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Novel estimates of the impact of hostility on the equilibrium stability in an arms race model

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Abstract. This article proposes a non-autonomous mathematical model of confrontation between two countries and examines the influence of the factor of hostility between countries on the stability of the equilibrium state. Namely, conditions for stability (asymptotic stability) at zero hostility have been established; an estimate was found for the total level of armament of the two countries, provided that the zero equilibrium state is stable in the linear approximation of the model; conditions for stability under parametric disturbances in the linear approximation of the model are established. For a non-autonomous quasi-linear model of confrontation between two countries, a new estimate of the deviation of the weapons vector from the equilibrium ray is obtained and stability conditions are indicated.

Keywords: non-autonomous mathematical models, arms race, hostility, stability, equilibrium.

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1 Introduction

Mathematical modeling of the arms race process goes back to the original work of Richardson [20] published in 1960. Over the past period of time, many researchers have developed the Richardson model or proposed alternative models. See, for example, [1,3,5,18,21–25,29,31] and the references therein.

These works are of theoretical importance, since they contain general conditions for which arms race can be controlled (stable) or can lead to real conflict between opposing countries. It is not by chance that the authors in [6] discussed the "responsibility of mathematicians and mathematics educators for achieving peace and dignity for mankind via mathematics (research and development) and mathematics education".

The question of reinterpreting and improving the mathematical models of the arms race has become very relevant recently against the background of the increasing number of conflicts between two countries. In fact, the relevance of research in the field of stability of the

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equilibrium of opposing countries and their alliances increases significantly, including analysis of the strategy of countries that own atomic weapons. This motivate numerous researchers to extend the existing models and advance the theory. For example, in [4] mathematical models of confrontation between two and n countries, including countries with nuclear weapons, have been proposed and analyzed. As a result of the analysis, stability conditions of the equilibrium state of the warring countries are established, and also the influence of hostility on the decrease (increase) in the norm of the vector of weapons n countries participating in alliances is considered. The edited by Gleditsch book [8] presents a very good overview of the recent research directed on the development of the Richardson arms race models. The paper [19] applied the fractional-order approach to extend the classical arms race model.

Most of the above papers investigated the stability behavior of the corresponding models. In fact, stability analysis is a fundamental approach in a wide range of fields in control theory and the qualitative theory of differential equations [11, 13, 26, 28]. This analysis is of a significant importance in the study of the patterns of many real world phenomena and opens the possibilities of additional knowledge about their behavior [7, 9, 10, 27].

One of the more strong and effective approaches applied in the stability analysis of systems is the second method of Lyapunov or the method of Lyapunov-like functions [2, 12, 30]. This interesting and fruitful technique is gaining more and more importance and gives a decisive impetus for the modern development of the stability theory of differential equations and their applications.

The Lyapunov-like function method has been applied in the recent paper [4] to establish conditions for the stability of the zero equilibrium state of a quasi-linear mathematical model of the confrontation between two countries.

The aim of this paper is to analyze the influence of the level of hostility between two opposing countries on the stability of the equilibrium state based on a non-autonomous quasi-linear model that generalizes the Richardson arms race model. In the article [4], such an analysis was started based on a quasi-linear model with an autonomous linear approximation.

In this case, the factor of hostility between opposing countries is filled with expanded content. In addition, it is noted that hostility between countries is formed by the media in order to control public opinion about the arms race of opposing countries. Thus, the dynamic properties of the model proposed in the article are rich and meaningful, which can motivate its future analysis, as well as the application of the obtained results in the study of arms race processes.

The analysis of the proposed mathematical model of confrontation is carried out using the Lyapunov function method and nonlinear integral inequalities [14–17].

The paper is arranged using the following agenda. Section 1 provides a generalized non-autonomous mathematical model of confrontation between two countries. Section 2 discusses the stability of the equilibrium state under zero hostility and under parametric perturbations of the weapon matrix. Section 3 discusses the mathematical model of confrontation with neutral stability of the linear approximation of the original Richardson's model of confrontation between countries. Section 4 examines the general problem of stability in a generalized Richardson's arms race model. In Section 5, for the non-autonomous quasi-linear model of confrontation between two countries conditions for the stability of weapon vector evasion from the beam balance of arms are established. The final Section 6 provides comments on the results obtained. Some future directions for research are also discussed.

2 Generalized Richardson model of arms race

The classical Richardson's model of a two country arms race is given by a system of differential equations:

$$\frac{dx}{dt} = ky - ax + g, \qquad \frac{dy}{dt} = lx - by + h, \tag{2.1}$$

where $x(t)$ and $y(t)$ represents the armament expenditures (or armament levels) of two countries (nations) at time t, $\frac{dx}{dt}$ or $\dot{x}$ is the rate of change in expenditures over time for the country N_1, and $\frac{dy}{dt}$ or $\dot{y}$ is the rate of change in expenditures over time for the country N_2, k, l, a, b are positive constants and g, h are real constants. The constants k and l describe mutual fear; the constants a and b measure proportionality factors for the "internal brakes" to further arms increases (economic burden parameters, called "fatigue and expense" initially by Richardson). Positive values for g and h correspond to underlying factors of ill will or distrust that would persist even if arms expenditures dropped to zero. Negative values for g and h represent a contribution based on good will [20].

The model (2.1) has been generalized and modified in different directions. In [4], the following generalization is studied:

$$\frac{dx}{dt} = ky - ax + g(t,x,y), \qquad \frac{dy}{dt} = lx - by + h(t,x,y), \tag{2.2}$$

where $t \in \mathbb{R}_\tau$, τ is a finite number or the symbol $+\infty$, $x, y \in \mathbb{R}_+$ represent measures of the levels of the countries' N_1 and N_2 armament, correspondingly, the parameters $k > 0$ and $l > 0$ describe the "threat" between the countries, parameters $a > 0$, $b > 0$ characterize the "expenses" of each from countries to maintain the level of their armament. The functions $g(t,x,y)$ and $h(t,x,y)$ characterize the factor of "hostility" between the countries N_1 and N_2, which depends on the level of public opinion about the confrontation between countries and their armament.

In this paper, we will extend the model (2.2) considering the non-autonomous quasi-linear system of differential equations:

$$\begin{aligned} \frac{dx}{dt} &= k(t)y - a(t)x + g(t,x,y), \qquad x(t_0) = x_0, \\ \frac{dy}{dt} &= l(t)x - b(t)y + h(t,x,y), \qquad y(t_0) = y_0, \end{aligned} \tag{2.3}$$

where $t_0 \in \mathbb{R}_\tau$, all the parameters $k(t) > 0$, $l(t) > 0$, $a(t) > 0$ and $b(t) > 0$ are supposed to be bounded for all $t \in \mathbb{R}_\tau$, and the rest of the parameters are as in [4].

Remark 2.1. The proposed model (2.3) extends and improve the classical Richardson's model [20] and numerous related models [1, 3, 5, 8, 25, 31] in which all the parameters $k(t)$, $l(t)$, $a(t)$ and $b(t)$ are positive constants and the hostility functions $g(t,x,y)$ and $h(t,x,y)$ are real constants, too. The model (2.3) extends also the recent model proposed in [4] in which the parameters $k(t)$, $l(t)$, $a(t)$ and $b(t)$ are positive constants.

Remark 2.2. If the countries N_1 and N_2 possess nuclear weapons, then the parameters $k(t)$ and $l(t)$ can take arbitrarily large values for any $t \in \mathbb{R}_\tau$ while remaining bounded.

From a practical point of view, we will also assume that hostility between countries N_1 and N_2 does not exceed some threshold level $\bar{b} = (\bar{b}_1, \bar{b}_2)^T$ on any finite time interval at any level of armament of the opposing countries. Thus, for the components of the hostility vector function $(g(t,x,y), h(t,x,y))^T$ there exist constants $\bar{b}_1 > 0$ and $\bar{b}_2 > 0$ such that

$$|g(t,x,y)| \leq \bar{b}_1, \qquad |h(t,x,y)| \leq \bar{b}_2 \tag{2.4}$$

for all $t \in \mathbb{R}_\tau$ and $x, y \in \mathbb{R}_+$.

Remark 2.3. As noted in [4], limit values of permissible hostility $\bar{b}_1$ and $\bar{b}_2$ between the countries N_1 and N_2 should be under the control of the civil society of these countries and/or international organizations, given that modern means of influence on the human psyche have reached the level of criminality.

3 Stability conditions for time-zero hostility

In this section, together with system (2.3) we will consider the following system

$$\frac{dx}{dt} = k(t)y - a(t)x, \qquad \frac{dy}{dt} = l(t)x - b(t)y, \tag{3.1}$$

with zero hostility $g(t,x,y) = 0$ and $h(t,x,y) = 0$ for all $t \in \mathbb{R}_\tau$ and $x, y \in \mathbb{R}_+$.

It is clear that $(x,y)^T = (0,0)^T$ is an equilibrium for the model (3.1).

Proposition 3.1. *If the model's parameters $k(t), a(t), l(t), b(t)$ in (3.1) are such that:*

(H_1) $$-a(t) \leq 0 \quad \text{for all } t \in \mathbb{R}_\tau,$$

(H_2) $$a(t)b(t) - (k(t) + l(t))^2 \geq 0 \quad \text{for all } t \in \mathbb{R}_\tau,$$

then its zero equilibrium $(x,y)^T = (0,0)^T$ is stable.

Proof. Consider a Lyapunov-like function

$$V(x,y) = \frac{1}{2}\left(x^2 + y^2\right).$$

For the derivative of the above function with respect to the system (3.1), we have

$$\frac{d}{dt}V(x,y) = \frac{dx}{dt}x + \frac{dy}{dt}y = x\left(k(t)y - a(t)x\right) + y\left(l(t)x - b(t)y\right). \tag{3.2}$$

Conditions (H_1) and (H_2) of Proposition 3.1 imply that $\frac{d}{dt}V(x,y)$ is non-negative, i.e., the zero equilibrium $(x,y)^T = (0,0)^T$ is stable by Lyapunov [12,30]. □

Proposition 3.2. *If for the model's parameters $k(t), a(t), l(t), b(t)$ in (3.1) there exist positive constants α, β such that:*

(H_3) $$-a(t) \leq -\alpha \quad \text{for all } t \in \mathbb{R}_\tau,$$

(H_4) $$a(t)b(t) - (k(t) + l(t))^2 \geq \beta \quad \text{for all } t \in \mathbb{R}_\tau,$$

then its zero equilibrium $(x,y)^T = (0,0)^T$ is asymptotically stable.

Proof. If we apply again the Lyapunov-like function

$$V(x,y) = \frac{1}{2}\left(x^2 + y^2\right),$$

its derivative with respect to the system (3.1) will be strictly negative under the conditions (H_3) and (H_4) of Proposition 3.2, which guarantees the asymptotic stability (by Lyapunov) of the zero equilibrium $(x,y)^T = (0,0)^T$ [12,30]. □

Next, we will rewrite system (3.1) in the matrix form $\frac{dw}{dt} = A(t)w$ and consider the initial value problem (IVP)

$$\frac{dw}{dt} = A(t)w, \;\; w(t_0) = w_0, \tag{3.3}$$

where the initial time $t_0 \in \mathbb{R}_\tau$, $w = (x,y)^T$ and $A(t)$ is a (2×2) bounded weapon matrix. We will study the stability of the equilibrium state $(x,y)^T = (0,0)^T$ of (3.3) for parametric perturbations of the weapons matrix. The initial value $w(t_0)$ denotes the levels of the two countries' armament at time $t = t_0$, the beginning of the interaction between the two actors.

Let at the moment $t = t^* \in \mathbb{R}_\tau$ the matrix $A(t^*)$ has an eigenvalue λ, that satisfies $Re\lambda \leq -\alpha$, i.e., the equilibrium state $(x,y)^T = (0,0)^T$ of (3.3) with the weapon matrix $A(t^*)$ is asymptotically stable.

System (3.3) can be represented in the form

$$\frac{dw}{dt} = A(t^*)w + \Delta A(t)w, \;\; w(t^*) = w^*, \tag{3.4}$$

where we denote $\Delta A(t) = A(t) - A(t^*)$.

For system (3.4) we will consider a Lyapunov-like function $V(w) = w^T w$ and the following conditions:

(H_5) $$w^T \left(A^T(t^*) + A(t^*)\right) w \leq -\lambda_M w^T w \quad \text{for all } t \in \mathbb{R}_\tau;$$

(H_6) $$w^T \left(\Delta A^T(t) + \Delta A(t)\right) w \leq \mu_M(t) w^T w \quad \text{for all } t \in \mathbb{R}_\tau,$$

where λ_M is the greatest eigenvalue of the matrix $A^T(t^*) + A(t^*)$ and $\mu_M(t)$ is the greatest eigenvalue of the matrix $\Delta A^T(t) + \Delta A(t)$ for all $t \in \mathbb{R}_\tau$.

Proposition 3.3. *If conditions (H_5) and (H_6) are satisfied for the model's parameters in (3.4), and for any $\varepsilon > 0$ there exists a $\delta = \delta(t^*, \varepsilon) > 0$ such that:*

$$\exp\left(-\lambda_M(t - t^*) + \int_{t^*}^{t} \mu_M(s)ds\right) < \frac{\varepsilon}{\delta} \;\; \textit{for all} \;\; t \in \mathbb{R}_\tau,$$

then the zero equilibrium $w = 0$ of (3.4) is stable.

Proof. For the total derivative of the positive definite Lyapunov function $V(w) = w^T w$ with respect to system (3.4) under the hypotheses (H_5) and (H_6), we have

$$\frac{d}{dt}V(w(t)) \leq (-\lambda_M + \mu_M(t))V(w(t)) \;\; \text{for all} \;\; t \in \mathbb{R}_\tau. \tag{3.5}$$

From estimate (3.5) it follows that

$$V(w(t)) \leq V(w((t^*)) \exp\left(\int_{t^*}^{t} (-\lambda_M + \mu_M(s))ds\right) \;\; \text{for all} \;\; t \in \mathbb{R}_\tau.$$

Hence, if the conditions of Proposition 3.3 are met, then the equilibrium state $w = 0$ of system (3.4) is stable under parametric perturbations of the weapons matrix $A(t^*)$. □

4 Stability conditions for a neutral linear approximation of the model

In this section, we will consider the case where at time $t_r \in \mathbb{R}_\tau$ the linear approximation of system (2.3) is neutral stable. In this case,

$$k(t_r)y - a(t_r)x = 0, \qquad l(t_r)x - b(t_r)y = 0, \tag{4.1}$$

and, the mathematical model of confrontation between two countries (2.3) takes the form

$$\frac{dx}{dt} = g(t,x,y), \quad x(t_r) = x_0^r; \qquad \frac{dy}{dt} = h(t,x,y), \quad y(t_r) = y_0^r \tag{4.2}$$

where $t \geq t_r$.

Let us assume that the hostility functions $g(t,x,y)$, $h(t,x,y)$ are essentially nonlinear (do not contain linear approximations). Denote by

$$(x(t), y(t))^T = (x(t;t_r,x_0^r), y(t;t_r,y_0^r))^T$$

the solution of the model (4.2).

System (4.2) will be unstable if its solutions $(x(t), y(t))^T$ increase for all $t > t_r$ and for arbitrarily small initial conditions $(x_0^r, y_0^r)^T$ at $t = t_r$.

We associate the positive and bounded Lyapunov function $V(x,y) = xy$, $x > 0$, $y > 0$ to the system of equations (4.2) and prove that the following statement is true.

Proposition 4.1. *Let the hostility functions in the mathematical model of confrontation (4.2) are such that $g(t,x,y) \neq 0$ and $h(t,x,y) \neq 0$ for all $t \geq t_r$. If there exists a positive function $\theta_0 = \theta_0(t)$, with $\int_0^\infty \theta(s)ds = \infty$ for which the inequality*

$$g(t,x,y)y + h(t,x,y)x \geq \theta_0(t)xy, \tag{4.3}$$

holds for any $t \geq t_r$, $x, y > 0$, then the system (4.2) is unstable.

Proof. When inequality (4.3) is satisfied, the total derivative of the Lyapunov function $V(x,y) = xy$ on the solutions of system (4.2) is an increasing function

$$V(x(t), y(t)) \geq V(x(t_r), y(t_r)) \exp\left(\int_{t_r}^t \theta_0(s)ds\right)$$

for any $t \geq t_r$.

It follows that for a given $\varepsilon > 0$, for any $\delta > 0$ such that $x(t_r)y(t_r) < \delta$, there exists a value $t^* > t_r$ such that $x(t^*)y(t^*) > \varepsilon$. Consequently, the system (4.2) is unstable according to the first theorem on Lyapunov instability (see [12]). □

Remark 4.2. If the condition $g(t,x,y)y + h(t,x,y)x > 0$ hold, then the system (4.2) may be stable.

Next, we consider the dynamics in model (4.2) for $(t,x,y) \in \mathbb{R}_\tau \times \mathbb{R}_+ \times \mathbb{R}_+$ under the following assumption:

(H_7) There exist non-negative integrable functions $\psi_1(t)$, $\psi_2(t)$ such that for $(t,x,y) \in \mathbb{R} \times \mathbb{R}_+ \times \mathbb{R}_+$ we have

(a) $$|g(t,x,y)y| \leq \psi_1(t)(xy)^p,$$

(b) $$|h(t,x,y)x| \leq \psi_2(t)(xy)^q,$$

(c) $$g(t,x_0^r,y_0^r) \neq 0,\ h(t,x_0^r,y_0^r) \neq 0,$$

where $1 < p$, $1 \leq q < \infty$.

Let us introduce the following definition (cf. [30]).

Definition 4.3. The arms race between two countries, described by the system of equations (4.2) is:

(a) β-bounded, if there exists a constant $\beta > 0$ such that

$$x(t;t_r,x_0^r)y(t;t_r,y_0^r) < \beta$$

for all $t \geq t_r$, where β may depend on each solution;

(b) equi-β-bounded, if for any $\alpha > 0$ and $t_r \in \mathbb{R}$ there exists a constant $\bar{\beta}(t_r,\alpha) > 0$ such that $x_0^r y_0^r < \alpha$ implies $x(t;t_r,x_0^r)y(t;t_r,y_0^r) < \bar{\beta}(t_r,\alpha)$ for all $t \geq t_r$;

(c) uniformly β-bounded, if $\bar{\beta}(t_r,\alpha)$ in Definition 4.3 (b) is independent on t_r.

For the Lyapunov function $V(x,y) = xy$ the following result holds.

Lemma 4.4. *Assume that for the model (4.2) the hypothesis (H_7) holds and the inequality*

$$N(t,t_r) = (p+q-2)\left[(V(x_0^r,y_0^r))^{p-1}\int_{t_r}^t \psi_1(s)ds + (V(x_0^r,y_0^r))^{q-1}\int_{t_r}^t \psi_2(s)ds\right] < 1 \tag{4.4}$$

is satisfied for $t \geq t_r$.

Then, for the Lyapunov function $V(x,y)$ with respect to the solutions of (4.2), the following estimate

$$V(x(t),y(t))) \leq V(x_0^r,y_0^r)\,(1 - N(t,t_r))^{-\frac{1}{p+q-2}} \tag{4.5}$$

holds for all $t \in \mathbb{R}_\tau$.

Proof. If conditions (a)–(c) of the hypothesis (H_7) are satisfied for the function $V(x,y)$, it is not difficult to obtain the integral inequality

$$V(x(t),y(t)) \leq V(x_0^r,y_0^r) + \int_{t_r}^t (\psi_1(s)V^p(x(s),\ y(s))ds + \psi_2(s)V^q(x(s),\ y(s)))ds, \quad t \geq t_r. \tag{4.6}$$

Next, we represent the resulting inequality in the pseudo-linear form

$$\begin{aligned} &V(x(t),y(t)) \\ &\quad\leq V(x_0^r,y_0^r) + \int_{t_r}^t \left(\psi_1(s)V(x(s),y(s))^{p-1} + \psi_2(s)V(x(s),y(s))^{q-1}\right)V(x(s),y(s))\ ds \end{aligned} \tag{4.7}$$

for $t \geq t_r$.

Applying the Gronwall–Bellman lemma [16,17] to inequality (4.7), we obtain the estimate

$$V(x(t),y(t)) \le V(x_0^r,y_0^r)\exp\left[\int_{t_r}^{t}(\psi_1(s)V(x(s),y(s))^{p-1}+\psi_2(s)V(x(s),y(s))^{q-1})ds\right] \tag{4.8}$$

for all $t \ge t_r$.

From inequality (4.8), we have

$$\begin{aligned}(V(x(t),y(t)))^{p-1} &\le (V(x_0^r,y_0^r))^{p-1}\exp\left[(p-1)\int_{t_0}^{t}(\psi_1(s)(V(x(s),y(s)))^{p-1}\right.\\ &\qquad\left.+\,\psi_2(s)(V(x(s),y(s)))^{q-1})ds\right],\\ (V(x(t),y(t)))^{q-1} &\le (V(x_0^r,y_0^r))^{q-1}\exp\left[(q-1)\int_{t_0}^{t}(\psi_1(s)(V(x(s),y(s)))^{p-1}\right.\\ &\qquad\left.+\,\psi_2(s)(V(x(s),y(s)))^{q-1})ds\right]\end{aligned} \tag{4.9}$$

for all $t \ge t_r$.

For $p > 1$ and $q \ge 1$, we have from (4.9) that

$$\begin{aligned}(V(x(t),y(t)))^{p-1} &\le (V(x_0^r,y_0^r))^{p-1}\exp\left[(p+q-2)\int_{t_0}^{t}(\psi_1(s)(V(x(s),y(s)))^{p-1}\right.\\ &\qquad\left.+\,\psi_2(s)(V(x(s),y(s)))^{q-1})ds\right],\\ (V(x(t),y(t)))^{q-1} &\le (V(x_0^r,y_0^r))^{q-1}\exp\left[(p+q-2)\int_{t_0}^{t}(\psi_1(s)(V(x(s),y(s)))^{p-1}\right.\\ &\qquad\left.+\,\psi_2(s)(V(x(s),y(s)))^{q-1})ds\right]\end{aligned} \tag{4.10}$$

for $t \ge t_r$.

Multiplying both sides of the first inequality from system (4.10) by $(p+q-2)\psi_1(t)$ and the second by $(p+q-2)\psi_2(t)$, we obtain

$$\begin{aligned}&-(V(x(t),y(t)))^{p-1}\psi_1(t)(p+q-2)\exp\left[-(p+q-2)\int_{t_r}^{t}(\psi_1(s)(V(x(s),y(s)))^{p-1}\right.\\ &\qquad\left.+\,\psi_2(s)(V(x(s),y(s)))^{q-1})ds\right] \ge -(V(x_0^r,y_0^r))^{p-1}(p+q-2)\psi_1(t),\\ &-(V(x(t),y(t)))^{q-1}\psi_2(t)(p+q-2)\exp\left[-(p+q-2)\int_{t_r}^{t}(\psi_1(s)(V(x(s),y(s)))^{p-1}\right.\\ &\qquad\left.+\,\psi_2(s)(V(x(s),y(s)))^{q-1})ds\right] \ge -(V(x_0^r,y_0^r))^{q-1}(p+q-2)\psi_2(t)\end{aligned} \tag{4.11}$$

for all $t \ge t_r$.

Hence,

$$\begin{aligned}&\frac{d}{dt}\exp\left[-(p+q-2)\int_{t_r}^{t}(\psi_1(s)(V(x(s),y(s)))^{p-1}+\psi_2(s)(V(x(s),y(s)))^{q-1})ds\right]\\ &\qquad\ge -(V(x_0^r,y_0^r))^{p-1}(p+q-2)\psi_1(t)-(V(x_0^r,y_0^r))^{q-1}(p+q-2)\psi_2(s)\end{aligned} \tag{4.12}$$

for all $t \geq t_r$.

Integrating both sides of inequality (4.12) from t_r to t, we obtain the estimate

$$\begin{aligned} \exp\left[(p+q-2)\int_{t_r}^{t}(\psi_1(s)(V(x(s),y(s)))^{p-1}+\psi_2(s)(V(x(s),y(s)))^{q-1})ds\right] \\ \leq (1-N(t,t_r))^{-1} \end{aligned} \tag{4.13}$$

for all $t \in \mathbb{R}_\tau$ such that $N(t,t_r) < 1$.

Taking into account the estimate (4.13), the estimate (4.8) implies

$$(V(x(t),y(t)))^{p+q-2}\,(V(x_0^r,y_0^r))^{-(p+q-2)} \leq (1-N(t,t_r))^{-1}$$

or, finally,

$$V(x(t),y(t)) \leq V(x_0^r,y_0^r)(1-N(t,t_r))^{-\frac{1}{p+q-2}} \tag{4.14}$$

for all $t \in \mathbb{R}_\tau$ for which $N(t,t_r) < 1$. □

Remark 4.5. The proof of Lemma 4.4 takes into account the estimation technique for nonlinear integral inequalities established in [16, 17]. The technique is applied to the nonlinear integral inequality (4.6).

Lemma 4.4 allows us to establish new boundedness conditions for the arms race of two opposite countries.

Proposition 4.6. *Assume that conditions of Lemma 4.4 for the model (4.2) are met. In addition, assume that for initial weapon levels x_0^r, y_0^r for any $\alpha > 0$, $x_0^r y_0^r < \alpha$ we have:*

(1) $\bar{N}(t,t_r) < 1$ *for all* $t \geq t_r$;

(2) *for a given* $\bar{\beta} = \bar{\beta}(\alpha) > 0$ *we have*

$$\left(1-\bar{N}(t,t_r)\right)^{-\frac{1}{p+q-2}} < \frac{\bar{\beta}(\alpha)}{\alpha}$$

for all $t \geq t_r$ *, where*

$$\bar{N}(t,t_r) = (p+q-2)\left[(\alpha)^{p-1}\int_{t_r}^{t}\psi_1(s)ds + (\alpha)^{q-1}\int_{t_r}^{t}\psi_2(s)ds\right] \tag{4.15}$$

Then the arms race of the two opposing countries is uniformly β-bounded.

Proof. Proposition 4.6 follows directly from estimate (4.5). □

5 Stability analysis of the generalized Richardson model

The unlimited growth of hostility between the countries N_1 and N_2 (lack of restrictions (2.4)) can lead to a real conflict with unpredictable consequences. Here, we will consider some cases of stability (instability) of the equilibrium state in model (2.3) with time-varying hostility.

We will consider the system of equations (2.3) in which the functions $g(t,x,y)$ and $h(t,x,y)$ are defined in the region $t > 0$ and $x \leq H$, $y \leq H$, where H is a positive constant, are continuous and satisfy the condition under which equations (2.3) have a unique solution under given initial conditions. The functions $g(t,x,y)$ and $h(t,x,y)$, in contrast to the linear approximation of system (2.3), always remain unknown. We will assume that they are small and satisfy the next condition:

(H_8) In the region $t > 0$ and $x \leq H$, $y \leq H$, the hostility functions $g(t,x,y)$, $h(t,x,y)$ are such that

$$g(t,x,y) \neq 0, \qquad h(t,x,y) \neq 0, \qquad t \in \mathbb{R}_\tau.$$

The linear approximation of system (2.3)

$$\frac{dx}{dt} = k(t)y - a(t)x, \quad x(t_0) = x_0, \qquad \frac{dy}{dt} = l(t)x - b(t)y, \quad y(t_0) = y_0, \tag{5.1}$$

has a zero solution, the stability (instability) of which we will study further under the influence of hostility functions.

Definition 5.1. The state $x = y = 0$ of the system (5.1) will be stable under the influence of hostility $g(t,x,y)$, $h(t,x,y)$ if for any $\varepsilon > 0$ and $t_0 \geq 0$ one can specify three numbers $\eta_1(t_0,\varepsilon)$, $\eta_2(t_0,\varepsilon)$, and $\eta_3(t_0,\varepsilon)$ such that any solution $(x(t,t_0,x_0), y(t,t_0,y_0))^T$ satisfying the initial condition $|x_0 + y_0| < \eta_1(t_0,\varepsilon)$ with hostility $g(t,x,y)$, $h(t,x,y)$, satisfying in the region $t > t_0$ and $|x+y| < H$ the condition $|g(t,x,y)| < \eta_2(t_0,\varepsilon)$, $|h(t,x,y)| < \eta_3(t_0,\varepsilon)$, satisfies the inequality $(x(t,t_0,x_0) + y(t,t_0,y_0)) < \varepsilon$ for all $t > t_0$.

We will show that the following statement holds.

Proposition 5.2. *Assume that for the model (2.3) hypothesis (H_8) holds and there exist positive function $\theta_1(t)$ such that for $t \in \mathbb{R}_\tau$, $x, y > 0$ the next inequality*

$$\begin{aligned} g(t,x,y)x + h(t,x,y)y + (k(t)+l(t))xy - a(t)x^2 - b(t)y^2 \leq \theta_1(t)(x^2+y^2), \\ \textit{holds for } \int_{t_0}^{\infty} \theta_1(s)ds = L_1 < \infty \end{aligned} \tag{5.2}$$

then the state equilibrium $(x,y)^T$=$(0,0)^T$ of the system (5.1) is stable under the influence of hostility.

Proof. The stability of the system (2.3) will be analyzed by the use of the Lyapunov function $2V(x,y) = \left(x^2+y^2\right)$.

The total derivative of the function $V(x,y)$ with respect to system (2.3) has the form

$$\frac{d}{dt}V(x,y) = x\frac{dx}{dt} + y\frac{dy}{dt} = g(t,x,y)x + h(t,x,y)y + (k(t)+l(t))xy - a(t)x^2 - b(t)y^2. \tag{5.3}$$

From this equality and condition (5.2) follows the estimate

$$\begin{aligned} V(x(t),y(t)) &\leq V(x(t_0),y(t_0)) \exp\left(\int_{t_0}^{t} 2\theta_1(s)ds\right) \\ &\leq V(x(t_0),y(t_0)) \exp^{2L_1}, \qquad t \geq t_0. \end{aligned}$$

The stability of the equilibrium state $(x,y)^T = (0,0)^T$ of the system (5.1) follows from the fact that the function $V(x(t),y(t))$ does not increase on the solutions of system (2.3) for any $t \geq t_0$. □

Corollary 5.3. *If the equilibrium state $(x,y)^T = (0,0)^T$ of the linear approximation (5.1) of system (2.3) is asymptotically stable, then it is stable under the influence of hostility $g(t,x,y)$, $h(t,x,y)$.*

Example 5.4. Let us consider the system of equations (2.3) with the linear approximation (5.1) which has the following parameters: $a(t) = b(t) = k(t) = l(t) = 1 + t$ for $t \geq t_\tau > 0$ and hostility functions $g(t,x,y)$ and $h(t,x,y)$ satisfying the conditions (2.4).

We introduce the notation $R(t,x,y) = g(t,x,y)x + h(t,x,y)y \leq \eta_2(t_0,\varepsilon)x + \eta_3(t_0,\varepsilon)y$ for $t \geq t_\tau > 0$ in the region $t > t_0$ and $|x+y| < H$.

The total derivative of the function $2V(x,y) = (x^2 + y^2)$ with respect to system (2.3) has the form

$$\begin{aligned} \frac{d}{dt}V(x(t),y(t))|_{(2.3)} &= x\frac{dx}{dt} + y\frac{dy}{dt} = \frac{d}{dt}V(x(t),y(t))|_{(5.1)} + R(t,x,y) \\ &= -(1+t)(x-y)^2 + R(t,x,y). \end{aligned} \tag{5.4}$$

We assume that the hypothesis (H_8) holds for the hostility functions $g(t,x,y)$ and $h(t,x,y)$ and calculate $\bar{\eta} = \max\{\eta_2(t_0,\varepsilon), \eta_3(t_0,\varepsilon)\}$. Then if $\bar{\eta} < \frac{(1+t)(x-y)^2}{x+y}$, then

$$\frac{d}{dt}V(x(t),y(t))|_{(2.3)} \leq -(1+t)(x-y)^2 + \bar{\eta}(x+y) < 0. \tag{5.5}$$

Hence, by Lyapunov's first theorem on stability it follows that the zero equilibrium of (5.1) is stable under the influence of hostility.

Next, we establish the conditions for the instability of the equilibrium state $(x,y)^T = (0,0)^T$ of the system of equations (5.1) under the assumption (H_8)

Proposition 5.5. *Assume that for the model* (2.3) *hypothesis* (H_8) *holds and there exist positive function* $\theta_2 = \theta_2(t)$ *such that for* $t \in \mathbb{R}_\tau$, $x, y > 0$ *the next inequality*

$$\begin{aligned} g(t,x,y)y + h(t,x,y)x - (a(t)+b(t))xy + l(t)x^2 + k(t)y^2 &\geq \theta_2(t)xy, \\ \textit{holds for } \int_{t_0}^{\infty} \theta_2(s)ds &= \infty, \end{aligned} \tag{5.6}$$

then the state equilibrium $(x,y)^T$=$(0,0)^T$ *of the system* (2.3) *is unstable under the influence of hostility.*

Proof. The instability of the system (2.3) will be analyzed by the use of the Lyapunov function $V(x,y) = xy$.

The total derivative of the function $V(x,y)$ with respect to system (2.3) has the form

$$\begin{aligned} \frac{d}{dt}V(x,y) &= \frac{dx}{dt}y + x\frac{dy}{dt} \\ &= g(t,x,y)y + h(t,x,y)x - (a(t)+b(t))xy + l(t)x^2 + k(t)y^2. \end{aligned} \tag{5.7}$$

If condition (5.6) is satisfied, then the equilibrium state $(x,y)^T = (0,0)^T$ of the system (5.1) is unstable under the influence of hostility due to the increase of the function $V(x,y) = xy$ on the solutions of system (2.3):

$$V(x(t),y(t)) \geq V(x(t_0),y(t_0)) \exp\left(\int_{t_0}^{t} \theta_2(s)ds\right),$$

for $t \to \infty$ (see [12] and Proposition 4.1). □

Example 5.6. Let us consider the system of equations (2.3) with the linear approximation (5.1) which has the following parameters: $a(t) = b(t) = 1/2$, $k(t) = l(t) = 1$ for $t \geq 0$.

Consider hostility functions $g(t,x,y)$ and $h(t,x,y)$ that satisfy hypothesis (H_8), and introduce the notation $Q(t,x,y) = g(t,x,y)y + h(t,x,y)x \le \eta_2(t_0,\varepsilon)x + \eta_3(t_0,\varepsilon)y$ for $t \ge t_\tau > 0$ in the region $t > t_0$ and $|x+y| < H$.

For the total derivative of the function $V(x,y) = xy$ with respect to system (2.3) we have

$$\begin{aligned} \frac{d}{dt}V(x(t),y(t))|_{(2.3)} &= y\frac{dx}{dt} + x\frac{dy}{dt} = \frac{d}{dt}V(x(t),y(t))|_{(5.1)} + Q(t,x,y) \\ &= x^2 + y^2 - xy + Q(t,x,y) \ge Q(t,x,y) + xy. \end{aligned} \tag{5.8}$$

If there exists a positive function $\theta_2 = \theta_2(t)$, such that $Q(t,x,y) \ge (\theta_2 - 1)xy$, then

$$\frac{d}{dt}V(x(t),y(t))|_{(2.3)} \ge \theta_2(t)xy \quad \text{for all } t \in \mathbb{R}_+,\ x,y > 0.$$

Hence, by Proposition 5.5 it follows that the zero equilibrium of (5.1) is unstable.

Next, we will consider the case, where at time $t_c \in \mathbb{R}_\tau$ the hostility functions $g(t,x,y)$, $h(t,x,y)$ are equivalent in influence to the level of threats, i.e. the following condition holds.

(H_9) The hostility functions $g(t,x,y)$, $h(t,x,y)$ are such that

$$g(t,x,y) - k(t)y \equiv 0, \quad h(t,x,y) - l(t)x \equiv 0, \quad t \in \mathbb{R}_\tau,\ x,y \in \mathbb{R}_+.$$

Proposition 5.7. *Assume that for the model* (2.3) *hypotheses* (H_8) *and* (H_9) *are satisfied.*

Then, the levels of armament of opposing countries are decreasing exponentially.

Proof. If the assumption (H_9) is satisfied, then the mathematical model (2.3) degenerates into the model

$$\frac{dx}{dt} = -a(t)x, \quad x(t_c) = x_{0c}; \qquad \frac{dy}{dt} = -b(t)y, \quad y(t_c) = y_{0c}. \tag{5.9}$$

Using a Lyapunov function of the form $V(x,y) = xy$ it is not difficult to show that

$$x(t;t_c,x_{0c})y(t;t_c,y_{0c}) = x_{0c}y_{0c}\exp\left(-(a(t)+b(t))(t-t_c)\right) \tag{5.10}$$

or

$$x(t;t_c,x_{0c}) = x_{0c}\exp\left(-a(t-t_c)\right), \qquad y(t;t_c,y_{0c}) = y_{0c}\exp\left(-b(t-t_c)\right) \tag{5.11}$$

for all $t \ge t_c \in \mathbb{R}_\tau$.

The statement of Proposition 5.7 follows from (5.7) and representations (5.8). □

Remark 5.8. The case considered in Proposition 5.7 may indicate some kind of emergency situation, since in the normal mode of coexistence of opposing countries such a phenomenon is not observed.

Finally, in this section we will consider the case where at time $t_c \in \mathbb{R}_\tau$ the hostility functions $g(t,x,y)$, $h(t,x,y)$ are equivalent in terms of the impact of the level of expenditures of each country on maintaining its armament, i.e., the following condition holds.

(H_{10}) The hostility functions $g(t,x,y)$, $h(t,x,y)$ are negative and

$$-g(t,x,y) + a(t)x \equiv 0, \qquad -h(t,x,y) + b(t)y \equiv 0, \qquad t \in \mathbb{R}_\tau,\ x,y \in \mathbb{R}_+.$$

Proposition 5.9. *Assume that for the model* (2.3) *hypotheses* (H_8) *and* (H_{10}) *are satisfied, and for all* $t \geq t_c \in \mathbb{R}_\tau$ *each country determines its level of armament based on the level of armament of the opposite side without taking into account their own economic expenses.*

Then, the zero equilibrium $(x, y)^T = (0, 0)^T$ *of* (2.3) *is unstable.*

Proof. If the assumption (H_{10}) is satisfied, then the mathematical model (2.3) degenerates into the model

$$\frac{dx}{dt} = k(t)y, \quad x(t^c) = x_0^c; \qquad \frac{dy}{dt} = l(t)x, \quad y(t^c) = y_0^c.$$

For the Lyapunov function $V(x, y) = \frac{1}{2}(x^2 + y^2)$ with respect to the system (5.9), we have

$$V(x(t), y(t)) = V(x_0^c, y_0^c) \exp(k(t) + l(t))(t - t_c)$$

for all $t \geq t_c \in \mathbb{R}_\tau$, which implies the statement of Proposition 5.9. □

Remark 5.10. The case considered in Proposition 5.9 may correspond to a global militarization of countries involved in the confrontation.

6 Stability of weapon vector evasion from the equilibrium vector

We rewrite the system of equations (2.3) in the form

$$\frac{dw}{dt} = A(t)w + B(t, w), \qquad w(t_0) = w_0, \tag{6.1}$$

where $w \in \mathbb{R}_+^2$ is the vector of weapons of the two countries, $A(t) = [a_{ij}(t)]$, is a (2×2) matrix with entries $a_{11}(t) = -a(t), a_{12}(t) = k(t), a_{21}(t) = l(t), a_{22}(t) = -b(t)$, which characterizes the armaments of the two countries changing at time t, $B(t, w) = (g(t, w), h(t, w))^T$ denotes a vector function, nonlinear in w, which characterizes the public opinion about the race weapons and hostility between the two countries.

Suppose that for the two opposing countries there is only one weapon balance beam (ray) (which is equilibrium vector of (6.1)) on the form $L = \{w_e \in K : w_e = \lambda e\}$, where $e = (1, 1) \in \mathbb{R}_+^2$, λ is a positive constant, K is a cone in $\mathbb{R}_+^2$.

Also, let us assume that the vector functions $A(t)w$ and $B(t, w)$ in the system of equations (6.1) meet the conditions.

(H_{11}) There exist nonnegative functions $f_1(t)$, $f_2(t)$ such that

(a) $\quad d[A(t)w, L] \leq f_1(t)d[w, L]$ for all $t \in \mathbb{R}_\tau$, and $w \in \mathbb{R}_+^2$;

(b) $\quad d[B(t, w), L] \leq f_2(t)d^\alpha[w, L]$ for all $t \in \mathbb{R}_\tau$ $\alpha > 1$ and $w \in \mathbb{R}_+^2$,

where $d[\cdot, \cdot]$ is the distance in $\mathbb{R}_+^2$.

(H_{12}) For $t \in \mathbb{R}_\tau$, we have

$$N(t, t_0) = (\alpha - 1)d^{\alpha-1}[w_0, L] \int_{t_0}^{t} f_2(s) \exp\left((\alpha - 1)\int_{t_0}^{s} f_1(\tau)d\tau\right) ds < 1. \tag{6.2}$$

In this section, using a nonlinear integral inequality, we will establish an estimate of the deviation of the weapon vector $w(t)$ from the equilibrium ray weapons L of system (6.1).

Lemma 6.1. *Assume that for the system* (6.1) *hypotheses* (H_{11}) *and* (H_{12}) *hold.*

Then, the deviation of the arms vector from the equilibrium ray is estimated by the inequality

$$d[w(t;t_0,w_0),L] \le d[w_0,L]\exp\int_{t_0}^{t} f_1(s)ds\,(1-N(t,t_0))^{-\frac{1}{\alpha-1}} \tag{6.3}$$

for $t \in \mathbb{R}_\tau$.

Proof. From (6.1), we have

$$w(t;t_0,w_0) = w_0 + \int_{t_0}^{t} (A(s)w(s) + B(s,w(s))ds. \tag{6.4}$$

Hence, using the estimates in (H_{11}), we obtain

$$\begin{aligned} d[w(t;t_0,w_0),L] &= d[w_0,L] + \int_{t_0}^{t} d[(A(s)w(s)+B(s,w(s)),L]ds \\ &\le d[w_0,L] + \int_{t_0}^{t} (f_1(s)d[w(s,t_0,w_0),L] + f_2(s)d^{\alpha}[w(s,t_0,w_0),L])ds. \end{aligned} \tag{6.5}$$

From inequality (6.5) it follows

$$d[w(t;t_0,w_0),L] \le d[w_0,L] + \int_{t_0}^{t} (f_1(s)d[w(s),L] + f_2(s)d^{\alpha}[w(s),L]))\,ds \tag{6.6}$$

for all $t \in \mathbb{R}_\tau$.

We will represent inequality (6.6) in the pseudo-linear form

$$d[w(t;t_0,w_0),L] \le d[w_0,L] + \int_{t_0}^{t} \left(f_1(s) + f_2(s)d^{\alpha-1}[w(s),L]\right) d[w(s;t_0,w_0),L]ds \tag{6.7}$$

for all $t \in \mathbb{R}_\tau$.

Applying the Gronwall–Bellman lemma to inequality (6.7), we obtain an estimate in the form

$$d[w(t;t_0,w_0),L] \le d[w_0,L]\exp\left(\int_{t_0}^{t} \left(f_1(s) + f_2(s)d^{\alpha-1}[w(s),L]\right) ds\right) \tag{6.8}$$

for all $t \in \mathbb{R}_\tau$.

Next, we will apply integral inequalities from [14,15] to evaluate

$$\exp\left(\int_{t_0}^{t} \left(f_1(s) + f_2(s)d^{\alpha-1}[w(s),L]\right) ds\right). \tag{6.9}$$

in (6.8).

Namely, we represent inequality (6.8) in the form

$$\begin{aligned} &d^{\alpha-1}[w(t;t_0,w_0),L] \\ &\quad\le d^{\alpha-1}[w_0,L]\exp\left((\alpha-1)\int_{t_0}^{t} \left(f_1(s) + f_2(s)d^{\alpha-1}[w(s;t_0,w_0),L]\right) ds\right). \end{aligned} \tag{6.10}$$

Multiplying both parts of inequality (6.10) by the negative factor

$$-(\alpha-1)f_2(t)\exp\left(-(\alpha-1)\int_{t_0}^{t} f_2(s)d^{\alpha-1}[w(s;t_0,w_0),L])ds\right),$$

we obtain

$$-d^{\alpha-1}[w(t;t_0,w_0),L](\alpha-1)f_2(t)\exp\left(-(\alpha-1)\int_{t_0}^t f_2(s)d^{\alpha-1}[w(s;t_0,w_0),L]ds\right)$$
$$\geq -(\alpha-1)d^{\alpha-1}[w_0,L]f_2(t)\exp\left(\alpha-1)\int_{t_0}^t f_1(s)ds\right). \tag{6.11}$$

The inequality (6.11) implies

$$\frac{d}{dt}\left\{\exp\left[-(\alpha-1)\int_{t_0}^t f_2(s)d^{\alpha-1}[w(s;t_0,w_0),L]ds\right]\right\}$$
$$\geq -(\alpha-1)d^{\alpha-1}[w_0,L]f_2(t)\exp\Big((\alpha-1)\int_{t_0}^t f_1(s)ds\Big). \tag{6.12}$$

Integrating the above inequality from t_0 to $t\in\mathbb{R}_\tau$, we have

$$\exp\left(-(\alpha-1)\int_{t_0}^t f_2(s)d^{\alpha-1}[w(s;t_0,w_0),L]ds\right)$$
$$\geq 1-(\alpha-1)d^{\alpha-1}[w_0,L]\int_{t_0}^t f_2(s)\exp\left(-(\alpha-1)\int_{t_0}^t f_1(s)d^{\alpha-1}[w(s;t_0,w_0),L]ds\right), \tag{6.13}$$

where $\exp\{t_0\}=1$.

Hence,

$$\begin{aligned}&\exp\left[(\alpha-1)\int_{t_0}^t f_2(s)d^{\alpha-1}[w(s;t_0,w_0),L]ds\right]\\ &\quad\leq\left\{1-(\alpha-1)d^{\alpha-1}[w_0,L]\int_{t_0}^t f_2(s)\exp\left[(\alpha-1)\int_{t_0}^s f_1(\tau)d\tau\right]ds\right\}^{-1}\end{aligned} \tag{6.14}$$

for all $t\in\mathbb{R}_\tau$.

Combining (6.10) and (6.14), we get

$$d^{\alpha-1}[w(t;t_0,w_0),L]\leq\frac{d^{\alpha-1}[w_0,L]\exp\left[(\alpha-1)\int_{t_0}^t f_1(s)ds\right]}{1-N(t,t_0)},$$

from which, taking into account (H_{12}), the deviation of the weapon vector $w(t)$ from the ray of equilibrium L is estimated by the inequality (6.3). □

Proposition 6.2. *Assume that for the system* (6.1) *hypotheses* (H_{11}) *and* (H_{12}) *hold. If for any* $\varepsilon>0$ *there exists a* $\delta=\delta(t_0,\varepsilon)>0$ *such that:*

(1) $d[w_0,L]<\delta$,

(2) $\overline{N}(t,t_0)=(\alpha-1)\delta^{\alpha-1}\int_{t_0}^t f_2(s)\exp\big((\alpha-1)\int_{t_0}^s f_1(\tau)d\tau\big)ds<1$ *for all* $t\in\mathbb{R}_\tau$,

(3) $\exp\big(\int_{t_0}^t f_1(s)ds\big)\big(1-\overline{N}(t,t_0)\big)^{-\frac{1}{\alpha-1}}<\frac{\varepsilon}{\delta}$, $t\in\mathbb{R}_\tau$,

then

$$d[w(t;t_0,w_0),L]<\varepsilon,\qquad t\in\mathbb{R}_\tau. \tag{6.15}$$

Proof. Under the conditions of Proposition 6.2 the deviation of the weapon vector from the equilibrium beam is estimated by inequality (6.3), from which the statement of Proposition 6.2 follows. □

Remark 6.3. Conditions (1)–(3) of Proposition 6.2 are sufficient for the stability of the deviation of the weapon vector of two countries in model (6.1) from the weapon equilibrium ray L.

7 Concluding remarks

In this paper, the influence of the hostility factor between countries on the stability of the equilibrium state is studied in detail for the non-autonomous model (2.3) of confrontation between two countries. Namely:

- the criteria for stability (asymptotic stability) are established under zero hostility;
- the criteria for stability (instability) of the arms race between two countries are established under the condition of neutral stability of the zero equilibrium state in the linear approximation of the proposed model (2.3);
- the criteria for stability of parametric disturbances of the armament matrix are established in the linear approximation of system (2.3);
- under various assumptions about the hostility between the two countries changing over time, the criteria for stability (instability) of the zero equilibrium state are derived;
- for the non-autonomous quasi-linear model of confrontation between two countries, a new estimate of the deviation of the armament vector from the equilibrium ray is obtained and the conditions for the stability of the equilibrium state are indicated.

The approaches to the analysis of the influence of hostility on the stability of the equilibrium state of the extended model (2.3) proposed in this paper are new and contribute to the development of the theory. It is also expected that this contribution will stimulate further improvement and analysis of arms race models.

Directions for expanding and increasing the flexibility of models include taking into account delay factors in the linear approximation of model (2.3), assessing the influence of control in functions describing hostility, including impulse disturbance. It is of interest to use fractional analysis or sequential modeling. An interesting direction for future research is also to consider an arms race model with more than two countries and with periodic coefficients in the linear approximation. Studying discrete versions of model (2.3) is also important for numerical modeling.

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