geq t_3.$$

Now

$$z(t) \geq \int_{t_3}^{t} \frac{r^{1/\alpha}(s)z'(s)}{r^{1/\alpha}(s)} ds = A(t)r^{1/\alpha}(t)z'(t),$$

which implies that

$$\left(\frac{z(t)}{A(t)}\right)' \leq 0,$$

for $t \geq t_4$ for some $t_4 > t_3$, i.e., $z(t)/A(t)$ is nonincreasing for $t \geq t_4$. The definition of z leads to

$$\begin{aligned} x(t) &= \frac{1}{p(\pi^{-1}(t))}\left[z(\pi^{-1}(t)) - x(\pi^{-1}(t))\right] \\ &\geq \frac{z(\pi^{-1}(t))}{p(\pi^{-1}(t))} - \frac{z(\pi^{-1}(\pi^{-1}(t)))}{p(\pi^{-1}(t))p(\pi^{-1}(\pi^{-1}(t)))}. \end{aligned} \tag{2.3}$$

By (C_3), we see that π^{-1} is increasing, $t \leq \pi^{-1}(t)$, and

$$\pi^{-1}(t) \leq \pi^{-1}(\pi^{-1}(t)). \tag{2.4}$$

(2.4) together with the noincreasing nature of z/A yields

$$z\left(\pi^{-1}(\pi^{-1}(t))\right) \leq \frac{A\left(\pi^{-1}(\pi^{-1}(t))\right) z(\pi^{-1}(t))}{A\left(\pi^{-1}(t)\right)}. \tag{2.5}$$

Using (2.5) in (2.3) gives

$$x(t) \geq \frac{z(\pi^{-1}(t))}{p(\pi^{-1}(t))}\left[1 - \frac{A(\pi^{-1}(\pi^{-1}(t)))}{A(\pi^{-1}(t))} \frac{1}{p(\pi^{-1}(\pi^{-1}(t)))}\right] \quad \text{for } t \geq t_4. \tag{2.6}$$

By (C_2) and that

$$\lim_{t\to\infty}\frac{A(\pi^{-1}(t))}{A(t)}=1,$$

we see that there exists $t_5 \geq t_4$ such that for any $\epsilon \in (0, \ell - 1)$ and $t \geq t_5$,

$$\frac{A(\pi^{-1}(\pi^{-1}(t)))}{A(\pi^{-1}(t))}\frac{1}{p(\pi^{-1}(\pi^{-1}(t)))}\leq\frac{1+\epsilon}{\ell}.$$

Using this in (2.6) gives

$$x(t)\geq d\frac{z(\pi^{-1}(t))}{p(\pi^{-1}(t))}\quad\text{for } t\geq t_5, \tag{2.7}$$

where $d = 1 - (1+\epsilon)/\ell > 0$. Substituting (2.7) into (1.1) yields

$$\left(r(z')^\alpha\right)'(t)+d^\beta k(t)p^{-\beta}(\pi^{-1}(\eta(t)))z^\beta\left(\pi^{-1}(\eta(t))\right)\leq 0, \tag{2.8}$$

for $t \geq t_6$. For $t \geq t_6$, we have $z(t) \geq z(t_6) := c > 0$, and so inequality (2.8) leads to

$$\left(r(z')^\alpha\right)'(t)+c^\beta d^\beta k(t)p^{-\beta}(b(t))\leq 0\quad\text{for } t\geq t_7\geq t_6. \tag{2.9}$$

Since λ is positive, decreasing and $\lim_{t\to\infty}\lambda(t) = 0$, we can choose a sufficiently large $t_* \in \mathcal{I}_0$ such that

$$0<\lambda(t)<1\quad\text{for } t\geq t_*. \tag{2.10}$$

Let $t_8 = \max\{t_7, t_*\}$. Integrating (2.9) from t_7 to ∞ and taking (2.10) into account, we obtain

$$\begin{aligned}r(t_7)(z'(t_7))^\alpha &\geq c^\beta d^\beta\int_{t_7}^{\infty}k(t)p^{-\beta}(b(t))dt\geq c^\beta d^\beta\int_{t_8}^{\infty}k(t)p^{-\beta}(b(t))dt\\ &\geq c^\beta d^\beta\int_{t_8}^{\infty}\lambda^\alpha(t)k(t)p^{-\beta}(b(t))dt,\end{aligned}$$

which is impossible in view of (2.1) and so $z'(t) < 0$ eventually and thus the proof ends. □

Lemma 2.2. *Suppose that x is an eventually positive solution of (1.1) on $\mathcal{I}_0$. If (2.1) holds, then the following are satisfied eventually:*

(i)
$$z(t)+\lambda(t)r^{1/\alpha}(t)z'(t)\geq 0;$$

(ii)
$$\frac{z(t)}{\lambda(t)}\quad\textit{is nondecreasing;}$$

(iii)
$$\left(\frac{z(b(t))}{r^{1/\alpha}(t)z'(t)}\right)^{\alpha-1}\geq\lambda^{\alpha-1}(t);$$

and

(iv)
$$\lim_{t\to\infty}z(t)=0.$$

Proof. Pick a $t_1 \in \mathcal{I}_0$ such that $x(t) > 0$, $x(\pi(t)) > 0$ and $x(\eta(t)) > 0$ for $t \geq t_1$. Proceeding similarly as in the proof of Lemma 2.1, we again see that the corresponding function $z(t)$ belongs to the class $\mathcal{N}_0$ eventually, say for $t \geq t_2$, for some $t_2 \geq t_1$. Since $r^{1/\alpha}z'$ is negative and decreasing,

$$z(t)\geq-\int_t^\infty\frac{r^{1/\alpha}(s)z'(s)}{r^{1/\alpha}(s)}ds\geq-\lambda(t)r^{1/\alpha}(t)z'(t), \tag{2.11}$$

which means that part (i) is fulfilled.

From (2.11),

$$\left(\frac{z(t)}{\lambda(t)}\right)' = \frac{\lambda(t)r^{1/\alpha}(t)z'(t) + z(t)}{\lambda^2(t)r^{1/\alpha}(t)} \geq 0,$$

i.e., part (ii) holds.

Since $b(t) \leq t$, we get

$$z(b(t)) \geq z(t). \tag{2.12}$$

By (2.11) and (2.12), we observe that

$$\left(\frac{z(b(t))}{r^{1/\alpha}(t)z'(t)}\right)^{\alpha-1} \geq \lambda^{\alpha-1}(t), \tag{2.13}$$

which proves part (iii).

Finally, we will show that part (iv) is true. From the monotonic properties of z on $[t_2, \infty)$, we can choose a constant $L_1 \geq 0$ such that

$$\lim_{t\to\infty} z(t) = L_1.$$

We will show that $L_1 = 0$. If $L_1 > 0$, then there exists a $T_1 \geq t_2$ such that for any $\epsilon > 0$,

$$L_1 < z(t) < L_1 + \epsilon \quad \text{for } t \geq T_1. \tag{2.14}$$

By (2.4) and $z'(t) < 0$,

$$z(\pi^{-1}(t)) \geq z(\pi^{-1}(\pi^{-1}(t))). \tag{2.15}$$

Using (2.15) in (2.3) leads to

$$x(t) \geq \frac{1}{p(\pi^{-1}(t))}\left[1 - \frac{1}{p(\pi^{-1}(\pi^{-1}(t)))}\right] z(\pi^{-1}(t)) \quad \text{for } t \geq t_2, \tag{2.16}$$

which together with (C_2) yields

$$x(t) \geq \kappa^{1/\beta}\frac{z(\pi^{-1}(t))}{p(\pi^{-1}(t))} \quad \text{for } t \geq t_2, \tag{2.17}$$

Substituting (2.17) into (1.1) gives

$$\left(r(z')^{\alpha}\right)'(t) + \kappa k(t)p^{-\beta}(b(t))z^{\beta}(b(t)) \leq 0, \tag{2.18}$$

for $t \geq t_3$, for some $t_3 \geq t_2$. In view of (2.2), we can rewrite (2.18) in the form

$$\left(r^{1/\alpha}z'\right)'(t) + \frac{\kappa}{\alpha}\frac{z^{\alpha-1}(b(t))}{\left(r^{1/\alpha}(t)z'(t)\right)^{\alpha-1}}k(t)p^{-\beta}(b(t))z^{\beta-\alpha+1}(b(t)) \leq 0. \tag{2.19}$$

By virtue of (2.13) and (2.19), we obtain

$$\left(r^{1/\alpha}z'\right)'(t) + \frac{\kappa}{\alpha}\lambda^{\alpha-1}(t)k(t)p^{-\beta}(b(t))z^{\beta-\alpha+1}(b(t)) \leq 0 \quad \text{for } t \geq t_3. \tag{2.20}$$

Let $t_4 = \max\{t_3, T_1\}$. Then, from (2.14) and (2.20), we observe that

$$\left(r^{1/\alpha}z'\right)'(t) + d\lambda^{\alpha-1}(t)k(t)p^{-\beta}(b(t)) \leq 0 \quad \text{for } t \geq t_4, \tag{2.21}$$

where $d = \kappa L_1^{1+\beta}/\alpha(L_1+\epsilon)^\alpha > 0$. Integrating (2.21) two times gives

$$\begin{aligned} z(t_4) &\geq d\int_{t_4}^{\infty}\frac{1}{r^{1/\alpha}(u)}\int_{t_4}^{u}\lambda^{\alpha-1}(s)k(s)p^{-\beta}(b(s))dsdu \\ &= d\int_{t_4}^{\infty}\lambda^{\alpha}(s)k(s)p^{-\beta}(b(s))ds = \infty, \end{aligned}$$

which is a contradiction. The contradiction obtained proves that $L_1 = 0$. The proof is over. □

Lemma 2.3. *Let x be an eventually positive solution of (1.1) on $\mathcal{I}_0$. If (2.1) holds, then the corresponding function z satisfies the linear inequalities*

$$\left(r^{1/\alpha}z'\right)'(t) + \frac{\kappa}{\alpha}\lambda^{\alpha-1}(t)k(t)p^{-\beta}(b(t))z(b(t)) \leq 0 \tag{2.22}$$

and

$$\left(z + \lambda r^{1/\alpha}z'\right)'(t) + \frac{\kappa}{\alpha}\lambda^{\alpha}(t)k(t)p^{-\beta}(b(t))z(b(t)) \leq 0, \tag{2.23}$$

eventually.

Proof. Following again the same arguments in the proof of Lemma 2.2, we arrive at (2.20) for $t \geq t_3$. Since z is positive, decreasing and $\lim_{t\to\infty} z(t) = 0$, there exists a sufficiently large $t_4 \geq t_3$ such that

$$0 < z(t) < 1 \quad \text{for } t \geq t_4. \tag{2.24}$$

By (2.24) and $\beta \leq \alpha$, we have

$$z^{\beta/\alpha}(t) \geq z(t) \quad \text{for } t \geq t_4. \tag{2.25}$$

From (2.24) and the fact that $\beta - \alpha \leq 0$, we obtain

$$z^{\beta-\alpha}(t) \geq z^{(\beta-\alpha)/\alpha}(t) \quad \text{for } t \geq t_4. \tag{2.26}$$

Using (2.26) in (2.20) gives

$$\left(r^{1/\alpha}z'\right)'(t) + \frac{\kappa}{\alpha}\lambda^{\alpha-1}(t)k(t)p^{-\beta}(b(t))z^{\beta/\alpha}(b(t)) \leq 0 \tag{2.27}$$

for $t \geq t_5$, for some $t_5 \geq t_4$. Using (2.25) in (2.27) leads to

$$\left(r^{1/\alpha}z'\right)'(t) + \frac{\kappa}{\alpha}\lambda^{\alpha-1}(t)k(t)p^{-\beta}(b(t))z(b(t)) \leq 0 \quad \text{for } t \geq t_5, \tag{2.28}$$

which proves (2.22).

From (2.28) and the fact that

$$\left(z + \lambda r^{1/\alpha}z'\right)'(t) = \lambda(t)\left(r^{1/\alpha}(t)z'(t)\right)', \tag{2.29}$$

we observe that

$$\left(z + \lambda r^{1/\alpha}z'\right)'(t) + \frac{\kappa}{\alpha}\lambda^{\alpha}(t)k(t)p^{-\beta}(b(t))z(b(t)) \leq 0 \quad \text{for } t \geq t_5, \tag{2.30}$$

i.e., (2.23) holds. The proof is complete. □

We are now ready to give the oscillation results for (1.1) in the case where $\beta \leq \alpha$ based on Lemmas 2.1–2.3.

Theorem 2.4. *If*

$$\limsup_{t\to\infty} W(t) > \frac{\alpha}{\kappa}, \tag{2.31}$$

where

$$W(t) := \lambda(t)\int_{t_0}^{t} \lambda^{\alpha-1}(s)k(s)p^{-\beta}(b(s))ds + \frac{1}{\lambda(b(t))}\int_{t}^{\infty} \lambda^{\alpha}(s)k(s)p^{-\beta}(b(s))\lambda(b(s))ds,$$

then equation (1.1) *is oscillatory.*

Proof. Let $x(t)$ be a nonoscillatory solution of (1.1), say $x(t) > 0$, $x(\pi(t)) > 0$ and $x(\eta(t)) > 0$ for $t \geq t_1$, for some $t_1 \geq t_0$. It follows from (2.31) that there exists a positive constant k such that

$$\limsup_{t\to\infty} W(t) \geq \mathrm{k}. \tag{2.32}$$

We now assert that (2.32) implies (2.1). Indeed, if not, then

$$\int^{\infty} \lambda^{\alpha}(t)k(t)p^{-\beta}(b(t))dt < \infty.$$

Thus, there exists a sufficiently large $t_2 \in [t_1, \infty)$ such that

$$\int_{t_2}^{\infty} \lambda^{\alpha}(t)k(t)p^{-\beta}(b(t))dt < \frac{\mathrm{k}}{4}. \tag{2.33}$$

Recalling again that λ is positive, decreasing and $\lim_{t\to\infty}\lambda(t) = 0$, we again see that there exists a sufficiently large $t_* \in [t_0, \infty)$ such that (2.10) holds. Let $t_3 = \max\{t_2, t_*\}$. Then, for $t \geq t_3$, it follows from (2.10) and (2.33) that

$$\begin{aligned}
\lambda(t)\int_{t_1}^{t} \lambda^{\alpha-1}(s)k(s)p^{-\beta}(b(s))ds &= \lambda(t)\int_{t_1}^{t_3} \lambda^{\alpha-1}(s)k(s)p^{-\beta}(b(s))ds \\
&\quad + \lambda(t)\int_{t_3}^{t} \lambda^{\alpha-1}(s)k(s)p^{-\beta}(b(s))ds \\
&\leq \lambda(t)\int_{t_1}^{t_3} \lambda^{\alpha-1}(s)k(s)p^{-\beta}(b(s))ds \\
&\quad + \int_{t_3}^{t} \lambda^{\alpha}(s)k(s)p^{-\beta}(b(s))ds \\
&\leq \lambda(t)\int_{t_1}^{t_3} \lambda^{\alpha-1}(s)k(s)p^{-\beta}(b(s))ds + \frac{\mathrm{k}}{4}.
\end{aligned} \tag{2.34}$$

Also, for $t \geq t_3$,

$$\begin{aligned}
\frac{1}{\lambda(b(t))}\int_{t}^{\infty} \lambda^{\alpha}(s)k(s)p^{-\beta}(b(s))\lambda(b(s))ds &\leq \int_{t}^{\infty} \lambda^{\alpha}(s)k(s)p^{-\beta}(b(s))ds \\
&\leq \int_{t_3}^{\infty} \lambda^{\alpha}(s)k(s)p^{-\beta}(b(s))ds < \frac{\mathrm{k}}{4}.
\end{aligned} \tag{2.35}$$

From (2.34) and (2.35), we deduce that

$$\limsup_{t\to\infty} W(t) \leq \frac{\mathrm{k}}{2},$$

which contradicts (2.32), and the contradiction obtained shows that (2.32) implies (2.1). Thereby, all results of Lemmas 2.1–2.3 are fulfilled. Proceeding similarly as in proof in Lemma 2.3, we again arrive at (2.28) and (2.30) for $t \geq t_5$. Integrating (2.30) from $t(\geq t_5)$ to ∞ leads to

$$(z + \lambda r^{1/\alpha} z')(t) \geq \frac{\kappa}{\alpha} \int_t^\infty \lambda^\alpha(s) k(s) p^{-\beta}(b(s)) z(b(s)) ds. \tag{2.36}$$

An integration of (2.28) from t_5 to t gives

$$-(\lambda r^{1/\alpha} z')(t) \geq \frac{\kappa}{\alpha} \lambda(t) \int_{t_5}^t \lambda^{\alpha-1}(s) k(s) p^{-\beta}(b(s)) z(b(s)) ds. \tag{2.37}$$

From (2.36) and (2.37), we obtain

$$\begin{aligned} z(t) \geq \frac{\kappa}{\alpha} \lambda(t) \int_{t_5}^t \lambda^{\alpha-1}(s) k(s) p^{-\beta}(b(s)) z(b(s)) ds \\ + \frac{\kappa}{\alpha} \int_t^\infty \lambda^\alpha(s) k(s) p^{-\beta}(b(s)) z(b(s)) ds. \end{aligned} \tag{2.38}$$

In view of the monotonicity properties of z, b and z/λ, we obtain

$$\int_{t_5}^t \lambda^{\alpha-1}(s) k(s) p^{-\beta}(b(s)) z(b(s)) ds \geq \left(\int_{t_5}^t \lambda^{\alpha-1}(s) k(s) p^{-\beta}(b(s)) ds \right) z(t), \tag{2.39}$$

and

$$\begin{aligned} \int_t^\infty \lambda^\alpha(s) k(s) p^{-\beta}(b(s)) z(b(s)) ds &= \int_t^\infty \lambda^\alpha(s) k(s) p^{-\beta}(b(s)) \lambda(b(s)) \frac{z(b(s))}{\lambda(b(s))} ds \\ &\geq \left(\frac{1}{\lambda(b(t))} \int_t^\infty \lambda^\alpha(s) k(s) p^{-\beta}(b(s)) \lambda(b(s)) ds \right) z(t). \end{aligned} \tag{2.40}$$

Using (2.39) and (2.40) in (2.38), we arrive at

$$\frac{\alpha}{\kappa} \geq \lambda(t) \int_{t_5}^t \lambda^{\alpha-1}(s) k(s) p^{-\beta}(b(s)) ds + \frac{1}{\lambda(b(t))} \int_t^\infty \lambda^\alpha(s) k(s) p^{-\beta}(b(s)) \lambda(b(s)) ds.$$

Taking lim sup as $t \to \infty$ in the latter inequality, we obtain a contradiction with (2.31). This completes the proof. □

Theorem 2.5. *If*

$$\limsup_{t \to \infty} \mathrm{H}(t) > \frac{\alpha}{\kappa}, \tag{2.41}$$

where

$$\begin{aligned} \mathrm{H}(t) := \lambda(b(t)) \int_{t_0}^{b(t)} \lambda^{\alpha-1}(s) k(s) p^{-\beta}(b(s)) ds + \int_{b(t)}^t \lambda^\alpha(s) k(s) p^{-\beta}(b(s)) ds \\ + \frac{1}{\lambda(b(t))} \int_t^\infty \lambda^\alpha(s) k(s) p^{-\beta}(b(s)) \lambda(b(s)) ds, \end{aligned}$$

then (1.1) *oscillates.*

Proof. Suppose that $x(t)$ is a nonoscillatory solution of (1.1) such that $x(t) > 0$, $x(\pi(t)) > 0$ and $x(\eta(t)) > 0$ for $t \geq t_1$, for some $t_1 \geq t_0$. As in the proof of Theorem 2.4, it is not difficult to see that (2.1) holds. Thereby, all results of Lemmas 2.1–2.3 are fulfilled. Using exactly the same arguments as in the proof of Theorem 2.4, we see that (2.38) holds and leads to

$$\begin{aligned} z(b(t)) \geq \frac{\kappa}{\alpha}\lambda(b(t)) \int_{t_5}^{b(t)} \lambda^{\alpha-1}(s)k(s)p^{-\beta}(b(s))z(b(s))ds \\ + \frac{\kappa}{\alpha}\int_{b(t)}^{t} \lambda^{\alpha}(s)k(s)p^{-\beta}(b(s))z(b(s))ds \\ + \frac{\kappa}{\alpha}\int_{t}^{\infty} \lambda^{\alpha}(s)k(s)p^{-\beta}(b(s))z(b(s))ds. \end{aligned}$$

In view of the monotonicity properties of z and z/λ, we conclude from the latter inequality that

$$\begin{aligned} \frac{\alpha}{\kappa} \geq \lambda(b(t)) \int_{t_5}^{b(t)} \lambda^{\alpha-1}(s)k(s)p^{-\beta}(b(s))ds + \int_{b(t)}^{t} \lambda^{\alpha}(s)k(s)p^{-\beta}(b(s))ds \\ + \frac{1}{\lambda(b(t))}\int_{t}^{\infty} \lambda^{\alpha}(s)k(s)p^{-\beta}(b(s))\lambda(b(s))ds. \end{aligned}$$

The remainder of the proof follows from that of Theorem 2.4. □

Theorem 2.6. *If*

$$\limsup_{t\to\infty} \int_{b(t)}^{t} \lambda^{\alpha}(s)k(s)p^{-\beta}(b(s))ds > \frac{\alpha}{\kappa}, \tag{2.42}$$

then equation (1.1) *is oscillatory.*

Proof. Suppose that $x(t)$ is a nonoscillatory solution of (1.1) such that $x(t) > 0$, $x(\pi(t)) > 0$ and $x(\eta(t)) > 0$ for $t \geq t_1$, for some $t_1 \geq t_0$. It is obvious that (2.42) implies (2.1). Thereby, all results of Lemmas 2.1–2.3 are fulfilled. By Lemma 2.3, we again have (2.30). Setting

$$y(t) := z(t) + \lambda(t)r^{1/\alpha}(t)z'(t),$$

and taking $0 \leq y(t) \leq z(t)$ into account, we observe from (2.30) that

$$y'(t) + \frac{\kappa}{\alpha}\lambda^{\alpha}(t)k(t)p^{-\beta}(b(t))y(b(t)) \leq 0.$$

Integrating the latter inequality from $b(t)$ to t gives

$$\begin{aligned} y(b(t)) &\geq \frac{\kappa}{\alpha}\int_{b(t)}^{t} \lambda^{\alpha}(s)k(s)p^{-\beta}(b(s))y(b(s))ds \\ &\geq \left(\frac{\kappa}{\alpha}\int_{b(t)}^{t} \lambda^{\alpha}(s)k(s)p^{-\beta}(b(s))ds\right) y(b(t)), \end{aligned}$$

from which

$$\frac{\alpha}{\kappa} \geq \int_{b(t)}^{t} \lambda^{\alpha}(s)k(s)p^{-\beta}(b(s))ds.$$

Taking the lim sup as $t \to \infty$, we find a contradiction to (2.42). The proof is complete. □

3 Oscillation results for (1.1) in the case where $\beta > \alpha$

In this section, we will establish the oscillation criteria for the case when $\beta > \alpha$. While in the previous section we have established the oscillation criteria via $z(t) \to 0$ as $t \to \infty$; in this section we will establish the oscillation criteria via $z(t)/\lambda(t) \to \infty$ as $t \to \infty$.

Lemma 3.1. *Let x be an eventually positive solution of (1.1) on $\mathcal{I}_0$. If*

$$\int_{t_0}^{\infty} \lambda^{\beta}(t)k(t)p^{-\beta}(b(t))dt = \infty, \tag{3.1}$$

then $z(t) \in \mathcal{N}_0$ and

$$\lim_{t\to\infty} \frac{z(t)}{\lambda(t)} = \infty.$$

Proof. Recalling again the monotonic properties of λ, we again see that there exists a sufficiently large $t_* \in [t_0, \infty)$ such that (2.10) holds. By (2.10) and $\beta > \alpha$, we have

$$\lambda^{\beta}(t) \le \lambda^{\alpha}(t) \quad \text{for } t \ge t_*, \tag{3.2}$$

from which,

$$\lambda^{\beta}(t)k(t)p^{-\beta}(b(t)) \le \lambda^{\alpha}(t)k(t)p^{-\beta}(b(t)). \tag{3.3}$$

Now, in view of (3.3), it is clear that (3.1) implies (2.1). So, all results of Lemma 2.1 and lemma 2.2 are fulfilled. Following similar arguments as in the proof of Lemma 2.2, we see that (2.18) holds. Since $\lambda(t) \to 0$ as $t \to \infty$ and $z(t) \to 0$ as $t \to \infty$, by L'Hôpital's rule,

$$\lim_{t\to\infty} \frac{z(t)}{\lambda(t)} = \lim_{t\to\infty} -r^{1/\alpha}(t)z'(t).$$

We now will show that $\lim_{t\to\infty} -r^{1/\alpha}(t)z'(t) = \infty$. Suppose to the contrary that positive and increasing function $-r^{1/\alpha}(t)z'(t)$ has finite limit. This assumption implies that there is a constant $b_1 > 0$ such that

$$-r^{1/\alpha}(t)z'(t) \le b_1 < \infty.$$

This together with an integration of (2.18) from t_3 to t yields

$$\begin{aligned} b_1^{\alpha} \ge -r(t)(z'(t))^{\alpha} &\ge \kappa \int_{t_3}^{t} k(s)p^{-\beta}(b(s))z^{\beta}(b(s))ds \\ &\ge \kappa \int_{t_3}^{t} \lambda^{\beta}(s)k(s)p^{-\beta}(b(s)) \left(\frac{z(s)}{\lambda(s)}\right)^{\beta} ds \\ &\ge \kappa \left(\frac{z(t_3)}{\lambda(t_3)}\right)^{\beta} \int_{t_3}^{t} \lambda^{\beta}(s)k(s)p^{-\beta}(b(s))ds, \end{aligned}$$

which contradicts (3.1) and we deduce that

$$\lim_{t\to\infty} \frac{z(t)}{\lambda(t)} = \lim_{t\to\infty} -r^{1/\alpha}(t)z'(t) = \infty.$$ □

Lemma 3.2. *Let x be an eventually positive solution of (1.1) on $\mathcal{I}_0$. If (3.1) holds, then*

$$\left(r^{1/\alpha}z'\right)'(t) + \frac{\kappa}{\alpha}\lambda^{\beta}(t)k(t)p^{-\beta}(b(t))\frac{z(b(t))}{\lambda(b(t))} \le 0 \tag{3.4}$$

and

$$\left(z + \lambda r^{1/\alpha}z'\right)'(t) + \frac{\kappa}{\alpha}\lambda^{\beta+1}(t)k(t)p^{-\beta}(b(t))\frac{z(b(t))}{\lambda(b(t))} \le 0, \tag{3.5}$$

eventually.

Proof. Since (3.1) implies (2.1), we again see that all the results of Lemma 2.1, Lemma 2.2 and Lemma 3.1 are fulfilled. Following similar arguments as in the proof of Lemma 2.2, we see that (2.20) for $t \geq t_3$ is satisfied. Since $z(t)/\lambda(t) \to \infty$ as $t \to \infty$, there exists a sufficiently large $T_1 \in [t_0, \infty)$ such that

$$\frac{z(t)}{\lambda(t)} \geq 1 \quad \text{for } t \geq T_1. \tag{3.6}$$

By (3.6) and $\beta > \alpha$,

$$\left(\frac{z(t)}{\lambda(t)}\right)^{\beta/\alpha} \geq \frac{z(t)}{\lambda(t)} \quad \text{for } t \geq T_1, \tag{3.7}$$

and

$$\beta - \alpha \geq \frac{\beta - \alpha}{\alpha}. \tag{3.8}$$

Now (3.6) and (3.8) leads to

$$\left(\frac{z(t)}{\lambda(t)}\right)^{\beta-\alpha} \geq \left(\frac{z(t)}{\lambda(t)}\right)^{(\beta-\alpha)/\alpha} \quad \text{for } t \geq T_1. \tag{3.9}$$

Since $\lim_{t\to\infty} b(t) = \infty$, we can choose $T_2 \geq T_1$ such that $b(t) \geq T_1$ for $t \geq T_2$ and so from (3.9)

$$\left(\frac{z(b(t))}{\lambda(b(t))}\right)^{\beta-\alpha} \geq \left(\frac{z(b(t))}{\lambda(b(t))}\right)^{(\beta-\alpha)/\alpha} \quad \text{for } t \geq T_2. \tag{3.10}$$

Let $t_4 = \max\{t_3, T_2\}$. Using (3.10) in (2.20) gives

$$\left(r^{1/\alpha} z'\right)'(t) + \frac{\kappa}{\alpha} \lambda^{\alpha-1}(t) k(t) p^{-\beta}(b(t)) \lambda^{\beta-\alpha+1}(b(t)) \left(\frac{z(b(t))}{\lambda(b(t))}\right)^{\beta/\alpha} \leq 0 \quad \text{for } t \geq t_4. \tag{3.11}$$

Using that λ is decreasing, $b(t) \leq t$ and $\beta > \alpha$, it follows from (3.11) that

$$\left(r^{1/\alpha} z'\right)'(t) + \frac{\kappa}{\alpha} \lambda^{\beta}(t) k(t) p^{-\beta}(b(t)) \left(\frac{z(b(t))}{\lambda(b(t))}\right)^{\beta/\alpha} \leq 0 \quad \text{for } t \geq t_4. \tag{3.12}$$

Using (3.7) in (3.12) yields

$$\left(r^{1/\alpha} z'\right)'(t) + \frac{\kappa}{\alpha} \lambda^{\beta}(t) k(t) p^{-\beta}(b(t)) \frac{z(b(t))}{\lambda(b(t))} \leq 0 \quad \text{for } t \geq t_4, \tag{3.13}$$

which proves (3.4). Recalling (2.29) again, we can rewrite (3.13) in the form

$$\left(z + \lambda r^{1/\alpha} z'\right)'(t) + \frac{\kappa}{\alpha} \lambda^{\beta+1}(t) k(t) p^{-\beta}(b(t)) \frac{z(b(t))}{\lambda(b(t))} \leq 0 \quad \text{for } t \geq t_4, \tag{3.14}$$

which proves (3.5). The proof is complete. □

Theorem 3.3. *If*

$$\limsup_{t\to\infty} \mathrm{K}(t) > \frac{\alpha}{\kappa}, \tag{3.15}$$

where

$$\mathrm{K}(t) := \lambda(t) \int_{t_0}^{t} \lambda^{\beta}(s) k(s) p^{-\beta}(b(s)) \lambda^{-1}(b(s)) ds + \frac{1}{\lambda(b(t))} \int_{t}^{\infty} \lambda^{\beta+1}(s) k(s) p^{-\beta}(b(s)) ds,$$

then (1.1) *oscillates.*

Proof. Pick $t_1 \in \mathcal{I}_0$ such that $x(t) > 0$, $x(\pi(t)) > 0$ and $x(\eta(t)) > 0$ for $t \geq t_1$. From (3.15), we can choose a $\delta > 0$ such that

$$\limsup_{t\to\infty} \mathrm{K}(t) \geq \delta. \tag{3.16}$$

We now claim that (3.16) implies (3.1). Indeed, if not, then

$$\int^{\infty} \lambda^{\beta}(t)k(t)p^{-\beta}(b(t))dt < \infty.$$

Thus, for a sufficiently large $t_2 \in [t_1, \infty)$,

$$\int_{t_2}^{\infty} \lambda^{\beta}(t)k(t)p^{-\beta}(b(t))dt < \frac{\delta}{4}. \tag{3.17}$$

Recalling again the monotonicity properties of λ, we see that there exists a sufficiently large $t_* \in [t_0, \infty)$ such that (2.10) holds. Let $t_3 = \max\{t_2, t_*\}$. Then, for $t \geq t_3$, (2.10) and (3.17) yields

$$\begin{aligned}
\lambda(t)\int_{t_1}^{t} \lambda^{\beta}(s)k(s)p^{-\beta}(b(s))\lambda^{-1}(b(s))ds &= \lambda(t)\int_{t_1}^{t_3} \lambda^{\beta}(s)k(s)p^{-\beta}(b(s))\lambda^{-1}(b(s))ds \\
&\quad + \lambda(t)\int_{t_3}^{t} \lambda^{\beta}(s)k(s)p^{-\beta}(b(s))\lambda^{-1}(b(s))ds \\
&\leq \lambda(t)\int_{t_1}^{t_3} \lambda^{\beta}(s)k(s)p^{-\beta}(b(s))\lambda^{-1}(b(s))ds \\
&\quad + \int_{t_3}^{t} \lambda^{\beta}(s)k(s)p^{-\beta}(b(s))ds \\
&\leq \lambda(t)\int_{t_1}^{t_3} \lambda^{\beta}(s)k(s)p^{-\beta}(b(s))\lambda^{-1}(b(s))ds + \frac{\delta}{4}.
\end{aligned} \tag{3.18}$$

Also, for $t \geq t_3$,

$$\begin{aligned}
\frac{1}{\lambda(b(t))}\int_{t}^{\infty} \lambda^{\beta+1}(s)k(s)p^{-\beta}(b(s))ds &\leq \int_{t}^{\infty} \lambda^{\beta}(s)k(s)p^{-\beta}(b(s))ds \\
&\leq \int_{t_3}^{\infty} \lambda^{\beta}(s)k(s)p^{-\beta}(b(s))ds < \frac{\delta}{4}.
\end{aligned} \tag{3.19}$$

From (3.18) and (3.19), we conclude that

$$\limsup_{t\to\infty} \mathrm{K}(t) \leq \frac{\delta}{2},$$

which contradicts (3.16). The contradiction obtained shows that (3.1) holds, and hence all results of Lemma 3.1 and Lemma 3.2 are fulfilled. Next, proceeding similarly as in proof in Lemma 3.2, we arrive at (3.13) and (3.14) for $t \geq t_4$. Integrating (3.14) from t to ∞ yields

$$\left(z + \lambda r^{1/\alpha} z'\right)(t) \geq \frac{\kappa}{\alpha}\int_{t}^{\infty} \lambda^{\beta+1}(s)k(s)p^{-\beta}(b(s))\frac{z(b(s))}{\lambda(b(s))}ds. \tag{3.20}$$

Integrating (3.13) from t_4 to t and then multiplying by $\lambda(t)$, we get

$$-\left(\lambda r^{1/\alpha} z'\right)(t) \geq \frac{\kappa}{\alpha}\lambda(t)\int_{t_4}^{t} \lambda^{\beta}(s)k(s)p^{-\beta}(b(s))\frac{z(b(s))}{\lambda(b(s))}ds. \tag{3.21}$$

It follows from (3.20) and (3.21) that

$$z(t) \geq \frac{\kappa}{\alpha}\lambda(t)\int_{t_4}^{t}\lambda^{\beta}(s)k(s)p^{-\beta}(b(s))\frac{z(b(s))}{\lambda(b(s))}ds + \frac{\kappa}{\alpha}\int_{t}^{\infty}\lambda^{\beta+1}(s)k(s)p^{-\beta}(b(s))\frac{z(b(s))}{\lambda(b(s))}ds. \tag{3.22}$$

In view of the monotonicity properties of z, b and z/λ, we obtain

$$\int_{t_4}^{t}\lambda^{\beta}(s)k(s)p^{-\beta}(b(s))\frac{z(b(s))}{\lambda(b(s))}ds \geq \left(\int_{t_4}^{t}\lambda^{\beta}(s)k(s)p^{-\beta}(b(s))\lambda^{-1}(b(s))ds\right)z(b(t)), \tag{3.23}$$

and

$$\int_{t}^{\infty}\lambda^{\beta+1}(s)k(s)p^{-\beta}(b(s))\frac{z(b(s))}{\lambda(b(s))}ds \geq \left(\frac{1}{\lambda(b(t))}\int_{t}^{\infty}\lambda^{\beta+1}(s)k(s)p^{-\beta}(b(s))ds\right)z(b(t)). \tag{3.24}$$

Using (3.23) and (3.24) in (3.22) yields

$$\frac{\alpha}{\kappa}z(t) \geq \left(\lambda(t)\int_{t_4}^{t}\lambda^{\beta}(s)k(s)p^{-\beta}(b(s))\lambda^{-1}(b(s))ds\right)z(b(t)) + \left(\frac{1}{\lambda(b(t))}\int_{t}^{\infty}\lambda^{\beta+1}(s)k(s)p^{-\beta}(b(s))ds\right)z(b(t)). \tag{3.25}$$

Using that z is decreasing and $b(t) \leq t$, we deduce from (3.25) that

$$\frac{\alpha}{\kappa} \geq \lambda(t)\int_{t_4}^{t}\lambda^{\beta}(s)k(s)p^{-\beta}(b(s))\lambda^{-1}(b(s))ds + \frac{1}{\lambda(b(t))}\int_{t}^{\infty}\lambda^{\beta+1}(s)k(s)p^{-\beta}(b(s))ds.$$

Now take the lim sup as $t \to \infty$, we get a contradiction to (3.15) and complete the proof. □

Theorem 3.4. *If*

$$\limsup_{t\to\infty} V(t) > \frac{\alpha}{\kappa}, \tag{3.26}$$

where

$$V(t) := \lambda(b(t))\int_{t_0}^{b(t)}\lambda^{\beta}(s)k(s)p^{-\beta}(b(s))\lambda^{-1}(b(s))ds + \int_{b(t)}^{t}\lambda^{\beta+1}(s)k(s)p^{-\beta}(b(s))\lambda^{-1}(b(s))ds + \frac{1}{\lambda(b(t))}\int_{t}^{\infty}\lambda^{\beta+1}(s)k(s)p^{-\beta}(b(s))ds,$$

then equation (1.1) is oscillatory.

Proof. Pick $t_1 \in \mathcal{I}_0$ such that $x(t) > 0$, $x(\pi(t)) > 0$ and $x(\eta(t)) > 0$ for $t \geq t_1$. As in the proof of Theorem 3.3, it is not difficult to see that (3.1). Thus, as in the proof of Theorem 3.3, we see that (3.22) holds and leads to

$$z(b(t)) \geq \frac{\kappa}{\alpha}\lambda(b(t))\int_{t_4}^{b(t)}\lambda^{\beta}(s)k(s)p^{-\beta}(b(s))\frac{z(b(s))}{\lambda(b(s))}ds + \frac{\kappa}{\alpha}\int_{b(t)}^{t}\lambda^{\beta+1}(s)k(s)p^{-\beta}(b(s))\frac{z(b(s))}{\lambda(b(s))}ds + \frac{\kappa}{\alpha}\int_{t}^{\infty}\lambda^{\beta+1}(s)k(s)p^{-\beta}(b(s))\frac{z(b(s))}{\lambda(b(s))}ds. \tag{3.27}$$

Using the monotonic properties of z and z/λ, we deduce from (3.27) that

$$\begin{aligned} z(b(t)) \geq & \left(\frac{\kappa}{\alpha}\lambda(b(t))\int_{t_4}^{b(t)}\lambda^{\beta}(s)k(s)p^{-\beta}(b(s))\lambda^{-1}(b(s))ds\right)z(b(t)) \\ & + \left(\frac{\kappa}{\alpha}\int_{b(t)}^{t}\lambda^{\beta+1}(s)k(s)p^{-\beta}(b(s))\lambda^{-1}(b(s))ds\right)z(b(t)) \\ & + \left(\frac{\kappa}{\alpha}\int_{t}^{\infty}\lambda^{\beta+1}(s)k(s)p^{-\beta}(b(s))ds\right)\frac{z(b(t))}{\lambda(b(t))}, \end{aligned}$$

from which

$$\begin{aligned} \frac{\alpha}{\kappa} \geq & \lambda(b(t))\int_{t_4}^{b(t)}\lambda^{\beta}(s)k(s)p^{-\beta}(b(s))\lambda^{-1}(b(s))ds \\ & + \int_{b(t)}^{t}\lambda^{\beta+1}(s)k(s)p^{-\beta}(b(s))\lambda^{-1}(b(s))ds \\ & + \frac{1}{\lambda(b(t))}\int_{t}^{\infty}\lambda^{\beta+1}(s)k(s)p^{-\beta}(b(s))ds. \end{aligned}$$

The rest of the proof is as that of Theorem 3.3. □

Theorem 3.5. *If*

$$\limsup_{t\to\infty}\int_{b(t)}^{t}\lambda^{\beta+1}(s)k(s)p^{-\beta}(b(s))\lambda^{-1}(b(s))ds > \frac{\alpha}{\kappa}, \tag{3.28}$$

then (1.1) *is oscillatory.*

Proof. Pick $t_1 \in \mathcal{I}_0$ such that $x(t) > 0$, $x(\pi(t)) > 0$ and $x(\eta(t)) > 0$ for $t \geq t_1$. It is clear that (3.28) implies that

$$\int_{t_0}^{\infty}\lambda^{\beta+1}(s)k(s)p^{-\beta}(b(s))\lambda^{-1}(b(s))ds = \infty. \tag{3.29}$$

Using the monotonicity properties of λ and b, we see that

$$\int_{b(t)}^{t}\lambda^{\beta+1}(s)k(s)p^{-\beta}(b(s))\lambda^{-1}(b(s))ds \leq \int_{b(t)}^{t}\lambda^{\beta}(s)k(s)p^{-\beta}(b(s))ds. \tag{3.30}$$

In view of (3.29) and (3.30), we observe that (3.28) implies (3.1). Thereby, all results of Lemmas 3.1–3.2 are fulfilled. Proceeding as in the proof of Lemma 3.2, we again arrive at (3.14). Integrating (3.14) from $b(t)$ to t yields

$$\begin{aligned} z(b(t)) + \lambda(b(t))r^{1/\alpha}(b(t))z'(b(t)) & \geq \frac{\kappa}{\alpha}\int_{b(t)}^{t}\lambda^{\beta+1}(s)k(s)p^{-\beta}(b(s))\frac{z(b(s))}{\lambda(b(s))}ds \\ & \geq \left(\frac{\kappa}{\alpha}\int_{b(t)}^{t}\lambda^{\beta+1}(s)k(s)p^{-\beta}(b(s))\lambda^{-1}(b(s))ds\right)z(b(t)), \end{aligned}$$

which, together with $z'(t) < 0$, gives

$$z(b(t)) \geq \left(\frac{\kappa}{\alpha}\int_{b(t)}^{t}\lambda^{\beta+1}(s)k(s)p^{-\beta}(b(s))\lambda^{-1}(b(s))ds\right)z(b(t)). \tag{3.31}$$

We now observe from (3.31) that

$$\frac{\alpha}{\kappa} \geq \int_{b(t)}^{t}\lambda^{\beta+1}(s)k(s)p^{-\beta}(b(s))\lambda^{-1}(b(s))ds.$$

Taking the lim sup as $t \to \infty$, we get a contradiction to (3.28) and ends the proof. □

4 Examples

Now the applicability of all theorems presented in this article will be demonstrated with examples for all cases of α and β, i.e., for $\alpha = \beta$, $\beta > \alpha$ and $\beta < \alpha$, respectively.

Example 4.1. Let us consider the half-linear equation

$$\left(t^{10}\left(\left(x(t)+2x\left(\frac{t}{2}\right)\right)'\right)^5\right)' + at^4x^5\left(\frac{t}{4}\right) = 0, \quad t \geq 1. \tag{4.1}$$

Here, $r(t) = t^{10}$, $p(t) = 2$, $\pi(t) = t/2$, $\eta(t) = t/4$, $\alpha = \beta = 5$, and $k(t) = at^4$ with $a > 0$. Then

$$\lambda(t) = 1/t, \quad \pi^{-1}(t) = 2t, \quad b(t) = t/2, \text{ and } \quad \lambda(b(t)) = 2/t.$$

Now, condition (2.31) becomes

$$\limsup_{t\to\infty} W(t) = \frac{a}{16},$$

condition (2.41) becomes

$$\limsup_{t\to\infty} H(t) = a2^{-5}\left(2+\ln 2\right),$$

condition (2.42) becomes

$$\limsup_{t\to\infty} \int_{t/2}^{t} \frac{a2^{-5}}{s} ds = a2^{-5}\ln 2.$$

Next, by applying Theorems 2.4–2.6 to equation (4.1), we deduce that (4.1) is oscillatory by Theorem 2.4 if $a > 2560$, Eq. (4.1) oscillates by Theorem 2.5 if $a > 1901.4$, and (4.1) is oscillatory by Theorem 2.6 if $a > 7386$.

Note that none of the results in [4,14,25] can be applied to equation (4.1) since the equation is half-linear and ($\alpha = \beta = 5$). Also, our findings required only one condition to obtain the result, but the approach in [14,25] requires two conditions.

Example 4.2. Consider the superlinear equation

$$\left(t^2\left(x(t)+4tx\left(\frac{t}{6}\right)\right)'\right)' + at^5x^3\left(\frac{t}{8}\right) = 0, \quad t \geq 1. \tag{4.2}$$

Here $r(t) = t^2$, $p(t) = 4t$, $\pi(t) = t/6$, $\eta(t) = t/8$, $\alpha = 1$, $\beta = 3$, and $k(t) = at^5$ with $a > 0$. Then

$$\lambda(t) = 1/t, \quad \pi^{-1}(t) = 6t, \quad b(t) = 3t/4, \quad \text{and} \quad \lambda(b(t)) = 4/3t.$$

Now, condition (3.15) becomes

$$\limsup_{t\to\infty} K(t) = \frac{a}{18},$$

condition (3.26) becomes

$$\limsup_{t\to\infty} V(t) = \frac{a}{36}(2+\ln\frac{4}{3}),$$

condition (3.28) becomes

$$\limsup_{t\to\infty} \int_{3t/4}^{t} \frac{a}{36s} ds = \frac{a}{36}\ln\frac{4}{3}.$$

Next, by applying Theorems 3.3–3.5 to equation (4.2), we deduce that (4.2) is oscillatory by Theorem 3.3 if $a > 128/3 = 42.667$, Eq. (4.2) oscillates by Theorem 3.4 if $a > 256/3(2 + \ln 4/3) = 37.301$, and (4.2) is oscillatory by Theorem 3.5 if $a > 296.62$.

Note that none of the results in [4,14,25] can be applied to equations (4.2) since the equation is superlinear ($\beta = 3 > \alpha = 1$).

Example 4.3. Consider the nonlinear equation

$$\left(t^{10}\left(\left(x(t)+10x\left(\frac{t}{2}\right)\right)'\right)^5\right)' + at^6x^7\left(\frac{t}{4}\right) = 0, \quad t \geq 1. \tag{4.3}$$

Here, $r(t) = t^{10}$, $p(t) = 10$, $\pi(t) = t/2$, $\eta(t) = t/4$, $\alpha = 5$, $\beta = 7$, and $k(t) = at^6$. Then

$$\lambda(t) = 1/t, \quad \pi^{-1}(t) = 2t, \quad b(t) = t/2, \quad \text{and} \quad \lambda(b(t)) = 2/t.$$

Now, condition (3.15) becomes

$$\limsup_{t\to\infty} \mathrm{K}(t) = a10^{-7},$$

condition (3.26) becomes

$$\limsup_{t\to\infty} \mathrm{V}(t) = a10^{-7}2^{-1}(2+\ln 2),$$

condition (3.28) becomes

$$\limsup_{t\to\infty} \int_{t/2}^{t} \frac{a10^{-7}}{2s} ds = a10^{-7}2^{-1}\ln 2.$$

Next, by applying Theorems 3.3–3.5 to equation (4.3), we deduce that (4.3) is oscillatory by Theorem 3.3 if $a > 10^{15}/2 \times 9^7$, Eq. (4.3) oscillates by Theorem 3.4 if $a > 10^{15}/9^7(2 + \ln 2)$, and (4.3) is oscillatory by Theorem 3.5 if $a > 10^{15}/9^7 \ln 2$.

Example 4.4. Consider the nonlinear equation

$$\left(t^{6}\left(\left(x(t)+2tx\left(\frac{t}{3}\right)\right)'\right)^3\right)' + at^3x\left(\frac{t}{6}\right) = 0, \quad t \geq 1. \tag{4.4}$$

Then

$$\lambda(t) = 1/t, \quad \pi^{-1}(t) = 3t, \quad b(t) = t/2, \text{ and } \quad \lambda(b(t)) = 2/t.$$

Now, condition (2.31) becomes

$$\limsup_{t\to\infty} \mathrm{W}(t) = 2a,$$

condition (2.41) becomes

$$\limsup_{t\to\infty} \mathrm{H}(t) = a(2+\ln 2),$$

(2.42) becomes

$$\limsup_{t\to\infty} \int_{t/2}^{t} \frac{a}{s} ds = a\ln 2.$$

Next, by applying Theorems 2.4–2.6 to equation (4.4), we deduce that (4.4) is oscillatory by Theorem 2.4 if $a > 3$, Eq. (4.4) oscillates by Theorem 2.5 if $a > 2.2279$, and (4.4) is oscillatory by Theorem 2.6 if $a > 8.6562$.

Note that none of the results in [4,14,25] can be applied to equations (4.3) and (4.4) since ($\alpha = 5$ and $\beta = 7$) and ($\alpha = 3$ and $\beta = 1$), respectively.

Remark 4.5. It would be of interest to extend the results in [8] to equations of type (1.1) using the technique in [8].

5 Conclusion

This study analyzed the oscillation of solutions to a class of second-order differential equations with bounded and unbounded neutral coefficients in noncanonical form. The oscillation results were obtained by a combination of the linearization technique and the monotonicity properties of the neutral term. Our findings advance the current understanding in this area by providing new results that can be applied to linear ($\alpha = \beta = 1$), superlinear ($\alpha = 1$ and $\beta > 1$), sublinear ($\alpha = 1$ and $\beta < 1$), half-linear ($\alpha = \beta \neq 1$) and the other nonlinear [$(1 \neq \alpha \neq \beta)$ and $(\alpha > 1$ and $\beta = 1)$] forms of the equation under study. Moreover, the oscillation results for all these cases are established by means of only one condition. Hence, results designed specifically for different classes are important in their own right and have both weak and strong points. In some cases, they improve and extend some existing results in the literature; and in some cases, they are interesting in their own right, i.e., they neither include nor are included by the existing results, and these differences are explained to some extent by examples.

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