Electronic Journal of Qualitative Theory of Differential Equations
2025, No. **63**, 1–22; https://doi.org/10.14232/ejqtde.2025.1.63 www.math.u-szeged.hu/ejqtde/

Sign-changing solution for critical Schrödinger–Poisson system with concave perturbation

Yu-Ru Li and **Yong-Yong Li** ✉

College of Mathematics and Statistics, Northwest Normal University, Lanzhou, 730070, China

Received 3 April 2025, appeared 27 October 2025
Communicated by Dimitri Mugnai

Abstract. In this paper, we consider the following Schrödinger–Poisson system with critical-concave nonlinearity:

$$\begin{cases} -\Delta u + l(x)\phi u = |u|^4 u + \mu h(x)|u|^{q-2}u & \text{in } \mathbb{R}^3, \\ -\Delta \phi = l(x)u^2 & \text{in } \mathbb{R}^3, \end{cases}$$

where $\mu > 0$, $1 < q < 2$ and the functions l, h satisfy some mild conditions. By constraining the energy functional of the above problem on a closed subset of the sign-changing Nehari manifold, we obtain a sign-changing solution with positive energy for $\mu > 0$ small enough.

Keywords: Schrödinger–Poisson system, critical-concave nonlinearity, Nehari manifold, sign-changing solution.

2020 Mathematics Subject Classification: 35J20, 35B33, 35B38, 35D30.

1 Introduction and main results

This paper focuses on standing waves of the following nonlinear Schrödinger–Poisson system:

$$\begin{cases} i\psi_t = -\Delta \psi + \phi(x)\psi - g(x,\psi), \\ -\Delta \phi = |\psi|^2, \end{cases} \tag{1.1}$$

where $g \in C(\mathbb{R}^3 \times \mathbb{C}, \mathbb{R})$ and $\psi : \mathbb{R}^3 \times [0,T] \to \mathbb{C}$. Such systems arise as electromagnetic field models in quantum mechanics and has a strong physical background. The first equation of system (1.1) describes the evolution of the quantum state of microscopic particles (such as electrons) over time, while the second equation (i.e. Poisson equation) is often used to describe the relationship between the electrostatic potential and the charge distribution. The interaction of charge particles with electromagnetic field can be described by coupling the

✉ Corresponding author. Email: mathliyy518@163.com

nonlinear Schrödinger equation and the Poisson equation. For more physical backgrounds of Schrödinger–Poisson system, we recommend readers to refer to [2,6].

In the past decades, many mathematicians studied the standing wave solution of type $\psi(x,t) = e^{i\lambda t}u(x)$ for system (1.1), which corresponds to a solution u of the elliptic system

$$\begin{cases} -\Delta u + u + \phi u = g(x,u) & \text{in } \mathbb{R}^3, \\ -\Delta \phi = u^2 & \text{in } \mathbb{R}^3. \end{cases} \tag{1.2}$$

For the case of $g(x,u) = |u|^{p-2}u$, D'Aprile and Mugnai [7] showed that system (1.2) has no nontrivial solution once $p \geq 6$ or $p \leq 2$ by establishing a related Pohožaev identity, Ruiz [18] showed that system (1.2) does not admit any nontrivial solution if $p \leq 3$ and has positive solution if $p \in (3,6)$, in this case ground state solution was further proved by Azzollini and Pomponio in [1]. Afterwards, Schrödinger–Poisson systems with Sobolev critical exponent and subcritical perturbation were investigated by many scholars. For example, see the cases of superlinear perturbations in [1,25], the cases of sublinear perturbations in [13] and the references therein. As we will recall below, various existence results of sign-changing solutions for Schrödinger–Poisson systems were widely studied in [4,5,8–12,15,17,19–21,23,24,26,27]. More precisely, for the Schrödinger–Poisson system

$$\begin{cases} -\Delta u + V(x)u + \phi u = K(x)f(u) & \text{in } \mathbb{R}^3, \\ -\Delta \phi = u^2 & \text{in } \mathbb{R}^3, \end{cases} \tag{1.3}$$

Ianni [10] studied the case of $V = K = 1$, $f(u) = |u|^{p-2}u$ and $p \in [4,6)$, by the heat flow method he got a radially nodal solution changing sign exactly $k-1$ times for every integer $k \geq 2$. Meanwhile, Kim and Seok [12] proved the similar results to [10] for $p \in (4,6)$ by using the variational methods. Wang and Zhou assumed $K = 1$, $f(u) = |u|^{p-2}u$, $p \in (4,6)$ and V satisfies compactness condition in [21], where they proved a sign-changing solution of system (1.3) by combining the constraint minimization argument and the Brouwer degree theory. In [19], by using a quantitative deformation lemma, Shuai and Wang extended the results in [21] to the case of general nonlinearity f, which satisfies an increasing condition. If V, K satisfy some vanishing conditions, Liang et al. [15] proved system (1.3) has a sign-changing solution. Chen and Tang [4] assumed that $f \in C(\mathbb{R}, \mathbb{R})$ satisfies a weaker condition than the increasing condition, by using a direct approach they obtained a ground state sign-changing solution with precisely two nodal domains. After this, Chen and Tang assumed that f satisfies asymptotically cubic or super-cubic conditions in [5], where they proved a sign-changing solution with precisely two nodal domains under mild assumptions on asymptotic behaviors of V, K. When $K = 1$ and f satisfies super-cubic conditions, via the method of invariant sets of descending flow, Liu et al. [17] proved that system (1.3) has infinitely many sign-changing solutions. In [26], Zhong and Tang studied the system

$$\begin{cases} -\Delta u + V(x)u + \phi u = |u|^2u + \mu h(x)u & \text{in } \mathbb{R}^3, \\ -\Delta \phi = u^2 & \text{in } \mathbb{R}^3, \end{cases}$$

where $V \in C(\mathbb{R}^3, \mathbb{R}^+)$ satisfies a compactness condition and $h \in L^{\frac{3}{2}}(\mathbb{R}^3) \setminus \{0\}$ is nonnegative, by using the sign-changing Nehari manifold they got a ground state sign-changing solution for any $\mu \in (0, \mu_1)$, with μ_1 denoting the first eigenvalue of $-\Delta u + u = \mu h(x)u$ in $H^1(\mathbb{R}^3)$. In

[8], Gu et al. considered the following system with subquadratic or quadratic nonlinearity:

$$\begin{cases} -\Delta u + V(x)u + \lambda\phi u = f(u) & \text{in } \mathbb{R}^3, \\ -\Delta\phi = u^2 & \text{in } \mathbb{R}^3, \end{cases}$$

where $V \in C(\mathbb{R}^3, \mathbb{R}^+)$ is radial if f is subquadratic and coercive if f is quadratic, by following the ideas of [17] they verified the existence and multiplicity of sign-changing solution for $\lambda > 0$ small enough.

As for the existence of sign-changing solutions for the critical Schrödinger–Poisson system

$$\begin{cases} -\Delta u + V(x)u + K(x)\phi u = |u|^4 u + g(x,u) & \text{in } \mathbb{R}^3, \\ -\Delta\phi = K(x)u^2 & \text{in } \mathbb{R}^3, \end{cases} \tag{1.4}$$

there are seldom results proved in [9, 11, 20, 27]. In [27], Zhong and Tang studied the case of $V = 1$ and $g(x,u) = \mu h(x)u$, under some mild conditions on K, g, by combining the variational methods and sign-changing Nehari manifold they obtained a ground state sign-changing solution of system (1.4) for any $\mu \in (0, \mu_1)$, where μ_1 is the first eigenvalue of $-\Delta u + u = \mu h(x)u$ in $H^1(\mathbb{R}^3)$. When $K = 1$, $f(x,u) = \mu f(u)$ satisfies an increasing condition and V satisfies compactness condition, Wang et al. [20] proved that system (1.4) has a least-energy sign-changing solution for $\mu > 0$ large enough. Later, Kang et al. [11] concerned with the following Schrödinger–Poisson system with steep well potential:

$$\begin{cases} -\Delta u + (\lambda V(x) + 1)u + \phi u = |u|^4 u + |u|^{p-2}u & \text{in } \mathbb{R}^3, \\ -\Delta\phi = u^2 & \text{in } \mathbb{R}^3, \end{cases}$$

where $p \in (5,6)$, by combining constrained minimization arguments and Hofer's deformation lemma, they proved the existence and concentration of ground sign-changing solution for $\lambda > 0$ large enough. Subsequently, Huang et al. [9] extended the results in [11] to the case of general perturbed nonlinearity.

Recently, sign-changing solutions on the Schrödinger–Poisson systems with concave-convex nonlinearity have attracted wide attention. Yang and Ou [24] studied the Schrödinger–Poisson system

$$\begin{cases} -\Delta u + \dfrac{1}{4\pi}\displaystyle\int_\Omega \frac{u^2(y)}{|x-y|}dyu = |u|^{q-2}u + \mu|u|^{p-2}u & \text{in } \Omega, \\ u(x) = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.5}$$

where $\Omega \subset \mathbb{R}^3$ is a bounded domain with smooth boundary $\partial\Omega$, $1 < p < 2$ and $4 < q < 6$. By using constrained variational method and quantitative deformation lemma, they proved that there exists some constant $\mu^* > 0$ such that system (1.5) has a sign-changing solution with positive energy for any $\mu < \mu^*$. After this work, Yang and Tang [23] studied the Schrödinger–Poisson system

$$\begin{cases} -\Delta u + V(x)u + \phi u = |u|^{p-2}u + \mu K(x)|u|^{q-2}u & \text{in } \mathbb{R}^3, \\ -\Delta\phi = u^2 & \text{in } \mathbb{R}^3, \end{cases}$$

where $1 < q < 2$, $4 < p < 6$, $K \in L^{\frac{6}{6-q}}(\mathbb{R}^3)$ and $V \in C(\mathbb{R}^3, \mathbb{R})$ satisfies a weakly coercive assumption. By constructing a nonempty closed subset of the sign-changing Nehari manifold, they obtained a sign-changing solution with positive energy provided $\mu < \mu^*$ for some $\mu^* > 0$. Inspired by the above works, we are interested in whether the critical-concave Schrödinger–Poisson system has sign-changing solution. In this paper, we consider sign-changing solution of the following Schrödinger–Poisson system:

$$\begin{cases} -\Delta u + l(x)\phi u = |u|^4 u + \mu h(x)|u|^{q-2}u & \text{in } \mathbb{R}^3, \\ -\Delta \phi = l(x)u^2 & \text{in } \mathbb{R}^3, \end{cases} \tag{1.6}$$

where $1 < q < 2$, $\mu > 0$, $0 < l(\cdot) \in L^2(\mathbb{R}^3) \cap L^3(\mathbb{R}^3)$ and the potential function h satisfies the condition

(H) $0 < h(\cdot) \in L^{\frac{6}{6-q}}(\mathbb{R}^3)$ and there exists some $\beta \in \left(3 - \frac{2q}{3}, 3 - \frac{q}{2}\right)$ such that $h(x) \geq \frac{1}{|x|^\beta}$ for all $|x| \leq 1$.

As is well known, the Poisson equation is uniquely solvable. Indeed, for any $u \in D^{1,2}(\mathbb{R}^3)$, define the linear functional

$$L_u(v) = \int_{\mathbb{R}^3} l(x)u^2 v dx, \qquad \forall\, v \in D^{1,2}(\mathbb{R}^3).$$

It follows from $l \in L^2(\mathbb{R}^3)$, the Hölder and Sobolev inequalities that

$$|L_u(v)| \leq |l|_2 |u|_6^2 |v|_6 \leq S^{-\frac{3}{2}} |l|_2 \|u\|^2 \|v\|.$$

Then, by the Lax–Milgram theorem, there exists a unique $\phi_u \in D^{1,2}\left(\mathbb{R}^3\right)$ such that $-\Delta \phi_u = l(x)u^2$ in $\left[D^{1,2}(\mathbb{R}^3)\right]^*$, where

$$\phi_u(x) = \frac{1}{4\pi} \int_{\mathbb{R}^3} \frac{l(y)u^2(y)}{|x-y|} dy.$$

Therefore, system (1.6) can be reduced to the semilinear Schrödinger equation with a non-local term

$$-\Delta u + l(x)\phi_u u = |u|^4 u + \mu h(x)|u|^{q-2}u \quad \text{in } \mathbb{R}^3. \tag{1.7}$$

Then, we say that $u \in D^{1,2}\left(\mathbb{R}^3\right)$ satisfying (1.7) is a weak solution of system (1.6) for short. Moreover, the solution u of system (1.6) is called as a sign-changing solution of system (1.6) once $u^\pm \neq 0$, where

$$u^+(x) := \max\{u, 0\} \quad \text{and} \quad u^-(x) := \min\{u, 0\}.$$

By the variational methods, weak solution of system (1.6) correspond to critical point of the functional

$$\mathcal{I}_\mu(u) = \frac{1}{2}\|u\|^2 + \frac{1}{4}\int_{\mathbb{R}^3} l(x)\phi_u u^2 dx - \frac{1}{6}\int_{\mathbb{R}^3} |u|^6 dx - \frac{\mu}{q}\int_{\mathbb{R}^3} h(x)|u|^q dx, \qquad \forall\, u \in D^{1,2}(\mathbb{R}^3).$$

Based on [14, Lemma 2.3], it is standard to verify that the functional $\mathcal{I}_\mu$ is well-defined in $D^{1,2}(\mathbb{R}^3)$ and is of class C^1. Naturally, we introduce the Nehari manifold of system (1.6) as follows:

$$\mathcal{N}_\mu = \left\{u \in D^{1,2}\left(\mathbb{R}^3\right) \setminus \{0\} : \left\langle \mathcal{I}_\mu'(u), u \right\rangle = 0\right\}.$$

As we will see, $\mathcal{N}_\mu$ is closely related to the behavior of the fiber map $\psi_u : t \to \mathcal{I}_\mu(tu)$, which is defined for any $u \in D^{1,2}(\mathbb{R}^3)$ and $t > 0$ by

$$\psi_u(t) = \frac{t^2}{2}\|u\|^2 + \frac{t^4}{4}\int_{\mathbb{R}^3} l(x)\phi_u u^2 dx - \frac{t^6}{6}|u|_6^6 - \frac{\mu t^q}{q}\int_{\mathbb{R}^3} h(x)|u|^q dx.$$

By a simple calculation, we obtain

$$\psi_u'(t) = t\|u\|^2 + t^3\int_{\mathbb{R}^3} l(x)\phi_u u^2 dx - t^5|u|_6^6 - \mu t^{q-1}\int_{\mathbb{R}^3} h(x)|u|^q dx$$

and

$$\psi_u''(t) = \|u\|^2 + 3t^2\int_{\mathbb{R}^3} l(x)\phi_u u^2 dx - 5t^4|u|_6^6 - (q-1)\mu t^{q-2}\int_{\mathbb{R}^3} h(x)|u|^q dx.$$

Clearly, for any $u \in D^{1,2}(\mathbb{R}^3) \setminus \{0\}$, there holds $\psi_u'(t) = 0$ iff $tu \in \mathcal{N}_\mu$. Specially, $\psi_u'(1) = 0$ iff $u \in \mathcal{N}_\mu$. To search for sign-changing solutions of system (1.6), we introduce the sign-changing Nehari manifold

$$\mathcal{M}_\mu = \left\{u \in D^{1,2}(\mathbb{R}^3) : u^\pm \neq 0,\ \langle \mathcal{I}_\mu'(u), u^\pm\rangle = 0\right\}.$$

Now we present our main result of this paper.

Theorem 1.1. *Assume (H) holds, then there exists some $\mu^* > 0$ such that system (1.6) possesses a sign-changing solution u_μ with positive energy for any $\mu \in (0, \mu^*)$.*

Remark 1.2. Our result extends the works [23, 24] to critical-concave Schrödinger–Poisson systems. The methods of studying subcritical concave-convex case in [23, 24] are invalid for critical-concave case, we need to search for a sign-changing Palais–Smale sequence and estimate the corresponding level less than a prescribed threshold to recover the loss of compactness. As we know, the Nehari manifold is a good choice to look for two positive solutions of system (1.6) by splitting the manifold into two parts and using minimization arguments on each parts. However, this idea cannot be directly used to prove sign-changing solutions, since we may not verify that the corresponding parts of the sign-changing Nehari manifold are complete. Thus we work in a closed subset of the one part of $\mathcal{M}_\mu$. Moreover, we propose a question whether system (1.6) has (ground state) sign-changing solution with negative energy.

The rest of this paper focus on proving Theorem 1.1 in Sect. 2. Throughout the present paper, we acknowledge the following notations:

- $L^p(\mathbb{R}^3)$ is the usual Lebesgue space with the norm $|u|_p = \left(\int_{\mathbb{R}^3} |u|^p dx\right)^{\frac{1}{p}}$ for $p \in [1, +\infty)$.
- $C_0^\infty(\mathbb{R}^3)$ consists of infinitely times differentiable functions with compact support in $\mathbb{R}^3$.
- $D^{1,2}(\mathbb{R}^3)$ is the completion of $C_0^\infty(\mathbb{R}^3)$ with respect to the norm $\|u\| = \left(\int_{\mathbb{R}^3} |\nabla u|^2 dx\right)^{\frac{1}{2}}$.
- The best Sobolev constant $S = \inf\left\{\|u\|^2 : u \in D^{1,2}(\mathbb{R}^3) \setminus \{0\}, |u|_6 = 1\right\}$.
- For $r > 0$ and $y \in \mathbb{R}^3$, $\mathbb{B}_r(y) := \left\{x \in \mathbb{R}^3 : |x - y| < r\right\}$ and $\mathbb{B}_r^c(y) = \mathbb{R}^3 \setminus \mathbb{B}_r(y)$.
- C_i $(i \in \mathbb{N}_+)$ denotes positive constant and $\mathbb{R}^+ = (0, +\infty)$.
- $o(1)$ denotes a quantity tending to 0 as $n \to \infty$.

2 Proof of Theorem 1.1

In this section, we will present some preliminary lemmas before completing the proof of our main result. For any $u \in D^{1,2}(\mathbb{R}^3)$ with $u^\pm \neq 0$, let the function f_u: $[0,+\infty) \times [0,+\infty) \to \mathbb{R}$ be defined as $f_u(s,t) = \mathcal{I}_\mu(su^+ + tu^-)$. From the Fubini theorem, we get

$$\begin{aligned} f_u(s,t) = {} & \frac{s^2}{2}\|u^+\|^2 + \frac{s^4}{4}\int_{\mathbb{R}^3} l(x)\phi_{u^+}|u^+|^2 dx - \frac{s^6}{6}|u^+|_6^6 - \frac{\mu s^q}{q}\int_{\mathbb{R}^3} h(x)|u^+|^q dx \\ & + \frac{t^2}{2}\|u^-\|^2 + \frac{t^4}{4}\int_{\mathbb{R}^3} l(x)\phi_{u^-}|u^-|^2 dx - \frac{t^6}{6}|u^-|_6^6 - \frac{\mu t^q}{q}\int_{\mathbb{R}^3} h(x)|u^-|^q dx \\ & + \frac{1}{2}s^2t^2\int_{\mathbb{R}^3} l(x)\phi_{u^+}|u^-|^2 dx. \end{aligned}$$

Through direct computations, we have

$$\begin{aligned} \frac{\partial f_u}{\partial s}(s,t) = {} & s\|u^+\|^2 + s^3\int_{\mathbb{R}^3} l(x)\phi_{u^+}|u^+|^2 dx + st^2\int_{\mathbb{R}^3} l(x)\phi_{u^+}|u^-|^2 dx \\ & - s^5|u^+|_6^6 - \mu s^{q-1}\int_{\mathbb{R}^3} h(x)|u^+|^q dx, \end{aligned} \tag{2.1}$$

$$\begin{aligned} \frac{\partial f_u}{\partial t}(s,t) = {} & t\|u^-\|^2 + t^3\int_{\mathbb{R}^3} l(x)\phi_{u^-}|u^-|^2 dx + ts^2\int_{\mathbb{R}^3} l(x)\phi_{u^+}|u^-|^2 dx \\ & - t^5|u^-|_6^6 - \mu t^{q-1}\int_{\mathbb{R}^3} h(x)|u^-|^q dx, \end{aligned} \tag{2.2}$$

$$\begin{aligned} \frac{\partial^2 f_u}{\partial s^2}(s,t) = {} & \|u^+\|^2 + 3s^2\int_{\mathbb{R}^3} l(x)\phi_{u^+}|u^+|^2 dx + t^2\int_{\mathbb{R}^3} l(x)\phi_{u^+}|u^-|^2 dx \\ & - 5s^4|u^+|_6^6 - \mu(q-1)s^{q-2}\int_{\mathbb{R}^3} h(x)|u^+|^q dx, \end{aligned} \tag{2.3}$$

$$\begin{aligned} \frac{\partial^2 f_u}{\partial t^2}(s,t) = {} & \|u^-\|^2 + 3t^2\int_{\mathbb{R}^3} l(x)\phi_{u^-}|u^-|^2 dx + s^2\int_{\mathbb{R}^3} l(x)\phi_{u^+}|u^-|^2 dx \\ & - 5t^4|u^-|_6^6 - \mu(q-1)t^{q-2}\int_{\mathbb{R}^3} h(x)|u^-|^q dx \end{aligned} \tag{2.4}$$

and

$$\frac{\partial^2 f_u}{\partial s \partial t}(s,t) = \frac{\partial^2 f_u}{\partial t \partial s}(s,t) = 2st\int_{\mathbb{R}^3} l(x)\phi_{u^+}|u^-|^2 dx. \tag{2.5}$$

It is easy to see that

$$\frac{\partial f_u}{\partial s}(s,t) = \left\langle \mathcal{I}'_\mu(su^+ + tu^-), u^+ \right\rangle \quad \text{and} \quad \frac{\partial f_u}{\partial t}(s,t) = \left\langle \mathcal{I}'_\mu(su^+ + tu^-), u^- \right\rangle,$$

which indicate that, for any $u \in D^{1,2}(\mathbb{R}^3)$ with $u^\pm \neq 0$, $su^+ + tu^- \in \mathcal{M}_\mu$ if and only if $\frac{\partial f_u}{\partial s}(s,t) = 0$ and $\frac{\partial f_u}{\partial t}(s,t) = 0$, that is

$$\mathcal{M}_\mu = \left\{ u \in D^{1,2}(\mathbb{R}^3) : u^\pm \neq 0, \ \frac{\partial f_u}{\partial s}(1,1) = 0, \ \frac{\partial f_u}{\partial t}(1,1) = 0 \right\}.$$

Moreover, for any $u \in \mathcal{M}_\mu$, by (2.1)–(2.4) we have

$$\begin{aligned}\frac{\partial^2 f_u}{\partial s^2}(1,1) = {} & \|u^+\|^2 + 3\int_{\mathbb{R}^3} l(x)\phi_{u^+}|u^+|^2dx + \int_{\mathbb{R}^3} l(x)\phi_{u^+}|u^-|^2dx \\ & - 5|u^+|_6^6 - \mu(q-1)\int_{\mathbb{R}^3} h(x)|u^+|^qdx \\ = {} & (2-q)\|u^+\|^2 + (4-q)\int_{\mathbb{R}^3} l(x)\phi_{u^+}|u^+|^2dx \\ & + (2-q)\int_{\mathbb{R}^3} l(x)\phi_{u^+}|u^-|^2dx - (6-q)|u^+|_6^6 \\ = {} & -4\|u^+\|^2 - 2\int_{\mathbb{R}^3} l(x)\phi_{u^+}|u^+|^2dx - 4\int_{\mathbb{R}^3} l(x)\phi_{u^+}|u^-|^2dx \\ & - \mu(q-6)\int_{\mathbb{R}^3} h(x)|u^+|^qdx \end{aligned} \tag{2.6}$$

and

$$\begin{aligned}\frac{\partial^2 f_u}{\partial t^2}(1,1) = {} & \|u^-\|^2 + 3\int_{\mathbb{R}^3} l(x)\phi_{u^-}|u^-|^2dx + \int_{\mathbb{R}^3} l(x)\phi_{u^+}|u^-|^2dx \\ & - 5|u^-|_6^6 - \mu(q-1)\int_{\mathbb{R}^3} h(x)|u^-|^qdx \\ = {} & (2-q)\|u^-\|^2 + (4-q)\int_{\mathbb{R}^3} l(x)\phi_{u^-}|u^-|^2dx \\ & + (2-q)\int_{\mathbb{R}^3} l(x)\phi_{u^+}|u^-|^2dx - (6-q)|u^-|_6^6 \\ = {} & -4\|u^-\|^2 - 2\int_{\mathbb{R}^3} l(x)\phi_{u^-}|u^-|^2dx - 4\int_{\mathbb{R}^3} l(x)\phi_{u^+}|u^-|^2dx \\ & - \mu(q-6)\int_{\mathbb{R}^3} h(x)|u^-|^qdx. \end{aligned} \tag{2.7}$$

To begin with, we give some useful properties for the non-local term in the following

Lemma 2.1. *For the unique solution ϕ_u of the Poisson equation, there hold*

(1) $\phi_u(\cdot) \geq 0$ *for any* $u \in D^{1,2}(\mathbb{R}^3)$;

(2) $\phi_{tu} = t^2\phi_u$ *for any* $t > 0$ *and* $u \in D^{1,2}(\mathbb{R}^3)$;

(3) $\int_{\mathbb{R}^3} l(x)\phi_u u^2dx \leq |\phi_u|_6\,|lu^2|_{\frac{6}{5}} \leq S^{-1}|l|_2^2|u|_6^4$;

(4) $\int_{\mathbb{R}^3} l(x)\phi_u u^2dx \leq |\phi_u|_6\,|lu^2|_{\frac{6}{5}} \leq S^{-1}|l|_3^2|u|_4^4$ *for any* $u \in L^4(\mathbb{R}^3)$;

(5) *if* $u_n \rightharpoonup u$ *in* $D^{1,2}(\mathbb{R}^3)$, *then* $\phi_{u_n} \rightharpoonup \phi_u$ *in* $D^{1,2}(\mathbb{R}^3)$;

(6) *if* $u_n \rightharpoonup u$ *in* $D^{1,2}(\mathbb{R}^3)$, *then* $\int_{\mathbb{R}^3} l(x)\phi_{u_n}u_n^2dx \to \int_{\mathbb{R}^3} l(x)\phi_u u^2dx$.

Proof. (1)–(4) are easily verified. (5) and (6) were proved in [14], we omit the details here. □

Lemma 2.2. *There exists some $\mu_1 > 0$ such that, for any $u \in D^{1,2}(\mathbb{R}^3)$ with $u^\pm \neq 0$, there holds that*

(1) *if $\mu \in (0, \mu_1)$, then $f_u(s,t)$ has exactly two critical points $0 < s_1(t) < s_2(t)$ for any fixed $t \geq 0$, where $s_1(t)$ is the minimum point and $s_2(t)$ is the maximum point;*

(2) *if $\mu \in (0, \mu_1)$, then $f_u(s,t)$ has exactly two critical points $0 < t_1(s) < t_2(s)$ for any fixed $s \geq 0$, where $t_1(s)$ is the minimum point and $t_2(s)$ is the maximum point.*

Proof. (1) For any $u \in D^{1,2}(\mathbb{R}^3)$ with $u^{\pm} \neq 0$, by (2.1) we get

$$\frac{\partial f_u}{\partial s}(s,t) = s^{q-1}\Big(s^{2-q}\|u^+\|^2 + s^{4-q}\int_{\mathbb{R}^3} l(x)\phi_{u^+}|u^+|^2dx \\ + s^{2-q}t^2\int_{\mathbb{R}^3} l(x)\phi_{u^+}|u^-|^2dx - s^{6-q}|u^+|_6^6 - \mu\int_{\mathbb{R}^3} h(x)|u^+|^q dx\Big).$$

Define the function $\beta_t(s) : [0,+\infty) \to \mathbb{R}$ by

$$\beta_t(s) = s^{2-q}\|u^+\|^2 + s^{4-q}\int_{\mathbb{R}^3} l(x)\phi_{u^+}|u^+|^2dx \\ + s^{2-q}t^2\int_{\mathbb{R}^3} l(x)\phi_{u^+}|u^-|^2dx - s^{6-q}|u^+|_6^6 - \mu\int_{\mathbb{R}^3} h(x)|u^+|^q dx.$$

For any fixed $t \geq 0$, we deduce $\frac{\partial f_u(s,t)}{\partial s} = 0$ iff $\beta_t(s) = 0$ on $(0,+\infty)$. By direct calculation, we derive

$$\begin{aligned}\beta_t'(s) &= (2-q)s^{1-q}\|u^+\|^2 + (4-q)s^{3-q}\int_{\mathbb{R}^3} l(x)\phi_{u^+}|u^+|^2dx \\ &\quad + (2-q)s^{1-q}t^2\int_{\mathbb{R}^3} l(x)\phi_{u^+}|u^-|^2dx - (6-q)s^{5-q}|u^+|_6^6 \\ &= s^{1-q}\Big[(2-q)\|u^+\|^2 + (4-q)s^2\int_{\mathbb{R}^3} l(x)\phi_{u^+}|u^+|^2dx \\ &\quad + (2-q)t^2\int_{\mathbb{R}^3} l(x)\phi_{u^+}|u^-|^2dx - (6-q)s^4|u^+|_6^6\Big].\end{aligned}$$

Then, for any fixed $t \geq 0$, it is easy to obtain that $\beta_t(s)$ has exactly one critical point $s_{\max} > 0$ and β_t is increasing in $(0, s_{\max})$ while decreasing in $(s_{\max}, +\infty)$. Moreover, we define $\gamma_t(s) : [0,+\infty) \to \mathbb{R}$ by

$$\gamma_t(s) = s^{2-q}\|u^+\|^2 - s^{6-q}|u^+|_6^6 - \mu\int_{\mathbb{R}^3} h(x)|u^+|^q dx.$$

By direct computation, we have

$$\gamma_t'(s) = (2-q)s^{1-q}\|u^+\|^2 - (6-q)s^{5-q}|u^+|_6^6.$$

Obviously, γ_t takes the maximum at $\bar{s} = \left[\frac{(2-q)\|u^+\|^2}{(6-q)|u^+|_6^6}\right]^{\frac{1}{4}}$ and

$$\max_{s>0}\gamma_t(s) = \gamma_t(\bar{s}) = \frac{4}{6-q}\left(\frac{2-q}{6-q}\right)^{\frac{2-q}{4}}\frac{\|u^+\|^{\frac{6-q}{2}}}{|u^+|_6^{\frac{6-3q}{2}}} - \mu\int_{\mathbb{R}^3} h(x)|u^+|^q dx.$$

Denote

$$\mu_1^+ = \frac{4}{6-q}\left(\frac{2-q}{6-q}\right)^{\frac{2-q}{4}}\inf_{u\in D^{1,2}(\mathbb{R}^3), u^+\neq 0}\frac{\|u^+\|^{\frac{6-q}{2}}}{|u^+|_6^{\frac{6-3q}{2}}\int_{\mathbb{R}^3} h(x)|u^+|^q dx}.$$

It follows from (H), the Hölder and Sobolev inequalities that

$$\mu_1^+ \geq \frac{4}{6-q}\left(\frac{2-q}{6-q}\right)^{\frac{2-q}{4}}\frac{S^{\frac{6-q}{4}}}{|h|_{\frac{6}{6-q}}} =: \mu_1. \tag{2.8}$$

Then, once $\mu \in (0, \mu_1)$, we can deduce $\beta_t(s_{\max}) \geq \beta_t(\bar{s}) > \gamma_t(\bar{s}) > 0$ for any $u \in D^{1,2}(\mathbb{R}^3)$ with $u^+ \neq 0$. As a consequence, for any fixed $t \geq 0$, if $\mu \in (0, \mu_1)$, there exist exactly $s_2(t) > s_1(t) > 0$ such that $\beta_t(s_1(t)) = 0$ and $\beta_t(s_2(t)) = 0$, which imply $\frac{\partial f_u}{\partial s}(s_1(t), t) = \frac{\partial f_u}{\partial s}(s_2(t), t) = 0$. Furthermore, we can easily deduce that f_u is decreasing in $s \in (0, s_1(t))$, increasing in $s \in (s_1(t), s_2(t))$ and decreasing in $s \in (s_2(t), +\infty)$. Thus, $s_1(t)$ is the minimum point and $s_2(t)$ is the maximum point, (1) is verified.

(2) Similar to the proof of (1). We define $\beta_s(t) : [0, +\infty) \to \mathbb{R}$ by

$$\begin{aligned}\beta_s(t) = t^{2-q}\|u^-\|^2 + t^{4-q}\int_{\mathbb{R}^3} l(x)\phi_{u^-}|u^-|^2 dx \\ + t^{2-q}s^2\int_{\mathbb{R}^3} l(x)\phi_{u^+}|u^-|^2 dx - t^{6-q}|u^-|_6^6 - \mu\int_{\mathbb{R}^3} h(x)|u^-|^q dx.\end{aligned}$$

For any fixed $s \geq 0$, we deduce $\frac{\partial f_u(s,t)}{\partial t} = 0$ iff $\beta_s(t) = 0$. It can be verified that $\beta_s(t)$ has exactly one critical point $t_{\max} > 0$ and β_s is increasing in $(0, t_{\max})$ while decreasing in $(t_{\max}, +\infty)$. Denote

$$\mu_1^- = \frac{4}{6-q}\left(\frac{2-q}{6-q}\right)^{\frac{2-q}{4}} \inf_{u\in D^{1,2}(\mathbb{R}^3), u^-\neq 0} \frac{\|u^-\|^{\frac{6-q}{2}}}{|u^-|_6^{\frac{6-3q}{2}} \int_{\mathbb{R}^3} h(x)|u^-|^q dx}.$$

It is clear that $\mu_1^+ = \mu_1^- \geq \mu_1$. Repeating the discussion in the proof of (1), we may obtain $\beta_s(t_{\max}) > 0$ for any $\mu \in (0, \mu_1)$. Then, for any fixed $s \geq 0$, if $\mu \in (0, \mu_1)$, there exist exactly $t_2(s) > t_1(s) > 0$ such that $\beta_s(t_1(s)) = \beta_s(t_2(s)) = 0$, which imply $\frac{\partial f_u}{\partial t}(s, t_1(s)) = \frac{\partial f_u}{\partial t}(s, t_2(s)) = 0$. Besides, it is easy to see that f_u is decreasing in $t \in (0, t_1(s))$, increasing in $t \in (t_1(s), t_2(s))$ and decreasing in $t \in (t_2(s), +\infty)$. Naturally, $t_1(s)$ is the minimum point and $t_2(s)$ is the maximum point. Thus, (2) is proved. □

Lemma 2.3. *If $0 < \mu < \mu_1$, then $\frac{\partial^2 f_u}{\partial s^2}(1,1) \neq 0$ and $\frac{\partial^2 f_u}{\partial t^2}(1,1) \neq 0$ for any $u \in \mathcal{M}_\mu$.*

Proof. For any $u \in \mathcal{M}_\mu$, according to Lemma 2.2-(1), $f_u(s, 1)$ has exactly two critical points $s_1(1)$ and $s_2(1)$, with $\frac{\partial^2 f_u}{\partial s^2}(s_1(1), 1) > 0$ and $\frac{\partial^2 f_u}{\partial s^2}(s_2(1), 1) < 0$. Since $u \in \mathcal{M}_\mu$, it follows that $\frac{\partial f_u}{\partial s}(1,1) = 0$. This implies either $s_1(1) = 1$ or $s_2(1) = 1$. Consequently, $\frac{\partial^2 f_u}{\partial s^2}(1,1) \neq 0$. Analogously, by Lemma 2.2-(2) we conclude $\frac{\partial^2 f_u}{\partial t^2}(1,1) \neq 0$. Thus the proof is completed. □

In order to find sign-changing solution of system (1.6), we introduce the following two sets

$$\mathcal{M}_\mu^- = \left\{u \in \mathcal{M}_\mu : \frac{\partial^2 f_u}{\partial s^2}(1,1) < 0,\ \frac{\partial^2 f_u}{\partial t^2}(1,1) < 0\right\}$$

and

$$\mathcal{M}_\mu^* = \left\{u \in \mathcal{M}_\mu : \frac{\partial^2 f_u}{\partial s^2}(1,1) < 0,\ \frac{\partial^2 f_u}{\partial t^2}(1,1) < 0,\ \psi_u''(1) < 0\right\}. \tag{2.9}$$

In the following, we will use the properties of f_u to show that the set $\mathcal{M}_\mu^* \neq \emptyset$ and $\mathcal{M}_\mu^* = \mathcal{M}_\mu^-$.

Lemma 2.4. *There exists some constant $c > 0$, independent of μ, such that $\|u^\pm\| \geq c$ for all $u \in \mathcal{M}_\mu^-$. Moreover, the functional $\mathcal{I}_\mu$ is coercive on $\mathcal{M}_\mu^*$.*

Proof. For any $u \in \mathcal{M}_\mu^-$, from $\frac{\partial^2 f_u}{\partial s^2}(1,1) < 0$, $\frac{\partial^2 f_u}{\partial t^2}(1,1) < 0$, (2.6), (2.7) and the Sobolev inequality, it follows that

$$\begin{aligned}(2-q)\|u^\pm\|^2 &< (2-q)\|u^\pm\|^2 + (4-q)\int_{\mathbb{R}^3} l(x)\phi_{u^\pm}|u^\pm|^2 dx + (2-q)\int_{\mathbb{R}^3} l(x)\phi_{u^+}|u^-|^2 dx \\ &< (6-q)|u^\pm|_6^6 \le (6-q)S^{-3}\|u^\pm\|^6,\end{aligned}$$

which shows

$$\|u^\pm\| > \left[\frac{(2-q)S^3}{6-q}\right]^{\frac{1}{4}} =: c > 0.$$

Moreover, for any $u \in \mathcal{M}_\mu^*$, by the Hölder inequality one has

$$\begin{aligned}\mathcal{I}_\mu(u) &= \mathcal{I}_\mu(u) - \frac{1}{4}\langle \mathcal{I}_\mu'(u), u\rangle \\ &= \frac{1}{4}\|u\|^2 + \frac{1}{12}|u|_6^6 - \mu\left(\frac{1}{q} - \frac{1}{4}\right)\int_{\mathbb{R}^3} h(x)|u|^q dx \\ &\ge \frac{1}{4}\|u\|^2 + \frac{1}{12}|u|_6^6 - \mu\left(\frac{1}{q} - \frac{1}{4}\right)|h|_{\frac{6}{6-q}}|u|_6^q \\ &\ge \frac{1}{4}\|u\|^2 + \min_{t\ge 0}\left[\frac{1}{12}t^6 - \mu\left(\frac{1}{q} - \frac{1}{4}\right)|h|_{\frac{6}{6-q}} t^q\right] \\ &\ge \frac{1}{4}\|u\|^2 - \mu^{\frac{6}{6-q}}\frac{6-q}{6}\left[\left(\frac{1}{q} - \frac{1}{4}\right)|h|_{\frac{6}{6-q}}(2q)^{\frac{q}{6}}\right]^{\frac{6}{6-q}}. \end{aligned} \tag{2.10}$$

As a consequence, we deduce that $\mathcal{I}_\mu$ is coercive on $\mathcal{M}_\mu^*$. Thus this lemma is proved. □

Lemma 2.5. *There exists $\mu_2 \in (0, \mu_1)$ such that, once $\mu \in (0, \mu_2)$, for any $u \in D^{1,2}(\mathbb{R}^3)$ with $u^\pm \neq 0$, there exists a unique pair $(s_u, t_u) \in \mathbb{R}^+ \times \mathbb{R}^+$ satisfying $s_u u^+ + t_u u^- \in \mathcal{M}_\mu^-$. Moreover,*

$$\mathcal{I}_\mu(s_u u^+ + t_u u^-) = \max_{s,t>0} \mathcal{I}_\mu(su^+ + tu^-).$$

Proof. Let $0 < \mu < \mu_1$. For any $u \in D^{1,2}(\mathbb{R}^3)$ with $u^\pm \neq 0$, by Lemma 2.2, we recall that $\frac{\partial f_u}{\partial s}$ satisfies

(*a*) $\frac{\partial f_u}{\partial t}(s, t_2(s)) = 0$ for all $s \ge 0$,

(*b*) $\frac{\partial^2 f_u}{\partial t^2}(s, t_2(s)) < 0$ for all $s \ge 0$,

(*c*) $\frac{\partial f_u}{\partial t}(s,t)$ is continuous and has continuous partial derivatives in $[0, +\infty) \times [0, +\infty)$.

Then, applying the implicit function theorem, we have that $\frac{\partial f_u}{\partial t}(s,t) = 0$ determines an implicit function $t_2(s)$ with continuous derivative on $[0, +\infty)$. Similarly, we may deduce that $\frac{\partial f_u}{\partial s}(s,t) = 0$ determines an implicit function $s_2(t)$ with continuous derivative on $[0, +\infty)$. For any $s \ge 0$, from $\frac{\partial f_u}{\partial t}(s, t_2(s)) = 0$ and $\frac{\partial f_u}{\partial t}(s,t) \le 0$ for sufficiently large $t > 0$, we claim that

$$t_2(s) < s \quad \text{for } s > 0 \text{ large enough.} \tag{2.11}$$

If not, for some sufficiently large s, it follows from the definition (2.2) that $\frac{\partial f_u}{\partial t}(s, t_2(s)) < 0$, which contradicts $\frac{\partial f_u}{\partial t}(s, t_2(s)) = 0$. Analogously, there holds that

$$s_2(t) < t \quad \text{for } t > 0 \text{ large enough.} \tag{2.12}$$

Hence, by (2.11), (2.12), $t_2(0) > 0$, $s_2(0) > 0$, the continuity of $t_2(s)$ and $s_2(t)$, we conclude the curves of $t_2(s)$ and $s_2(t)$ intersect at some point $(s_u, t_u) \in \mathbb{R}^+ \times \mathbb{R}^+$. Naturally, $\frac{\partial f_u}{\partial s}(s_u, t_u) = \frac{\partial f_u}{\partial t}(s_u, t_u) = 0$. In addition, from (2.5) and (b) it follows that

$$t_2'(s) = -\frac{\frac{\partial^2 f_u}{\partial t \partial s}(s, t_2(s))}{\frac{\partial^2 f_u}{\partial t^2}(s, t_2(s))} > 0 \quad \text{for any } s > 0.$$

Therefore, for any $s > 0$, we obtain that the function t_2 is increasing in $s \in (0, +\infty)$. Similarly, the function s_2 is increasing in $t \in (0, +\infty)$. Hence, there exists unique pair $(s_u, t_u) \in \mathbb{R}^+ \times \mathbb{R}^+$ such that

$$\frac{\partial f_u}{\partial s}(s_u, t_u) = \frac{\partial f_u}{\partial t}(s_u, t_u) = 0$$

and

$$\frac{\partial^2 f_u}{\partial s^2}(s_u, t_u) < 0, \qquad \frac{\partial^2 f_u}{\partial t^2}(s_u, t_u) < 0,$$

that is, $s_u u^+ + t_u u^- \in \mathcal{M}_\mu^-$. Moreover, we prove (s_u, t_u) is the unique maximum point of f_u on $(\mathbb{R}^+)^2$. For this, it suffices to show $(1, 1)$ is the unique maximum point of f_u on $(\mathbb{R}^+)^2$ for any $u \in \mathcal{M}_\mu^-$. Denote

$$\bar{\mu}_2 = \frac{4S^{\frac{q}{2}}}{(6-q)q|h|_{\frac{6}{6-q}}} \left(\frac{qc^2}{2-q} \right)^{\frac{2-q}{2}}.$$

Set $\mu_2 = \min\{\mu_1, \bar{\mu}_2\}$ and let $\mu \in (0, \mu_2)$. For any $u \in \mathcal{M}_\mu^-$, we consider the Hessian matrix

$$H(u) = \begin{pmatrix} \frac{\partial^2 f_u}{\partial s^2}(1,1) & \frac{\partial^2 f_u}{\partial s \partial t}(1,1) \\ \frac{\partial^2 f_u}{\partial t \partial s}(1,1) & \frac{\partial^2 f_u}{\partial t^2}(1,1) \end{pmatrix}.$$

With (2.5)$-$(2.7) in hand, we deduce from Lemma 2.4 and the Hölder inequality that the determinant

$$\begin{aligned}
\det H(u) = & \left[4\|u^+\|^2 + \int_{\mathbb{R}^3} \left(2l(x)\phi_{u^+}|u^+|^2 + 4l(x)\phi_{u^+}|u^-|^2 \right) dx - \mu(6-q) \int_{\mathbb{R}^3} h(x)|u^+|^q dx \right] \\
& \times \left[4\|u^-\|^2 + \int_{\mathbb{R}^3} \left(2l(x)\phi_{u^-}|u^-|^2 + 4l(x)\phi_{u^+}|u^-|^2 \right) dx - \mu(6-q) \int_{\mathbb{R}^3} h(x)|u^-|^q dx \right] \\
& - 4 \left(\int_{\mathbb{R}^3} l(x)\phi_{u^+}|u^-|^2 dx \right)^2 \\
\geq & \left[4\|u^+\|^2 + 4\int_{\mathbb{R}^3} l(x)\phi_{u^+}|u^-|^2 dx - \mu(6-q)S^{-\frac{q}{2}}|h|_{\frac{6}{6-q}}\|u^+\|^q \right] \\
& \times \left[4\|u^-\|^2 + 4\int_{\mathbb{R}^3} l(x)\phi_{u^+}|u^-|^2 dx - \mu(6-q)S^{-\frac{q}{2}}|h|_{\frac{6}{6-q}}\|u^-\|^q \right] \\
& - 4 \left(\int_{\mathbb{R}^3} l(x)\phi_{u^+}|u^-|^2 dx \right)^2 \\
\geq & \left\{ 2\|u^+\|^2 + 4\int_{\mathbb{R}^3} l(x)\phi_{u^+}|u^-|^2 dx - \frac{4-2q}{q} \left[\frac{\mu(6-q)q|h|_{\frac{6}{6-q}}}{4S^{\frac{q}{2}}} \right]^{\frac{2}{2-q}} \right\}
\end{aligned}$$

$$
\begin{aligned}
&\times \left\{ 2\|u^-\|^2 + 4\int_{\mathbb{R}^3} l(x)\phi_{u^+}|u^-|^2dx - \frac{4-2q}{q}\left[\frac{\mu(6-q)q|h|_{\frac{6}{6-q}}}{4S^{\frac{q}{2}}}\right]^{\frac{2}{2-q}} \right\} \\
&- 4\left(\int_{\mathbb{R}^3} l(x)\phi_{u^+}|u^-|^2dx\right)^2 \\
\geq\ & 4\int_{\mathbb{R}^3} l(x)\phi_{u^+}|u^-|^2dx \left\{ 2c^2 - \frac{4-2q}{q}\left[\frac{\mu(6-q)q|h|_{\frac{6}{6-q}}}{4S^{\frac{q}{2}}}\right]^{\frac{2}{2-q}} \right\} \\
&+ 4\int_{\mathbb{R}^3} l(x)\phi_{u^+}|u^-|^2dx \left\{ 2c^2 - \frac{4-2q}{q}\left[\frac{\mu(6-q)q|h|_{\frac{6}{6-q}}}{4S^{\frac{q}{2}}}\right]^{\frac{2}{2-q}} \right\} \\
&+ \left\{ 2c^2 - \frac{4-2q}{q}\left[\frac{\mu(6-q)q|h|_{\frac{6}{6-q}}}{4S^{\frac{q}{2}}}\right]^{\frac{2}{2-q}} \right\} \times \left\{ 2c^2 - \frac{4-2q}{q}\left[\frac{\mu(6-q)q|h|_{\frac{6}{6-q}}}{4S^{\frac{q}{2}}}\right]^{\frac{2}{2-q}} \right\} \\
>\ & 0,
\end{aligned}
$$

which together with $\frac{\partial^2 f_u}{\partial s^2}(1,1) < 0$ implies that $H(u)$ is negative definite. Consequently, there holds that $\mathcal{I}_\mu(u^+ + u^-) = \max_{s,t>0} \mathcal{I}_\mu(su^+ + tu^-)$ for any $u \in \mathcal{M}_\mu^-$. Therefore, the proof is completed. □

Lemma 2.6. *Once $\mu \in (0, \mu_2)$, then $\mathcal{M}_\mu^* \neq \emptyset$ and $\mathcal{M}_\mu^* = \mathcal{M}_\mu^-$.*

Proof. It is obvious that $\mathcal{M}_\mu^* \subset \mathcal{M}_\mu^-$. Moreover, for any $u \in \mathcal{M}_\mu^-$, it follows from the proof of Lemma 2.5 that $(1,1)$ is the unique maximum point of $f_u(s,t)$. Observe $f_u(r,r) = \psi_u(r)$, we know that 1 is a maximum point of ψ_u. Naturally, $\psi_u''(1) < 0$. By the arbitrariness of $u \in \mathcal{M}_\mu^-$, we derive $\mathcal{M}_\mu^- \subset \mathcal{M}_\mu^*$. That is, $\mathcal{M}_\mu^- = \mathcal{M}_\mu^*$. Then, due to Lemma 2.5, we know $\mathcal{M}_\mu^* \neq \emptyset$. Thus the proof is completed. □

In the forthcoming lemmas, we construct a minimizing sequence of Palais–Smale type for

$$m_\mu^* = \inf_{u \in \mathcal{M}_\mu^*} \mathcal{I}_\mu(u).$$

First, we estimate the level m_μ^*. As we know, the extremal functions of the best Sobolev constant S are

$$U_\varepsilon(\cdot) = \frac{3^{\frac{1}{4}}\varepsilon^{\frac{1}{2}}}{(\varepsilon^2 + |\cdot|^2)^{\frac{1}{2}}}, \qquad \varepsilon > 0.$$

Choose two cut-off functions φ, $\psi \in C_0^\infty(\mathbb{R}^3, [0,1])$ such that,

$$\varphi(x) = \begin{cases} 1, & x \in \mathbb{B}_1(0), \\ 0, & x \in \mathbb{B}_2^c(0), \end{cases}$$

and

$$\psi(x) = \begin{cases} 0, & x \in \mathbb{B}_2(0), \\ 1, & x \in \mathbb{B}_4(0) \setminus \mathbb{B}_3(0), \\ 0, & x \in \mathbb{B}_5^c(0). \end{cases}$$

Define $u_\varepsilon(x) = \varphi(x)U_\varepsilon(x)$, $v_\varepsilon(x) = -\psi(x)U_\varepsilon(x)$ and $w_\varepsilon(x) = u_\varepsilon(x) + v_\varepsilon(x)$. Clearly, $w_\varepsilon^+(x) = u_\varepsilon(x)$ and $w_\varepsilon^-(x) = v_\varepsilon(x)$. Following the similar calculations to [3], we conclude that, for $\varepsilon > 0$ small enough,

$$\begin{cases} \int_{\mathbb{R}^3} |\nabla u_\varepsilon|^2 dx = S^{\frac{3}{2}} + O(\varepsilon), \\ \int_{\mathbb{R}^3} |u_\varepsilon|^6 dx = S^{\frac{3}{2}} + O(\varepsilon^3), \\ \int_{\mathbb{R}^3} |u_\varepsilon|^4 dx = O(\varepsilon), \\ \int_{\mathbb{R}^3} |\nabla v_\varepsilon|^2 dx = O(\varepsilon), \\ \int_{\mathbb{R}^3} |v_\varepsilon|^6 dx \in \left[C_1\varepsilon^3, C_2\varepsilon^3\right]. \end{cases} \tag{2.13}$$

Lemma 2.7. *There exists $\mu_3 \in (0, \mu_2)$ such that $0 < m_\mu^* < \frac{1}{3}S^{\frac{3}{2}} - D\mu^{\frac{6}{6-q}}$ for any $\mu \in (0, \mu_3)$, where $D = \frac{6-q}{6}\left[\left(\frac{1}{q} - \frac{1}{4}\right)|h|_{\frac{6}{6-q}}(2q)^{\frac{q}{6}}\right]^{\frac{6}{6-q}}$.*

Proof. Firstly, let

$$\mu < \min\left\{\mu_2, \frac{4q}{(4-q)|h|_{\frac{6}{6-q}}(2q)^{\frac{q}{6}}}\left[\frac{3c^2}{4(6-q)}\right]^{\frac{6-q}{6}}\right\}.$$

From (2.10) and Lemma 2.4, it follows that

$$\mathcal{I}_\mu(u) \geq \frac{1}{4}c^2 - \mu^{\frac{6}{6-q}}\frac{6-q}{6}\left[\left(\frac{1}{q} - \frac{1}{4}\right)|h|_{\frac{6}{6-q}}(2q)^{\frac{q}{6}}\right]^{\frac{6}{6-q}} \geq \frac{1}{8}c^2, \quad \forall\, u \in \mathcal{M}_\mu^*,$$

which implies $m_\mu^* > 0$. Due to Lemma 2.5, there exist some $s_\varepsilon, t_\varepsilon > 0$ such that $s_\varepsilon u_\varepsilon + t_\varepsilon v_\varepsilon \in \mathcal{M}_\mu^*$. Observe spt $u_\varepsilon \cap$ spt $v_\varepsilon = \emptyset$, we have

$$\begin{aligned} \mathcal{I}_\mu(s_\varepsilon u_\varepsilon + t_\varepsilon v_\varepsilon) = {}& \frac{s_\varepsilon^2}{2}\|u_\varepsilon\|^2 + \frac{s_\varepsilon^4}{4}\int_{\mathbb{R}^3} l(x)\phi_{u_\varepsilon}u_\varepsilon^2 dx - \frac{s_\varepsilon^6}{6}|u_\varepsilon|_6^6 - \frac{\mu s_\varepsilon^q}{q}\int_{\mathbb{R}^3} h(x)|u_\varepsilon|^q dx \\ & + \frac{t_\varepsilon^2}{2}\|v_\varepsilon\|^2 + \frac{t_\varepsilon^4}{4}\int_{\mathbb{R}^3} l(x)\phi_{v_\varepsilon}v_\varepsilon^2 dx - \frac{t_\varepsilon^6}{6}|v_\varepsilon|_6^6 - \frac{\mu t_\varepsilon^q}{q}\int_{\mathbb{R}^3} h(x)|v_\varepsilon|^q dx \\ & + \frac{1}{2}\int_{\mathbb{R}^3} l(x)\phi_{s_\varepsilon u_\varepsilon}|t_\varepsilon v_\varepsilon|^2 dx. \end{aligned} \tag{2.14}$$

By the semigroup property of convolution (see [16, Theorem 5.9]) and the Young inequality, we have

$$\begin{aligned} &\int_{\mathbb{R}^3} l(x)\phi_{s_\varepsilon u_\varepsilon}|t_\varepsilon v_\varepsilon|^2 dx \\ &\quad = \frac{1}{4\pi}\int_{\mathbb{R}^3}\left(\frac{1}{|x|} * l(x)|s_\varepsilon u_\varepsilon|^2\right) l(x)|t_\varepsilon v_\varepsilon|^2 dx \\ &\quad = \frac{1}{4\pi}\int_{\mathbb{R}^3}\left(\frac{1}{|x|^2} * l(x)|s_\varepsilon u_\varepsilon|^2\right)\left(\frac{1}{|x|^2} * l(x)|t_\varepsilon v_\varepsilon|^2\right) dx \\ &\quad \leq \frac{1}{8\pi}\int_{\mathbb{R}^3}\left(\frac{1}{|x|^2} * l(x)|s_\varepsilon u_\varepsilon|^2\right)^2 + \left(\frac{1}{|x|^2} * l(x)|t_\varepsilon v_\varepsilon|^2\right)^2 dx \\ &\quad = \frac{1}{8\pi}\int_{\mathbb{R}^3}\left(\frac{1}{|x|} * l(x)|s_\varepsilon u_\varepsilon|^2\right) l(x)|s_\varepsilon v_\varepsilon|^2 + \left(\frac{1}{|x|} * l(x)|t_\varepsilon u_\varepsilon|^2\right) l(x)|t_\varepsilon v_\varepsilon|^2 dx \\ &\quad = \frac{s_\varepsilon^4}{2}\int_{\mathbb{R}^3} l(x)\phi_{u_\varepsilon}u_\varepsilon^2 dx + \frac{t_\varepsilon^4}{2}\int_{\mathbb{R}^3} l(x)\phi_{v_\varepsilon}v_\varepsilon^2 dx. \end{aligned}$$

Then, (2.14) can be reduced to

$$\begin{aligned}\mathcal{I}_\mu(s_\varepsilon u_\varepsilon + t_\varepsilon v_\varepsilon) \leq \frac{s_\varepsilon^2}{2}\int_{\mathbb{R}^3}|\nabla u_\varepsilon|^2dx + \frac{s_\varepsilon^4}{2}\int_{\mathbb{R}^3} l(x)\phi_{u_\varepsilon}|u_\varepsilon|^2dx - \frac{s_\varepsilon^6}{6}|u_\varepsilon|_6^6 - \frac{\mu s_\varepsilon^q}{q}\int_{\mathbb{R}^3} h(x)|u_\varepsilon|^q dx \\ + \frac{t_\varepsilon^2}{2}\int_{\mathbb{R}^3}|\nabla v_\varepsilon|^2dx + \frac{t_\varepsilon^4}{2}\int_{\mathbb{R}^3} l(x)\phi_{v_\varepsilon}|v_\varepsilon|^2dx - \frac{t_\varepsilon^6}{6}|v_\varepsilon|_6^6. \end{aligned} \tag{2.15}$$

For convenience, we introduce

$$\begin{aligned}\mathcal{J}_1(\varepsilon) &= \frac{s_\varepsilon^2}{2}\int_{\mathbb{R}^3}|\nabla u_\varepsilon|^2dx + \frac{s_\varepsilon^4}{2}\int_{\mathbb{R}^3} l(x)\phi_{u_\varepsilon}u_\varepsilon^2dx - \frac{s_\varepsilon^6}{6}|u_\varepsilon|_6^6 - \frac{\mu s_\varepsilon^q}{q}\int_{\mathbb{R}^3} h(x)|u_\varepsilon|^q dx, \\ \mathcal{J}_2(\varepsilon) &= \frac{t_\varepsilon^2}{2}\int_{\mathbb{R}^3}|\nabla v_\varepsilon|^2dx + \frac{t_\varepsilon^4}{2}\int_{\mathbb{R}^3} l(x)\phi_{v_\varepsilon}v_\varepsilon^2dx - \frac{t_\varepsilon^6}{6}|v_\varepsilon|_6^6.\end{aligned}$$

By Lemma 2.1-(3) and (2.13), for $\varepsilon > 0$ small enough, we get

$$\begin{aligned}\mathcal{J}_1(\varepsilon) &\leq s_\varepsilon^2 S^{\frac{3}{2}} + s_\varepsilon^4|l|_2^2 - \frac{s_\varepsilon^6}{12}S^{\frac{3}{2}}, \\ \mathcal{J}_2(\varepsilon) &\leq C_3 t_\varepsilon^2\varepsilon + C_4 t_\varepsilon^4\varepsilon^2 - C_1 t_\varepsilon^6\varepsilon^3 \\ &= C_3\left(t_\varepsilon\sqrt{\varepsilon}\right)^2 + C_4\left(t_\varepsilon\sqrt{\varepsilon}\right)^4 - C_1\left(t_\varepsilon\sqrt{\varepsilon}\right)^6.\end{aligned} \tag{2.16}$$

Define $\mathcal{G}(t) = C_3t^2 + C_4t^4 - C_1t^6$ for $t \geq 0$. We infer that $\mathcal{G}$ has exactly one critical point $t_{\max} > 0$ and $\mathcal{G}(t_{\max}) > \mathcal{G}(0) = 0$. Hence,

$$\mathcal{J}_2(\varepsilon) \leq \max_{t\geq 0}\mathcal{G}(t) = \mathcal{G}(t_{\max}). \tag{2.17}$$

Noting $m_\mu^* \leq \mathcal{I}_\mu(s_\varepsilon u_\varepsilon + t_\varepsilon v_\varepsilon)$, we deduce from (2.15)–(2.17) that s_ε is bounded from above and below by positive constants. Namely, there exist $S_1, S_2 > 0$ such that $S_1 \leq s_\varepsilon \leq S_2$ for $\varepsilon > 0$ small enough. From Lemma 2.5, we know $t_\varepsilon = t_2(s_\varepsilon)$ and t_ε is continuous on $[0, +\infty)$. Therefore, t_ε is also bounded from above. Then, by (2.15), (2.13) and Lemma 2.1-(4) we can obtain that, for $\varepsilon > 0$ small enough,

$$\begin{aligned}&\mathcal{I}_\mu(s_\varepsilon u_\varepsilon + t_\varepsilon v_\varepsilon) \\ &\quad\leq \max_{s>0}\left(\frac{s^2}{2}\|u_\varepsilon\|^2 - \frac{s^6}{6}|u_\varepsilon|_6^6\right) + \frac{S_2^4}{2}\int_{\mathbb{R}^3} l(x)\phi_{u_\varepsilon}u_\varepsilon^2dx - \frac{\mu S_1^q}{q}\int_{\mathbb{R}^3} h(x)|u_\varepsilon|^q dx + O(\varepsilon) \\ &\quad\leq \frac{\|u_\varepsilon\|^3}{3|u_\varepsilon|_6^3} + \frac{1}{2}S_2^4 S^{-1}|l|_3^2|u_\varepsilon|_4^4 - \frac{\mu S_1^q}{q}\int_{\mathbb{R}^3} h(x)|u_\varepsilon|^q dx + O(\varepsilon) \\ &\quad\leq \frac{1}{3}\frac{\left[S^{\frac{3}{2}} + O(\varepsilon)\right]^{\frac{3}{2}}}{\left[S^{\frac{3}{2}} + O(\varepsilon^3)\right]^{\frac{1}{2}}} - \frac{\mu S_1^q}{q}\int_{\mathbb{R}^3} h(x)|u_\varepsilon|^q dx + O(\varepsilon) \\ &\quad\leq \frac{1}{3}S^{\frac{3}{2}} + C_5\varepsilon - \frac{\mu S_1^q}{q}\int_{\mathbb{R}^3} h(x)|u_\varepsilon|^q dx.\end{aligned}$$

Recalling that $h(x) \geq \frac{1}{|x|^\beta}$ for all $|x| \leq 1$ and some $\beta > 3 - \frac{2q}{3}$, if letting $\varepsilon = \mu^{\frac{6}{6-q}}$, we conclude

$$\begin{aligned}
&C_5\varepsilon - \frac{\mu S_1^q}{q}\int_{\mathbb{R}^3} h(x)|u_\varepsilon|^q dx \\
&\leq C_5\varepsilon - \frac{\mu S_1^q}{q}\int_{\mathbb{B}_1(0)} \frac{3^{\frac{q}{4}}\varepsilon^{\frac{q}{2}}}{|x|^\beta(\varepsilon^2+|x|^2)^{\frac{q}{2}}}dx \\
&= C_5\varepsilon - \mu\varepsilon^{3-\frac{q}{2}-\beta}\frac{S_1^q}{q}\int_{\mathbb{B}_{\frac{1}{\varepsilon}}(0)} \frac{3^{\frac{q}{4}}}{|x|^\beta(1+|x|^2)^{\frac{q}{2}}}dx \\
&\leq C_5\varepsilon - C_6\mu\varepsilon^{3-\frac{q}{2}-\beta} \\
&= \mu^{\frac{6}{6-q}}\left[C_5 - C_6\mu^{1+\frac{6}{6-q}(2-\frac{q}{2}-\beta)}\right] \\
&< -D\mu^{\frac{6}{6-q}}
\end{aligned}$$

for $\mu > 0$ small enough. Thus, there exists some small

$$\mu_3 < \min\left\{\mu_2, \frac{4q}{(4-q)|h|_{\frac{6}{6-q}}(2q)^{\frac{q}{6}}}\left[\frac{3c^2}{4(6-q)}\right]^{\frac{6-q}{6}}\right\}$$

such that $m_\mu^* \leq \mathcal{I}_\mu(s_\varepsilon u_\varepsilon + t_\varepsilon v_\varepsilon) < \frac{1}{3}S^{\frac{3}{2}} - D\mu^{\frac{2}{2-q}}$ for any $\mu \in (0, \mu_3)$, the proof is completed. $\square$

Lemma 2.8. *There exists some $\mu^* \in (0, \mu_3)$ such that m_μ^* has a minimizing sequence $\{u_n\} \subset \mathcal{M}_\mu^*$ satisfying $\mathcal{I}'_\mu(u_n) \xrightarrow{n} 0$ in $[D^{1,2}(\mathbb{R}^3)]^*$ for any $\mu \in (0, \mu^*)$.*

Proof. Observe that $\mathcal{I}_\mu$ is coercive on $\mathcal{M}_\mu^*$ due to Lemma 2.4, so $\mathcal{I}_\mu$ is bounded from blow on $\mathcal{M}_\mu^*$. It follows from Lemmas 2.3 and 2.4 that $\mathcal{M}_\mu^*$ is complete. Then, by the Ekeland variational principle, there exists a minimizing sequence $\{u_n\} \subset \mathcal{M}_\mu^*$ such that

$$\begin{aligned}
m_\mu^* \leq \mathcal{I}_\mu(u_n) &\leq m_\mu^* + \frac{1}{n}, \\
\mathcal{I}_\mu(v) \geq \mathcal{I}_\mu(u_n) &- \frac{1}{n}\|v - u_n\|, \qquad \forall\, v \in M_\mu^*.
\end{aligned} \tag{2.18}$$

For each n and $\varphi \in D^{1,2}(\mathbb{R}^3)$, we define the functions $h_n^\pm : \mathbb{R}^3 \to \mathbb{R}$ by

$$\begin{aligned}
h_n^\pm(t,s,\tau) = &\int_{\mathbb{R}^3} \left|\nabla(u_n + t\varphi + su_n^+ + \tau u_n^-)^\pm\right|^2 dx \\
&+ \int_{\mathbb{R}^3} l(x)\phi_{u_n+t\varphi+su_n^++\tau u_n^-}|(u_n + t\varphi + su_n^+ + \tau u_n^-)^\pm|^2 dx \\
&- \int_{\mathbb{R}^3} \left|(u_n + t\varphi + su_n^+ + \tau u_n^-)^\pm\right|^6 dx - \mu\int_{\mathbb{R}^3} h(x)\left|(u_n + t\varphi + su_n^+ + \tau u_n^-)^\pm\right|^q dx.
\end{aligned}$$

It is clear that $h_n^\pm(0,0,0) = 0$ and $h_n^\pm(t,s,\tau)$ are of class C^1. Let $v_n = u_n + t\varphi + su_n^+ + \tau u_n^-$. We have

$$\begin{aligned}
\frac{\partial h_n^+(t,s,\tau)}{\partial s} &= 2\int_{\mathbb{R}^3} \nabla v_n^+ \cdot \nabla u_n^+ dx + 2\int_{\mathbb{R}^3} l(x)\phi_{v_n} v_n^+ u_n^+ dx \\
&\quad + 2\int_{\mathbb{R}^3} l(x)\phi_{v_n^+} v_n u_n^+ dx - 6\int_{\mathbb{R}^3} |v_n^+|^5 u_n^+ dx - \mu q\int_{\mathbb{R}^3} h(x)|v_n^+|^{q-1}u_n^+ dx.
\end{aligned}$$

Because of $\{u_n\} \subset \mathcal{M}_\mu^*$, there holds

$$\frac{\partial h_n^+(0,0,0)}{\partial s} = 2\int_{\mathbb{R}^3} l(x)\phi_{u_n^+}|u_n^+|^2 dx - 4|u_n^+|_6^6 + \mu(2-q)\int_{\mathbb{R}^3} h(x)|u_n^+|^q dx.$$

Through direct calculation, we have

$$\begin{aligned}\frac{\partial h_n^+(t,s,\tau)}{\partial \tau} &= 2\int_{\mathbb{R}^3} \nabla v_n^+ \cdot \nabla u_n^- dx + 2\int_{\mathbb{R}^3} l(x)\phi_{v_n} v_n^+ u_n^- dx \\ &\quad + 2\int_{\mathbb{R}^3} l(x)\phi_{v_n^+} v_n u_n^- dx - 6\int_{\mathbb{R}^3} |v_n^+|^5 u_n^- dx - \mu q \int_{\mathbb{R}^3} h(x)|v_n^+|^{q-1} u_n^- dx,\end{aligned}$$

which implies

$$\frac{\partial h_n^+(0,0,0)}{\partial \tau} = 2\int_{\mathbb{R}^3} l(x)\phi_{u_n^+}|u_n^-|^2 dx > 0. \tag{2.19}$$

Analogously, we can obtain

$$\frac{\partial h_n^-(0,0,0)}{\partial \tau} = 2\int_{\mathbb{R}^3} l(x)\phi_{u_n^-}|u_n^-|^2 dx - 4|u_n^-|_6^6 + \mu(2-q)\int_{\mathbb{R}^3} h(x)|u_n^-|^q dx \tag{2.20}$$

and

$$\frac{\partial h_n^-(0,0,0)}{\partial s} = 2\int_{\mathbb{R}^3} l(x)\phi_{u_n^-}|u_n^+|^2 dx > 0.$$

Consider the Jacobi determinant

$$\det M = \begin{vmatrix} \frac{\partial h^+(0,0,0)}{\partial s} & \frac{\partial h^+(0,0,0)}{\partial \tau} \\ \frac{\partial h^-(0,0,0)}{\partial s} & \frac{\partial h^-(0,0,0)}{\partial \tau} \end{vmatrix}.$$

It is easy to see that

$$\begin{aligned}\det M &= \left[2\int_{\mathbb{R}^3} l(x)\phi_{u_n^+}|u_n^+|^2 dx - 4|u_n^+|_6^6 + \mu(2-q)\int_{\mathbb{R}^3} h(x)|u_n^+|^q dx\right] \\ &\quad \times \left[2\int_{\mathbb{R}^3} l(x)\phi_{u_n^-}|u_n^-|^2 dx - 4|u_n^-|_6^6 + \mu\,(2-q)\int_{\mathbb{R}^3} h(x)|u_n^-|^q dx\right] \\ &\quad - 4\left(\int_{\mathbb{R}^3} l(x)\phi_{u_n^-}|u_n^+|^2 dx\right)^2.\end{aligned}$$

Since $\{u_n\} \subset \mathcal{M}_\mu^*$, there results that

$$\begin{aligned}&\|u_n^+\|^2 + \int_{\mathbb{R}^3} l(x)\phi_{u_n^+}|u_n^+|^2 dx + \int_{\mathbb{R}^3} l(x)\phi_{u_n^-}|u_n^+|^2 dx - |u_n^+|_6^6 - \mu\int_{\mathbb{R}^3} h(x)|u_n^+|^q dx = 0,\\ &\|u_n^-\|^2 + \int_{\mathbb{R}^3} l(x)\phi_{u_n^-}|u_n^-|^2 dx + \int_{\mathbb{R}^3} l(x)\phi_{u_n^+}|u_n^-|^2 dx - |u_n^-|_6^6 - \mu\int_{\mathbb{R}^3} h(x)|u_n^-|^q dx = 0.\end{aligned}$$

Therefore,

$$\begin{aligned}\det M &= \left[-2\|u_n^+\|^2 - 2\int_{\mathbb{R}^3} l(x)\phi_{u_n^+}|u_n^-|^2 dx - 2|u_n^+|_6^6 + \mu(4-q)\int_{\mathbb{R}^3} h(x)|u_n^+|^q dx\right] \\ &\quad \times \left[-2\|u_n^-\|^2 - 2\int_{\mathbb{R}^3} l(x)\phi_{u_n^+}|u_n^-|^2 dx - 2|u_n^-|_6^6 + \mu(4-q)\int_{\mathbb{R}^3} h(x)|u_n^-|^q dx\right] \\ &\quad - 4\left(\int_{\mathbb{R}^3} l(x)\phi_{u_n^-}|u_n^+|^2 dx\right)^2 \\ &\geq 4\|u_n^+\|^2\|u_n^-\|^2 - 2\mu(4-q)\left(\|u_n^-\|^2 + |u_n^-|_6^6 + \int_{\mathbb{R}^3} l(x)\phi_{u_n^+}|u_n^-|^2 dx\right)\int_{\mathbb{R}^3} h(x)|u_n^+|^q dx \\ &\quad - 2\mu(4-q)\left(\|u_n^+\|^2 + |u_n^+|_6^6 + \int_{\mathbb{R}^3} l(x)\phi_{u_n^+}|u_n^-|^2 dx\right)\int_{\mathbb{R}^3} h(x)|u_n^-|^q dx.\end{aligned} \tag{2.21}$$

Since Lemma 2.7 and (2.10) imply

$$\frac{1}{3}S^{\frac{3}{2}} \geq \frac{1}{4}\|u_n\|^2 - \mu_3^{\frac{6}{6-q}}\frac{6-q}{6}\left[\left(\frac{1}{q}-\frac{1}{4}\right)|h|_{\frac{6}{6-q}}(2q)^{\frac{q}{6}}\right]^{\frac{6}{6-q}},$$

there exists some $C_7 > 0$, independent of μ, such that $\sup_n \|u_n\| \leq C_7$. Then, by the Hölder inequality and Lemma 2.1-(3), we have

$$|u_n^{\pm}|_6^6 \leq S^{-3}\|u_n^{\pm}\|^6 \leq S^{-3}C_7^6$$

and

$$\int_{\mathbb{R}^3} h(x)|u_n^{\pm}|^q dx \leq |h|_{\frac{6}{6-q}}|u_n^{\pm}|_6^q \leq S^{-\frac{q}{2}}|h|_{\frac{6}{6-q}}\|u_n^{\pm}\|^q \leq S^{-\frac{q}{2}}|h|_{\frac{6}{6-q}}C_7^q,$$

$$\int_{\mathbb{R}^3} l(x)\phi_{u_n^+}|u_n^-|^2 dx \leq \left|\phi_{u_n^+}\right|_6 \left|l(x)|u_n^-|^2\right|_{\frac{6}{5}} \leq S^{-\frac{1}{2}}|l|_2\left\|\phi_{u_n^+}\right\||u_n^-|_6^2 \leq S^{-3}|l|_2^2 C_7^4.$$

Then, if letting $\mu < \mu^*$, where

$$\mu^* = \frac{c^4 S^{\frac{q}{2}}}{2(4-q)|h|_{\frac{6}{6-q}}C_7^{2+q}(1+S^{-3}C_7^4+S^{-3}|l|_2^2C_7^2)},$$

by (2.21) and Lemma 2.4, we obtain

$$\det M \geq 2c^4 > 0. \tag{2.22}$$

Therefore, by the implicit function theorem, there exist δ_n and two functions $s_n(t), \tau_n(t) \in C^1((-\delta_n, \delta_n), \mathbb{R})$ such that $s_n(0) = \tau_n(0)$ and for any $t \in (-\delta_n, \delta_n)$,

$$h_n^{\pm}(t, s_n(t), \tau_n(t)) = 0, \tag{2.23}$$

which means that

$$v_n = u_n + t\varphi + s_n(t)u_n^+ + \tau_n(t)u_n^- \in \mathcal{M}_\mu \quad \text{for } t \in (-\delta_n, \delta_n).$$

Further, according to (2.3) and (2.4), by the continuity, there exists some $\delta_n' \in (0, \delta_n)$ such that $\frac{\partial^2 f_{v_n}}{\partial s^2}(1,1) < 0$ and $\frac{\partial^2 f_{v_n}}{\partial t^2}(1,1) < 0$, i.e.

$$v_n = u_n + t\varphi + s_n(t)u_n^+ + \tau_n(t)u_n^- \in \mathcal{M}_\mu^* \quad \text{for } t \in (-\delta_n', \delta_n'). \tag{2.24}$$

We assert that both $\left\{\frac{s_n'(0)}{\|\varphi\|}\right\}$ and $\left\{\frac{\tau_n'(0)}{\|\varphi\|}\right\}$ are bounded for any $\varphi \in D^{1,2}(\mathbb{R}^3) \setminus \{0\}$. It follows from (2.23) that

$$\begin{cases} \dfrac{\partial h_n^+(0,0,0)}{\partial t} + \dfrac{\partial h_n^+(0,0,0)}{\partial s}s_n'(0) + \dfrac{\partial h_n^+(0,0,0)}{\partial \tau}\tau_n'(0) = 0, \\ \dfrac{\partial h_n^-(0,0,0)}{\partial t} + \dfrac{\partial h_n^-(0,0,0)}{\partial s}s_n'(0) + \dfrac{\partial h_n^-(0,0,0)}{\partial \tau}\tau_n'(0) = 0, \end{cases}$$

where

$$\begin{aligned} \frac{\partial h_n^{\pm}(0,0,0)}{\partial t} &= 2\int_{\mathbb{R}^3} \nabla u_n^{\pm} \cdot \nabla\varphi dx + 2\int_{\mathbb{R}^3} l(x)\phi_{u_n^{\pm}}u_n\varphi dx \\ &\quad + 2\int_{\mathbb{R}^3} l(x)\phi_{u_n}u_n^{\pm}\varphi dx - 6\int_{\mathbb{R}^3}|u_n^{\pm}|^5\varphi dx - \mu q\int_{\mathbb{R}^3} h(x)|u_n^{\pm}|^{q-1}\varphi dx. \end{aligned} \tag{2.25}$$

By Cramer's rule, there results

$$s_n'(0) = \frac{1}{\det M}\left(\frac{\partial h_n^-}{\partial t}\frac{\partial h_n^+}{\partial \tau} - \frac{\partial h_n^+}{\partial t}\frac{\partial h_n^-}{\partial \tau}\right)\Big|_{(0,0,0)}. \tag{2.26}$$

Recalling $\sup_n \|u_n\| \le C_7$, from (2.19), (2.20), (2.25), Lemma 2.1 and the Hölder inequality we deduce

$$\left|\left(\frac{\partial h_n^+}{\partial t}\frac{\partial h_n^-}{\partial \tau} - \frac{\partial h_n^-}{\partial t}\frac{\partial h_n^+}{\partial \tau}\right)\Big|_{(0,0,0)}\right| \le C_8\|\varphi\|.$$

Then, (2.22) and (2.26) imply that $|s_n'(0)| \le C_9\|\varphi\|$ for any $\varphi \in D^{1,2}(\mathbb{R}^3)\setminus\{0\}$. Similarly, we deduce that $|\tau_n'(0)| \le C_{10}\|\varphi\|$ for any $\varphi \in D^{1,2}(\mathbb{R}^3)\setminus\{0\}$. Due to (2.18), there holds that

$$\mathcal{I}_\mu(u_n) - \mathcal{I}_\mu\left(u_n + t\varphi + s_n(t)u_n^+ + \tau_n(t)u_n^-\right) \le \frac{1}{n}\|t\varphi + s_n(t)u_n^+ + \tau_n(t)u_n^-\|.$$

Dividing t in both sides of the above inequality and letting $t \to 0^+$, by $\sup_n \|u_n\| \le C_7$, $|s_n'(0)| \le C_9\|\varphi\|$ and $|\tau_n'(0)| \le C_{10}\|\varphi\|$ we have

$$\langle \mathcal{I}_\mu'(u_n), \varphi\rangle \le \frac{1}{n}\left\|\varphi + s_n'(0)u_n^+ + \tau_n'(0)u_n^-\right\| \le \frac{C_{11}}{n}\|\varphi\|, \quad \forall\, \varphi \in D^{1,2}(\mathbb{R}^3),$$

which implies

$$\left|\langle \mathcal{I}_\mu'(u_n), \varphi\rangle\right| \le \frac{C_{11}}{n}\|\varphi\|, \quad \forall\, \varphi \in D^{1,2}(\mathbb{R}^3).$$

Consequently, we obtain that $\mathcal{I}_\mu'(u_n) \to 0$ as $n \to \infty$. Thus, the proof is completed. □

Lemma 2.9. *For any $\mu \in (0,\mu^*)$, every sequence $\{u_n\} \subset \mathcal{M}_\mu^*$ such that $\mathcal{I}_\mu(u_n) \xrightarrow{n} m_\mu^*$ and $\mathcal{I}_\mu'(u_n) \xrightarrow{n} 0$ has a convergent subsequence. Moreover, up to a subsequence, $u_n^\pm \xrightarrow{n} u_\mu^\pm$ in $D^{1,2}(\mathbb{R}^3)$.*

Proof. Since Lemma 2.4 implies that $\mathcal{I}_\mu$ is coercive on $\mathcal{M}_\mu^*$ and Lemma 2.7 implies $m_\mu^* < +\infty$, we deduce that $\{u_n\}$ is bounded in $D^{1,2}(\mathbb{R}^3)$. Up to a subsequence, there exists some $u_\mu \in D^{1,2}(\mathbb{R}^3)$ such that, as $n \to \infty$,

$$u_n \rightharpoonup u_\mu \quad \text{in } D^{1,2}(\mathbb{R}^3)\cap L^6(\mathbb{R}^3), \tag{2.27}$$

$$u_n(x) \to u_\mu(x) \quad \text{a.e. in } \mathbb{R}^3. \tag{2.28}$$

By using standard arguments, we derive $\mathcal{I}_\mu'(u_\mu) = 0$. From (2.28), it easily follows that $|u_n|^q \rightharpoonup |u_\mu|^q$ in $L^{\frac{6}{q}}(\mathbb{R}^3)$ in the sense of subsequence, which and $h \in L^{\frac{6}{6-q}}(\mathbb{R}^3)$ imply

$$\int_{\mathbb{R}^3} h(x)|u_n|^q dx \xrightarrow{n} \int_{\mathbb{R}^3} h(x)|u_\mu|^q dx. \tag{2.29}$$

Let $v_n = u_n - u_\mu$. By the Brézis–Lieb lemma [22, Lemma 1.32], (2.29) and Lemma 2.1-(6), we have

$$\begin{aligned} m_\mu^* + o(1) = \mathcal{I}_\mu(u_n) &= \frac{1}{2}\|u_n\|^2 + \frac{1}{4}\int_{\mathbb{R}^3} l(x)\phi_{u_n}u_n^2 dx - \frac{1}{6}|u_n|_6^6 - \frac{\mu}{q}\int_{\mathbb{R}^3} h(x)|u_n|^q dx \\ &= \frac{1}{2}\|v_n\|^2 - \frac{1}{6}|v_n|_6^6 + \mathcal{I}_\mu(u_\mu) \end{aligned} \tag{2.30}$$

and

$$o(1) = \left\langle \mathcal{I}'_\mu(u_n), u_n \right\rangle = \|u_n\|^2 + \int_{\mathbb{R}^3} l(x)\phi_{u_n} u_n^2 dx - |u_n|_6^6 - \mu \int_{\mathbb{R}^3} h(x)|u_n|^q dx$$
$$= \|v_n\|^2 - |v_n|_6^6. \tag{2.31}$$

Denote $\limsup_{n\to\infty} \|v_n\|^2 = a$. By (2.31) and the definition of the best Sobolev constant, we get $a \le S^{-3}a^3$, which implies either $a = 0$ or $a \ge S^{\frac{3}{2}}$. We claim that $a \ge S^{\frac{3}{2}}$ can't occur. From $\mathcal{I}'_\mu(u_\mu) = 0$, (2.30), (2.31) and (2.10), it follows that

$$m^*_\mu = \lim_{n\to\infty} \left[\frac{1}{3}\|v_n\|^2 + \mathcal{I}_\mu(u_\mu) - \frac{1}{4}\langle \mathcal{I}'_\mu(u_\mu), u_\mu \rangle \right] \ge \frac{1}{3}S^{\frac{3}{2}} - D\mu^{\frac{6}{6-q}},$$

which together with Lemma 2.7 gives a contradiction. Therefore, $a = 0$. That is, $u_n \to u_\mu$ in $D^{1,2}(\mathbb{R}^3)$.

Further, we prove $u_n^\pm \to u_\mu^\pm$ in $D^{1,2}(\mathbb{R}^3)$. It follows from $\{u_n\} \subset \mathcal{M}^*_\mu$ and $\mathcal{I}'_\mu(u_\mu) = 0$ that

$$\begin{cases} \|u_n^\pm\|^2 + \int_{\mathbb{R}^3} l(x)\phi_{u_n}|u_n^\pm|^2 dx = |u_n^\pm|_6^6 + \mu \int_{\mathbb{R}^3} h(x)|u_n^\pm|^q dx, \\ \|u_\mu^\pm\|^2 + \int_{\mathbb{R}^3} l(x)\phi_{u_\mu}|u_\mu^\pm|^2 dx = |u_\mu^\pm|_6^6 + \mu \int_{\mathbb{R}^3} h(x)|u_\mu^\pm|^q dx. \end{cases} \tag{2.32}$$

It is easy to verify that

$$|u_n^\pm(x) - u_\mu^\pm(x)| \le |u_n(x) - u_\mu(x)|, \qquad \forall\, x \in \mathbb{R}^3, \tag{2.33}$$

which and $u_n \to u_\mu$ in $D^{1,2}(\mathbb{R}^3)$ lead to

$$\left| |u_n^\pm|_6 - |u_\mu^\pm|_6 \right|^6 \le \left| u_n^\pm - u_\mu^\pm \right|_6^6 \le |u_n - u_\mu|_6^6 \le S^{-3}\|u_n - u_\mu\|^6 \xrightarrow{n} 0. \tag{2.34}$$

Noting $u_n \to u_\mu$ in $L^6(\mathbb{R}^3)$, we deduce $|u_n^\pm|^q \rightharpoonup |u_\mu^\pm|^q$ in $L^{\frac{6}{q}}(\mathbb{R}^3)$ in the sense of subsequence. Therefore,

$$\int_{\mathbb{R}^3} h(x)|u_n^\pm|^q dx \xrightarrow{n} \int_{\mathbb{R}^3} h(x)|u_\mu^\pm|^q dx. \tag{2.35}$$

Moreover, since $l \in L^2(\mathbb{R}^3)$ and $|u_n - u_\mu|_6 \to 0$ implies that $\left||u_n^\pm|^2 - |u_\mu^\pm|^2\right|^{\frac{6}{5}} \rightharpoonup 0$ in $L^{\frac{5}{2}}(\mathbb{R}^3)$ up to a subsequence, by the Hölder inequality and Lemma 2.1-(3) we have

$$\begin{aligned} \left| \int_{\mathbb{R}^3} l(x)\phi_{u_n}(|u_n^\pm|^2 - |u_\mu^\pm|^2) dx \right| &\le |\phi_{u_n}|_6 \left(\int_{\mathbb{R}^3} |l(x)|^{\frac{6}{5}} \left||u_n^\pm|^2 - |u_\mu^\pm|^2\right|^{\frac{6}{5}} dx \right)^{\frac{5}{6}} \\ &\le S^{-1}|l|_2|u_n|_6^2 \left(\int_{\mathbb{R}^3} |l(x)|^{\frac{6}{5}} \left||u_n^\pm|^2 - |u_\mu^\pm|^2\right|^{\frac{6}{5}} dx \right)^{\frac{5}{6}} \\ &\xrightarrow{n} 0. \end{aligned} \tag{2.36}$$

Observe $|lu_\mu^\pm|^2 \in L^{\frac{6}{5}}(\mathbb{R}^3)$, we deduce from (2.27) and Lemma 2.1-(5) that

$$\left| \int_{\mathbb{R}^3} l(x)(\phi_{u_n} - \phi_{u_\mu})|u_\mu^\pm|^2 dx \right| \xrightarrow{n} 0,$$

by which and (2.36) we get

$$\int_{\mathbb{R}^3} l(x)\phi_{u_n}|u_n^{\pm}|^2 dx \xrightarrow{n} \int_{\mathbb{R}^3} l(x)\phi_{u_\mu}|u_\mu^{\pm}|^2 dx. \tag{2.37}$$

Now, by combining (2.32), (2.34), (2.35) and (2.37) we conclude $\|u_n^{\pm}\|^2 \to \|u_\mu^{\pm}\|^2$, which together with (2.27) implies $u_n^{\pm} \xrightarrow{n} u_\mu^{\pm}$ in $D^{1,2}(\mathbb{R}^3)$. Thus the lemma is proved. □

Next, with the above preliminary lemmas in hand, we end the proof of our main result.

Proof of Theorem 1.1. For any $\mu \in (0, \mu^*)$, due to Lemmas 2.8 and 2.7, there exists a sequence $\{u_n\} \subset \mathcal{M}_\mu^*$ such that, as $n \to \infty$,

$$\mathcal{I}_\mu(u_n) \to m_\mu^* \in \left(0, \frac{1}{3}S^{\frac{3}{2}} - D\mu^{\frac{6}{6-q}}\right) \quad \text{and} \quad \mathcal{I}'_\mu(u_n) \to 0 \text{ in } \left[D^{1,2}(\mathbb{R}^3)\right]^*.$$

According to Lemma 2.9, up to a subsequence, there exists some u_μ satisfying $u_n \to u_\mu$ and $u_n^{\pm} \to u_\mu^{\pm}$ in $D^{1,2}(\mathbb{R}^3)$ as $n \to \infty$. Therefore, we get $\mathcal{I}'_\mu(u_\mu) = 0$ and $\mathcal{I}_\mu(u_\mu) = m_\mu^*$. Moreover, Lemma 2.4 says $\|u_n^{\pm}\| \geq c > 0$, which implies $u_\mu^{\pm} \neq 0$. By combining the above arguments, we know that u_μ is a sign-changing solution of system (1.6) with positive energy. Thus Theorem 1.1 is proved. □

Acknowledgements

This work was partially supported by National Natural Science Foundation of China (No. 12301143 and No. 12261079), Natural Science Foundation of Gansu Province, China (No. 24JRRA141) and NWNU-LKQN2022-02. The authors would like to thank the referees and editors for carefully reading this paper and making valuable comments and suggestions, which greatly improve the original manuscript.

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