Electronic Journal of Qualitative Theory of Differential Equations
2025, No. **41**, 1–15; https://doi.org/10.14232/ejqtde.2025.1.41 www.math.u-szeged.hu/ejqtde/

Global existence of strong solutions to nonhomogeneous MHD system in thin three-dimensional domains

Felipe W. Cruz✉[1], **Exequiel Mallea-Zepeda**[2] and **Marko A. Rojas-Medar**[3]

[1]Departamento de Matemática, Universidade Federal de Pernambuco, Recife - PE, 50740-560, Brazil
[2]Departamento de Matemáticas, Universidad Católica del Norte, Antofagasta, Chile
[3]Departamento de Matemática, Universidad de Tarapacá, Arica, Chile

Received 6 April 2025, appeared 29 August 2025
Communicated by Maria Alessandra Ragusa

Abstract. We establish the global existence of strong solutions to the nonhomogeneous incompressible magnetohydrodynamics (MHD) equations in a thin three-dimensional domain $\Omega = \mathbb{R}^2 \times (0, \epsilon)$, with $\epsilon \in (0, 1]$, subject to Dirichlet boundary conditions on the top and bottom boundaries. Global well-posedness may hold for large initial data, provided the vertical thickness ϵ is sufficiently small. Moreover, when $\epsilon \to 0^+$, both the velocity and magnetic field tend to vanish away from the initial time. The analysis is based on *a priori* H^2 estimates of the solutions, with particular attention to the dependence on the vertical parameter ϵ.

Keywords: MHD system, thin domains, global existence.

2020 Mathematics Subject Classification: 35Q35, 76D03, 76W05.

1 Introduction

This work investigates nonhomogeneous incompressible magnetohydrodynamic (MHD) flows in thin 3D domains. The equations involve a coupling between the nonhomogeneous Navier–Stokes system and the Maxwell equations in $Q := \Omega \times (0, \infty)$, where $\Omega := \mathbb{R}^2 \times (0, \epsilon)$, with $0 < \epsilon \leq 1$, is a thin domain of $\mathbb{R}^3$. The governing equations, described extensively in textbooks such as [8, 13, 20], are as follows:

$$\begin{cases} \rho \boldsymbol{u}_t + \rho(\boldsymbol{u} \cdot \nabla)\boldsymbol{u} - \mu \Delta \boldsymbol{u} + \nabla\left(P + \frac{1}{2}|\boldsymbol{b}|^2\right) = (\boldsymbol{b} \cdot \nabla)\boldsymbol{b}, \\ \boldsymbol{b}_t + (\boldsymbol{u} \cdot \nabla)\boldsymbol{b} - \eta \Delta \boldsymbol{b} = (\boldsymbol{b} \cdot \nabla)\boldsymbol{u}, \\ \rho_t + \boldsymbol{u} \cdot \nabla \rho = 0, \\ \operatorname{div} \boldsymbol{u} = 0, \quad \operatorname{div} \boldsymbol{b} = 0. \end{cases} \tag{1.1}$$

✉Corresponding author. Email: felipe.wcruz@ufpe.br

Here, $\boldsymbol{u}(\boldsymbol{x},t) \in \mathbb{R}^3$ and $\boldsymbol{b}(\boldsymbol{x},t) \in \mathbb{R}^3$ denote the velocity and magnetic field of the fluid at each point $(\boldsymbol{x},t) \in Q$, respectively. The real-valued functions $\rho(\boldsymbol{x},t) \in \mathbb{R}^+$ and $P(\boldsymbol{x},t) \in \mathbb{R}$ represent the fluid density and hydrostatic pressure, respectively. The quantity $|\boldsymbol{b}|^2/2$ corresponds to the *magnetic pressure*. Accordingly, we define the total pressure of the fluid as $p := P + \frac{1}{2}|\boldsymbol{b}|^2$. The positive constants μ and η denote the viscosity and the resistivity coefficient, respectively. The latter is inversely proportional to the electrical conductivity and serves as the magnetic diffusivity of the magnetic field.

We complete the system (1.1) with the following initial and boundary conditions:

$$\begin{cases} (\rho,\boldsymbol{u},\boldsymbol{b})\,\big|_{t=0} = (\rho_0,\boldsymbol{u}_0,\boldsymbol{b}_0) & \text{in } \Omega, \\ (\boldsymbol{u}(\boldsymbol{x},t),\boldsymbol{b}(\boldsymbol{x},t)) = (\boldsymbol{0},\boldsymbol{0}) & \text{for all } (\boldsymbol{x},t) \in \Gamma \times (0,\infty), \end{cases} \tag{1.2}$$

where $\Gamma := \big\{(x_1,x_2,x_3) \,/\, (x_1,x_2) \in \mathbb{R}^2, x_3 = 0 \text{ or } x_3 = \epsilon\big\}$.

The magnetohydrodynamic (MHD) equations with variable density have been studied extensively in recent years; see [1,9,14,19,25] and the references therein (see also [3,4,6,10–12] for further results concerning the MHD equations). These results are closely related to those obtained for the variable-density Navier-Stokes equations. The study of the classical Navier–Stokes system in thin domains was initiated by Raugel and Sell in [22] and subsequently generalized in several directions; see, for example, [16] and the references therein. The first work addressing the variable-density Navier-Stokes equations in thin domains was carried out by Liao in [18].

In this work, we show that results analogous to those of Liao [18] are valid for the system (1.1)–(1.2). Specifically, we prove that global well-posedness may be achieved for large initial data when the vertical thickness ϵ is sufficiently small. For additional related work on fluid mechanics equations in thin domains, see, for instance, [5,7,17,23,24].

The paper is organized as follows. In Section 2, we introduce the basic notation and state the main result. In Section 3, we provide the proof of our main theorem (see Theorem 2.6 below).

2 Preliminaries and main result

In this section, we introduce the function spaces, definitions, and the main result that will be used throughout this work.

2.1 Functional framework

Throughout this paper, we employ the standard Lebesgue space $L^p(\Omega)$, for $1 \le p \le \infty$, and the Sobolev space

$$W^{m,p}(\Omega) := \Big\{ f \in L^p(\Omega) \;:\; \|\partial^k f\|_p < \infty \text{ for all } |k| \le m \Big\}, \quad m \in \mathbb{N} \cup \{0\},$$

with norms denoted by $\|\cdot\|_p$ and $\|\cdot\|_{W^{m,p}}$, respectively. In particular, we denote the L^2 norm and the L^2 inner product by $\|\cdot\|$ and $\langle\cdot,\cdot\rangle$, respectively.

Corresponding Sobolev spaces of vector-valued functions are denoted in boldface. For example, we write $\boldsymbol{L}^2(\Omega) := (L^2(\Omega))^n$ and $\boldsymbol{W}^{m,p}(\Omega) := (W^{m,p}(\Omega))^n$, where $n \in \mathbb{N}$.

Furthermore, we denote by $\boldsymbol{W}_0^{m,p}(\Omega)$ the closure of $\boldsymbol{C}_0^\infty(\Omega)$ in $\boldsymbol{W}^{m,p}(\Omega)$. When $p = 2$, we set

$$\boldsymbol{H}^m(\Omega) := \boldsymbol{W}^{m,2}(\Omega) \quad \text{and} \quad \boldsymbol{H}_0^m(\Omega) := \boldsymbol{W}_0^{m,2}(\Omega).$$

We define

$$\mathcal{V}(\Omega) := \{\boldsymbol{v} \in \boldsymbol{C}_0^\infty(\Omega) \,:\, \operatorname{div} \boldsymbol{v} = 0 \text{ in } \Omega\},$$

and denote by $\boldsymbol{H}$ and $\boldsymbol{V}$ the closures of $\mathcal{V}(\Omega)$ in $\boldsymbol{L}^2(\Omega)$ and $\boldsymbol{H}_0^1(\Omega)$, respectively.

Let X be a Banach space and $0 < T \leq \infty$. We denote by $L^p(0,T;X)$, or alternatively $L_T^p(X)$, the Banach space of X-valued (equivalence classes of) functions defined almost everywhere on $[0,T]$ that are L^p-integrable in the sense of Bochner. The associated norm is denoted by $\|\cdot\|_{L_T^p(X)}$.

We will frequently work with spaces of the form $L^p(0,T;X)$ where $X = \boldsymbol{W}^{m,q}(\Omega)$. In such cases, for any $\boldsymbol{v} \in L^p(0,T;\boldsymbol{W}^{m,q}(\Omega))$, the notation $\boldsymbol{v}(t)$ refers to the spatial function $\boldsymbol{v}(\cdot,t)$. We also use the notation $C^k([0,T);X)$ for the space of k times continuously differentiable functions on $[0,T)$ with values in X. Moreover, as usual, the letters $C, K, C_1, K_1, \ldots$ are positive constants, independents of $(\rho, \boldsymbol{u}, \boldsymbol{b})$, but their values may change from line to line.

Notation

As usual, we set

$$\|(\varphi,\psi)\|_X^2 := \|\varphi\|_X^2 + \|\psi\|_X^2.$$

In addition, we introduce the pair $\boldsymbol{z}(t) := (\boldsymbol{u}(t), \boldsymbol{b}(t))$ with initial data $\boldsymbol{z}_0 := \boldsymbol{z}(0) = (\boldsymbol{u}_0, \boldsymbol{b}_0)$, and the following norms:

$$\begin{aligned}
\|\boldsymbol{z}(t)\|^2 &:= \|\boldsymbol{u}(t)\|^2 + \|\boldsymbol{b}(t)\|^2, & \|\boldsymbol{z}_t(t)\|^2 &:= \|\boldsymbol{u}_t(t)\|^2 + \|\boldsymbol{b}_t(t)\|^2,\\
\|\nabla \boldsymbol{z}(t)\|^2 &:= \|\nabla \boldsymbol{u}(t)\|^2 + \|\nabla \boldsymbol{b}(t)\|^2, & \|\Delta \boldsymbol{z}(t)\|^2 &:= \|\Delta \boldsymbol{u}(t)\|^2 + \|\Delta \boldsymbol{b}(t)\|^2,\\
\|\nabla^2 \boldsymbol{z}(t)\|^2 &:= \|\nabla^2 \boldsymbol{u}(t)\|^2 + \|\nabla^2 \boldsymbol{b}(t)\|^2, & \|\nabla \boldsymbol{z}_t(t)\|^2 &:= \|\nabla \boldsymbol{u}_t(t)\|^2 + \|\nabla \boldsymbol{b}_t(t)\|^2.
\end{aligned}$$

2.2 Auxiliary results

Following the approach in [18, Theorem 1.1], we obtain the following result concerning strong solutions to the system (1.1)–(1.2).

Theorem 2.1. *Assume that the initial data $(\rho_0, \boldsymbol{u}_0, \boldsymbol{b}_0)$ satisfy*

$$0 < \alpha \leq \rho_0(\boldsymbol{x}) \leq \beta < \infty \quad \text{in } \Omega, \text{ with } \alpha,\ \beta \in \mathbb{R}^+, \tag{2.1}$$

$$\boldsymbol{u}_0,\ \boldsymbol{b}_0 \in \boldsymbol{V}. \tag{2.2}$$

Then, there exists a time $T^* > 0$ and a unique strong solution $(\rho, \boldsymbol{u}, p, \boldsymbol{b})$ in $\Omega \times [0,T^*]$ for the problem (1.1)–(1.2) satisfying

$$\begin{gathered}
\alpha \leq \rho(x,t) \leq \beta,\\
\boldsymbol{u},\ \boldsymbol{b} \in L^\infty(0,T^*;\boldsymbol{V}) \cap L^2(0,T^*;\boldsymbol{V} \cap \boldsymbol{H}^2(\Omega)),\\
\nabla \boldsymbol{u},\ \nabla \boldsymbol{b} \in L^2(0,T^*;\boldsymbol{L}^\infty(\Omega) \cap \boldsymbol{W}^{1,6}(\Omega)),\\
\boldsymbol{u}_t,\ \boldsymbol{b}_t \in L^2(0,T^*;\boldsymbol{H}),\\
\nabla p \in L^\infty(0,T^*;\boldsymbol{L}^2(\Omega)) \cap L^2(0,T^*;\boldsymbol{L}^6(\Omega)).
\end{gathered}$$

Moreover, if $\boldsymbol{u}_0,\ \boldsymbol{b}_0 \in \boldsymbol{V} \cap \boldsymbol{H}^2(\Omega)$, then the unique solution satisfies

$$\boldsymbol{u},\ \boldsymbol{b} \in L^\infty(0,T^*;\boldsymbol{V} \cap \boldsymbol{H}^2(\Omega)) \quad \text{and} \quad \boldsymbol{u}_t,\ \boldsymbol{b}_t \in L^\infty(0,T^*;\boldsymbol{H}) \cap L^2(0,T^*;\boldsymbol{V}).$$

We will show that the unique solution to the problem (1.1)–(1.2) exists globally in time, provided that the initial data $\boldsymbol{u}_0$ and $\boldsymbol{b}_0$ satisfy the "smallness" condition

$$\epsilon^{\frac{1}{2}}\left(\|\nabla \boldsymbol{u}_0\|^2+\|\nabla \boldsymbol{b}_0\|^2\right)^{\frac{1}{2}} \leq c_0, \tag{2.3}$$

for some constant $c_0>0$ sufficiently small, depending only on α and β.

To this end, the following estimates will be instrumental.

Lemma 2.2 (See [18], Lemma 1.6, p. 174)**.** *If $\boldsymbol{v} \in \boldsymbol{H}_0^1(\Omega)$, then the following Poincaré and Sobolev's inequalities hold:*

$$\|\boldsymbol{v}\| \leq \epsilon\|\partial_3 \boldsymbol{v}\|, \qquad \|\boldsymbol{v}\|_6 \leq C\|\nabla \boldsymbol{v}\|.$$

Remark 2.3. Observe that if $\boldsymbol{u}_0, \boldsymbol{b}_0$ satisfy (2.2)–(2.3), then

$$\begin{aligned}
&\|\boldsymbol{u}_0\|\|\nabla \boldsymbol{u}_0\| \leq \epsilon\|\partial_3 \boldsymbol{u}_0\|\|\nabla \boldsymbol{u}_0\| \leq c_0^2, \qquad \|\boldsymbol{u}_0\|\|\nabla \boldsymbol{b}_0\| \leq \epsilon\|\partial_3 \boldsymbol{u}_0\|\|\nabla \boldsymbol{b}_0\| \leq c_0^2,\\
&\|\boldsymbol{b}_0\|\|\nabla \boldsymbol{u}_0\| \leq \epsilon\|\partial_3 \boldsymbol{b}_0\|\|\nabla \boldsymbol{u}_0\| \leq c_0^2, \qquad \|\boldsymbol{b}_0\|\|\nabla \boldsymbol{b}_0\| \leq \epsilon\|\partial_3 \boldsymbol{b}_0\|\|\nabla \boldsymbol{b}_0\| \leq c_0^2.
\end{aligned}$$

In particular, we have

$$\left(\|\boldsymbol{u}_0\|^2+\|\boldsymbol{b}_0\|^2\right)\left(\|\nabla \boldsymbol{u}_0\|^2+\|\nabla \boldsymbol{b}_0\|^2\right) \leq 4c_0^4.$$

Remark 2.4. From smallness condition (2.3), applying the Poincaré's inequality in the vertical direction on $\boldsymbol{u}_0$ and $\boldsymbol{b}_0$, we get the following smallness assumption:

$$\|\boldsymbol{u}_0\| \leq \epsilon\|\partial_3 \boldsymbol{u}_0\| \leq \epsilon\|\nabla \boldsymbol{u}_0\| \leq c_0\epsilon^{\frac{1}{2}} \quad \text{and} \quad \|\boldsymbol{b}_0\| \leq \epsilon\|\partial_3 \boldsymbol{b}_0\| \leq \epsilon\|\nabla \boldsymbol{b}_0\| \leq c_0\epsilon^{\frac{1}{2}}.$$

Remark 2.5. The "smallness" assumption (2.3) closely resembles the one proposed in [21]:

$$\|\boldsymbol{u}_0\|\|\nabla \boldsymbol{u}_0\| \leq \vartheta, \quad \text{with } \vartheta>0 \text{ sufficiently small.}$$

In fact, from Remark 2.2, we have

$$\|\boldsymbol{u}_0\|\|\nabla \boldsymbol{u}_0\| \leq c_0^2 \quad \text{and} \quad \|\boldsymbol{b}_0\|\|\nabla \boldsymbol{b}_0\| \leq c_0^2.$$

Therefore, under the smallness condition

$$\|\boldsymbol{u}_0\|\|\nabla \boldsymbol{u}_0\|+\|\boldsymbol{b}_0\|\|\nabla \boldsymbol{b}_0\| \leq 2c_0^2,$$

and following the approach in [21], we conclude that the strong solution given in Theorem 2.1 is global in time (i.e., $T^*=T$, and T could be $+\infty$).

2.3 Statement of the main result

Initially, we define

$$\begin{aligned}
\mathcal{U}(t) &:= \left\|(\boldsymbol{u}, \boldsymbol{b})\right\|^2_{L_t^\infty(\boldsymbol{L}^2(\Omega))}+\left\|(\sqrt{\mu}\,\nabla \boldsymbol{u}, \sqrt{\eta}\,\nabla \boldsymbol{b})\right\|^2_{L_t^2(\boldsymbol{L}^2(\Omega))},\\
\mathcal{V}(t) &:= \left\|(\nabla \boldsymbol{u}, \nabla \boldsymbol{b})\right\|^2_{L_t^\infty(\boldsymbol{L}^2(\Omega))}+\left\|(\boldsymbol{u}_t, \boldsymbol{b}_t, \Delta \boldsymbol{u}, \Delta \boldsymbol{b})\right\|^2_{L_t^2(\boldsymbol{L}^2(\Omega))},\\
\mathcal{W}(t) &:= \left\|(\boldsymbol{u}_t, \boldsymbol{b}_t, \nabla^2 \boldsymbol{u}, \nabla^2 \boldsymbol{b}, \nabla p)\right\|^2_{L_t^\infty(\boldsymbol{L}^2(\Omega))}+\left\|(\nabla \boldsymbol{u}_t, \nabla \boldsymbol{b}_t)\right\|^2_{L_t^2(\boldsymbol{L}^2(\Omega))},
\end{aligned} \tag{2.4}$$

and

$$\mathcal{U}_0 := \|\boldsymbol{u}_0\|^2+\|\boldsymbol{b}_0\|^2, \quad \mathcal{V}_0 := \|\nabla \boldsymbol{u}_0\|^2+\|\nabla \boldsymbol{b}_0\|^2, \quad \mathcal{W}_0 := \|\nabla^2 \boldsymbol{u}_0\|^2+\|\nabla^2 \boldsymbol{b}_0\|^2. \tag{2.5}$$

The main result of this paper is the following.

Theorem 2.6. *Suppose that the initial data $(\rho_0, \boldsymbol{u}_0, \boldsymbol{b}_0)$ satisfy (2.1)–(2.3). Then the problem (1.1)–(1.2) has a unique global in time strong solution $(\rho, \boldsymbol{u}, p, \boldsymbol{b})$ such that, for all $T \in (0, \infty)$, the following inequalities hold*

$$\alpha \le \rho(\boldsymbol{x}, t) \le \beta, \qquad \mathcal{U}(T)C \le C\,\mathcal{U}_0, \qquad \mathcal{V}(T) \le \widetilde{C}\,\mathcal{V}_0,$$
$$\|(\boldsymbol{u}, \boldsymbol{b})\|^2_{L^\infty_T(\boldsymbol{L}^2(\Omega))} \le C\,\mathcal{U}_0\, e^{-\gamma T/\epsilon^2},$$
$$\|(\nabla \boldsymbol{u}, \nabla \boldsymbol{b})\|^2_{L^\infty_T(\boldsymbol{L}^2(\Omega))} \le \widetilde{C}\,\mathcal{V}_0\, e^{-\overline{\gamma} T/\epsilon^2},$$

where $C = C(\alpha,\, \beta) > 0$, $\widetilde{C} = \widetilde{C}(\alpha,\, \beta,\, \mu,\, \eta,\, c_0) > 0$, $\gamma := \min\left\{\frac{2\mu}{\beta}, 2\eta\right\}$ and $\overline{\gamma} := \min\left\{\frac{\widetilde{\gamma}}{\mu}, \frac{\widetilde{\gamma}}{\eta}\right\}$, with $\widetilde{\gamma} := \min\left\{\frac{\mu^2}{16\beta}, \frac{\eta^2}{16}\right\}$. Moreover, if $\boldsymbol{u}_0,\ \boldsymbol{b}_0 \in \boldsymbol{V} \cap \boldsymbol{H}^2(\Omega)$, then

$$\mathcal{W}(T) \le \widetilde{C}\,\mathcal{W}_0 \quad \text{and} \quad \|(\nabla^2 \boldsymbol{u}, \nabla^2 \boldsymbol{b})\|^2_{L^\infty_T(\boldsymbol{L}^2(\Omega))} \le \widetilde{C}\,\mathcal{W}_0\, e^{-\sigma T/\epsilon^2},$$

where $\sigma := \min\{\gamma, \overline{\gamma}\}$. In particular, for all $t_\star > 0$, we conclude that

$$\lim_{\epsilon \to 0^+} (\boldsymbol{u}, \boldsymbol{b}) = (\boldsymbol{0}, \boldsymbol{0}) \quad \text{uniformly in } C\big([t_\star, \infty); \boldsymbol{H}^2(\Omega)\big).$$

3 Proof of the main result

Theorem 2.6 follows as a consequence of Theorem 2.1 above, together with Propositions 3.1 and 3.3 established below.

A priori bounds

From now on, we denote by C, K, and $\widetilde{C}$ generic positive constants, which may vary from line to line. More specifically, C denotes a constant depending only on α and β; K depends solely on α, β, μ, and η; and $\widetilde{C}$ depends on α, β, μ, η, and c_0.

In the following, we derive *a priori* estimates for the strong solution $(\rho, \boldsymbol{u}, p, \boldsymbol{b})$ to the problem (1.1)–(1.2), defined on $\Omega \times [0, T]$, for $T \in (0, \infty)$.

Proposition 3.1 (Estimates in the $\boldsymbol{H}^1(\Omega)$ norm)**.** *Assume that the initial data $(\rho_0, \boldsymbol{u}_0, \boldsymbol{b}_0)$ satisfy (2.1)–(2.3). Let $(\rho, \boldsymbol{u}, p, \boldsymbol{b})$ be the strong solution furnished by Theorem 2.1 defined on $\Omega \times [0, T]$. There exist constants $C = C(\alpha,\, \beta) > 0$ and $\widetilde{C} = \widetilde{C}(\alpha,\, \beta,\, \mu,\, \eta,\, c_0) > 0$, such that*

$$\alpha \le \rho(\boldsymbol{x}, t) \le \beta, \qquad \forall\, (\boldsymbol{x}, t) \in \Omega \times [0, T], \tag{3.1}$$
$$\|\boldsymbol{u}(t)\|^2 + \|\boldsymbol{b}(t)\|^2 \le C\left(\|\boldsymbol{u}_0\|^2 + \|\boldsymbol{b}_0\|^2\right), \qquad \forall\, t \in [0, T], \tag{3.2}$$
$$\int_0^t \left(\mu\|\nabla \boldsymbol{u}(s)\|^2 + \eta\|\nabla \boldsymbol{b}(s)\|^2\right) ds \le C\left(\|\boldsymbol{u}_0\|^2 + \|\boldsymbol{b}_0\|^2\right), \qquad \forall\, t \in [0, T], \tag{3.3}$$
$$\|\boldsymbol{u}(t)\|^2 + \|\boldsymbol{b}(t)\|^2 \le C\left(\|\boldsymbol{u}_0\|^2 + \|\boldsymbol{b}_0\|^2\right) e^{-\gamma t/\epsilon^2}, \qquad \forall\, t \in [0, T], \tag{3.4}$$
$$\|\nabla \boldsymbol{u}(t)\|^2 + \|\nabla \boldsymbol{b}(t)\|^2 \le \widetilde{C}\left(\|\nabla \boldsymbol{u}_0\|^2 + \|\nabla \boldsymbol{b}_0\|^2\right), \qquad \forall\, t \in [0, T], \tag{3.5}$$
$$\int_0^t \left(\|\boldsymbol{u}_t(s)\|^2 + \|\boldsymbol{b}_t(s)\|^2\right) ds \le \widetilde{C}\left(\|\nabla \boldsymbol{u}_0\|^2 + \|\nabla \boldsymbol{b}_0\|^2\right), \qquad \forall\, t \in [0, T], \tag{3.6}$$
$$\int_0^t \left(\|\Delta \boldsymbol{u}(s)\|^2 + \|\Delta \boldsymbol{b}(s)\|^2\right) ds \le \widetilde{C}\left(\|\nabla \boldsymbol{u}_0\|^2 + \|\nabla \boldsymbol{b}_0\|^2\right), \qquad \forall\, t \in [0, T], \tag{3.7}$$
$$\|\nabla \boldsymbol{u}(t)\|^2 + \|\nabla \boldsymbol{b}(t)\|^2 \le \widetilde{C}\left(\|\nabla \boldsymbol{u}_0\|^2 + \|\nabla \boldsymbol{b}_0\|^2\right) e^{-\overline{\gamma} t/\epsilon^2}, \qquad \forall\, t \in [0, T], \tag{3.8}$$

with $\gamma := \min\left\{\frac{2\mu}{\beta}, 2\eta\right\}$ and $\overline{\gamma} := \min\left\{\frac{\widetilde{\gamma}}{\mu}, \frac{\widetilde{\gamma}}{\eta}\right\}$, where $\widetilde{\gamma} := \min\left\{\frac{\mu^2}{16\beta}, \frac{\eta^2}{16}\right\}$.

Proof. For a given $\boldsymbol{u} = \boldsymbol{u}(\boldsymbol{x}, t)$, one gets ρ, solution of $(1.1)_3$, through the classical method of characteristics. Indeed, if

$$\Pi(s) := \rho(\boldsymbol{x} + s\boldsymbol{u},\, t + s), \quad \forall\, s \in [-t, \infty) \qquad (\text{with } t \geq 0),$$

then

$$\frac{d}{ds}\Pi(s) = \nabla\rho(\boldsymbol{x} + s\boldsymbol{u},\, t + s) \cdot \boldsymbol{u}(\boldsymbol{x} + s\boldsymbol{u},\, t + s) + \rho_t(\boldsymbol{x} + s\boldsymbol{u},\, t + s) = 0, \quad \forall\, s \in (-t, \infty).$$

Thus, $\Pi(\cdot)$ is a constant function of s. Hence,

$$\rho(\boldsymbol{x}, t) = \Pi(0) = \Pi(-t) = \rho(\boldsymbol{x} - t\boldsymbol{u},\, 0) = \rho_0(\boldsymbol{x} - t\boldsymbol{u}).$$

Therefore, from the hypothesis (2.1), it follows that ρ satisfies bound (3.1). Testing equation $(1.1)_1$ by $\boldsymbol{u}$ and equation $(1.1)_2$ by $\boldsymbol{b}$, one obtains, respectively

$$\frac{1}{2}\frac{d}{dt}\left(\|\sqrt{\rho}\boldsymbol{u}\|^2\right) + \mu\|\nabla\boldsymbol{u}\|^2 = \langle(\boldsymbol{b} \cdot \nabla)\boldsymbol{b}, \boldsymbol{u}\rangle$$

and

$$\frac{1}{2}\frac{d}{dt}\left(\|\boldsymbol{b}\|^2\right) + \eta\|\nabla\boldsymbol{b}\|^2 = \langle(\boldsymbol{b} \cdot \nabla)\boldsymbol{u}, \boldsymbol{b}\rangle,$$

since $\rho_t = -\boldsymbol{u} \cdot \nabla\rho$ and $\operatorname{div}\boldsymbol{u} = 0$. Adding the above identities and noting that $\langle(\boldsymbol{b} \cdot \nabla)\boldsymbol{b}, \boldsymbol{u}\rangle + \langle(\boldsymbol{b} \cdot \nabla)\boldsymbol{u}, \boldsymbol{b}\rangle = 0$, because $\operatorname{div}\boldsymbol{b} = 0$, we get the energy equation

$$\frac{d}{dt}\left(\|\sqrt{\rho}\boldsymbol{u}\|^2 + \|\boldsymbol{b}\|^2\right) + 2\left(\mu\|\nabla\boldsymbol{u}\|^2 + \eta\|\nabla\boldsymbol{b}\|^2\right) = 0. \tag{3.9}$$

Integrating (3.9) in time and using (3.1), we find

$$\alpha\|\boldsymbol{u}(t)\|^2 + \|\boldsymbol{b}(t)\|^2 + 2\int_0^t \left(\mu\|\nabla\boldsymbol{u}(s)\|^2 + \eta\|\nabla\boldsymbol{b}(s)\|^2\right) ds \leq \beta\|\boldsymbol{u}_0\|^2 + \|\boldsymbol{b}_0\|^2, \tag{3.10}$$

which proves the estimates (3.2) and (3.3).

Furthermore, applying Poincaré's inequality (see Lemma 2.2) to identity (3.9), we obtain

$$\frac{d}{dt}\left(\|\sqrt{\rho}\boldsymbol{u}\|^2 + \|\boldsymbol{b}\|^2\right) + \frac{\gamma}{\epsilon^2}\left(\|\sqrt{\rho}\boldsymbol{u}\|^2 + \|\boldsymbol{b}\|^2\right) \leq 0, \tag{3.11}$$

where $\gamma := \min\{2\mu/\beta, 2\eta\}$. Multiplying the above inequality by the integrating factor $e^{\gamma\epsilon^{-2}t}$, we get

$$\frac{d}{dt}\left(e^{\gamma\epsilon^{-2}t}\left[\|\sqrt{\rho}\boldsymbol{u}\|^2 + \|\boldsymbol{b}\|^2\right]\right) \leq 0.$$

Integrating from 0 to t, one obtains

$$\|\boldsymbol{u}(t)\|^2 + \|\boldsymbol{b}(t)\|^2 \leq C\left(\|\boldsymbol{u}_0\|^2 + \|\boldsymbol{b}_0\|^2\right) e^{-\gamma t/\epsilon^2}. \tag{3.12}$$

This proves bound (3.4).

On the other hand, taking the $\boldsymbol{L}^2$ scalar product of first and second equations in (1.1) by $\boldsymbol{u}_t$ and $\boldsymbol{b}_t$ respectively, and adding the resulting identities, one gets

$$\begin{aligned}\|\sqrt{\rho}\boldsymbol{u}_t\|^2 + \|\boldsymbol{b}_t\|^2 &+ \frac{1}{2}\frac{d}{dt}\left(\mu\|\nabla\boldsymbol{u}\|^2 + \eta\|\nabla\boldsymbol{b}\|^2\right) \\ &= \langle(\boldsymbol{b} \cdot \nabla)\boldsymbol{b}, \boldsymbol{u}_t\rangle + \langle\rho(\boldsymbol{u} \cdot \nabla)\boldsymbol{u}, -\boldsymbol{u}_t\rangle + \langle(\boldsymbol{b} \cdot \nabla)\boldsymbol{u}, \boldsymbol{b}_t\rangle + \langle(\boldsymbol{u} \cdot \nabla)\boldsymbol{b}, -\boldsymbol{b}_t\rangle.\end{aligned} \tag{3.13}$$

Analogously, making the $\boldsymbol{L}^2$ inner product of first and second equations in (1.1) by $-\kappa\Delta\boldsymbol{u}$ and $-\widetilde{\kappa}\Delta\boldsymbol{b}$ respectively, with κ, $\widetilde{\kappa} > 0$ to be chosen later on, we obtain, after adding the resulting equations, that

$$\kappa\mu\|\Delta\boldsymbol{u}\|^2 + \widetilde{\kappa}\eta\|\Delta\boldsymbol{b}\|^2 = \kappa\langle\rho\boldsymbol{u}_t, \Delta\boldsymbol{u}\rangle + \kappa\langle\rho(\boldsymbol{u}\cdot\nabla)\boldsymbol{u}, \Delta\boldsymbol{u}\rangle + \kappa\langle(\boldsymbol{b}\cdot\nabla)\boldsymbol{b}, -\Delta\boldsymbol{u}\rangle$$
$$+ \widetilde{\kappa}\langle\boldsymbol{b}_t, \Delta\boldsymbol{b}\rangle + \widetilde{\kappa}\langle(\boldsymbol{u}\cdot\nabla)\boldsymbol{b}, \Delta\boldsymbol{b}\rangle + \widetilde{\kappa}\langle(\boldsymbol{b}\cdot\nabla)\boldsymbol{u}, -\Delta\boldsymbol{b}\rangle. \tag{3.14}$$

Then applying Cauchy–Schwarz and Young's inequalities, we bound each term on the right-hand side of equalities (3.13) and (3.14) as follows:

$$|\langle(\boldsymbol{b}\cdot\nabla)\boldsymbol{b}, \boldsymbol{u}_t\rangle| \leq \frac{1}{6}\|\sqrt{\rho}\boldsymbol{u}_t\|^2 + \frac{3}{2\alpha}\|(\boldsymbol{b}\cdot\nabla)\boldsymbol{b}\|^2,$$
$$|\langle\rho(\boldsymbol{u}\cdot\nabla)\boldsymbol{u}, -\boldsymbol{u}_t\rangle| \leq \frac{1}{6}\|\sqrt{\rho}\boldsymbol{u}_t\|^2 + \frac{3\beta}{2}\|(\boldsymbol{u}\cdot\nabla)\boldsymbol{u}\|^2,$$
$$|\langle(\boldsymbol{b}\cdot\nabla)\boldsymbol{u}, \boldsymbol{b}_t\rangle| \leq \frac{1}{6}\|\boldsymbol{b}_t\|^2 + \frac{3}{2}\|(\boldsymbol{b}\cdot\nabla)\boldsymbol{u}\|^2,$$
$$|\langle(\boldsymbol{u}\cdot\nabla)\boldsymbol{b}, -\boldsymbol{b}_t\rangle| \leq \frac{1}{6}\|\boldsymbol{b}_t\|^2 + \frac{3}{2}\|(\boldsymbol{u}\cdot\nabla)\boldsymbol{b}\|^2,$$
$$\kappa|\langle\rho\boldsymbol{u}_t, \Delta\boldsymbol{u}\rangle| \leq \frac{1}{6}\|\sqrt{\rho}\boldsymbol{u}_t\|^2 + \frac{3}{2}\beta\kappa^2\|\Delta\boldsymbol{u}\|^2,$$
$$\kappa|\langle\rho(\boldsymbol{u}\cdot\nabla)\boldsymbol{u}, \Delta\boldsymbol{u}\rangle| \leq \frac{\beta}{4}\|(\boldsymbol{u}\cdot\nabla)\boldsymbol{u}\|^2 + \frac{3}{2}\beta\kappa^2\|\Delta\boldsymbol{u}\|^2,$$
$$\kappa|\langle(\boldsymbol{b}\cdot\nabla)\boldsymbol{b}, -\Delta\boldsymbol{u}\rangle| \leq \frac{1}{4\alpha}\|(\boldsymbol{b}\cdot\nabla)\boldsymbol{b}\|^2 + \beta\kappa^2\|\Delta\boldsymbol{u}\|^2,$$
$$\widetilde{\kappa}|\langle\boldsymbol{b}_t, \Delta\boldsymbol{b}\rangle| \leq \frac{1}{6}\|\boldsymbol{b}_t\|^2 + \frac{3}{2}\widetilde{\kappa}^2\|\Delta\boldsymbol{b}\|^2,$$
$$\widetilde{\kappa}|\langle(\boldsymbol{u}\cdot\nabla)\boldsymbol{b}, \Delta\boldsymbol{b}\rangle| \leq \frac{1}{6}\|(\boldsymbol{u}\cdot\nabla)\boldsymbol{b}\|^2 + \frac{3}{2}\widetilde{\kappa}^2\|\Delta\boldsymbol{b}\|^2,$$
$$\widetilde{\kappa}|\langle(\boldsymbol{b}\cdot\nabla)\boldsymbol{u}, -\Delta\boldsymbol{b}\rangle| \leq \frac{1}{4}\|(\boldsymbol{b}\cdot\nabla)\boldsymbol{u}\|^2 + \widetilde{\kappa}^2\|\Delta\boldsymbol{b}\|^2.$$

Consequently, applying the estimates above to resulting equation of the sum of the identities (3.13) and (3.14) and choosing $\kappa = \mu/8\beta$ and $\widetilde{\kappa} = \eta/8$, one finds

$$\|\sqrt{\rho}\boldsymbol{u}_t\|^2 + \|\boldsymbol{b}_t\|^2 + \frac{d}{dt}\left(\mu\|\nabla\boldsymbol{u}\|^2 + \eta\|\nabla\boldsymbol{b}\|^2\right) + \frac{1}{8}\left(\frac{\mu^2}{\beta}\|\Delta\boldsymbol{u}\|^2 + \eta^2\|\Delta\boldsymbol{b}\|^2\right)$$
$$\leq C\left(\|(\boldsymbol{b}\cdot\nabla)\boldsymbol{b}\|^2 + \|(\boldsymbol{u}\cdot\nabla)\boldsymbol{u}\|^2 + \|(\boldsymbol{b}\cdot\nabla)\boldsymbol{u}\|^2 + \|(\boldsymbol{u}\cdot\nabla)\boldsymbol{b}\|^2\right). \tag{3.15}$$

Now, using Hölder, standard interpolation, Sobolev and Young's inequalities (see [2]), we have

$$C\|(\boldsymbol{b}\cdot\nabla)\boldsymbol{b}\|^2 \leq C\|\boldsymbol{b}\|_6^2\,\|\nabla\boldsymbol{b}\|_3^2 \leq C\|\boldsymbol{b}\|_6^2\,\|\nabla\boldsymbol{b}\|\,\|\Delta\boldsymbol{b}\| \leq C\|\nabla\boldsymbol{b}\|^3\,\|\Delta\boldsymbol{b}\|$$
$$\leq \frac{\eta^2}{32}\|\Delta\boldsymbol{b}\|^2 + K\|\nabla\boldsymbol{b}\|^6, \tag{3.16}$$
$$C\|(\boldsymbol{u}\cdot\nabla)\boldsymbol{u}\|^2 \leq \frac{\mu^2}{32\beta}\|\Delta\boldsymbol{u}\|^2 + K\|\nabla\boldsymbol{u}\|^6, \tag{3.17}$$
$$C\|(\boldsymbol{b}\cdot\nabla)\boldsymbol{u}\|^2 \leq \frac{\mu^2}{32\beta}\|\Delta\boldsymbol{u}\|^2 + K\|\nabla\boldsymbol{b}\|^4\|\nabla\boldsymbol{u}\|^2, \tag{3.18}$$
$$C\|(\boldsymbol{u}\cdot\nabla)\boldsymbol{b}\|^2 \leq \frac{\eta^2}{32}\|\Delta\boldsymbol{b}\|^2 + K\|\nabla\boldsymbol{u}\|^4\|\nabla\boldsymbol{b}\|^2. \tag{3.19}$$

Thus, applying the estimates (3.16)–(3.19) to inequality (3.15), we find

$$\begin{aligned}\|\sqrt{\rho}\boldsymbol{u}_t\|^2 + \|\boldsymbol{b}_t\|^2 + \frac{d}{dt}\left(\mu\|\nabla\boldsymbol{u}\|^2 + \eta\|\nabla\boldsymbol{b}\|^2\right) + \widetilde{\gamma}\left(\|\Delta\boldsymbol{u}\|^2 + \|\Delta\boldsymbol{b}\|^2\right)\\ \le K\left(\|\nabla\boldsymbol{u}\|^4 + \|\nabla\boldsymbol{b}\|^4\right)\left(\|\nabla\boldsymbol{u}\|^2 + \|\nabla\boldsymbol{b}\|^2\right),\end{aligned} \tag{3.20}$$

where $\widetilde{\gamma} := \min\left\{\frac{\mu^2}{16\beta}, \frac{\eta^2}{16}\right\}$. By integration of the above inequality over $[0,t]$, we obtain

$$\|\nabla\boldsymbol{z}(t)\|^2 + \int_0^t \left(\|\boldsymbol{z}_t(s)\|^2 + \|\Delta\boldsymbol{z}(s)\|^2\right) ds \le K\|\nabla\boldsymbol{z}_0\|^2 + K\int_0^t \|\nabla\boldsymbol{z}(s)\|^4\|\nabla\boldsymbol{z}(s)\|^2 ds.$$

So we deduce from estimate (3.10) that

$$\begin{aligned}\|\nabla\boldsymbol{z}(t)\|^2 + \int_0^t \left(\|\boldsymbol{z}_t(s)\|^2 + \|\Delta\boldsymbol{z}(s)\|^2\right) ds &\le K\left(\|\nabla\boldsymbol{z}_0\|^2 + \|\nabla\boldsymbol{z}\|^4_{L_t^\infty(\boldsymbol{L}^2(\Omega))}\|\nabla\boldsymbol{z}\|^2_{L_t^2(\boldsymbol{L}^2(\Omega))}\right)\\ &\le K\left(\|\nabla\boldsymbol{z}_0\|^2 + \|\boldsymbol{z}_0\|^2\|\nabla\boldsymbol{z}\|^4_{L_t^\infty(\boldsymbol{L}^2(\Omega))}\right).\end{aligned}$$

Having in mind the identities (2.4) and (2.5), one concludes that

$$\mathcal{V}(t) \le K\left(\mathcal{V}_0 + \mathcal{U}_0\,\mathcal{V}(t)^2\right).$$

Hence, if c_0 is sufficiently small, we conclude that

$$\mathcal{V}(t) \le \widetilde{C}\mathcal{V}_0,$$

for some positive constant $\widetilde{C}$ depending of α, β, μ, η and c_0. This proves inequalities (3.5)–(3.7).

Lastly, we will focus on the estimate (3.8). Notice that, by bound (3.20), one has

$$\frac{d}{dt}(\mu\|\nabla\boldsymbol{u}\|^2 + \eta\|\nabla\boldsymbol{b}\|^2) + \widetilde{\gamma}(\|\Delta\boldsymbol{u}\|^2 + \|\Delta\boldsymbol{b}\|^2) \le K\Big(\|\nabla\boldsymbol{u}\|^4 + \|\nabla\boldsymbol{b}\|^4\Big)(\|\nabla\boldsymbol{u}\|^2 + \|\nabla\boldsymbol{b}\|^2). \tag{3.21}$$

Then similarly as we got estimate (3.11), we deduce, from bound (3.21) and Poincaré's inequality (see Lemma 2.2), that

$$\frac{d}{dt}(\mu\|\nabla\boldsymbol{u}\|^2 + \eta\|\nabla\boldsymbol{b}\|^2) + \frac{\overline{\gamma}}{\epsilon^2}\Big(\mu\|\nabla\boldsymbol{u}\|^2 + \eta\|\nabla\boldsymbol{b}\|^2\Big) \le K\Big(\|\nabla\boldsymbol{u}\|^4 + \|\nabla\boldsymbol{b}\|^4\Big)(\|\nabla\boldsymbol{u}\|^2 + \|\nabla\boldsymbol{b}\|^2),$$

where $\overline{\gamma} := \min\left\{\frac{\widetilde{\gamma}}{\mu}, \frac{\widetilde{\gamma}}{\eta}\right\}$. Multiplying by the integrating factor $e^{\overline{\gamma}\epsilon^{-2}t}$ gives

$$\frac{d}{dt}\left\{e^{\overline{\gamma}\epsilon^{-2}t}\left(\mu\|\nabla\boldsymbol{u}\|^2 + \eta\|\nabla\boldsymbol{b}\|^2\right)\right\} \le K\,e^{\overline{\gamma}\epsilon^{-2}t}\left(\|\nabla\boldsymbol{u}\|^4 + \|\nabla\boldsymbol{b}\|^4\right)\left(\|\nabla\boldsymbol{u}\|^2 + \|\nabla\boldsymbol{b}\|^2\right).$$

Thus, integrating in time and using estimate (3.3), one finds

$$\begin{aligned}\|\nabla\boldsymbol{z}(t)\|^2 &\le K\|\nabla\boldsymbol{z}_0\|^2\,e^{-\overline{\gamma}\epsilon^{-2}t} + K\int_0^t e^{-\overline{\gamma}\epsilon^{-2}(t-s)}\|\nabla\boldsymbol{z}(s)\|^4\|\nabla\boldsymbol{z}(s)\|^2 ds\\ &\le K\|\nabla\boldsymbol{z}_0\|^2\,e^{-\overline{\gamma}\epsilon^{-2}t} + K\|\nabla\boldsymbol{z}\|^4_{L_t^\infty(\boldsymbol{L}^2(\Omega))}\|\nabla\boldsymbol{z}\|^2_{L_t^2(\boldsymbol{L}^2(\Omega))}\\ &\le K\|\nabla\boldsymbol{z}_0\|^2\,e^{-\overline{\gamma}\epsilon^{-2}t} + K\|\boldsymbol{z}_0\|^2\|\nabla\boldsymbol{z}\|^4_{L_t^\infty(\boldsymbol{L}^2(\Omega))}.\end{aligned}$$

Consequently, if c_0 is small enough, then we deduce that

$$\|\nabla\boldsymbol{z}(t)\|^2 \le \widetilde{C}\|\nabla\boldsymbol{z}_0\|^2\,e^{-\overline{\gamma}t/\epsilon^2},$$

where $\widetilde{C} = \widetilde{C}(\alpha,\beta,\mu,\eta,c_0) > 0$. The proposition follows. □

Remark 3.2 (Estimates for the "initial data" $\boldsymbol{u}_t(0)$ and $\boldsymbol{b}_t(0)$)**.** Multiplying the first equation in (1.1) by $\boldsymbol{u}_t$ in $\boldsymbol{L}^2(\Omega)$ and applying Cauchy–Schwarz and Young's inequalities, we get

$$\|\sqrt{\rho}\boldsymbol{u}_t\|^2 \leq 3\beta\|(\boldsymbol{u}\cdot\nabla)\boldsymbol{u}\|^2 + \frac{3\mu^2}{\alpha}\|\Delta\boldsymbol{u}\|^2 + \frac{3}{\alpha}\|(\boldsymbol{b}\cdot\nabla)\boldsymbol{b}\|^2,$$

since $\operatorname{div}\boldsymbol{u}_t = 0$. It follows from Gagliardo–Nirenberg inequalities that

$$\|\boldsymbol{u}_t\|^2 \leq K_1\left(\|\boldsymbol{u}\|\|\nabla\boldsymbol{u}\|\|\nabla^2\boldsymbol{u}\|^2 + \|\Delta\boldsymbol{u}\|^2 + \|\boldsymbol{b}\|\|\nabla\boldsymbol{b}\|\|\nabla^2\boldsymbol{b}\|^2\right),$$

where $K_1 > 0$ depends only on the constants α, β and μ. Due to Remark 2.3, we have

$$\begin{aligned}\|\boldsymbol{u}_t(0)\|^2 &\leq K_1\left(\|\boldsymbol{u}_0\|\|\nabla\boldsymbol{u}_0\|\|\nabla^2\boldsymbol{u}_0\|^2 + \|\Delta\boldsymbol{u}_0\|^2 + \|\boldsymbol{b}_0\|\|\nabla\boldsymbol{b}_0\|\|\nabla^2\boldsymbol{b}_0\|^2\right)\\ &\leq K_1\left(c_0^2\|\nabla^2\boldsymbol{u}_0\|^2 + \|\Delta\boldsymbol{u}_0\|^2 + c_0^2\|\nabla^2\boldsymbol{b}_0\|^2\right)\\ &\leq \widetilde{C}_1\|\nabla^2\boldsymbol{z}_0\|^2,\end{aligned} \tag{3.22}$$

where $\widetilde{C}_1 = \widetilde{C}_1(\alpha, \beta, \mu, c_0) > 0$. With similar arguments, we can find

$$\|\boldsymbol{b}_t(0)\|^2 \leq \widetilde{C}_2\|\nabla^2\boldsymbol{z}_0\|^2, \tag{3.23}$$

where $\widetilde{C}_2 = \widetilde{C}_2(\eta, c_0) > 0$. Hence, by bounds (3.22) and (3.23), one concludes that

$$\|\boldsymbol{u}_t(0)\|^2 + \|\boldsymbol{b}_t(0)\|^2 \leq \widetilde{C}\left(\|\nabla^2\boldsymbol{u}_0\|^2 + \|\nabla^2\boldsymbol{b}_0\|^2\right), \tag{3.24}$$

where $\widetilde{C} > 0$ is a constant depending only on α, β, μ, η and c_0.

Proposition 3.3 (Bounds in the $\boldsymbol{H}^2(\Omega)$ norm)**.** *Suppose that the initial data $(\rho_0, \boldsymbol{u}_0, \boldsymbol{b}_0)$ satisfy (2.1) and (2.3), with $\boldsymbol{u}_0$, $\boldsymbol{b}_0 \in \boldsymbol{V}\cap\boldsymbol{H}^2(\Omega)$. Let $(\rho, \boldsymbol{u}, p, \boldsymbol{b})$ be the strong solution furnished by Theorem 2.1 defined on $\Omega\times[0,T]$. There exists a constant $\widetilde{C} = \widetilde{C}(\alpha, \beta, \mu, \eta, c_0) > 0$ such that*

$$\|\boldsymbol{u}_t(t)\|^2 + \|\boldsymbol{b}_t(t)\|^2 \leq \widetilde{C}\left(\|\nabla^2\boldsymbol{u}_0\|^2 + \|\nabla^2\boldsymbol{b}_0\|^2\right), \tag{3.25}$$

$$\int_0^t \left(\|\nabla\boldsymbol{u}_t(s)\|^2 + \|\nabla\boldsymbol{b}_t(s)\|^2\right) ds \leq \widetilde{C}\left(\|\nabla^2\boldsymbol{u}_0\|^2 + \|\nabla^2\boldsymbol{b}_0\|^2\right), \tag{3.26}$$

$$\left\|(\nabla^2\boldsymbol{u}, \nabla^2\boldsymbol{b}, \nabla p)\right\|^2_{L_t^\infty(\boldsymbol{L}^2(\Omega))} \leq \widetilde{C}\left(\|\nabla^2\boldsymbol{u}_0\|^2 + \|\nabla^2\boldsymbol{b}_0\|^2\right), \tag{3.27}$$

$$\|\boldsymbol{u}_t(t)\|^2 + \|\boldsymbol{b}_t(t)\|^2 \leq \widetilde{C}\left(\|\nabla^2\boldsymbol{u}_0\|^2 + \|\nabla^2\boldsymbol{b}_0\|^2\right)e^{-\sigma t/\epsilon^2}, \tag{3.28}$$

$$\left\|(\nabla^2\boldsymbol{u}, \nabla^2\boldsymbol{b}, \nabla p)\right\|^2_{L_t^\infty(\boldsymbol{L}^2(\Omega))} \leq \widetilde{C}\left(\|\nabla^2\boldsymbol{u}_0\|^2 + \|\nabla^2\boldsymbol{b}_0\|^2\right)e^{-\sigma t/\epsilon^2}, \tag{3.29}$$

for all $t \in [0,T]$, with $\sigma := \min\{\widehat{\gamma}, \overline{\gamma}\}$, where $\widehat{\gamma} := \min\left\{\frac{\mu}{\beta}, \eta\right\}$, $\overline{\gamma} := \min\left\{\frac{\widetilde{\gamma}}{\mu}, \frac{\widetilde{\gamma}}{\eta}\right\}$ and $\widetilde{\gamma} := \min\left\{\frac{\mu^2}{16\beta}, \frac{\eta^2}{16}\right\}$.

Proof. Differentiating the equations $(1.1)_1$ and $(1.1)_2$ with respect to t, taking the dot product with $\boldsymbol{u}_t$ and $\boldsymbol{b}_t$, respectively, integrating the resulting identities on Ω and adding both results up, we get

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\left(\|\sqrt{\rho}\boldsymbol{u}_t\|^2 + \|\boldsymbol{b}_t\|^2\right) + \mu\|\nabla\boldsymbol{u}_t\|^2 + \eta\|\nabla\boldsymbol{b}_t\|^2\\ &\quad= \langle(\boldsymbol{b}_t\cdot\nabla)\boldsymbol{b}, \boldsymbol{u}_t\rangle + 2\langle\rho(\boldsymbol{u}\cdot\nabla)\boldsymbol{u}_t, -\boldsymbol{u}_t\rangle + \langle\rho_t(\boldsymbol{u}\cdot\nabla)\boldsymbol{u}, -\boldsymbol{u}_t\rangle\\ &\qquad+ \langle\rho(\boldsymbol{u}_t\cdot\nabla)\boldsymbol{u}, -\boldsymbol{u}_t\rangle + \langle(\boldsymbol{b}_t\cdot\nabla)\boldsymbol{u}, \boldsymbol{b}_t\rangle + \langle(\boldsymbol{u}_t\cdot\nabla)\boldsymbol{b}, -\boldsymbol{b}_t\rangle,\end{aligned} \tag{3.30}$$

since $\rho_t = -\boldsymbol{u}\cdot\nabla\rho$ and $\operatorname{div}\boldsymbol{u} = \operatorname{div}\boldsymbol{u}_t = \operatorname{div}\boldsymbol{b} = 0$.

Now, we estimate each term on the right-hand side of identity (3.30). Using Hölder, Sobolev, interpolation and Young's inequalities, we find

$$
\begin{aligned}
|\langle(\boldsymbol{b}_t\cdot\nabla)\boldsymbol{b},\boldsymbol{u}_t\rangle| &\le \|\boldsymbol{b}_t\|_6\|\nabla\boldsymbol{b}\|\|\boldsymbol{b}_t\|_3\\
&\le \delta_1\|\nabla\boldsymbol{b}_t\|^2 + C_{\delta_1}\|\nabla\boldsymbol{b}\|^4\|\boldsymbol{b}_t\|^2,\\
2|\langle\rho(\boldsymbol{u}\cdot\nabla)\boldsymbol{u}_t,-\boldsymbol{u}_t\rangle| &\le 2\|\rho\|_\infty\|\boldsymbol{u}\|_6\|\nabla\boldsymbol{u}_t\|\|\boldsymbol{u}_t\|_3\\
&\le \delta_2\|\nabla\boldsymbol{u}_t\|^2 + C_{\delta_2}\|\nabla\boldsymbol{u}\|^4\|\boldsymbol{u}_t\|^2,\\
|\langle\rho_t(\boldsymbol{u}\cdot\nabla)\boldsymbol{u},-\boldsymbol{u}_t\rangle| &= |\langle\operatorname{div}(\rho\boldsymbol{u})(\boldsymbol{u}\cdot\nabla)\boldsymbol{u},\boldsymbol{u}_t\rangle|\\
&\le C\|\nabla\boldsymbol{u}\|^2\|\nabla^2\boldsymbol{u}\|\|\nabla\boldsymbol{u}_t\|\\
&\le \delta_2\|\nabla\boldsymbol{u}_t\|^2 + C_{\delta_2}\|\nabla\boldsymbol{u}\|^4\|\nabla^2\boldsymbol{u}\|^2,\\
|\langle\rho(\boldsymbol{u}_t\cdot\nabla)\boldsymbol{u},-\boldsymbol{u}_t\rangle| &\le \|\rho\|_\infty\|\boldsymbol{u}_t\|_6\|\nabla\boldsymbol{u}\|\|\boldsymbol{u}_t\|_3\\
&\le \delta_2\|\nabla\boldsymbol{u}_t\|^2 + C_{\delta_2}\|\nabla\boldsymbol{u}\|^4\|\boldsymbol{u}_t\|^2,\\
|\langle(\boldsymbol{b}_t\cdot\nabla)\boldsymbol{u},\boldsymbol{b}_t\rangle| &\le \|\boldsymbol{b}_t\|_6\|\nabla\boldsymbol{u}\|\|\boldsymbol{b}_t\|_3\\
&\le \delta_1\|\nabla\boldsymbol{b}_t\|^2 + C_{\delta_1}\|\nabla\boldsymbol{u}\|^4\|\boldsymbol{b}_t\|^2,\\
|\langle(\boldsymbol{u}_t\cdot\nabla)\boldsymbol{b},-\boldsymbol{b}_t\rangle| &\le \|\boldsymbol{u}_t\|_6\|\nabla\boldsymbol{b}\|\|\boldsymbol{b}_t\|_3\\
&\le C\|\nabla\boldsymbol{u}_t\|\|\nabla\boldsymbol{b}\|\|\boldsymbol{b}_t\|^{1/2}\|\nabla\boldsymbol{b}_t\|^{1/2}\\
&\le \delta_1\|\nabla\boldsymbol{b}_t\|^2 + \delta_2\|\nabla\boldsymbol{u}_t\|^2 + C_{\delta_1,\delta_2}\|\nabla\boldsymbol{b}\|^4\|\boldsymbol{b}_t\|^2.
\end{aligned}
$$

So, applying the estimates above to equality (3.30) and taking $\delta_1 = \eta/6$ and $\delta_2 = \mu/8$, we obtain

$$
\begin{aligned}
&\frac{d}{dt}\left(\|\sqrt{\rho}\boldsymbol{u}_t\|^2 + \|\boldsymbol{b}_t\|^2\right) + \mu\|\nabla\boldsymbol{u}_t\|^2 + \eta\|\nabla\boldsymbol{b}_t\|^2\\
&\quad\le K\left(\|\nabla\boldsymbol{u}\|^4\|\boldsymbol{u}_t\|^2 + \|\nabla\boldsymbol{u}\|^4\|\boldsymbol{b}_t\|^2 + \|\nabla\boldsymbol{b}\|^4\|\boldsymbol{b}_t\|^2 + \|\nabla\boldsymbol{u}\|^4\|\nabla^2\boldsymbol{u}\|^2\right)\\
&\quad\le K\left[\left(\|\nabla\boldsymbol{u}\|^4 + \|\nabla\boldsymbol{b}\|^4\right)\left(\|\boldsymbol{u}_t\|^2 + \|\boldsymbol{b}_t\|^2\right) + \|\nabla\boldsymbol{u}\|^4\|\nabla^2\boldsymbol{u}\|^2\right]. \qquad (3.31)
\end{aligned}
$$

Moreover, since $\alpha \le \rho(\boldsymbol{x},t) \le \beta$ for all $(\boldsymbol{x},t)\in\Omega\times[0,T]$, we deduce that

$$
\begin{aligned}
&\frac{d}{dt}\left(\|\sqrt{\rho}\boldsymbol{u}_t\|^2 + \|\boldsymbol{b}_t\|^2\right) + \mu\|\nabla\boldsymbol{u}_t\|^2 + \eta\|\nabla\boldsymbol{b}_t\|^2\\
&\quad\le K\left(\|\nabla\boldsymbol{u}\|^4 + \|\nabla\boldsymbol{b}\|^4\right)\left(\frac{1}{\alpha}\|\sqrt{\rho}\boldsymbol{u}_t\|^2 + \|\boldsymbol{b}_t\|^2\right) + K\|\nabla\boldsymbol{u}\|^4\|\nabla^2\boldsymbol{u}\|^2\\
&\quad\le C\left(\|\nabla\boldsymbol{u}\|^4 + \|\nabla\boldsymbol{b}\|^4\right)\left(\|\sqrt{\rho}\boldsymbol{u}_t\|^2 + \|\boldsymbol{b}_t\|^2\right) + C\|\nabla\boldsymbol{u}\|^4\|\nabla^2\boldsymbol{u}\|^2. \qquad (3.32)
\end{aligned}
$$

Then, applying a generalized Gronwall inequality (see, for instance, [15, Lemma 4, p. 656]) and taking into account (3.24), we have

$$
\begin{aligned}
\|\boldsymbol{z}_t(t)\|^2 &= \|\boldsymbol{u}_t(t)\|^2 + \|\boldsymbol{b}_t(t)\|^2\\
&\le \exp\left(C\int_0^t\left(\|\nabla\boldsymbol{u}(s)\|^4 + \|\nabla\boldsymbol{b}(s)\|^4\right)ds\right)\left(\widetilde{C}\|\nabla^2\boldsymbol{z}_0\|^2 + C\int_0^t\|\nabla\boldsymbol{u}(s)\|^4\|\nabla^2\boldsymbol{u}(s)\|^2\,ds\right)\\
&\le \exp\left(C\|\nabla\boldsymbol{u}\|^2_{L^2_t(\boldsymbol{L}^2(\Omega))}\|\nabla\boldsymbol{u}\|^2_{L^\infty_t(\boldsymbol{L}^2(\Omega))} + C\|\nabla\boldsymbol{b}\|^2_{L^2_t(\boldsymbol{L}^2(\Omega))}\|\nabla\boldsymbol{b}\|^2_{L^\infty_t(\boldsymbol{L}^2(\Omega))}\right)\\
&\quad\times\left(\widetilde{C}\|\nabla^2\boldsymbol{z}_0\|^2 + C\|\nabla\boldsymbol{u}\|^2_{L^2_t(\boldsymbol{L}^2(\Omega))}\|\nabla\boldsymbol{u}\|^2_{L^\infty_t(\boldsymbol{L}^2(\Omega))}\|\nabla^2\boldsymbol{z}\|^2_{L^\infty_t(\boldsymbol{L}^2(\Omega))}\right).
\end{aligned}
$$

This estimate, together with (3.3), (3.5), and Remark 2.3, implies that

$$\begin{aligned}\|z_t(t)\|^2 &\le \exp\Big(\widetilde{C}\|z_0\|^2\|\nabla z_0\|^2\Big)\Big(\widetilde{C}\|\nabla^2 z_0\|^2 + \widetilde{C}\|z_0\|^2\|\nabla z_0\|^2\|\nabla^2 z\|^2_{L^\infty_t(L^2(\Omega))}\Big)\\ &\le \widetilde{C}\exp\Big(\widetilde{C}c_0^4\Big)\Big(\|\nabla^2 z_0\|^2 + c_0^4\|\nabla^2 z\|^2_{L^\infty_t(L^2(\Omega))}\Big)\\ &\le \widetilde{C}\Big(\|\nabla^2 z_0\|^2 + c_0^4\|\nabla^2 z\|^2_{L^\infty_t(L^2(\Omega))}\Big).\end{aligned} \tag{3.33}$$

Now, integrating (3.32) in time, and using again (3.3), (3.5), Remark 2.3, as well as (3.33), we obtain

$$\begin{aligned}&\|z_t(t)\|^2 + \int_0^t \|\nabla z_t(s)\|^2\,ds\\ &\le \widetilde{C}\|\nabla^2 z_0\|^2 + C\int_0^t \Big(\|\nabla u(s)\|^4 + \|\nabla b(s)\|^4\Big)\|z_t(s)\|^2\,ds + C\int_0^t \|\nabla u(s)\|^4\|\nabla^2 u(s)\|^2\,ds\\ &\le \widetilde{C}\|\nabla^2 z_0\|^2 + C\|\nabla z\|^2_{L^2_t(L^2(\Omega))}\|\nabla z\|^2_{L^\infty_t(L^2(\Omega))}\|z_t\|^2_{L^\infty_t(L^2(\Omega))}\\ &\quad + C\|\nabla z\|^2_{L^2_t(L^2(\Omega))}\|\nabla z\|^2_{L^\infty_t(L^2(\Omega))}\|\nabla^2 z\|^2_{L^\infty_t(L^2(\Omega))}\\ &\le \widetilde{C}\|\nabla^2 z_0\|^2 + \widetilde{C}\|z_0\|^2\|\nabla z_0\|^2\|z_t\|^2_{L^\infty_t(L^2(\Omega))} + \widetilde{C}\|z_0\|^2\|\nabla z_0\|^2\|\nabla^2 z\|^2_{L^\infty_t(L^2(\Omega))}\\ &\le \widetilde{C}\|\nabla^2 z_0\|^2 + \widetilde{C}c_0^4\|z_t\|^2_{L^\infty_t(L^2(\Omega))} + \widetilde{C}c_0^4\|\nabla^2 z\|^2_{L^\infty_t(L^2(\Omega))}\\ &\le \widetilde{C}\|\nabla^2 z_0\|^2 + \widetilde{C}c_0^4\|\nabla^2 z_0\|^2 + \widetilde{C}c_0^8\|\nabla^2 z\|^2_{L^\infty_t(L^2(\Omega))} + \widetilde{C}c_0^4\|\nabla^2 z\|^2_{L^\infty_t(L^2(\Omega))}\\ &\le \widetilde{C}\Big(\|\nabla^2 z_0\|^2 + c_0^4\|\nabla^2 z\|^2_{L^\infty_t(L^2(\Omega))}\Big),\end{aligned} \tag{3.34}$$

since $c_0 > 0$ is sufficiently small.

On the other side, by identity (3.14), one concludes that

$$\|\nabla^2 z(t)\|^2 \le K\big(\|z_t\|^2 + \|(u\cdot\nabla)u\|^2 + \|(b\cdot\nabla)b\|^2 + \|(u\cdot\nabla)b\|^2 + \|(b\cdot\nabla)u\|^2\big). \tag{3.35}$$

In addition, since $\operatorname{div} u = 0$, applying the divergence operator to the first in (1.1), we obtain

$$\Delta p = -\operatorname{div}\big(\rho u_t + \rho(u\cdot\nabla)u - (b\cdot\nabla)b\big),$$

and therefore the total pressure p may be recovered by

$$\nabla p = -\nabla\Delta^{-1}\operatorname{div}\big(\rho u_t + \rho(u\cdot\nabla)u - (b\cdot\nabla)b\big). \tag{3.36}$$

Hence, we have

$$\|\nabla p(t)\|^2 \le C_\beta\left(\|u_t\|^2 + \|(u\cdot\nabla)u\|^2 + \|(b\cdot\nabla)b\|^2\right).$$

So, from bounds (3.35) and (3.36), we get

$$\|(\nabla^2 u, \nabla^2 b, \nabla p)\|^2 \le K\left(\|u_t\|^2 + \|b_t\|^2 + F(t)\right), \tag{3.37}$$

where $F(t) := \|(u\cdot\nabla)u\|^2 + \|(u\cdot\nabla)b\|^2 + \|(b\cdot\nabla)b\|^2 + \|(b\cdot\nabla)u\|^2$. To obtain the estimate of $\big\|(\nabla^2 u, \nabla^2 b, \nabla p)\big\|^2_{L^\infty_t(L^2(\Omega))}$, it remains to control $F(t)$. Using Hölder, standard interpolation and Sobolev's inequalities (see [2]), we find, from estimates (3.2) and (3.5), and Remark 2.3, that

$$\begin{aligned}\|(\boldsymbol{u}\cdot\nabla)\boldsymbol{u}\|^2 &\le \|\boldsymbol{u}\|_3^2\,\|\nabla\boldsymbol{u}\|_6^2 \le C\|\boldsymbol{u}\|\,\|\nabla\boldsymbol{u}\|\,\|\nabla^2\boldsymbol{u}\|^2\\ &\le \widetilde{C}\|\boldsymbol{z}_0\|\,\|\nabla\boldsymbol{z}_0\|\,\|\nabla^2\boldsymbol{u}\|^2\\ &\le \widetilde{C}c_0^2\,\|\nabla^2\boldsymbol{u}\|^2, \qquad (3.38)\\ \|(\boldsymbol{u}\cdot\nabla)\boldsymbol{b}\|^2 &\le \|\boldsymbol{u}\|_3^2\,\|\nabla\boldsymbol{b}\|_6^2 \le C\|\boldsymbol{u}\|\,\|\nabla\boldsymbol{u}\|\,\|\nabla^2\boldsymbol{b}\|^2\\ &\le \widetilde{C}\|\boldsymbol{z}_0\|\,\|\nabla\boldsymbol{z}_0\|\,\|\nabla^2\boldsymbol{b}\|^2\\ &\le \widetilde{C}c_0^2\,\|\nabla^2\boldsymbol{b}\|^2. \qquad (3.39)\end{aligned}$$

Analogously, one has

$$\|(\boldsymbol{b}\cdot\nabla)\boldsymbol{b}\|^2 \le \widetilde{C}c_0^2\,\|\nabla^2\boldsymbol{b}\|^2, \tag{3.40}$$

$$\|(\boldsymbol{b}\cdot\nabla)\boldsymbol{u}\|^2 \le \widetilde{C}c_0^2\,\|\nabla^2\boldsymbol{u}\|^2. \tag{3.41}$$

Consequently, by inequality (3.37), one obtains

$$\|(\nabla^2\boldsymbol{z},\nabla p)\|^2 \le \widetilde{C}\big(\|\boldsymbol{z}_t\|^2 + c_0^2\,\|\nabla^2\boldsymbol{z}\|^2\big). \tag{3.42}$$

Since $c_0 > 0$ is sufficiently small, one gets

$$\|(\nabla^2\boldsymbol{z},\nabla p)\|^2 \le \widetilde{C}\|\boldsymbol{z}_t\|^2. \tag{3.43}$$

Furthermore, having in mind the identities (2.4)–(2.5) and summing the bounds (3.34) and (3.43), one concludes that

$$\mathcal{W}(t) \le \widetilde{C}\Big(\mathcal{W}_0 + c_0^4\,\mathcal{W}(t)\Big).$$

Hence, the smallness of c_0 ensures that

$$\mathcal{W}(t) \le \widetilde{C}\mathcal{W}_0.$$

This proves estimates (3.25)–(3.27). Finally, using Poincaré's inequality (see Lemma 2.2), we get

$$\|\nabla\boldsymbol{u}_t\| \ge \epsilon^{-1}\|\boldsymbol{u}_t\| \quad \text{and} \quad \|\nabla\boldsymbol{b}_t\| \ge \epsilon^{-1}\|\boldsymbol{b}_t\|.$$

Thus, by estimate (3.31), we have

$$\begin{aligned}&\frac{d}{dt}\big(\|\sqrt{\rho}\boldsymbol{u}_t\|^2 + \|\boldsymbol{b}_t\|^2\big) + \frac{\widehat{\gamma}}{\epsilon^2}\big(\|\sqrt{\rho}\boldsymbol{u}_t\|^2 + \|\boldsymbol{b}_t\|^2\big)\\ &\qquad\le K\left[\Big(\|\nabla\boldsymbol{u}\|^4 + \|\nabla\boldsymbol{b}\|^4\Big)\big(\|\boldsymbol{u}_t\|^2 + \|\boldsymbol{b}_t\|^2\big) + \|\nabla\boldsymbol{u}\|^4\|\nabla^2\boldsymbol{u}\|^2\right], \qquad (3.44)\end{aligned}$$

where $\widehat{\gamma} := \min\{\mu/\beta, \eta\}$. Multiplying inequality (3.44) by integrating factor $e^{\widehat{\gamma}\epsilon^{-2}t}$ yields

$$\begin{aligned}&\frac{d}{dt}\left\{e^{\widehat{\gamma}\epsilon^{-2}t}\big(\|\sqrt{\rho}\boldsymbol{u}_t\|^2 + \|\boldsymbol{b}_t\|^2\big)\right\}\\ &\qquad\le K\,e^{\widehat{\gamma}\epsilon^{-2}t}\left[\Big(\|\nabla\boldsymbol{u}\|^4 + \|\nabla\boldsymbol{b}\|^4\Big)\big(\|\boldsymbol{u}_t\|^2 + \|\boldsymbol{b}_t\|^2\big) + \|\nabla\boldsymbol{u}\|^4\|\nabla^2\boldsymbol{u}\|^2\right].\end{aligned}$$

Then, we deduce from estimate (3.24) that

$$\|\sqrt{\rho}\boldsymbol{u}_t\|^2 + \|\boldsymbol{b}_t\|^2 \le \widetilde{C}\|\nabla^2 \boldsymbol{z}_0\|^2 e^{-\widehat{\gamma}\epsilon^{-2}t}$$
$$+ K\int_0^t e^{-\widehat{\gamma}\epsilon^{-2}(t-s)}\left[\left(\|\nabla \boldsymbol{u}(s)\|^4 + \|\nabla \boldsymbol{b}(s)\|^4\right)\|\boldsymbol{z}_t(s)\|^2 + \|\nabla \boldsymbol{u}(s)\|^4\|\nabla^2 \boldsymbol{u}(s)\|^2\right] ds.$$

Consequently, we have

$$\begin{aligned}\|\boldsymbol{z}_t(t)\|^2 \le \widetilde{C}\|\nabla^2 \boldsymbol{z}_0\|^2 e^{-\widehat{\gamma}\epsilon^{-2}t} &+ K\|\nabla \boldsymbol{u}\|^2_{L^2_t(\boldsymbol{L}^2(\Omega))}\|\nabla \boldsymbol{u}\|^2_{L^\infty_t(\boldsymbol{L}^2(\Omega))}\|\boldsymbol{z}_t\|^2_{L^\infty_t(\boldsymbol{L}^2(\Omega))}\\ &+ K\|\nabla \boldsymbol{b}\|^2_{L^2_t(\boldsymbol{L}^2(\Omega))}\|\nabla \boldsymbol{b}\|^2_{L^\infty_t(\boldsymbol{L}^2(\Omega))}\|\boldsymbol{z}_t\|^2_{L^\infty_t(\boldsymbol{L}^2(\Omega))}\\ &+ K\|\nabla \boldsymbol{u}\|^2_{L^2_t(\boldsymbol{L}^2(\Omega))}\|\nabla \boldsymbol{u}\|^2_{L^\infty_t(\boldsymbol{L}^2(\Omega))}\|\nabla^2 \boldsymbol{u}\|^2_{L^\infty_t(\boldsymbol{L}^2(\Omega))}.\end{aligned}$$

Let us recall that estimates (3.3), (3.8), (3.25), and (3.27) yield the following bounds:

$$\|(\sqrt{\mu}\,\nabla \boldsymbol{u}, \sqrt{\eta}\,\nabla \boldsymbol{b})\|^2_{L^2_t(\boldsymbol{L}^2(\Omega))} \le C\,\|\boldsymbol{z}_0\|^2,$$
$$\|(\nabla \boldsymbol{u}, \nabla \boldsymbol{b})\|^2_{L^\infty_t(\boldsymbol{L}^2(\Omega))} \le \widetilde{C}\,\|\nabla \boldsymbol{z}_0\|^2 e^{-\overline{\gamma}\,t/\epsilon^2},$$
$$\|(\boldsymbol{u}_t, \boldsymbol{b}_t)\|^2_{L^\infty_t(\boldsymbol{L}^2(\Omega))} \le \widetilde{C}\,\|\nabla^2 \boldsymbol{z}_0\|^2$$

and

$$\|(\nabla^2 \boldsymbol{u}, \nabla^2 \boldsymbol{b})\|^2_{L^\infty_t(\boldsymbol{L}^2(\Omega))} \le \widetilde{C}\,\|\nabla^2 \boldsymbol{z}_0\|^2,$$

respectively. These imply that

$$\|\boldsymbol{z}_t(t)\|^2 \le \widetilde{C}\|\nabla^2 \boldsymbol{z}_0\|^2 e^{-\widehat{\gamma}\epsilon^{-2}t} + \widetilde{C}\|\boldsymbol{z}_0\|^2\|\nabla \boldsymbol{z}_0\|^2\|\nabla^2 \boldsymbol{z}_0\|^2 e^{-\overline{\gamma}\epsilon^{-2}t},$$

where $\overline{\gamma} := \min\left\{\frac{\widetilde{\gamma}}{\mu}, \frac{\widetilde{\gamma}}{\eta}\right\}$ and $\widetilde{\gamma} := \min\left\{\frac{\mu^2}{16\beta}, \frac{\eta^2}{16}\right\}$. Moreover, by Remark 2.3, we have

$$\|\boldsymbol{z}_0\|^2\|\nabla \boldsymbol{z}_0\|^2 \le 4c_0^4.$$

Hence,

$$\|\boldsymbol{z}_t(t)\|^2 \le \widetilde{C}\|\nabla^2 \boldsymbol{z}_0\|^2 e^{-\widehat{\gamma}\epsilon^{-2}t} + 4\widetilde{C}c_0^4\,\|\nabla^2 \boldsymbol{z}_0\|^2 e^{-\overline{\gamma}\epsilon^{-2}t}.$$

Therefore, choosing $\sigma := \min\{\widehat{\gamma}, \overline{\gamma}\}$, we infer

$$\|\boldsymbol{z}_t(t)\|^2 \le \widetilde{C}\|\nabla^2 \boldsymbol{z}_0\|^2 e^{-\sigma t/\epsilon^2},$$

which together with inequality (3.43) implies that

$$\|(\nabla^2 \boldsymbol{z}, \nabla p)\|^2 \le \widetilde{C}\|\nabla^2 \boldsymbol{z}_0\|^2 e^{-\sigma t/\epsilon^2}.$$

This proves estimates (3.28)–(3.29). The proof of Proposition 3.3 is finished. □

Acknowledgements

F. W. Cruz was partially supported by the Brazilian funding agency CNPq, under grants No. 304605/2022-0 and No. 200039/2024-5. E. Mallea-Zepeda has been supported by ANID-Chile through of project research Fondecyt de Iniciación 11200208. M. A. Rojas-Medar was partially funded by the National Agency for Research and Development (ANID) through the Fondecyt project 1240152 (Chile). The authors express their gratitude to the anonymous referee for their valuable suggestions, which improved the presentation of this manuscript.

References

[1] H. Abidi, M. Paicu, Global existence for the magnetohydrodynamic system in critical spaces, *Proc. Roy. Soc. Edinburgh Sect. A* **138**(2008), No. 3, 447–476. https://doi.org/10.1017/S0308210506001181; MR2418123; Zbl 1148.35066

[2] R. A. Adams, *Sobolev spaces*, Academic Press, New York, 1975. MR0450957; Zbl 0314.46030

[3] R. P. Agarwal, S. Gala, M. A. Ragusa, A regularity criterion of the 3D MHD equations involving one velocity and one current density component in Lorentz space, *Z. Angew. Math. Phys.* **71**(2020), No. 3, Paper No. 95, 11 pp. https://doi.org/10.1007/s00033-020-01318-4; MR4106811; Zbl 1440.35258

[4] A. M. Alghamdi, S. Gala, M. A. Ragusa, Z. Zhang, A regularity criterion for the 3D density-dependent MHD equations, *Bull. Braz. Math. Soc. (N.S.)* **52**(2021), No. 2, 241–251. https://doi.org/10.1007/s00574-020-00199-5; MR4253404; Zbl 1473.35431

[5] P. Braz e Silva, F. W. Cruz, M. A. Rojas-Medar, Global strong solutions for variable density incompressible asymmetric fluids in thin domains, *Nonlinear Anal. Real World Appl.* **55**(2020), Article ID 103125, 14 pp. https://doi.org/10.1016/j.nonrwa.2020.103125; MR4079061; Zbl 1451.35127

[6] F. Chen, B. Guo, X. Zhai, Global solution to the 3-D inhomogeneous incompressible MHD system with discontinuous density, *Kinet. Relat. Mod.* **12**(2019), No. 1, 37–58. https://doi.org/10.3934/krm.2019002; MR3835531; Zbl 1410.35110

[7] F. W. Cruz, E. E. Ortega-Torres, M. A. Rojas-Medar, Strong solutions to the equations of the unconsolidated porous media in 3D thin domains, *Appl. Anal.* **103**(2024), No. 5, 970–984. https://doi.org/10.1080/00036811.2023.2217825; MR4711411; Zbl 1545.35104

[8] P. A. Davidson, *An introduction to magnetohydrodynamics*, Cambridge University Press, Cambridge, 2001. https://doi.org/10.1017/CBO9780511626333; MR1825486; Zbl 0974.76002

[9] B. Desjardins, C. Le Bris, Remarks on a nonhomogeneous model of magnetohydrodynamics, *Differential Integral Equations* **11**(1998), No. 3, 377–394. https://doi.org/10.57262/die/1367341058; MR1745545; Zbl 1067.76097

[10] S. Gala, M. A. Ragusa, A logarithmic regularity criterion for the two-dimensional MHD equations, *J. Math. Anal. Appl.* **444**(2016), No. 2, 1752–1758. https://doi.org/10.1016/j.jmaa.2016.07.001; MR3535787; Zbl 1349.35294

[11] S. Gala, M. A. Ragusa, On the regularity criterion of weak solutions for the 3D MHD equations, *Z. Angew. Math. Phys.* **68**(2017), No. 6, Paper No. 140, 13 pp. https://doi.org/10.1007/s00033-017-0890-9; MR3735569; Zbl 1379.35240

[12] S. Gala, M. A. Ragusa, A new regularity criterion for the 3D incompressible MHD equations via partial derivatives, *J. Math. Anal. Appl.* **481**(2020), No. 2, 123497. https://doi.org/10.1016/j.jmaa.2019.123497; MR4007203; Zbl 1426.35058

[13] J. F. Gerbeau, C. Le Bris, T. Lelièvre, *Mathematical methods for the magnetohydrodynamics of liquid metals*, Numerical Mathematics and Scientific Computation, Oxford University

Press, Oxford, 2006. https://doi.org/10.1093/acprof:oso/9780198566656.001.0001; MR2289481; Zbl 1107.76001

[14] H. Gong, J. Li, Global existence of strong solutions to incompressible MHD, *Commun. Pure Appl. Anal.* **13**(2014), No. 4, 1553–1561. https://doi.org/10.3934/cpaa.2014.13.1553; MR3177746; Zbl 1282.35300

[15] J. G. Heywood, The Navier–Stokes equations: On the existence, regularity and decay of solutions, *Indiana Univ. Math. J.* **29**(1980), No. 5, 639–681. https://doi.org/10.1512/iumj.1980.29.29048; MR0589434; Zbl 0494.35077

[16] I. Kukavica, M. Ziane, On the regularity of the Navier–Stokes equation in a thin periodic domain, *J. Differential Equations* **234**(2007), No. 2, 485–506. https://doi.org/10.1016/j.jde.2006.11.020; MR2300665; Zbl 1118.35029

[17] S. Li, Y. Sun, Global solutions to 2D compressible isothermal Navier–Stokes equations on thin domains, *J. Math. Fluid Mech.* **25**(2023), No. 1, Paper No. 3, 18 pp. https://doi.org/10.1007/s00021-022-00746-8; MR4514951; Zbl 1510.76139

[18] X. Liao, On the strong solutions of the inhomogeneous incompressible Navier–Stokes equations in a thin domain, *Differential Integral Equations* **29**(2016), No. 1–2, 167–182. https://doi.org/10.57262/die/1448323258; MR3450754; Zbl 1363.35276

[19] B. Lü, Z. Xu, X. Zhong, Global existence and large time asymptotic behavior of strong solutions to the Cauchy problem of 2D density-dependent magnetohydrodynamic equations with vacuum, *J. Math. Pures Appl.* (9) **108**(2017), No. 1, 41–62. https://doi.org/10.1016/j.matpur.2016.10.009; MR3660768; Zbl 1364.35279

[20] R. Moreau, *Magnetohydrodynamics*, Fluid Mechanics and its Applications, Vol. 3. Kluwer Acad. Pub., Dordrecht, 1990. https://doi.org/10.1007/978-94-015-7883-7; MR1144019; Zbl 0714.76003

[21] M. Paicu, P. Zhang, Z. Zhang, Global unique solvability of inhomogeneous Navier–Stokes equations with bounded density, *Comm. Partial Differential Equations* **38**(2013), No. 7, 1208–1234. https://doi.org/10.1080/03605302.2013.780079; MR3169743; Zbl 1314.35086

[22] G. Raugel, G. R. Sell, Navier–Stokes equations on thin 3D domains. I: Global attractors and global regularity of solutions, *J. Amer. Math. Soc.* **6**(1993), No. 3, 503–568. https://doi.org/10.2307/2152776; MR1179539; Zbl 0787.34039

[23] Y. Sun, S. Wang, Inhomogeneous incompressible Navier–Stokes equations on thin domains, *Commun. Math. Stat.* **8**(2020), No. 2, 239–253. https://doi.org/10.1007/s40304-019-00202-6; MR4102362; Zbl 1442.35312

[24] S. Wang, Global solution to inhomogeneous incompressible Navier–Stokes equations on thin domain, *J. Math. Anal. Appl.* **482**(2020), No. 2, Article ID 123556, 11 pp. https://doi.org/10.1016/j.jmaa.2019.123556; MR4015692; Zbl 1471.35224

[25] Y. Zhou, J. Fan, A regularity criterion for the density-dependent magnetohydrodynamic equations, *Math. Meth. Appl. Sci.* **33**(2010), No. 11, 1350–1355. https://doi.org/10.1002/mma.1255; MR2674777; Zbl 1193.35160