



Resonance phenomena in the second differential order equations with nonlinear impulsive effects

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Abstract. In this paper, we discuss the resonance phenomena in linear and weakly nonlinear impulsive differential equations. Using a topological approach, we obtain the Landesman–Lazer type necessary and sufficient conditions for the existence of periodic solutions and extend the Fredholm alternative for linear equations with impulses. Moreover, we show the coexistence of periodic and unbounded solutions of weakly nonlinear impulsive differential equations through the use of the successor map and a Lyapunov function.

Keywords: impulsive differential equations, resonance phenomena, periodic solutions, Fredholm alternative, Lyapunov function.

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1 Introduction

In this paper, we focus on the second-order impulsive ordinary differential equations

$$\begin{cases} u'' + f(t, u) = 0, \\ \Delta u(t_j) = J_j(u(t_j^-), u'(t_j^-)), \\ \Delta u'(t_j) = Q_j(u(t_j^-), u'(t_j^-)), \quad j = \pm 1, \pm 2, \dots, \end{cases} \quad (1.1)$$

where $f : \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$ is continuous, 2π -periodic with respect to first variable, $0 \leq t_1 < \dots < t_k < 2\pi$, $\Delta u(t_j) = u(t_j^+) - u(t_j^-)$, $\Delta u'(t_j) = u'(t_j^+) - u'(t_j^-)$, $J_j, Q_j : \mathbb{R}^2 \rightarrow \mathbb{R}$ are continuous for $j = \pm 1, \pm 2, \dots$. Moreover, we assume that the impulsive time is 2π -periodic, that is, $t_{j+k} = t_j + 2\pi$, for $j = \pm 1, \pm 2, \dots$.

There are many interesting results on the existence and multiplicity of periodic solutions of impulsive differential equations. Various methodologies include fixed point theory [12–14], the Leray–Schauder continuation method of topological degree [5, 19], the variational method

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[4, 15, 21, 22], the Poincaré–Birkhoff twist theorem [8, 18], and Moser’s twist theorem for the Lagrange stability of impulsive differential equations [16, 20].

Regarding the existence of periodic solutions for impulsive differential equations, several significant results have been achieved. However, compared to non-impulsive differential equations, many unresolved issues remain that are worthy of further investigation. For the linear equation

$$u'' + n^2 u = p(t), \quad (1.2)$$

where $p(t) \in C([0, 2\pi])$ is 2π -period. The so-called phenomenon of linear resonance arises whenever n is a positive integer. In other words, periodic solutions and unbounded solutions cannot coexist for the linear equation (1.2). This implies that all solutions of (1.2) are either periodic or unbounded. This is also known as the Fredholm alternative. However, slightly different equations do not have such strong conclusions. For instance, Alonso and Ortega [2] proved that similar results fail to hold for piecewise linear equations

$$u'' + au^+ - bu^- = p(t),$$

where $u^+ = \max\{u, 0\}$, $u^- = \max\{-u, 0\}$ and a, b are different and satisfy

$$\frac{1}{\sqrt{a}} + \frac{1}{\sqrt{b}} \in \mathbb{Q}. \quad (1.3)$$

They deduced that periodic and unbounded solutions can coexist when (1.3) holds and $a \neq b$ in [2].

Meanwhile, under the influence of impulses, the resonance phenomenon in linear equations has attracted significant attention. Similarly, the coexistence of periodic solutions with unbounded solutions in weakly linear equations has been a focus of recent studies. Recently, Drábek and Langerová [6] obtained the Fredholm alternative type results for linear equations with bounded impulses. They applied critical point theory to provide the necessary and sufficient conditions for the existence of a solution when solutions satisfying the impulse conditions in the derivative:

$$\Delta u(t_j) = 0, \quad \Delta u'(t_j) = \tilde{I}_j(u(t_j^-)),$$

$$\lim_{s \rightarrow \pm\infty} \tilde{I}_j(s) = \tilde{I}_j(\pm\infty), \quad \tilde{I}_j(+\infty) < \tilde{I}_j(s) < \tilde{I}_j(-\infty), \quad j = 1, \dots, k.$$

In their work [15], they also mentioned employing similar approaches to analyze the necessary and sufficient conditions for the periodic solutions of linear equations. However, we note that when applying variational methods or critical point theory to impulsive equations, due to the limitations of these methods, it is necessary to transform the periodic solution or boundary value problem into a functional critical point. In such cases, the impulses must satisfy

$$\Delta u(t_j) = 0, \quad \Delta u'(t_j) = \tilde{I}_j(u(t_j^-)).$$

An interesting question is whether we can overcome this limitation and obtain the necessary and sufficient conditions for the existence of periodic solutions of linear equations under more general impulsive conditions. One of the main results of this paper is to relax the impulse restrictions established in [6], via topological framework, allowing for impulses in both the function and its derivative.

We consider the following impulsive equation

$$\begin{cases} u''(t) + n^2 u(t) = p(t), & t \in [0, 2\pi], t \neq t_j; \\ \Delta u(t_j) = J_j(u'(t_j^-)), & \Delta u'(t_j) = Q_j(u(t_j^-)), \quad j = 1, \dots, k; \\ u(0) = u(2\pi), & u'(0) = u'(2\pi). \end{cases} \quad (1.4)$$

Suppose that

(H₁) For $j = 1, \dots, k$, there exist limits

$$\lim_{s \rightarrow \pm\infty} J_j(s) = J_j(\pm\infty), \quad \lim_{s \rightarrow \pm\infty} Q_j(s) = Q_j(\pm\infty),$$

and the following inequalities hold:

$$J_j(+\infty) < J_j(s) < J_j(-\infty), \quad Q_j(-\infty) < Q_j(s) < Q_j(+\infty), \quad s \in \mathbb{R}.$$

That gives our Landesman–Lazer type condition.

(H₂) For any $\theta \in \mathbb{R}$, $p(t)$ satisfies

$$\begin{aligned} & n \sum_{j=1}^k J_j(-\infty) \cos^+(nt_j + \theta) - n \sum_{j=1}^k J_j(+\infty) \cos^-(nt_j + \theta) + \sum_{j=1}^k Q_j(+\infty) \sin^-(nt_j + \theta) \\ & - \sum_{j=1}^k Q_j(-\infty) \sin^+(nt_j + \theta) > \int_0^{2\pi} \sin(nt + \theta) p(t) dt > n \sum_{j=1}^k J_j(+\infty) \cos^+(nt_j + \theta) \\ & - n \sum_{j=1}^k J_j(-\infty) \cos^-(nt_j + \theta) - \sum_{j=1}^k Q_j(+\infty) \sin^+(nt_j + \theta) + \sum_{j=1}^k Q_j(-\infty) \sin^-(nt_j + \theta). \end{aligned}$$

Theorem 1.1. Assume that (H₁) holds. Then (1.4) has a 2π -periodic solution if and only if $p(t)$ satisfies (H₂). Moreover, if (1.4) has no periodic solutions then every solution must be unbounded.

Remark 1.2. Theorem 1.1 provides the necessary and sufficient conditions for impulses in both the function and its derivative, and also discusses unbounded solutions, extending the results in [6].

And we illustrate our results with an example.

Example 1.3. Let $n = 2$ in (1.4), the impulse time $t_1 = \frac{\pi}{4}$, $t_2 = \frac{\pi}{2}$ and the impulse condition is of the form $J_j(u') = -\arctan u'$, $Q_j(u) = \arctan u$, $j = 1, 2$. Then (1.4) has a 2π -periodic solution if $p(t) = \cos 2t + \sin 2t$.

Furthermore, the Fredholm alternative does not hold for nonlinear equations. There is no similar theorem even for piecewise linear equations, as shown by Alonso and Ortega [2]. Fonda and Mawhin [7] proved all solutions of $J\dot{u} = \nabla \mathcal{H}(u) + f(u) + p(t)$ are bounded in the past, and those with sufficiently large amplitude are unbounded in the future. Qian [17] showed that the coexistence of periodic solution and unbounded solution, the infinity of large amplitude subharmonics for asymmetric weakly nonlinear oscillator. There is no similar theorem even for piecewise linear equations by Alonso and Ortega [2]. Fonda and Mawhin [7] proved all solutions of $J\dot{u} = \nabla \mathcal{H}(u) + f(u) + p(t)$ are bounded in the past, and those with sufficiently large amplitude are unbounded in the future. Qian [17] showed that the coexistence

of periodic solution and unbounded solution, the infinity of large amplitude subharmonics for asymmetric weakly nonlinear oscillator

$$x'' + a^2x^+ - b^2x^- + h(x) = p(t),$$

with $h(\pm\infty) = 0$ and $xh(x) \rightarrow +\infty (x \rightarrow \infty)$. For the other discussion of the coexistence of unbounded solutions and periodic solutions, we refer the reader to [3, 9, 10]. Thus the coexistence of periodic solution and unbounded solution is a typical nonlinear phenomenon. We consider the following weakly nonlinear impulsive equation

$$\begin{cases} u''(t) + n^2u(t) + h(u(t)) = p(t), & t \in [0, 2\pi], t \neq t_j; \\ \Delta u(t_j) = J_j(u(t_j^-), u'(t_j^-)), & \Delta u'(t_j) = Q_j(u(t_j^-), u'(t_j^-)), \quad j = 1, \dots, k, \end{cases} \quad (1.5)$$

where $0 < t_1 < \dots < t_k < 2\pi$, $h : C[0, 2\pi] \rightarrow C[0, 2\pi]$, $p(t) \in C([0, 2\pi])$ is 2π -periodic and $J_j, Q_j : \mathbb{R}^2 \rightarrow \mathbb{R}$ are continuous, $j = 1, \dots, k$. Suppose that

(D₁) $h(u)$ satisfies

- (i) $h(\pm\infty) = \lim_{u \rightarrow \pm\infty} h(u) = 0$,
- (ii) $\lim_{|u| \rightarrow \infty} uh(u) = +\infty$.

(D₂) Impulsive terms satisfy

$$J_j(u, u') = O\left(\frac{1}{r}\right), \quad Q_j(u, u') = O\left(\frac{1}{r}\right), \quad j = 1, \dots, k,$$

where $r = \sqrt{|u|^2 + |u'|^2}$. $f = O(\frac{1}{r})$ for $r \rightarrow \infty$ means that $f/(\frac{1}{r})$ is bounded for all sufficiently large r .

Theorem 1.4. Assume that (D₁) and (D₂) hold. Then there exists $q_0 > 0$, such that (1.5) has both 2π -periodic solution and unbounded solution provided that $p(t)$ satisfies

$$0 < M_p < q_0,$$

where

$$M_p = \max_{\theta \in [0, 2\pi]} \left| \int_0^{2\pi} p(t) \sin(nt + \theta) dt \right|.$$

Example 1.5. Let $h(u) = u^{-\frac{1}{3}}$ and the impulse condition is given by $Q_j(u, u') = J_j(u, u') = \frac{1}{\sqrt{u^2 + u'^2}}$. Then (1.5) has both 2π -periodic solution and unbounded solution.

The rest of the paper is organized as follows. In Section 2, we discuss the necessary and sufficient conditions for the existence of periodic solutions of linear impulsive equations via constant variation method and degree theory. Moreover, we prove that all solutions are unbounded when the periodic solution does not exist by the method of variation of constants. In Section 3, we consider weakly nonlinear impulsive equations and introduce the successor map to prove the existence of a periodic solution. The proof of the existence of the unbounded solution is mainly using Poincaré map to construct a Liapunov function that makes the solution grow infinitely along the orbit. So that the solution of the equation is unbounded.

2 Resonance and the existence of periodic solutions

In this section, we discuss a linear equation with impulses and provide necessary and sufficient conditions for the existence of a 2π -periodic solution under assumptions (H_1) and (H_2) . Furthermore, we prove the existence of unbounded solutions and then complete the proof of Theorem 1.1.

Consider the linear periodic boundary value problem

$$\begin{cases} u''(t) + n^2 u(t) = p(t), & t \in [0, 2\pi], t \neq t_j; \\ \Delta u(t_j) = J_j(u(t_j^-), u'(t_j^-)), & \Delta u'(t_j) = Q_j(u(t_j^-), u'(t_j^-)), j = 1, \dots, k; \\ u(0) = u(2\pi), & u'(0) = u'(2\pi). \end{cases} \quad (2.1)$$

where $0 < t_1 < \dots < t_k < 2\pi$, $p(t) \in C(\mathbb{R}, \mathbb{R})$, $p(t)$ is 2π -periodic and the impulses $J_j, Q_j : \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}, j = 1, \dots, k$, are continuous.

2.1 The necessary condition

For a general case, we consider the following impulsive equation

$$\begin{cases} x'(t) = Ax(t) + P(t), & t \neq t_j; \\ \Delta x(t_j) = I_j(x(t_j^-)), & j = 1, \dots, k; \\ x(0) = x(2\pi), \end{cases} \quad (2.2)$$

where 2×2 matrix A is constant, two-dimensional vector $P(t)$ is continuous, $\Delta x(t_j) = x(t_j^+) - x(t_j^-)$, I_j are continuous, $j = 1, \dots, k$.

For simplicity, we set $t_0 = 0$ and $t_{k+1} = 2\pi$. It is well known that the solutions of (2.2) which include instantaneous impulses result in jump discontinuities in both velocity and position at times $t = t_j, j = 1, \dots, k$. In other words, any motion of the solution is the same as the motion of the corresponding equation without impulses until it meets the next impulse time. Hence, we first consider the differential equation without impulses

$$x'(t) = Ax(t) + P(t). \quad (2.3)$$

Fixed any $j \in \{0, 1, \dots, k\}$. Suppose that equation (2.3) satisfies the initial condition $x(t_j) = x(t_j^+)$. By the method of variation of parameters, we have

$$x(t) = \Phi(t) \left(\Phi^{-1}(t_j^+) x(t_j^+) + \int_{t_j}^t \Phi^{-1}(s) P(s) ds \right), \quad t \in (t_j^+, t_{j+1}^-),$$

where $\Phi(t)$ is the fundamental matrix of (2.3) that satisfies $\Phi(0) = \Phi(2\pi) = E$. In particular,

$$x(t_{j+1}^-) = \Phi(t_{j+1}^-) \left(\Phi^{-1}(t_j^+) x(t_j^+) + \int_{t_j}^{t_{j+1}^-} \Phi^{-1}(s) P(s) ds \right).$$

Next, we present the expression for the solutions of the impulsive differential equation (2.2). Note that $\Phi(t_j^+) = \Phi(t_j) = \Phi(t_j^-)$, since $\Phi(t)$ is continuous. Substituting $x(t_1^-)$ into the

expression for $x(t_2^-)$ yields

$$\begin{aligned} x(t_2^-) &= \Phi(t_2^-) \left(\Phi^{-1}(t_1^+) x(t_1^+) + \int_{t_1^+}^{t_2^-} \Phi^{-1}(s) P(s) ds \right) \\ &= \Phi(t_2^-) \left(\Phi^{-1}(t_1^+) (x(t_1^-) + x(t_1^+) - x(t_1^-)) + \int_{t_1^+}^{t_2^-} \Phi^{-1}(s) P(s) ds \right) \\ &= \Phi(t_2^-) \left(x(0) + \int_0^{t_2^-} \Phi^{-1}(s) P(s) ds + \Phi^{-1}(t_1^+) I_1(x(t_1^-)) \right). \end{aligned}$$

Repeating the above steps by substituting $x(t_2^-)$ into the expression for $x(t_3^-)$, and so on, we obtain

$$x(t) = \Phi(t) \left(x(0) + \int_0^t \Phi^{-1}(s) P(s) ds + \sum_{t > t_j} \Phi^{-1}(t_j) I_j(x(t_j^-)) \right). \quad (2.4)$$

In particular,

$$\begin{aligned} x(2\pi) &= \Phi^{-1}(2\pi) \left(x(0) + \int_0^{2\pi} \Phi^{-1}(s) P(s) ds + \sum_{j=1}^k \Phi^{-1}(t_j) I_j(x(t_j^-)) \right) \\ &= x(0) + \int_0^{2\pi} \Phi^{-1}(s) P(s) ds + \sum_{j=1}^k \Phi^{-1}(t_j) I_j(x(t_j^-)). \end{aligned} \quad (2.5)$$

Since $x(0) = x(2\pi)$, it follows that

$$\int_0^{2\pi} \Phi^{-1}(s) P(s) ds + \sum_{j=1}^k \Phi^{-1}(t_j) I_j(x(t_j^-)) = 0. \quad (2.6)$$

Recall that the equation (2.1). We set $x_1(t) = u(t)$, $x_2(t) = u'(t)$, $x(t) = (x_1(t), x_2(t))^T$, $P(t) = (0, p(t))^T$, $I_j = (J_j, Q_j)^T$, $j = 1, \dots, k$, and $A = \begin{pmatrix} 0 & 1 \\ -n^2 & 0 \end{pmatrix}$. Obviously,

$$\Phi(t) = \begin{pmatrix} \cos nt & \frac{1}{n} \sin nt \\ -n \sin nt & \cos nt \end{pmatrix}.$$

Then, plugging these expression back into (2.6), we obtain

$$\begin{aligned} \int_0^{2\pi} p(t) \sin nt dt &= n \sum_{j=1}^k J_j(u(t_j^-), u'(t_j^-)) \cos nt_j - \sum_{j=1}^k Q_j(u(t_j^-), u'(t_j^-)) \sin nt_j, \\ \int_0^{2\pi} p(t) \cos nt dt &= -n \sum_{j=1}^k J_j(u(t_j^-), u'(t_j^-)) \sin nt_j - \sum_{j=1}^k Q_j(u(t_j^-), u'(t_j^-)) \cos nt_j. \end{aligned}$$

We now assume that the impulses affect only one component of the system. Consider the following linear periodic boundary value problem

$$\begin{cases} u''(t) + n^2 u(t) = p(t), & t \in [0, 2\pi], t \neq t_j; \\ \Delta u(t_j) = J_j(u'(t_j^-)), & \Delta u'(t_j) = Q_j(u(t_j^-)), \quad j = 1, \dots, k; \\ u(0) = u(2\pi), & u'(0) = u'(2\pi). \end{cases} \quad (2.7)$$

where the impulses satisfies (H_1) .

Similarly, we have

$$\begin{aligned}\int_0^{2\pi} p(t) \sin nt dt &= n \sum_{j=1}^k J_j(u'(t_j^-)) \cos nt_j - \sum_{j=1}^k Q_j(u(t_j^-)) \sin nt_j, \\ \int_0^{2\pi} p(t) \cos nt dt &= -n \sum_{j=1}^k J_j(u'(t_j^-)) \sin nt_j - \sum_{j=1}^k Q_j(u(t_j^-)) \cos nt_j.\end{aligned}$$

Choose any $\theta \in \mathbb{R}$. Multiplying the above two expressions by $\cos \theta$ and $\sin \theta$, respectively, and adding them together, we obtain

$$\begin{aligned}\int_0^{2\pi} p(t) \sin(nt + \theta) dt &= n \sum_{j=1}^k J_j(u'(t_j^-)) \cos(nt_j + \theta) - \sum_{j=1}^k Q_j(u(t_j^-)) \sin(nt_j + \theta) \\ &= n \sum_{j=1}^k J_j(u'(t_j^-)) (\cos^+(nt_j + \theta) - \cos^-(nt_j + \theta)) \\ &\quad - \sum_{j=1}^k Q_j(u(t_j^-)) (\sin^+(nt_j + \theta) - \sin^-(nt_j + \theta)),\end{aligned}$$

where $f^+ = \max\{f, 0\}$ and $f^- = \max\{0, -f\}$.

By combining (H_1) , we finally obtain

$$\begin{aligned}& n \sum_{j=1}^k J_j(-\infty) \cos^+(nt_j + \theta) - n \sum_{j=1}^k J_j(+\infty) \cos^-(nt_j + \theta) \\ & \quad + \sum_{j=1}^k Q_j(+\infty) \sin^-(nt_j + \theta) - \sum_{j=1}^k Q_j(-\infty) \sin^+(nt_j + \theta) \\ & > \int_0^{2\pi} \sin(nt + \theta) p(t) dt \\ & > n \sum_{j=1}^k J_j(+\infty) \cos^+(nt_j + \theta) - n \sum_{j=1}^k J_j(-\infty) \cos^-(nt_j + \theta) \\ & \quad - \sum_{j=1}^k Q_j(+\infty) \sin^+(nt_j + \theta) + \sum_{j=1}^k Q_j(-\infty) \sin^-(nt_j + \theta),\end{aligned}$$

which is exactly assumption (H_2) .

2.2 The sufficient condition

For the Banach space

$$\begin{aligned}X = \{x : [0, 2\pi] \rightarrow \mathbb{R}^2 : x(t) \text{ is continuous for } t \neq t_j, x(t_j^-), x(t_j^+) \text{ exist,} \\ \text{and } x(t_j^-) = x(t_j), j = 1, \dots, k, x(0) = x(2\pi)\},\end{aligned}$$

we define the norm as

$$\|x\|_X = \sup_{t \in B} \left(\sum_{i=1}^2 x_i(t)^2 \right)^{\frac{1}{2}},$$

where $B = [0, 2\pi] \setminus \{t_1, t_2, \dots, t_k\}$.

The Banach space X is reflexive. Indeed, for any $x \in X$, since the interval $[0, 2\pi]$ is measurable, $x(t)$ is a measurable function in it. It is obvious that $\int_0^{2\pi} x^2(t)dt < \infty$ which implies $X \subset L^2[0, 2\pi]$. $L^2[0, 2\pi]$ is a reflexive Banach space and so is X .

To establish the existence of a periodic solution to (2.7), we make use the homotopy invariance property of Leray–Schauder degree. Consider the following equation

$$\begin{cases} u''(t) + (n^2 - (1 - \lambda)\delta)u(t) = \lambda p(t), & t \in [0, 2\pi], t \neq t_j; \\ \Delta u(t_j) = \lambda J_j(u'(t_j^-)), & \Delta u'(t_j) = \lambda Q_j(u(t_j^-)), \quad j = 1, \dots, k; \\ u(0) = u(2\pi), & u'(0) = u'(2\pi). \end{cases} \quad (2.8)$$

and the corresponding homotopy $F_\lambda : X \rightarrow X$, defined as

$$\begin{aligned} F_\lambda x = x(0) + \frac{2\pi - t}{2\pi} & \left(\int_0^{2\pi} (\lambda P(s) + A_\lambda x(s))ds + \sum_{j=1}^k \lambda I_j(x(t_j^-)) \right) \\ & + \int_0^t (\lambda P(s) + A_\lambda x(s))ds + \sum_{t > t_j}^k \lambda I_j(x(t_j^-)), \end{aligned}$$

where $A_\lambda = \begin{pmatrix} 0 \\ -(n^2 - (1 - \lambda)\delta) & 1 \\ 0 \end{pmatrix}$, $\lambda \in [0, 1]$ and $\delta \in (0, \frac{1}{2})$ is sufficiently small so that

$$(n - 1)^2 < n^2 - (1 - \lambda)\delta < n^2.$$

It is well known that the 2π -periodic solutions of (2.8) correspond to the fixed points of the operator F_λ .

Indeed, $F_\lambda(M)$ is an equicontinuous family of functions. Let $M \subset X$ be a bounded set. For any $x \in M$, $F_\lambda(x(t))$ is continuous, for $t \neq t_j$, $j = 1, \dots, k$. Thus, $F_\lambda(x(t))$ is uniformly continuous on each closed subinterval with $0, t_1, \dots, t_k, 2\pi$ as the end point. For any $x \in M$, $x(t)$ is bounded since $x(t)$ is continuous in $[0, 2\pi]$, $t \neq t_j$. Clearly, $P(t)$ is bounded. Note that the boundedness of $J_j(x(t))$ and $Q_j(x(t))$ implies the boundedness of $I_j(x(t))$, $j = 1, \dots, k$. We conclude from (2.8) that $F_\lambda x$ is bounded and hence uniformly bounded. By the Arzelà–Ascoli's theorem, we conclude that $F_\lambda(M)$ is relatively compact and, in consequence, the operator F_λ is completely continuous.

It remains to show that there exists $R_0 > 0$, such that for all $R \geq R_0$ and $\lambda \in [0, 1]$, if $x \in X$, $\|x\| = R$, then $x - F_\lambda x \neq 0$.

We perform the proof via contradiction. Assume that there exist sequences $x_m \in X$, $\|x_m\| \rightarrow \infty$, $m \rightarrow \infty$ and $\lambda_m \in [0, 1]$ such that $x_m = F_{\lambda_m} x_m$. Define $y_m = \frac{x_m}{\|x_m\|}$. Then the equation is equivalent to

$$\begin{aligned} F_{\lambda_m} y_m = \frac{x(0)}{\|x_m\|} + \frac{2\pi - t}{2\pi} & \left(\int_0^{2\pi} \left(\lambda_m \frac{P(s)}{\|x_m\|} + A_{\lambda_m} y_m(s) \right) ds + \sum_{j=1}^k \lambda_m \frac{I_j(x_m(t_j^-))}{\|x_m\|} \right) \\ & + \int_0^t \left(\lambda_m \frac{P(s)}{\|x_m\|} + A_{\lambda_m} y_m(s) \right) ds + \sum_{t > t_j}^k \lambda_m \frac{I_j(x_m(t_j^-))}{\|x_m\|} = y_m. \end{aligned} \quad (2.9)$$

Clearly, $\frac{x(0)}{\|x_m\|} \rightarrow 0$ and $\frac{P(t)}{\|x_m\|} \rightarrow 0$. By (H_1) , we see that $\frac{J_j(u'_m(t_j^-))}{\|x_m\|} \rightarrow 0$, $\frac{Q_j(u_m(t_j^-))}{\|x_m\|} \rightarrow 0$ and so $\frac{I_j(x_m(t_j^-))}{\|x_m\|} \rightarrow 0$.

Since the Banach space X is reflexive and the boundedness of $\{y_m\}$, there exists a weakly convergence subsequence which is still denoted by $\{y_m\}$. Assume that $y_m \rightharpoonup y \in X$ and $\lambda_m \rightarrow \lambda \in [0, 1]$. Since F_λ is compact, we have $F_\lambda y_m \rightarrow F_\lambda y$. It then follows from (2.9) that $y_m \rightarrow y$ strongly in X .

Passing to the limit as $m \rightarrow \infty$, we obtain from (2.9) that

$$F_\lambda y = \frac{2\pi - t}{2\pi} \int_0^{2\pi} A_\lambda y(t) dt + \int_0^t A_\lambda y(t) dt = y,$$

and equivalent to

$$\begin{cases} u'' + (n^2 - (1 - \lambda)\delta)u = 0, \\ u(0) = u(2\pi), \quad u'(0) = u'(2\pi). \end{cases}$$

where $y = (u, u')$, $\|y\| = 1$, $\lambda \in [0, 1]$. By the fact that $(n - 1)^2 < n^2 - (1 - \lambda)\delta < n^2$, the above equation has a solution if and only if $\lambda = 1$ and $u(t) = c_0 \sin(nt + \theta_0)$ where $c_0 > 0$, $\theta_0 \in \mathbb{R}$. The condition $c_0 > 0$ can be ensured by choosing an appropriate θ_0 . Therefore, we obtain

$$\frac{u_m}{\|x_m\|} \rightarrow c_0 \sin(nt + \theta_0), \quad m \rightarrow \infty.$$

We have

$$u_m''(t) + (n^2 - (1 - \lambda_m)\delta)u_m(t) = \lambda_m p(t), \quad (2.10)$$

$$u_m(t_j^+) - u_m(t_j^-) = \lambda_m J_j(u_m'(t_j^-)), \quad u_m'(t_j^+) - u_m'(t_j^-) = \lambda_m Q_j(u_m(t_j^-)), \quad j = 1, \dots, k, \quad (2.11)$$

which follows from $x_m - F_{\lambda_m} x_m = 0$.

Choose the test function $v : [0, 2\pi] \rightarrow \mathbb{R}$ that is continuously differentiable and satisfies $v(0) = v(2\pi)$, $v'(0) = v'(2\pi)$. Multiplying equation (2.10) by $v(t)$ and integrating over B , we get

$$\int_0^{2\pi} u_m''(t)v(t)dt + (n^2 - (1 - \lambda_m)\delta) \int_0^{2\pi} u_m(t)v(t)dt = \int_0^{2\pi} \lambda_m p(t)v(t)dt, \quad (2.12)$$

where

$$\int_0^{2\pi} u_m''(t)v(t)dt = \int_0^{t_1^-} u_m''(t)v(t)dt + \sum_{j=1}^{k-1} \int_{t_j^+}^{t_{j+1}^-} u_m''(t)v(t)dt + \int_{t_k^+}^{2\pi} u_m''(t)v(t)dt.$$

Then, using integration by parts, we obtain

$$\begin{aligned} \int_{t_j^+}^{t_{j+1}^-} u_m''(t)v(t)dt &= u_m'(t)v(t) \Big|_{t_j^+}^{t_{j+1}^-} - \int_{t_j^+}^{t_{j+1}^-} u_m'(t)v'(t)dt \\ &= u_m'(t_{j+1}^-)v(t_{j+1}) - u_m'(t_j^+)v(t_j) - u_m(t_{j+1}^-)v'(t_{j+1}) \\ &\quad + u_m(t_j^+)v'(t_j) + \int_{t_j^+}^{t_{j+1}^-} u_m(t)v''(t)dt. \end{aligned} \quad (2.13)$$

Combining (2.11), (2.12) and (2.13), we obtain

$$\begin{aligned} \int_0^{2\pi} u_m(t)v''(t)dt + (n^2 - (1 - \lambda_m)\delta) \int_0^{2\pi} u_m(t)v(t)dt + \lambda_m \sum_{j=1}^k J_j(u_m'(t_j^-))v'(t_j) \\ - \lambda_m \sum_{j=1}^k Q_j(u_m(t_j^-))v(t_j) = \int_0^{2\pi} \lambda_m p(t)v(t)dt. \end{aligned}$$

Now, choose $v(t) = c_0 \sin(nt + \theta_0)$, then,

$$\begin{aligned} & - (1 - \lambda_m) \delta \int_0^{2\pi} u_m(t) \sin(nt + \theta_0) dt + n \lambda_m \sum_{j=1}^k J_j(u'_m(t_j^-)) \cos(nt_j + \theta_0) \\ & - \lambda_m \sum_{j=1}^k Q_j(u_m(t_j^-)) \sin(nt_j + \theta_0) = \int_0^{2\pi} \lambda_m p(t) \sin(nt + \theta_0) dt. \end{aligned} \quad (2.14)$$

Notice that $\frac{u_m}{\|x_m\|} \rightarrow c_0 \sin(nt + \theta_0)$, $c_0 > 0$, which implies that

$$\int_0^{2\pi} u_m(t) \sin(nt + \theta_0) dt > 0, \quad m \gg 1. \quad (2.15)$$

From (2.14) and (2.15), it follows that

$$\int_0^{2\pi} p(t) \sin(nt + \theta_0) dt \leq n \sum_{j=1}^k J_j(u'_m(t_j^-)) \cos(nt_j + \theta_0) - \sum_{j=1}^k Q_j(u_m(t_j^-)) \sin(nt_j + \theta_0).$$

Passing to the limit as $m \rightarrow \infty$, we finally obtain

$$\begin{aligned} \int_0^{2\pi} p(t) \sin(nt + \theta_0) dt & \leq n \sum_{j=1}^k J_j(+\infty) \cos^+(nt_j + \theta_0) - n \sum_{j=1}^k J_j(-\infty) \cos^-(nt_j + \theta_0) \\ & \quad - \sum_{j=1}^k Q_j(+\infty) \sin^+(nt_j + \theta_0) + \sum_{j=1}^k Q_j(-\infty) \sin^-(nt_j + \theta_0), \end{aligned}$$

a contradiction with the assumption (H_2) .

We can now choose an appropriate $R > 0$, such that $x - F_\lambda x \neq 0$, for $x \in X$ with $\|x\| = R$. Define $B_R = \{x \in X \mid \|x\| < R\}$. We say that $x(t) \in \partial B_R$ if there exists a $t_0 \in [0, 2\pi]$, such that $x(t_0) = R$.

Let $x_1(t) = r \cos \varphi$, $x_2(t) = r \sin \varphi$, we have

$$|r'(t)| = \left| \frac{x_1 x'_1 + x_2 x'_2}{\sqrt{x_1^2 + x_2^2}} \right| = \left| \frac{(1 - n^2 + (1 - \lambda)\delta)x_1(t)x_2(t) + \lambda p(t)x_2(t)}{\sqrt{x_1^2 + x_2^2}} \right| \leq n^2 r(t) + P,$$

where $P = \max_{t \in [0, 2\pi]} |p(t)|$, which implies that for sufficiently large $x_1^2 + x_2^2$, there exists a $B_P > 0$, such that

$$(r(t_0) + B_P)e^{-n^2(t-t_0)} \leq r(t) + B_P \leq (r(t_0) + B_P)e^{n^2(t-t_0)}, \quad t \in [t_0, t_0 + 2\pi].$$

This shows that $x(t_0) \rightarrow +\infty \Leftrightarrow \|x\| \rightarrow +\infty$.

To summarize: there exists a $B_R > 0$ such that for each $x(t) \in \partial B_R$, $x - F_\lambda x \neq 0$, $\lambda \in [0, 1]$. Now, by the Theorem 3 of [11], we have

$$\deg(I - F_L; B_R, 0) = \deg(I - F_0; B_R, 0) = (-1)^2 \deg_B(A_0|_{\mathbb{R}^2}; B_R \cap \mathbb{R}^2, 0),$$

where $\deg(I - F_0; B_R, 0)$ denotes the Leray–Schauder degree, and $\deg_B(A_0|_{\mathbb{R}^2}; B_R \cap \mathbb{R}^2, 0)$ denotes the Brouwer degree.

Let

$$A_0 x : \mathbb{R}^2 \rightarrow \mathbb{R}^2, \quad (x_1, x_2) \mapsto (x_2, -(n^2 - \delta)x_1),$$

where $x = (x_1, x_2)^T$.

It is easy to verify that $A_0 x$ has exactly one zero in $B_R \cap \mathbb{R}^2$. Therefore,

$$\deg_B(A_0|_{\mathbb{R}^2}; B_R \cap \mathbb{R}^2, 0) = \operatorname{sgn}(\det(A_0)) = \operatorname{sgn}(n^2 - \delta) = 1,$$

and consequently,

$$\deg(I - F_1; B_R, 0) = \deg(I - F_0; B_R, 0) = \deg_B(I - F_0; B_R, 0) = 1.$$

Hence, the original system has at least one 2π -periodic solution.

Finally, we have the following result.

Proposition 2.1. *Assume that (H_1) holds. Then (2.7) has a 2π -periodic solution if and only if $p(t)$ satisfies (H_2) .*

Remark 2.2. Suppose that

(H'_1) For $j = 1, \dots, k$, there exist limits

$$\lim_{s \rightarrow \pm\infty} J_j(s) = J_j(\pm\infty), \quad \lim_{s \rightarrow \pm\infty} Q_j(s) = Q_j(\pm\infty),$$

and the following inequalities hold:

$$J_j(-\infty) < J_j(s) < J_j(+\infty), \quad Q_j(+\infty) < Q_j(s) < Q_j(-\infty).$$

(H'_2) For any $\theta \in \mathbb{R}$, $p(t)$ satisfies

$$\begin{aligned} & n \sum_{j=1}^k J_j(-\infty) \cos^+(nt_j + \theta) - n \sum_{j=1}^k J_j(+\infty) \cos^-(nt_j + \theta) \\ & + \sum_{j=1}^k Q_j(+\infty) \sin^-(nt_j + \theta) - \sum_{j=1}^k Q_j(-\infty) \sin^+(nt_j + \theta) \\ & < \int_0^{2\pi} \sin(nt + \theta) p(t) dt \\ & < n \sum_{j=1}^k J_j(+\infty) \cos^+(nt_j + \theta) - n \sum_{j=1}^k J_j(-\infty) \cos^-(nt_j + \theta) \\ & - \sum_{j=1}^k Q_j(+\infty) \sin^+(nt_j + \theta) + \sum_{j=1}^k Q_j(-\infty) \sin^-(nt_j + \theta). \end{aligned}$$

Proposition 2.3. *Assume that (H'_1) holds. Then (2.7) has a 2π -periodic solution if and only if $p(t)$ satisfies (H'_2) .*

The proof of this proposition can be completed using a method similar to the one presented above.

2.3 The existence of unbounded solutions

This section is to prove that all solutions of (2.7) are unbounded when the periodic solution does not exist, namely

Proposition 2.4. Assume that (H_1) holds. There exists $\theta_0 \in \mathbb{R}$ such that $p(t)$ satisfies either

$$\begin{aligned} \int_0^{2\pi} p(t) \sin(nt + \theta_0) dt &\geq n \sum_{j=1}^k J_j(-\infty) \cos^+(nt_j + \theta_0) - n \sum_{j=1}^k J_j(+\infty) \cos^-(nt_j + \theta_0) \\ &\quad + \sum_{j=1}^k Q_j(+\infty) \sin^-(nt_j + \theta_0) - \sum_{j=1}^k Q_j(-\infty) \sin^+(nt_j + \theta_0), \end{aligned}$$

or

$$\begin{aligned} \int_0^{2\pi} p(t) \sin(nt + \theta_0) dt &\leq n \sum_{j=1}^k J_j(+\infty) \cos^+(nt_j + \theta_0) - n \sum_{j=1}^k J_j(-\infty) \cos^-(nt_j + \theta_0) \\ &\quad - \sum_{j=1}^k Q_j(+\infty) \sin^+(nt_j + \theta_0) + \sum_{j=1}^k Q_j(-\infty) \sin^-(nt_j + \theta_0). \end{aligned}$$

Then all solutions of (2.7) are unbounded (in the phase plane).

Proof. We denote by $u(t)$ a solution of (2.7). Setting $x(t) = (u(t), u'(t))^T$. Then for any $m \in \mathbb{N}$, by (2.4) we get

$$\begin{aligned} nu(2m\pi) &= nu(0) - \int_0^{2m\pi} p(t) \sin nt dt + n \sum_{j=1}^{mk} J_j(u'(t_j^-)) \cos nt_j - \sum_{j=1}^{mk} Q_j(u(t_j^-)) \sin nt_j, \\ u'(2m\pi) &= u'(0) + \int_0^{2m\pi} p(t) \cos nt dt + n \sum_{j=1}^{mk} J_j(u'(t_j^-)) \sin nt_j + \sum_{j=1}^{mk} Q_j(u(t_j^-)) \cos nt_j. \end{aligned}$$

Multiply the two equations above by $-\cos \theta_0, \sin \theta_0$, respectively, and add them together to get

$$\begin{aligned} &u'(2m\pi) \sin \theta_0 - nu(2m\pi) \cos \theta_0 \\ &= u'(0) \sin \theta_0 - nu(0) \cos \theta_0 + \int_0^{2m\pi} p(t) \sin(nt + \theta_0) dt \\ &\quad - n \sum_{j=1}^{mk} J_j(u'(t_j^-)) \cos(nt_j + \theta_0) + \sum_{j=1}^{mk} Q_j(u(t_j^-)) \sin(nt_j + \theta_0). \end{aligned}$$

Hence, we immediately deduce that

$$\begin{aligned} &|u'(2m\pi) \sin \theta_0 - nu(2m\pi) \cos \theta_0| \\ &\geq \left| \int_0^{2m\pi} p(t) \sin(nt + \theta_0) dt - n \sum_{j=1}^{mk} J_j(u'(t_j^-)) \cos(nt_j + \theta_0) + \sum_{j=1}^{mk} Q_j(u(t_j^-)) \sin(nt_j + \theta_0) \right| \\ &\quad - |u'(0) \sin \theta_0 - nu(0) \cos \theta_0|. \end{aligned}$$

Note that

$$\int_0^{2m\pi} p(t) \sin(nt + \theta_0) dt = m \int_0^{2\pi} p(t) \sin(nt + \theta_0) dt,$$

since $p(t) \sin(nt + \theta_0)$ is 2π -periodic.

Assume that the $u(t)$ is bounded. By (H_1) , for all $t_j, j = 1, 2, \dots$, we have

$$J_j(+\infty) < J_j(u'(t_j^-)) < J_j(-\infty), \quad Q_j(-\infty) < Q_j(u(t_j^-)) < Q_j(+\infty).$$

and

$$\xi_0 = \min \left\{ n \sum_{j=1}^k |J_j(\pm\infty) - J_j(u'(t_j^-))|, \sum_{j=1}^k |Q_j(\pm\infty) - Q_j(u(t_j^-))| \right\} > 0.$$

For $t \in [2l\pi, 2(l+1)\pi]$, $l = 1, 2, \dots, m-1$,

$$\begin{aligned} \int_0^{2\pi} \sin(nt + \theta_0) p(t) dt &= \int_{2l\pi}^{2(l+1)\pi} \sin(nt + \theta_0) p(t) dt \\ &\geq n \sum_{j=(l-1)k}^{lk} J_j(-\infty) \cos^+(nt_j + \theta_0) - n \sum_{j=(l-1)k}^{lk} J_j(+\infty) \cos^-(nt_j + \theta_0) \\ &\quad + \sum_{j=(l-1)k}^{lk} Q_j(+\infty) \sin^-(nt_j + \theta_0) - \sum_{j=(l-1)k}^{lk} Q_j(-\infty) \sin^+(nt_j + \theta_0) \\ &> n \sum_{j=(l-1)k}^{lk} J_j(u'(t_j^-)) \cos(nt_j + \theta_0) - \sum_{j=(l-1)k}^{lk} Q_j(u(t_j^-)) \sin(nt_j + \theta_0), \end{aligned}$$

it follows that

$$\int_0^{2\pi} p(t) \sin(nt + \theta_0) dt - n \sum_{j=(l-1)k}^{lk} J_j(u'(t_j^-)) \cos(nt_j + \theta_0) + \sum_{j=(l-1)k}^{lk} Q_j(u(t_j^-)) \sin(nt_j + \theta_0) > \xi_0.$$

Adding the m inequalities above, we get

$$m \int_0^{2\pi} p(t) \sin(nt + \theta_0) dt - n \sum_{j=1}^{mk} J_j(u'(t_j^-)) \cos(nt_j + \theta_0) + \sum_{j=1}^{mk} Q_j(u(t_j^-)) \sin(nt_j + \theta_0) > m\xi_0,$$

therefore,

$$|u'(2m\pi) \sin \theta_0 - nu(2m\pi) \cos \theta_0| \geq m\xi_0 - |u'(0) \sin \theta_0 - nu(0) \cos \theta_0|.$$

Since

$$n\|x(2m\pi)\|_X \geq n(u^2(2m\pi) + u'^2(2m\pi))^{\frac{1}{2}} \geq |u'(2m\pi) \sin \theta_0 - nu(2m\pi) \cos \theta_0|,$$

which yields

$$n\|x(2m\pi)\|_X \geq m\xi_0 - |u'(0) \sin \theta_0 - nu(0) \cos \theta_0|,$$

hence

$$n\|x(2m\pi)\|_X \rightarrow +\infty, \quad m \rightarrow +\infty,$$

which implies that $x(t)$ is unbounded in $[0, +\infty)$. $\square$

By combining Proposition 2.1 and Proposition 2.4, we have thus proved the Theorem 1.1.

3 Coexistence of periodic and unbounded solutions

In this section, we consider the coexistence of periodic and unbounded solutions of the following weakly nonlinear impulsive differential equations

$$\begin{cases} u''(t) + n^2 u(t) + h(u(t)) = p(t), & t \in [0, 2\pi], t \neq t_j; \\ \Delta u(t_j) = J_j(u(t_j^-), u'(t_j^-)), & \Delta u'(t_j) = Q_j(u(t_j^-), u'(t_j^-)), \quad j = 1, \dots, k; \end{cases} \quad (3.1)$$

where $0 < t_1 < \dots < t_k < 2\pi$, $p(t)$ is 2π -periodic and continuous, h is continuous and $J_j, Q_j: \mathbb{R}^2 \rightarrow \mathbb{R}$, $j = 1, \dots, k$, are continuous. Assume that $(D_1), (D_2)$ hold.

Let $P(t) = (0, p(t) - h(u(t)))^T$ and initial value $(u(\tau; \tau, w), u'(\tau; \tau, w)) = (0, w)$. We conclude from (2.4) that

$$\begin{aligned} u(t) &= \frac{w}{n} \sin(nt - n\tau) + \frac{1}{n} \int_{\tau}^t \sin(nt - ns)(p(s) - h(u(s)))ds \\ &\quad + \sum_{t > t_j} J_j(u(t_j^-), u'(t_j^-)) \cos(nt - nt_j) + \frac{1}{n} \sum_{t > t_j} Q_j(u(t_j^-), u'(t_j^-)) \sin(nt - nt_j), \\ u'(t) &= w \cos(nt - n\tau) + \int_{\tau}^t \cos(nt - ns)(p(s) - h(u(s)))ds \\ &\quad - n \sum_{t > t_j} J_j(u(t_j^-), u'(t_j^-)) \sin(nt - nt_j) + \sum_{t > t_j} Q_j(u(t_j^-), u'(t_j^-)) \cos(nt - nt_j). \end{aligned} \quad (3.2)$$

The proof of Theorem 1.4 is presented as following. We divide the proof in two steps. The first one is to prove the existence of the periodic solution using the successor map. The second step is to construct an auxiliary Lyapunov function that makes the solution grow infinitely along the orbit, which is motivated by [1]. So that the solution is unbounded.

3.1 The existence of a periodic solution

We assume that $h(u)$ is locally Lipschitzian which guarantees the uniqueness of the solutions for the initial value problem associated with (3.1), if necessary. Write $g(u) = n^2 u + h(u)$, $G(u) = \int_0^u g(s)ds$. Consider the following equation

$$\begin{cases} u' = v, \\ v' = -g(u) + \lambda p(t), \quad t \in [0, 2\pi], t \neq t_j; \\ \Delta u(t_j) = \lambda J_j(u(t_j^-), v(t_j^-)), \\ \Delta v(t_j) = \lambda Q_j(u(t_j^-), v(t_j^-)), \quad j = 1, \dots, k, \end{cases} \quad (3.3)$$

where $\lambda \in [0, 1]$. Notice that (3.3) is equivalent to (3.1) when $\lambda = 1$.

Let $(u(t; \tau, w), v(t; \tau, w))$ denote the solution of (3.3) with the initial point $(0, w)$, $w \gg 1$. We define τ_1 as the time when $(u(t; \tau, w), v(t; \tau, w))$ intersects the positive v -axis again, and write $w(\tau_1; \tau, w) = w_1$.

The successor map is defined as $S: (\tau, v) \mapsto (\tau_1, w_1)$. S is well defined for $|w| \gg 1$, and the uniqueness of the solution for the initial value problem guarantees that S is continuous and one to one. The periodicity of $p(t)$ implies that $S(\tau + 2\pi, w) = S(\tau, w) + (2\pi, 0)$.

By the boundedness of J_j, Q_j and h , we have

Lemma 3.1. $\tau_1 - \tau = \frac{2\pi}{n} + O(\frac{1}{r})$.

Now, let $u(t) = u(t; \tau, v)$, $v(t) = v(t; \tau, v)$, $H(t) = \frac{1}{2}v^2(t) + G(u(t))$, and $l(t) = \sqrt{2H(t)}$. Then we have

$$|l'(t)| = \left| \frac{v(t)}{l(t)} p(t) \right| \leq |p(t)| < P, \quad t \neq t_j, j = 1, \dots, k,$$

where $P = \max_{t \in [0, 2\pi]} \{|p(t)|\}$.

From this, it follows that

$$|l(t) - l(t')| \leq P|t - t'|,$$

when $t, t' \in (t_j, t_{j+1})$, $j = 1, \dots, k-1$, or $t, t' \in (0, t_1)$, or $t, t' \in (t_k, 2\pi)$.

When $t = t_j$, by (D_2) , $u(t_j^+) - u(t_j^-) = O(\frac{1}{r})$ and $v(t_j^+) - v(t_j^-) = O(\frac{1}{r})$ for large w . Then for sufficiently small $\varepsilon > 0$, there exists $w'_0 > 0$, such that for $w > w'_0$, $u(t_j^+) = u(t_j^-) \pm \varepsilon$, $v(t_j^+) = v(t_j^-) \pm \varepsilon$. Therefore,

$$\begin{aligned} H(t_j^+) &= \frac{1}{2}v^2(t_j^+) + G(u(t_j^+)) = \frac{1}{2}(v(t_j^-) \pm \varepsilon)^2 + G(u(t_j^-) \pm \varepsilon) \\ &= H(t_j^-) + \varepsilon_1, \end{aligned}$$

where $\varepsilon_1 = \pm v(t_j^-)\varepsilon + \int_{u(t_j^-)}^{u(t_j^+) \pm \varepsilon} g(s)ds + \varepsilon^2$.

In conclusion, for sufficiently large $|l(\tau)|$, we have

$$|l(t) - l(\tau)| \leq 2(P+1)|t - \tau|.$$

Thus, we obtain the following

Lemma 3.2.

$$|l(t) - w| \leq 2(P+1)T, \quad |t - \tau| \leq T, \quad w \geq w_0.$$

Further, we can prove the following lemma.

Lemma 3.3.

$$\begin{aligned} w_1 - w &= \int_{\tau}^{\tau + \frac{2\pi}{n}} \cos(ns - n\tau)p(s) + O\left(\frac{1}{r}\right), \quad w \gg 1; \\ \tau_1 - \tau &= \frac{\pi}{n} - \frac{1}{nw} \int_{\tau}^{\tau + \frac{2\pi}{n}} \sin(n\tau - ns)(h - p(s))ds \\ &\quad - \frac{1}{w} \lambda \sum_{t > t_j} J_j(u(t_j^-), u'(t_j^-)) \cos(n\tau_1 - nt_j) \\ &\quad - \frac{1}{nw} \lambda \sum_{t > t_j} Q_j(u(t_j^-), u'(t_j^-)) \sin(nt - nt_j) + O\left(\frac{1}{r^2}\right), \quad w \gg 1. \end{aligned}$$

Proof. Without loss of generality, we can assume that there are $k_1 \leq k$ impulses in the interval $[\tau, \tau_1]$. The corresponding impulse times are denoted by $t_1, t_2, \dots, t_{k_1}$.

Step 1. We estimate $w_1 - w$.

By the nature of definite integral, we have

$$w_1 - w = l(\tau_1) - l(\tau) = \int_{\tau}^{\tau_1} \frac{v(s)p(s)}{l(s)} ds + \sum_{j=1}^{k_1} l(t_j^+) - l(t_j^-).$$

We now begin by estimating $\sum_{j=1}^{k_1} l(t_j^+) - l(t_j^-)$. By definition, we have

$$\begin{aligned} l(t_j^+) - l(t_j^-) &= \sqrt{v^2(t_j^+) + 2G(u(t_j^+))} - \sqrt{v^2(t_j^-) + 2G(u(t_j^-))} \\ &= \frac{(v(t_j^+) + v(t_j^-))(v(t_j^+) - v(t_j^-)) + 2 \int_{u(t_j^-)}^{u(t_j^+)} g(s)ds}{\sqrt{v^2(t_j^+) + 2G(u(t_j^+))} + \sqrt{v^2(t_j^-) + 2G(u(t_j^-))}}. \end{aligned}$$

According to the mean value theorem for integrals and (i) of (D_1) , there exists M_j such that $\int_{u(t_j^-)}^{u(t_j^+)} h(s)ds = M_j(u(t_j^+) - u(t_j^-))$. Hence, we have

$$\int_{u(t_j^-)}^{u(t_j^+)} g(s)ds = \int_{u(t_j^-)}^{u(t_j^+)} n^2 u + h(s)ds = \left(\frac{1}{2} n^2 (u(t_j^+) + u(t_j^-)) + M_j \right) (u(t_j^+) - u(t_j^-)),$$

it follows that

$$\begin{aligned} \sum_{j=1}^{k_1} l(t_j^+) - l(t_j^-) &= \sum_{j=1}^{k_1} \frac{(v(t_j^+) + v(t_j^-)) \lambda Q_j(u(t_j^-), u'(t_j^-))}{\sqrt{v^2(t_j^+) + 2G(u(t_j^+))} + \sqrt{v^2(t_j^-) + 2G(u(t_j^-))}} \\ &\quad + \frac{2(\frac{1}{2} n^2 (u(t_j^+) + u(t_j^-)) + M_j) \lambda J_j(u(t_j^-), u'(t_j^-))}{\sqrt{v^2(t_j^+) + 2G(u(t_j^+))} + \sqrt{v^2(t_j^-) + 2G(u(t_j^-))}}. \end{aligned}$$

This implies that

$$\sum_{j=1}^{k_1} l(t_j^+) - l(t_j^-) = O\left(\frac{1}{r}\right), \quad w \gg 1, \quad (3.4)$$

by (D_2) .

Combining (3.4), Lemma 3.1 and Lemma 3.2, the further estimate for $w_1 - w$ is given by

$$\begin{aligned} w_1 - w &= \int_{\tau}^{\tau_1} \frac{v(s)p(s)}{l(s)} ds + O\left(\frac{1}{r}\right) \\ &= \int_{\tau}^{\tau_1} \left(w \cos(ns - n\tau) + \int_{\tau}^s (\cos(ns - n\xi)(p(\xi) - h)) d\xi \right) \frac{p(s)}{l(s)} ds + O\left(\frac{1}{r}\right) \\ &= \int_{\tau}^{\tau_1} \frac{w \cos(ns - n\tau)p(s)}{l(s)} ds + O\left(\frac{1}{r}\right) \\ &= \int_{\tau}^{\tau + \frac{2\pi}{n}} \cos n(s - \tau) p(s) ds + O\left(\frac{1}{r}\right), \quad w \gg 1. \end{aligned}$$

Step 2. The estimation of $\tau_1 - \tau$ is given below.

By the definition of τ_1 and (3.2), we have

$$\begin{aligned} \frac{w}{n} \sin(n\tau_1 - n\tau) &= \frac{1}{n} \int_{\tau}^{\tau_1} \sin(n\tau_1 - ns)(h - p(s)) ds \\ &\quad - \lambda \sum_{t > t_j} J_j(u(t_j^-), u'(t_j^-)) \cos(n\tau_1 - nt_j) \\ &\quad - \frac{\lambda}{n} \sum_{t > t_j} Q_j(u(t_j^-), u'(t_j^-)) \sin(nt - nt_j). \end{aligned}$$

Using the Taylor expansion, we obtain

$$\begin{aligned} \frac{w}{n} \sin(n\tau_1 - n\tau) &= \frac{w}{n} \sin n \left(\frac{\pi}{n} - \tau_1 + \tau \right) \\ &= w \left(\frac{\pi}{n} - \tau_1 + \tau + O\left(\frac{1}{r^3}\right) \right). \end{aligned}$$

It follows from Lemma 3.1 that

$$\begin{aligned} \tau_1 - \tau &= \frac{\pi}{n} - \frac{1}{nw} \int_{\tau}^{\tau + \frac{2\pi}{n}} \sin(n\tau - ns)(h - p(s)) ds - \frac{1}{w} \lambda \sum_{t > t_j} J_j(u(t_j^-), u'(t_j^-)) \cos(n\tau_1 - nt_j) \\ &\quad - \frac{1}{nw} \lambda \sum_{t > t_j} Q_j(u(t_j^-), u'(t_j^-)) \sin(nt - nt_j) + O\left(\frac{1}{r^2}\right), \quad w \gg 1. \end{aligned}$$

The proof of Lemma 3.2 is then completed. $\square$

Let

$$\begin{aligned}\phi(t - \tau) &= \frac{1}{n} \sin(nt - n\tau), \\ \psi(t) &= \frac{1}{n} \int_{\tau}^t \sin(nt - ns)(p(s) - h)ds + \sum_{t > t_j} J_j(u(t_j^-), u'(t_j^-)) \cos(nt - nt_j) \\ &\quad + \frac{1}{n} \sum_{t > t_j} Q_j(u(t_j^-), u'(t_j^-)) \sin(nt - nt_j).\end{aligned}$$

It is easy to observe that $\psi(t)$ is bounded by (D_1) and (D_2) .

Lemma 3.4. Assume that (ii) of (D_1) holds. For any $K > 0$, there exists $w_0 = w(K) > 0$, such that

$$\int_{\tau}^{\tau + \frac{2\pi}{n}} h(w\phi(s - \tau) + \psi(s)) \sin n(s - \tau) ds \geq \frac{K}{w}, \quad w \geq w_0.$$

Proof. Without loss of generality, we take $\tau = 0$. It is sufficient to prove the inequality:

$$\int_0^{\frac{2\pi}{n}} h\left(\frac{\omega}{n} \sin ns + \psi(s)\right) \sin ns ds \geq \frac{K}{w}, \quad w \geq w_0.$$

By definition of ψ , $\psi(t)$ is bounded and $\psi(t) \rightarrow 0$ as $t \rightarrow 0$.

By condition (D_2) , there exists $\delta = O(\frac{1}{r}) > 0$, such that, for $\frac{2\pi}{n} > t > \frac{2\pi}{n} - \delta$, $h(\frac{\omega}{n} \sin nt + \psi(t)) \sin nt > 0$, which together with (D_1) yields

$$\int_0^{\frac{2\pi}{n}} h\left(\frac{\omega}{n} \sin ns + \psi(s)\right) \sin ns ds \geq \int_{\delta}^{\frac{2\pi}{n} - \delta} h\left(\frac{\omega}{n} \sin ns + \psi(s)\right) \sin ns ds - O\left(\frac{1}{r}\right).$$

Introducing the transformation $\frac{\omega}{n} \sin nt + \psi(t) = \frac{\omega}{n} \sin n\tilde{\xi}$. Then, there exists $\omega \gg 1$, such that

$$h\left(\frac{\omega}{n} \sin n\tilde{\xi}\right) \frac{\omega}{n} \sin n\tilde{\xi} \geq 4M_h.$$

Therefore,

$$\begin{aligned}\int_{\delta}^{\frac{2\pi}{n} - \delta} h\left(\frac{\omega}{n} \sin ns + \psi(s)\right) \sin ns ds &= \frac{1}{\omega} \int_{\tilde{\xi}^-}^{\tilde{\xi}^+} h\left(\frac{\omega}{n} \sin n\tilde{\xi}\right) \frac{\omega \sin n\tilde{\xi} - n\psi(t)}{\omega \cos nt + \psi'(t)} \omega \cos n\tilde{\xi} d\tilde{\xi} \\ &\geq \frac{n}{4\omega} \int_{\tilde{\xi}^-}^{\tilde{\xi}^+} h\left(\frac{\omega}{n} \sin n\tilde{\xi}\right) \frac{\omega}{n} \sin n\tilde{\xi} \cos n\tilde{\xi} d\tilde{\xi} \\ &\geq \frac{\bar{M}M_h}{\omega} := \frac{K}{\omega}\end{aligned}$$

where $\bar{M} = \int_{\frac{\pi}{8}}^{\frac{3\pi}{8}} \cos n\tilde{\xi} d(n\tilde{\xi})$. $\square$

Introduce the Banach space

$$\begin{aligned}X = \{x : [0, 2\pi] \rightarrow \mathbb{R}^2 : x(t) \text{ is continuous for } t \neq t_j, x(t_j^-), x(t_j^+) \text{ exist,} \\ \text{and } x(t_j^-) = x(t_j^+) j = 1, \dots, k, x(0) = x(2\pi)\},\end{aligned}$$

and the operator $F_\lambda : X \rightarrow X$

$$F_\lambda x = x(0) + \frac{2\pi - t}{2\pi} \left(\int_0^{2\pi} (\lambda P(t) + Ax(t) - H(x(t))) dt + \lambda \sum_{j=1}^k I_j(x(t_j^-)) \right) \\ + \int_0^t (\lambda P(s) + Ax(s) - H(x(s))) ds + \lambda \sum_{t > t_j} I_j(x(t_j^-)),$$

where $H(x(t)) = (0, h(u(t)))^T$. According to the above we can know F_λ is completely continuous.

It is clear that (3.3) is equivalent to a fixed point problem, that is $x = F_\lambda x$. We now apply the topological degree to prove the existence of periodic solutions. Let

$$\Sigma = \{(x, \lambda) \in X \times [0, 1] \mid F_\lambda x = x\}.$$

Lemma 3.5. *If there is a bounded open set $\Omega \subset X$, such that*

$$\Sigma \cap \partial\Omega = \emptyset,$$

then $\deg(F_\lambda, \Omega, 0) = \deg(F_0, \Omega, 0) = 1$, and thus, the equation (3.3) has at least one 2π -periodic solution.

Similar to the proof of proposition 2.1, we have $\deg(F_0, \Omega, 0) = 1$. We now seek an appropriate bounded open set. Let

$$\Omega(k) = \left\{ (u, v) \in X \mid \omega(u, v) = \frac{1}{2}v^2(t) + G(u(t)) < k \right\}.$$

On the one hand, $(u(t), v(t)) \in \Sigma \cap \partial(\Omega(k))$, for sufficiently large k , then there exists $t_0 \in [0, 2\pi]$, such that $\frac{1}{2}v^2(t_0) + G(u(t_0)) = k$. By Lemma 3.2, $(u(t), v(t)) \in \mathbb{R}^2$ is in a bounded set by

$$\left\{ (u, v) \in X \mid k_- < \frac{1}{2}v^2(t) + G(u(t)) < k_+ \right\},$$

where $\sqrt{k_\pm} = \sqrt{k} \pm 4(P+1)\pi$. Thus $(u(t), v(t))$ starts from $u(\tau) = 0, v(\tau) = w, \tau \in [0, 2\pi]$, $w \in [k_-, k_+]$. The 2π -periodicity implies that $u(\tau + 2\pi) = 0, v(\tau + 2\pi) = w$. Thus,

$$\tau_m = \tau + 2\pi,$$

for some $m \in \mathbb{N}$.

On the other hand, by using the variation of constant formula, we obtain $u(t) = w\phi(t) + \psi(t)$. Combining this with Lemma 3.4, we have

$$\tau_n - \tau = 2\pi - \frac{1}{nw} \int_\tau^{\tau+2\pi} \sin(n\tau_1 - ns)(h - p(s)) ds + O\left(\frac{1}{w^2}\right) \\ \leq 2\pi - \frac{K}{w^2} + \frac{M'_p}{w} + O\left(\frac{1}{w^2}\right) < 2\pi,$$

where K is sufficiently large, M'_p is sufficiently small, such that for some k ,

$$\frac{K}{w^2} - \frac{M'_p}{w} - O\left(\frac{1}{w^2}\right) > 0,$$

for $w \in [k_-, k_+]$.

Thus, we find that the bounded open set $\Omega(k)$ satisfies the condition of Lemma 3.5 and we have

Proposition 3.6. Assume that (D_1) and (D_2) hold. Then there exists $q_0 > 0$, such that the equation (1.5) has 2π -periodic solution provided that $p(t)$ satisfies

$$M_p < q_0,$$

where

$$M_p = \max_{\theta \in [0, 2\pi]} \left| \int_0^{2\pi} p(t) \sin(nt + \theta) dt \right|.$$

3.2 The existence of unbounded solutions

Let us continue our analysis about the above equation. Given a function $p(t)$, we shown in the previous section that there exists 2π -periodic solution with

$$\max \left\{ \int_0^{2\pi} p(t) \sin(nt + \phi) dt : \phi \in \mathbb{R} \right\} > 0,$$

within a sufficiently large disk in the phase plane. Accordingly, we fix some $\phi \in \mathbb{R}$ such that

$$d_* := \frac{1}{2} \int_0^{2\pi} p(t) \sin(nt + \phi) dt > 0.$$

Let $(u(t; \zeta), u'(t; \zeta))$ be the solution of (3.1) with the initial condition $\zeta = (\omega, v) \in \mathbb{R}^2$. The associated Poincaré map is defined by $\mathcal{P} : (\omega, v) \rightarrow (u(2\pi; \zeta), u'(2\pi; \zeta))$. To investigate the existence of unbounded solutions, we introduce the auxiliary function

$$V : \mathbb{R}^2 \rightarrow \mathbb{R}, \quad V(\zeta) := v \sin \phi - n\omega \cos \phi.$$

First, we verify the following property of V : there exists a constant $r_* > 0$, such that, for $|\zeta| \geq r_*$,

$$V(\mathcal{P}(\zeta)) \geq V(\zeta) + d_*. \quad (3.5)$$

Multiplying both sides of (3.1) by $\sin(nt + \phi)$ and integrating both sides at $[0, 2\pi]$, we get

$$\begin{aligned} \int_0^{2\pi} u''(t; \zeta) \sin(nt + \phi) dt &= \int_0^{2\pi} p(t) \sin(nt + \phi) dt - n^2 \int_0^{2\pi} u(t; \zeta) \sin(nt + \phi) dt \\ &\quad - \int_0^{2\pi} h(u(t; \zeta)) \sin(nt + \phi) dt. \end{aligned}$$

Using integration by parts we obtain

$$\begin{aligned} \int_0^{2\pi} u''(t; \zeta) \sin(nt + \phi) dt &= V(F(\zeta)) - V(\zeta) + n \sum_{j=1}^k J_j(u(t_j^-), u'(t_j^-)) \cos(nt_j + \phi) \\ &\quad - \sum_{j=1}^k Q_j(u(t_j^-), u'(t_j^-)) \sin(nt_j + \phi) - n^2 \int_0^{2\pi} u(t; \zeta) \sin(nt + \phi) dt. \end{aligned}$$

It follows that

$$\begin{aligned} V(F(\zeta)) - V(\zeta) &= \int_0^{2\pi} p(t) \sin(nt + \phi) dt - \int_0^{2\pi} h(u(t; \zeta)) \sin(nt + \phi) dt \\ &\quad - n \sum_{j=1}^k J_j(u(t_j^-), u'(t_j^-)) \cos(nt_j + \phi) + \sum_{j=1}^k Q_j(u(t_j^-), u'(t_j^-)) \sin(nt_j + \phi). \end{aligned}$$

We now estimate the second term on the right-hand side of the above equation.

Lemma 3.7. *There exists a sufficiently large $r_* > 0$, such that $\zeta = (\omega, v) > r_*$ satisfying*

$$\left| \int_0^{2\pi} h(u(t; \zeta)) \sin(nt + \phi) dt \right| < \frac{d_*}{2}. \quad (3.6)$$

Proof. By (D_1) , there exists $u_* > 0$, such that for $|u(t; \zeta)| > u_*$, $|h(u(t; \zeta))| < \frac{d_*}{8\pi}$. Denote the upper bound of $|h|$ by M_h .

We take the transformation

$$u(t; \zeta) = r(t; \zeta) \cos \theta(t; \zeta), \quad u'(t; \zeta) = r(t; \zeta) \sin \theta(t; \zeta).$$

For $r(t; \zeta) \geq 1$, we have

$$|r'(t; \zeta)| \leq (n^2 + M_h)r(t; \zeta), \quad \frac{1}{2} \leq \theta'(t; \zeta) \leq (n^2 + M_h), \quad t \neq t_1, \dots, t_k.$$

Analysis similar to that in the proof of Lemma 3.1 shows that, for $t \in [0, 2\pi], t \neq t_1, \dots, t_k$,

$$e^{-2(n^2 + M_h)\pi} r(0; \zeta) \leq r(t; \zeta) \leq e^{2(n^2 + M_h)\pi} r(0; \zeta), \quad (3.7)$$

$$-2(n^2 + M_h)\pi < \theta(2\pi; \zeta) - \theta(0; \zeta) < 0. \quad (3.8)$$

Let $\delta_* > 0$ be sufficiently small such that $|\cos \theta(t; \zeta)| \leq \cos(\frac{\pi}{2} - \delta_*)$ for $\theta(t; \zeta) \in [m\pi + \frac{\pi}{2} - \delta_*, m\pi + \frac{\pi}{2} + \delta_*], m \in \mathbb{Z}$. Define

$$E_* = \left\{ t \in [0, 2\pi] : \theta(t; \zeta) \in \left[m\pi + \frac{\pi}{2} - \delta_*, m\pi + \frac{\pi}{2} + \delta_* \right], m \in \mathbb{Z} \right\}.$$

By (3.8), we have $\text{mes}(E_*) \leq 8(n^2 + M_h)\delta_*$ and

$$|u(t; \zeta)| \geq r(t; \zeta) \cos\left(\frac{\pi}{2} - \delta_*\right), \quad t \in [0, 2\pi] \setminus E_*.$$

By (3.7), we take

$$\delta_* = \frac{d_*}{32(n^2 + M_h)M_h}, \quad r = |\zeta| \geq r_* = \frac{u_* e^{2(n^2 + M_h)\pi}}{\cos(\frac{\pi}{2} - \delta_*)},$$

then

$$\begin{aligned} & \left| \int_0^{2\pi} h(u(t; \zeta)) \sin(nt + \phi) dt \right| \\ & \leq \left| \int_{t \in E_*} h(u(t; \zeta)) \sin(nt + \phi) dt \right| + \left| \int_{t \in [0, 2\pi] \setminus E_*} h(u(t; \zeta)) \sin(nt + \phi) dt \right| \\ & \leq \text{mes}(E_*)M_h + 2\pi \cdot \frac{d_*}{8\pi} \leq \frac{d_*}{4} + \frac{d_*}{4} = \frac{d_*}{2}. \end{aligned} \quad \square$$

On the other hand, from (D_2) , it follows that

$$\left| -n \sum_{j=1}^k J_j(u(t_j^-), u'(t_j^-)) \cos(nt_j + \phi) + \sum_{j=1}^k Q_j(u(t_j^-), u'(t_j^-)) \sin(nt_j + \phi) \right| \leq \frac{d_*}{2},$$

for $r(t; \zeta)$ large enough.

Combining (3.7) and the boundedness of impulses, we can take $|\zeta| \geq r_*$ large enough such that the above inequality holds. Then

$$V(\mathcal{P}(\zeta)) - V(\zeta) \geq 2d_* - \frac{d_*}{2} - \frac{d_*}{2} = d_*.$$

Furthermore, we notice that V is bounded above on the bounded set $|\zeta| \leq r_*$, i.e.

$$\overline{V}_* := \sup\{V(\zeta) : |\zeta| \leq r_*\} < +\infty.$$

Write $r^* = n\overline{V}_*^{-1/2}e^{4(n^2+M_h)\pi}$. For $|\zeta_0| \geq r^*$ and $\zeta_0 = (-\cos\phi, \sin\phi)|\zeta_0|$, we can see (3.5) holds. It follows that

$$V(\mathcal{P}(\zeta_0)) \geq V(\zeta_0) + d_* > V(\zeta_0) = \cos^2\phi + n\sin^2\phi > \overline{V}_*,$$

hence that $|\mathcal{P}(\zeta_0)| \geq r_*$ by the definition of $\overline{V}_*$.

Then let us denote by $\zeta_1 = \mathcal{P}(\zeta_0)$ the new initial point. Likewise,

$$V(\mathcal{P}(\zeta_1)) \geq V(\zeta_1) + d_* \geq V(\zeta_0) + 2d_*.$$

Similar arguments apply to the case $m \in \mathbb{N}$. Then we have

$$V(\mathcal{P}(\zeta_{m+1})) \geq V(\zeta_m) + d_* \geq V(\zeta_0) + (m+1)d_*,$$

which implies that $\zeta_m \rightarrow \infty$, $m \rightarrow \infty$. If otherwise, then this contradicts the fact that V is bounded above on the bounded set. From this, we conclude that if $|\zeta_0| \geq r^*$ and $\zeta_0 = (-\cos\phi, \sin\phi)|\zeta_0|$, then $(u(t; \zeta_0), u'(t; \zeta_0))$ is an unbounded solution of (3.1).

If $|\zeta_0| \geq r^*$ but $\zeta_0 \neq (-\cos\phi, \sin\phi)|\zeta_0|$, by $\theta'(t; \zeta_0) \geq \frac{1}{2}$ and $r^* = n\overline{V}_*^{-1/2}e^{4(n^2+M_h)\pi}$, then we find a $\tau_0 \in [0, 4\pi)$ such that $(u(\tau_0; \zeta_0), u'(\tau_0; \zeta_0)) = (-\cos\phi, \sin\phi)r(\tau_0; \zeta_0)$ and $r(\tau_0; \zeta_0) \geq r^*$. Similarly, let $\tilde{\zeta}_m = \mathcal{P}^m(\tilde{\zeta}_0)$. By the same method as shown before, we have $\tilde{\zeta}_m \rightarrow \infty$, $m \rightarrow \infty$, which implies that $(u(t; \zeta_0), u'(t; \zeta_0))$ is also an unbounded solution. Therefore, we have the following assertion.

Proposition 3.8. *Assume that (D_1) and (D_2) hold. Then there exists $q_0 > 0$, such that the equation (3.1) has an unbounded solution provided that $p(t)$ satisfies*

$$0 < M_p < q_0.$$

Combining Proposition 3.6 and Proposition 3.8, we have thus proved the Theorem 1.4.

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