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On uniform asymptotic h-stability for linear nonautonomous systems

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Abstract. A classical result in the theory of stability of nonautonomous linear systems is that the uniform exponential decay of its corresponding transition matrix is equivalent to the uniform asymptotic stability of the trivial solution. The aim of this article is to extend this equivalence for non-exponential decays $h(t)$. In this article we prove that, in some suitable cases, the function $h(t)$ allows the construction of a topological abelian group that makes possible to formulate a more general definition of uniform stability which is equivalent with a decay dominated by $h(t)$. Moreover, we can use this group to establish the necessary elements to develop a theory of h-stable systems. As a first step in this direction, we provide integral conditions on the solutions of a uniformly asymptotically h-stable system.

Keywords: linear nonautonomous systems, uniform asymptotic h-stability, group theory.

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1 Introduction and motivation

1.1 Preliminaries

This article provides new results about the property of uniform asymptotic h-stability of the nonautonomous linear system of ordinary differential equations:

$$\dot{x} = A(t)x \quad \text{for any } t \in J := (a, +\infty), \tag{1.1}$$

where a can be either a real number or $a = -\infty$, $A\colon J \to M_n(\mathbb{R})$ is a piecewise continuous matrix valued function. Moreover, $\Phi(t)$ and $\Phi(t,s) = \Phi(t)\Phi^{-1}(s)$ will denote respectively a fundamental matrix of (1.1) and its corresponding transition matrix. The solution of (1.1) passing through x_0 at time $t = t_0 > a$ will be denoted by $t \mapsto x(t, t_0, x_0) := \Phi(t, t_0)x_0$. For a

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given norm $\|\cdot\|$ on $\mathbb{R}^n$ and any matrix $U \in M_n(\mathbb{R})$, the induced operator norm is defined as $\|U\|_{\mathcal{L}} := \sup\{\|Ux\| \colon \|x\| \leq 1\}$.

The stability and asymptotic stability of the nonautonomous linear systems (1.1) are classic research topics and there exists a myriad of asymptotic stabilities whose classification and description has been carried out in authoritative monographs as [5, 9, 14]. In particular, the uniform asymptotic h-stability is defined as follows:

Definition 1.1. The nonautonomous system (1.1) is *uniformly asymptotically h-stable* if there exist constants $K \geq 1$ and $\alpha > 0$ and a function $h \colon J \to (0, +\infty)$ such that:

$$\|\Phi(t, t_0)\|_{\mathcal{L}} \leq K \left(\frac{h(t_0)}{h(t)}\right)^{\alpha} \quad \text{for any } t \geq t_0 > a, \tag{1.2}$$

where $h(\cdot)$ is continuous, surjective and strictly increasing.

To the best of our knowledge, the above definition has been introduced by M. Pinto in [13] in a more general context, including the nonlinear case and considering $\alpha = 1$. This definition of stability has been explored by the current research, either by considering a fixed function $h(\cdot)$ such as in the cases of polynomial stability [7] or algebraic dichotomy [11], or by considering a large family of functions, such as in [12].

The previous definition is a particular case of the *uniform stability* and encompasses a wide range of well known stabilities, for example, the classical property of *uniform asymptotic stability*, which is ubiquitous in theoretical and applied research and will be recalled to make this article self contained:

Definition 1.2 ([2, Ch. III.1])**.** The nonautonomous linear system (1.1) is *uniformly stable* if any solution $t \mapsto x(t) := x(t, t_0, x_0)$ satisfies the following property:

a) For each $\varepsilon > 0$ there is a $\delta_\varepsilon > 0$ such that if $\|x(t_0)\| < \delta_\varepsilon$ for some $t_0 \in J$ then $\|x(t)\| < \varepsilon$ for all $t \geq t_0$.

The nonautonomous system (1.1) is *uniformly asymptotically stable* if it is uniformly stable and in addition:

b) There is a $\delta_0 > 0$ such that for all $\varepsilon > 0$ there exists a constant $T_\varepsilon \in (0, +\infty)$ such that if $\|x(t_0)\| < \delta_0$ then $\|x(t)\| < \varepsilon$ for all $t \geq t_0 + T_\varepsilon$.

The proof that the uniform asymptotic stability described in Definition 1.2 is a particular case of the uniform asymptotic h-stability follows from the following results considering $h(t) = e^t$:

Proposition 1.3 ([9, Prop.3.3.1])**.** *The nonautonomous linear system* (1.1) *is uniformly stable if and only if there exists a constant* $K \geq 1$ *such that:*

$$\|\Phi(t, s)\|_{\mathcal{L}} \leq K \quad \text{for all } t \geq s > a.$$

Proposition 1.4 ([10, Th.4.11])**.** *The nonautonomous linear system* (1.1) *is uniformly asymptotically stable if and only if it is uniformly exponentially stable, namely, there exist constants* $K \geq 1$ *and* $\alpha > 0$ *such that:*

$$\|\Phi(t, s)\|_{\mathcal{L}} \leq K e^{-\alpha(t-s)} \quad \text{for all } t \geq s > a. \tag{1.3}$$

Another equivalent characterization of uniform asymptotic stability in terms of the transition matrix is given by:

Proposition 1.5. *The nonautonomous linear system* (1.1) *is uniformly asymptotically stable if and only if it is uniformly stable and there exists a constant* $T \in (0, +\infty)$ *such that*

$$\sup_{s \in J} \|\Phi(T + s, s)\|_{\mathcal{L}} < 1. \tag{1.4}$$

The step proving the necessity of the above result can be found as an intermediate step in the proof of Proposition 1.4 given by [2, Ch. III.1] and as Lemma 5 in [6], while the sufficiency step of the is an immediate consequence of Proposition 1.4.

1.2 Novelty of the article and main results

We stress that the equivalence between the ε–δ characterization of uniform asymptotic stability stated in Definition 1.2 and the estimations for the transition matrices given by Propositions 1.4 and 1.5 is a classical topic in stability theory. Nevertheless, a generalization for decays described by $h(\cdot)$ functions has not been addressed and is the aim of this article.

A possible way to study the uniform asymptotic h-stability is to introduce the change of time variables

$$t = h^{-1}(e^{\tilde{t}}) \quad \text{and} \quad \tilde{t} = \log(h(t)) \quad \text{with } \log(h(a^+)) < \tilde{t} < +\infty, \tag{1.5}$$

and the matrix $\Phi_h(\tilde{u})$:

$$\Phi_h(\tilde{u}) = \Phi(h^{-1}(e^{\tilde{u}})) \quad \text{for any } \tilde{u} > \log(h(a^+)). \tag{1.6}$$

These change of variables allow to reinterpret the estimation (1.2) as follows:

$$\|\Phi_h(\tilde{t}, \tilde{t}_0)\|_{\mathcal{L}} \leq K e^{-\alpha(\tilde{t} - \tilde{t}_0)} \quad \text{for any } \tilde{t} \geq \tilde{t}_0 > \log(h(a^+)). \tag{1.7}$$

Notice that the right-hand side of the inequalities (1.3) and (1.7) describes a similar exponential decay. Nevertheless, we warn that this analogy could induce the erroneous belief that the uniform asymptotic h-stability can be trivially revisited as a uniform exponential stability via a suitable change of variables. To support this warning, we state that the columns of the transformed matrix $\Phi_h(\cdot)$ are not a basis of solutions and the left side of (1.7) as well as (1.5)–(1.6) do not have an intuitively clear meaning. These issues deserve a meticulous study and a big contribution of this article is to show that any function $h(\cdot)$ allows to construct a topological group $(J, *)$ such that the change of variables described by (1.5) can be seen as the commutative diagrams of isomorphisms of topological groups such that

$$\begin{array}{ccc} (\mathbb{R}, +) & \xrightarrow{\;e\;} & (\mathbb{R}^+, \cdot) \\ & & \big\downarrow h^{-1} \\ & & (J, *) \end{array} \quad \text{and} \quad \begin{array}{ccc} (J, *) & \xrightarrow{\;h\;} & (\mathbb{R}^+, \cdot) \\ & & \big\downarrow \log \\ & & (\mathbb{R}, +), \end{array}$$

where the isomorphism $h \colon (J, *) \to (\mathbb{R}^+, \cdot)$ verifies $h(t * s) = h(t)h(s)$, which mimics the exponential identity $e^{t+s} = e^t e^s$.

The above described approach will enable to state the main results of this article:

- To provide an ε–δ definition of uniform asymptotic h-stability generalizing the one described in Definition 1.2.
- To generalize the Proposition 1.5 for the uniformly asymptotically h-stable case.

- To deduce integral characterizations for the uniform asymptotic h-stability which generalizes those obtained for the uniform asymptotic stability.

We stress that Definition 1.2 and Proposition 1.5 are stated in term of the additive group $(\mathbb{R},+)$ whereas its generalization will be carried out in terms of the *additive* structure of $(J,*)$. We stress the originality of this approach, its remarkable simplicity and its potential application for the study of h-dichotomies.

1.3 Structure of the article

In Section 2 we provide examples of systems which are uniformly h-stable but not uniformly asymptotically stable. Section 3 is devoted to the construction the topological group $(J,*)$, an invariant measure on J under the action of this group, and the study of their properties. In Section 4 we state and prove the main results, namely, the generalization of the equivalences between Definition 1.2 and Propositions 1.4 and 1.5 to the uniform h-stability framework. Finally, in Section 5 we generalize, to uniform h-stable systems, a pair of results characterizing the uniform exponentially stable systems in terms integral conditions.

2 Examples of uniform asymptotic h-stability

We will consider simple h-stable scalar equations and use them to show the existence of an equation that is logarithmically stable but not exponentially.

Example 2.1. Given $h : J \to (0,+\infty)$ continuously differentiable, surjective and strictly increasing, together with a constant $\alpha > 0$, let us consider the equation:

$$x' = -\alpha\left(\frac{h'(t)}{h(t)}\right)x,$$

which provides a simple example of h-stability. Indeed, the transition matrix is given by

$$\Phi(t,s) = \left(\frac{h(s)}{h(t)}\right)^{\alpha},$$

and the Definition 1.1 is verified with $K = 1$.

Example 2.2. The above example allows us to construct a nonautonomous linear equation whose solutions $t \mapsto x(t,s,x_0)$ asymptotically approach the origin but are not uniformly exponentially stable. In fact, when considering $h(t) = \log(1+t)$, the previous example says that the corresponding transition matrix is given by

$$\Phi(t,s) = \left(\frac{\log(1+s)}{\log(1+t)}\right)^{\alpha},$$

this system is uniformly asymptotically log-stable and for any $s \in J := (0,+\infty)$ and $x_0 \in \mathbb{R}$ it follows that

$$\lim_{t\to+\infty}\Phi(t,s)x_0 = \lim_{t\to+\infty}\left(\frac{\log(1+s)}{\log(1+t)}\right)^{\alpha}x_0 = 0.$$

However, for any pair of positive numbers α and $\beta > 0$, we have the estimation

$$\lim_{t\to+\infty}\log(1+t)e^{-t\frac{\beta}{\alpha}} = 0.$$

Hence for a fixed $s \in J$ and $K > 1$ there exists t large enough such that

$$\log(1+t)e^{-t^{\frac{\beta}{\alpha}}} < K^{-1}\log(1+s)e^{-s^{\frac{\beta}{\alpha}}},$$

and we can rearrange this inequality to obtain

$$Ke^{\beta(s-t)} < \left(\frac{\log(1+s)}{\log(1+t)}\right)^{\alpha},$$

which shows that the inequality (1.3) is not true for all pairs $t \geq s$ and the uniform exponential stability described by (1.3) cannot be verified.

3 A topological group induced by $h(\cdot)$

The conditions we have given to the function $h(\cdot)$ on Definition 1.1 are enough to carry out a deep study of the solutions of (1.1). In fact, as any continuous bijection between two open intervals is an homeomorphism, the function $h : (a, +\infty) \to (0, +\infty)$ has a continuous inverse $h^{-1}(\cdot)$. Then, we introduce the operation

$$t * s := h^{-1}(h(t)h(s)), \tag{3.1}$$

on $J = (a, +\infty)$, namely, the domain of h. To distinguish $*$ from the usual operations on $\mathbb{R}$, we will denote the integer powers of t as t^{*n}. This operation is associative and commutative, it has an identity element defined by

$$e_* = h^{-1}(1)$$

and each $t \in J$ has an inverse

$$t^{*-1} = h^{-1}\left(\frac{1}{h(t)}\right). \tag{3.2}$$

These properties, alongside the continuity of h and h^{-1}, imply that $(J, *)$, with the standard real open interval topology, is a locally compact abelian topological group.

Example 3.1. Let $h(t) = \log(t)$ on the interval $(1, +\infty)$. The corresponding group operation is

$$t * s = e^{\log(t)\log(s)} = t^{\log(s)} = s^{\log(t)},$$

and the identity on this group is $h^{-1}(1) = e^1 = e$.

It is straightforward to deduce the following properties as a consequence of (3.1) and (3.2):

$$\begin{aligned} h(t*s) &= h(t)h(s) \\ h^{-1}(t) * h^{-1}(s) &= h^{-1}(ts) \\ h(s * t^{*-1}) &= \frac{h(s)}{h(t)}. \end{aligned} \tag{3.3}$$

The first two identities of (3.3) imply that the function $h : J \mapsto \mathbb{R}^+$ is an isomorphism of topological groups between $(J, *)$ and the multiplicative group of positive real numbers $(\mathbb{R}^+, \cdot)$. Due to this isomorphism, we may understand properties of $(J, *)$ by first considering the analogous case on $(\mathbb{R}^+, \cdot)$ and passing from one group to the other by using the functions h and h^{-1}.

Remark 3.2. In the case of uniform exponential stability, namely, when $h(t) = e^t$, this operation becomes $t * s = \log(e^t e^s) = t + s$ and, consequently, the group $(\mathbb{R}, *)$ coincides with the additive group $(\mathbb{R}, +)$.

Remark 3.3. Note that inequality (1.2), which defines uniform asymtptotic h-stability, can be seen from another perspective by using the identity (3.3) as follows:

$$\|\Phi(t,s)\|_{\mathcal{L}} \leq Kh(s * t^{*-1})^{\alpha} \quad \text{for all } t \geq s > a.$$

We also can define a measure μ_* on the interval J, which is invariant under the action of elements of $(J, *)$. This is an absolutely continuous measure whose Radon–Nikodym derivative is the logarithmic derivative of h, that is, for a Borel measurable set $A \subset J$ we define its measure μ_* as

$$\mu_*(A) = \int_A \frac{h'(u)}{h(u)} \mathrm{d}m(u), \tag{3.4}$$

where $m(\cdot)$ is the Lebesgue measure. To see that this is an invariant measure, it is enough to prove that it is invariant for compact intervals, as these sets generate the Borel σ-algebra.

Lemma 3.4. *Given a compact interval $[s,t] \subset J$, its measure is invariant under translation by elements of $(J, *)$, that is, for any $\gamma \in J$*

$$\mu_*([s,t]) = \mu_*([\gamma * s, \gamma * t])$$

Proof. The measure of the interval $[s,t] \subset J$ is

$$\mu_*([s,t]) = \int_s^t \frac{h'(u)}{h(u)} \mathrm{d}m(u) = \log\left(\frac{h(t)}{h(s)}\right). \tag{3.5}$$

Notice that, given any constant $\gamma \in J$, by (3.3) we have

$$\begin{aligned} \mu_*([\gamma * s, \gamma * t]) &= \log\left(\frac{h(\gamma * t)}{h(\gamma * s)}\right) \\ &= \log\left(\frac{h(t)}{h(s)}\right) \\ &= \mu_*([s,t]). \end{aligned}$$

□

Remark 3.5. The identity (3.5) allows an additional characterization of the uniform asymptotic h-stability in terms of the measure μ_* as follows:

$$\|\Phi(t,s)\|_{\mathcal{L}} \leq Ke^{-\alpha\mu_*([s,t])} \quad \text{for all } t \geq s \geq a.$$

Remark 3.6. It will be of interest to understand the behavior of the iterates of elements on $(J, *)$ in the sense

$$\begin{aligned} J \times J &\to J \\ (\gamma, \gamma) &\mapsto \gamma^{*2} := \gamma * \gamma. \end{aligned}$$

Firstly, by using the isomorphism between $(J, *)$ and $(\mathbb{R}^+, \cdot)$, it follows that the n-the power of γ is

$$\gamma^{*n} = h^{-1} \circ h(\gamma^{*n}) = h^{-1}(h(\gamma)^n). \tag{3.6}$$

Secondly, given an element $\gamma > e_*$ we have that $h(\gamma) > 1$, which implies that $h(\gamma)^n \to +\infty$ and $h(\gamma)^{-n} \to 0$ as $n \to +\infty$. This means that any forward orbit of the action $J \to \gamma * J$ approaches $+\infty$ while any backward orbit approaches a.
Analogously, for $\gamma < e_*$ forward orbits approach a while backward orbits approach $+\infty$.

Ref.	$h(t)$	J	$c * d$	$\frac{h'(t)}{h(t)}$
[2]	e^t	$\mathbb{R}$	$c + d$	1
[4,7]	t	$\mathbb{R}^+$	cd	$\frac{1}{t}$
	$\log(1+t)$	$\mathbb{R}^+$	$(1+c)^{\log(1+d)} - 1$	$\frac{1}{(t+1)\log(1+t)}$
[11]	$t + \sqrt{t^2+1}$	$\mathbb{R}$	$c\sqrt{d^2+1} + d\sqrt{c^2+1}$	$\frac{1}{\sqrt{t^2+1}}$

Table 3.1: Some examples for functions $h(\cdot)$ with its respective interval of stability, group operation, and logarithmic derivative.

A useful construction on $\mathbb{R}$ is the partition on intervals $[k, k+1)$ of equal length, where $k \in \mathbb{Z}$. In proofs of exponential stability theory such as [8], this partition allows us to approximate the continuous system (1.1) by a discretization of $\Phi(t,s)$. However, when the decay rate of $\Phi(t,s)$ is given by a function $h(\cdot)$ rather than the exponential function, the notion of length, given by the invariant measure, changes. For this reason, we propose the following partition:

Lemma 3.7. *For any $\gamma \in (e_*, +\infty)$ the intervals $J_k = [\gamma^{*k}, \gamma^{*(k+1)})$ with $k \in \mathbb{Z}$ define a partition of J into sets of constant μ_* measure.*

Proof. The function h^{-1} is strictly increasing and $h(\gamma) > 1$, hence the sequence $\{\gamma^{*k}\}_{k\in\mathbb{Z}}$ which is defined by (3.6), that is, by the identity $\gamma^{*k} = h^{-1}(h(\gamma)^k)$, is strictly increasing, which implies that the intervals J_k are disjoint. It remains to show that the intervals J_k form a covering of J. Without loss of generality we assume that $t \in [e_*, +\infty)$. As discussed in Remark 3.6, for any $t \in [e_*, +\infty)$ there exists $n \in \mathbb{N} \cup \{0\}$ such that $\gamma^{*(n+1)} > t$. Since the inequality $\gamma^{*(n+1)} > t$ is satisfied by a set of natural numbers, there must be a least element of this set, which we call m. The number m is the minimal natural such that $\gamma^{*(m+1)} > t$, hence $\gamma^{*m} \leq t$, that is, $t \in J_m$. □

A noticeable byproduct of the above result is:

Corollary 3.8. *The measure μ_* is σ-finite.*

Example 3.9. In the case corresponding to $h(t) = t$, which defines polynomial stability in [7] and [1], the operation is defined by $t * s = ts$, the domain of h is the interval $(0, +\infty)$ and, for any $\gamma > 1$, the subintervals $\{[1,\gamma), [\gamma,\gamma^2), [\gamma^2,\gamma^3), \ldots\}$, and $\{[\gamma^{-1},1), [\gamma^{-2},\gamma^{-1}), [\gamma^{-3},\gamma^{-2}), \ldots\}$ define a uniform partition of $(0,+\infty)$. Moreover, given a constant $C > 0$ and considering the measure defined by (3.4), we have $\mu_*([C\gamma^k, C\gamma^{k+1})) = \mu_*([\gamma^k, \gamma^{k+1})) = \log(\gamma)$ for any $k \in \mathbb{Z}$.

4 Characterizations of uniform asymptotic *h*-stability

The main result of this work is a characterization of the uniform asymptotic h-stability which emulates and generalizes the equivalence between uniform asymptotic stability with Proposition 1.4 and Proposition 1.5. The group $(J, *)$ introduced in the previous section, as well as its properties, will play an essential role on the proof.

Theorem 4.1. *Given a nonautonomous system (1.1) with a transition matrix $\Phi(t,s)$, the following statements are equivalent:*

i) The system (1.1) is uniformly asymptotically h-stable.

ii) The system (1.1) is uniformly stable and there exists a constant $T \in (e_*, +\infty)$ *such that*

$$\sup_{s \in J} \|\Phi(T * s, s)\|_{\mathcal{L}} < 1. \tag{4.1}$$

iii) The solutions $x(t) = \Phi(t, t_0)x_0$ *of the system (1.1) satisfy the following two properties:*

a) For each $\varepsilon > 0$ *there is a* $\delta_\varepsilon > 0$ *such that if* $\|x(t_0)\| < \delta_\varepsilon$ *for some* $t_0 \in J$ *then* $\|x(t)\| < \varepsilon$ *for all* $t \geq t_0$*.*

b) There is a $\delta_0 > 0$ *such that for all* $\varepsilon > 0$ *there exists a constant* $T_\varepsilon \in (e_*, +\infty)$ *such that if* $\|x(t_0)\| < \delta_0$ *then* $\|x(t)\| < \varepsilon$ *for all* $t \geq t_0 * T_\varepsilon$*.*

Proof. We begin with i) $\Rightarrow$ ii): Suppose that the system (1.1) is uniformly asymptotically h-stable. By using (1.2) it follows that an uniformly asymptotically h-stable system always satisfies the bound $\|\Phi(t,s)\|_{\mathcal{L}} \leq K$ since $\frac{h(s)}{h(t)} \leq 1$ for all pairs s, t with $a < s \leq t < +\infty$, that is, for any h an uniformly asymptotically h-stable system is also uniformly stable, due to Proposition 1.3.

Now, note that for all $s \in J$ and $T \in (e_*, +\infty)$, by using (1.2) and (3.3) we have

$$\begin{aligned}
\|\Phi(T * s, s)\|_{\mathcal{L}} &\leq K \left(\frac{h(s)}{h(T * s)} \right)^\alpha \\
&= K \left(\frac{h(s)}{h(T)h(s)} \right)^\alpha \\
&= K \left(\frac{1}{h(T)} \right)^\alpha,
\end{aligned}$$

thus, by choosing $T \in (e_*, +\infty)$ large enough such that $K\left(\frac{1}{h(T)}\right)^\alpha < 1$, we obtain (4.1) since the right hand of the above inequality is independent of s.

We proceed to prove ii) $\Rightarrow$ i): Let us consider a pair of values (t, s) such that $a < s \leq t < +\infty$, by following a construction analogue to that of Lemma 3.7, we construct a partition of the interval J into the sets $s * A_n := s * [T^{*n}, T^{*(n+1)})$ for $n \in \mathbb{Z}$. As $\{s * A_n\}_{n \in \mathbb{Z}}$ is a partition of J, there must be an unique integer k such that $t \in s * A_k$. We can then rewrite $\Phi(t, s)$ as

$$\Phi(t, s) = \Phi(t, T^{*k} * s)\Phi(T^{*k} * s, T^{*(k-1)} * s) \cdots \Phi(T * s, s). \tag{4.2}$$

On the other hand, due to inequality (4.1) and the uniform stability of the system, we know that there exists constants $\theta < 1$ and $1 \leq K$ respectively, such that

$$\|\Phi(T * s, s)\|_{\mathcal{L}} < \theta \quad \text{and} \quad \|\Phi(t, T^{*k} * s)\|_{\mathcal{L}} \leq K.$$

Using these two bounds along with the submultiplicative property of the operator norm applied to (4.2), we get

$$\|\Phi(t, s)\|_{\mathcal{L}} \leq K\theta^k = Ke^{-k|\log(\theta)|}, \tag{4.3}$$

where $\log(\theta) = -|\log(\theta)|$ because $\theta < 1$. As the integer k depends on the values of T, t and s, we can bound it by a function of these numbers, to do this we notice that $t \in s * A_k$ is equivalent to the inequalities

$$s * T^{*k} \leq t < s * T^{*(k+1)}. \tag{4.4}$$

For any $\gamma \in J$, the group action on J, $\gamma * (\cdot)$, is an strictly increasing function, as can be seen by noting that $\gamma * (\cdot) = h^{-1}(h(\gamma)h(\cdot))$, where the left hand side is a composition of increasing functions. Hence, the action of s^{*-1} on (4.4) mantains the inequalities

$$T^{*k} \leq t * s^{*-1} < T^{*(k+1)}.$$

By passing the above inequality through the monotone function $u \mapsto \log(h(u))$ combined with the identities (3.2) and (3.3), we get

$$k \log(h(T)) \leq \log\left(\frac{h(t)}{h(s)}\right) < (k+1)\log(h(T)).$$

Notice that $\log\left(\frac{h(t)}{h(s)}\right)\log(h(T))^{-1} - 1 < k$, then by (4.3) and, due to the fact that the exponent $-k|\log(\theta)| < 0$, we deduce

$$\begin{aligned} \|\Phi(t,s)\|_{\mathcal{L}} &\leq K\theta^k \\ &\leq Ke^{-[\log(\frac{h(t)}{h(s)})\log(h(T))^{-1}-1]|\log(\theta)|} \\ &= Ke^{|\log(\theta)|}\left(\frac{h(s)}{h(t)}\right)^{\frac{|\log(\theta)|}{\log(h(T))}}, \end{aligned}$$

as this is true for any pair (t,s), we have shown that $\Phi(t,s)$ is uniformly asymptotically h-stable with constants $Ke^{|\log(\theta)|} \geq 1$ and $\frac{|\log(\theta)|}{\log(h(T))} > 0$.

We now prove i) $\Rightarrow$ iii): Suppose that the system (1.1) is uniformly asymptotically h-stable. The property a) of iii) is immediately verfied since, as we said in the step i) $\Rightarrow$ ii), the uniform h-stability implies the uniform stability from Definition 1.2 which is equivalent to iii).

In order to prove that the solutions $x(t)$ satisfy property b), let us consider $\delta_0 = 1$, $T \in (e_*, +\infty)$ and $t_0 \in (a, +\infty)$. Now, if $x(t)$ is a solution of (1.1) such that $\|x_0\| < \delta_0$ and $t \geq T * t_0$ by using the fact that (1.1) is uniformly asymptotically h-stable combined with (3.3) we can deduce that

$$\begin{aligned} \|x(t)\| &= \|\Phi(t, T * t_0)\Phi(T * t_0, t_0)x_0\| \\ &\leq \|\Phi(t, T * t_0)\|_{\mathcal{L}} \cdot \|\Phi(T * t_0, t_0)\|_{\mathcal{L}} \cdot \|x_0\| \\ &\leq K^2 \left(\frac{h(t_0)}{h(T * t_0)}\right)^{\alpha} \delta_0 \\ &= K^2 \left(\frac{1}{h(T)}\right)^{\alpha}. \end{aligned}$$

As the function h is surjective onto the interval $(0, +\infty)$, for any given $\varepsilon > 0$ we can choose a T_ε such that

$$K^2 \left(\frac{1}{h(T_\varepsilon)}\right)^{\alpha} < \varepsilon,$$

which proves property b).

Finally, we need to prove iii) $\Rightarrow$ ii): If (1.1) satisfies iii) then it has property a), which is the uniform stability described by Definition 1.2.

We now prove (4.1). Let δ_0 be as in property b) of statement iii). Given any $0 < \varepsilon < \delta_0/2$ we define $\theta = \varepsilon(2/\delta_0) < 1$. There exists $T_\varepsilon > e_*$ such that

$$\|\Phi(T_\varepsilon * s, s)\|_{\mathcal{L}} = \sup_{\|x\|=1} \frac{2}{\delta_0}\|\Phi(T_\varepsilon * s, s)\frac{\delta_0}{2}x\| < \theta,$$

which proves iii). □

5 Integral conditions

The results proved in the previous sections will allow us to carry out a wider study of the properties of uniform asymptotic h-stability, we will extend two existing results regarding necessary and sufficient conditions for uniform exponential stability. One of these theorems is proved in [9, Th. 3.3.15] in a finite dimensional framework. Besides the case of uniform exponential stability, these results were proved for uniform polynomial stability in [7], within the more general framework of evolutionary processes in Banach spaces.

Theorem 5.1. *A nonautonomous linear system* (1.1) *with a transition matrix* $\Phi(t,s)$ *is uniformly asymptotically h-stable if and only if it is uniformly stable and*

$$\sup_{s\in J}\int_s^{+\infty}\|\Phi(u,s)x\|\mathrm{d}\mu_*(u)<+\infty \quad \textit{for all } x\in\mathbb{R}^n. \tag{5.1}$$

Proof. We begin with the forward direction of the equivalence. Let $\Phi(t,s)$ be a transition matrix of an uniformly asymptotically h-stable system. As a uniformly asymptotically h-stable system is also uniformly stable, we only need to prove (5.1). To do this we can use (1.2) to bound the integral as follows

$$\begin{aligned}\int_s^{+\infty}\|\Phi(u,s)x\|\mathrm{d}\mu_*(u) &\leq K\|x\|\int_s^{+\infty}\left(\frac{h(s)}{h(u)}\right)^{\alpha}\frac{h'(u)}{h(u)}\mathrm{d}m(u)\\ &= K\|x\|h(s)^{\alpha}\int_s^{+\infty}\frac{h'(u)}{h(u)^{\alpha+1}}\mathrm{d}m(u)\\ &= K\alpha^{-1}\|x\|\lim_{t\to+\infty}\left(1-\frac{h(s)^{\alpha}}{h(t)^{\alpha}}\right)\\ &= K\alpha^{-1}\|x\|,\end{aligned}$$

which gives, for each $x\in\mathbb{R}^n$, a finite bound that is independent from s and the property (5.1) is verified.

To prove the backwards direction of the equivalence we will use Theorem 4.1. Given a compact interval $[c,d]\subset J$, the fact that $\Phi(t,s)$ is the transition matrix of an uniformly stable system implies that there exists a constant $K>1$ such that

$$\begin{aligned} rcl\|\Phi(d,c)\|_{\mathcal{L}} &\leq \|\Phi(d,u)\|_{\mathcal{L}}\cdot\|\Phi(u,c)\|_{\mathcal{L}}\\ &\leq K\|\Phi(u,c)\|_{\mathcal{L}} \text{ for } c\leq u\leq d.\end{aligned}$$

Given that the integral in (5.1) has a non-negative integrand we have the inequalities

$$\begin{aligned}+\infty &> \sup_{s\in J}\int_s^{+\infty}\|\Phi(u,s)x\|\mathrm{d}\mu_*(u)\\ &\geq \int_c^d\|\Phi(u,c)x\|\mathrm{d}\mu_*(u)\\ &\geq \int_c^d K^{-1}\|\Phi(d,c)x\|\mathrm{d}\mu_*(u)\\ &= K^{-1}\|\Phi(d,c)x\|\mu_*([c,d]).\end{aligned}$$

As the above inequality is satisfied by all $x\in\mathbb{R}^n$ and $[c,d]\subset J$, the Banach–Steinhaus theorem implies the existence of a constant M such that

$$\mu_*([c,d])K^{-1}\|\Phi(d,c)\|_{\mathcal{L}}\leq M$$

for any compact interval $[c,d] \subset J$.

There exists $T \in (e_*, +\infty)$ large enough such that $\frac{MK}{\mu_*([e_*,T])} < 1$. Then, by the invariance of μ_* under $*$-translations given for intervals in Lemma 3.4 we have:

$$\|\Phi(T*s,s)\|_{\mathcal{L}} \leq \frac{MK}{\mu_*([s,T*s])} = \frac{MK}{\mu_*([e_*,T])} < 1,$$

which due to Theorem 4.1 proves that $\Phi(t,s)$ is uniformly asymptotically h-stable. □

Theorem 5.2. *A nonautonomous system* (1.1) *with a transition matrix* $\Phi(t,s)$ *is uniformly asymptotically h-stable if and only if it is uniformly stable and*

$$\sup_{t\in J}\int_a^t \|\Phi(t,u)\|_{\mathcal{L}} \mathrm{d}\mu_*(u) < \infty. \tag{5.2}$$

Proof. We begin with the forward direction of the equivalence. As uniform asymptotic h-stability implies uniform stability, we only need to prove that $\Phi(t,s)$ satisfies (5.2). For any $t \in J = (a,+\infty)$ we replace $\|\Phi(t,s)\|_{\mathcal{L}}$ with its bound (1.2) to obtain the inequality

$$\begin{aligned}
\int_a^t \|\Phi(t,u)\|\mathrm{d}\mu_*(u) &\leq K\int_a^t \left(\frac{h(u)}{h(t)}\right)^{\alpha} \mathrm{d}\mu_*(u) \\
&= K\int_a^t \left(\frac{h(u)}{h(t)}\right)^{\alpha} \frac{h'(u)}{h(u)}\mathrm{d}m(u) \\
&= \frac{K}{h(t)^{\alpha}} \lim_{s\to a^+}\int_s^t h(u)^{\alpha-1}h'(u)\mathrm{d}m(u) \\
&= K\alpha^{-1},
\end{aligned}$$

which proves the first part of the theorem.

Now we prove the backward direction of the equivalence. Let $K > 0$ be the bound of $\|\Phi(t,s)\|_{\mathcal{L}}$ obtained by the uniform stability and $M > 0$ a bound of (5.2). For all $a < s \leq t < +\infty$ it is true that

$$\begin{aligned}
KM &> K\int_s^t \|\Phi(t,u)\|_{\mathcal{L}}\mathrm{d}\mu_*(u) \\
&> \int_s^t \|\Phi(t,u)\|_{\mathcal{L}}\|\Phi(u,s)\|_{\mathcal{L}}\mathrm{d}\mu_*(u) \\
&\geq \int_s^t \|\Phi(t,s)\|_{\mathcal{L}}\mathrm{d}\mu_*(u) \\
&= \log\left(\frac{h(t)}{h(s)}\right)\|\Phi(t,s)\|_{\mathcal{L}}.
\end{aligned} \tag{5.3}$$

The function h is upper unbounded, hence there exists $T \in (e_*, +\infty)$ such that $\log(h(T)) > KM$. Given any $s \in J$, the inequality (5.3) for the pair $(T*s,s)$ is

$$\|\Phi(T*s,s)\|_{\mathcal{L}} \leq \frac{KM}{\log(h(T))} < 1.$$

It follows from Theorem 4.1 that $\Phi(t,s)$ is uniformly h-stable. □

6 Discussion

In this work, we revisited the property of uniform h-stability for nonautonomous linear systems and proved a set of properties emulating the classical results for the uniform asymptotic stability and its equivalence with the uniform exponential one. The main tool was the characterization of h as an isomorphism of topological abelian groups $h : (J, *) \to (\mathbb{R}, \cdot)$ which, in the specific case $h(t) = e^t$ with $(J, *) = (\mathbb{R}, +)$, recovers the uniform exponential stability and its characterizations.

To the best of knowledge, there are few results on stability for nonautonomous linear systems based on group theory. However, the methods we have presented in this paper can be applied to generalize other existing theorems of the theory of exponential stability to uniform asymptotic h-stability. In particular, we have obtained preliminary results in the following topics:

a) The problem of $(\mathcal{B}, \mathcal{D})$-admissibility, that is, if for each $f \in \mathcal{B}$ the system

$$\dot{x} = A(t)x + f(t), \tag{6.1}$$

has a solution $x \in \mathcal{D}$, where $\mathcal{B}$ and $\mathcal{D}$ are suitable function spaces. An affirmative answer to this problem, allows us to give a functional characterization of stability properties of the linear system associated to (6.1).

b) The roughness of uniform asymptotic h-stability, which concerns sufficient smallness conditions for a matrix valued function $B(\cdot)$ such that the system

$$\dot{x} = A(t)x + B(t)x,$$

is uniformly asymptotically h-stable, provided that (1.1) is a uniformly asymptotically h-stable system.

To further emphasize the usefulness of the ideas developed in this article, we will also mention that the function $\frac{h'(t)}{h(t)}$ has been used by other authors to prove theorems regarding h-stability and dichotomy, see for example [7] and [3]. Despite this, the way it relates to the algebraic structure of these problems, such as the fact that it defines an invariant measure, has neither been established nor employed to its further extent in previous works.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal internships that could have appeared to influence the work reported on the paper.

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