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Decreasing solutions of cyclic second-order difference systems

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Abstract. The existence and asymptotic behavior of positive decreasing solutions to the cyclic second-order nonlinear difference system

$$\Delta(p_i(n)|\Delta x_i(n)|^{\alpha_i-1}\Delta x_i(n)) = q_i(n)|x_{i+1}(n+1)|^{\beta_i-1}x_{i+1}(n+1), \quad i=\overline{1,N},$$

are studied, where $x_{N+1} = x_1$, $p_i = \{p_i(n)\}$ and $q_i = \{q_i(n)\}$ are positive real sequences, and the constants α_i and β_i, $i = \overline{1,N}$ are positive and satisfy the sublinear condition $\alpha_1\alpha_2 \cdot \ldots \cdot \alpha_N > \beta_1\beta_2 \cdot \ldots \cdot \beta_N$. Two distinct types of positive decreasing solutions are considered, depending on whether the series $\sum_{n=1}^{\infty} p_i(n)^{-1/\alpha_i}$ is divergent or convergent. In the first case, necessary and sufficient conditions for the existence of solutions tending to a positive constant as well as solutions tending to zero, while their associated quasi-differences approach a nonzero limit, are rigorously derived using fixed point techniques. In the second case, the analysis is focused on solutions whose components and quasi-differences both tend to zero. Under the additional assumption that the coefficient sequences are regularly varying, necessary and sufficient conditions for the existence of such solutions are obtained, and their precise asymptotic behavior is determined using the theory of discrete regular variation.

Keywords: cyclic system of difference equations, Emden–Fowler type difference equation, nonlinear difference equations, decreasing solutions, asymptotic behavior, regularly varying sequence.

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1 Introduction

The cyclic second-order nonlinear system of difference equations examined in this paper is of the following form:

$$\Delta(p_i(n)|\Delta x_i(n)|^{\alpha_i-1}\Delta x_i(n)) = q_i(n)|x_{i+1}(n+1)|^{\beta_i-1}x_{i+1}(n+1), \tag{SE}$$

where $i = \overline{1,N},\ x_{N+1} = x_1,\ n \in \mathbb{N}$, and following conditions hold:

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(a) α_i and β_i, $i=\overline{1,N}$ are positive constants such that

$$\alpha_1\alpha_2\cdot\ldots\cdot\alpha_N > \beta_1\beta_2\cdot\ldots\cdot\beta_N;$$

(b) $p_i=\{p_i(n)\}$ and $q_i=\{q_i(n)\}$ are positive real sequences;

(c) All p_i, $i=\overline{1,N}$ simultaneously satisfy either

$$(I) \qquad S_i=\sum_{n=1}^{\infty}\frac{1}{p_i(n)^{1/\alpha_i}}=\infty,$$

or

$$(II) \qquad S_i=\sum_{n=1}^{\infty}\frac{1}{p_i(n)^{1/\alpha_i}}<\infty.$$

If (I) holds, the following notation will be used

$$P_i(n)=\sum_{k=1}^{n-1}\frac{1}{p_i(k)^{1/\alpha_i}},\quad i=\overline{1,N} \tag{1.1}$$

while if (II) holds, the following notation will be used

$$\pi_i(n)=\sum_{k=n}^{\infty}\frac{1}{p_i(k)^{1/\alpha_i}},\quad i=\overline{1,N} \tag{1.2}$$

System (SE) is referred to as sublinear if the condition (a) holds. When the opposite inequality is satisfied, the system is termed superlinear. If $\alpha_1\alpha_2\cdot\ldots\cdot\alpha_N=\beta_1\beta_2\cdot\ldots\cdot\beta_N$, then system is called half-linear.

Over the past fifty years, the application of difference equations in solving problems throughout statistics, engineering, and various scientific disciplines has significantly expanded. The development of high-speed digital computer technology has motivated the application of difference equations to ordinary and partial differential equations. Apart from this, difference equations are very useful for analyzing electrical, mechanical, thermal and other systems, the behavior of electric-wave filters and other filters, insulator strings, crankshafts of multi-cylinder engines among others.

As the generalization of the most studied second-order nonlinear differential equation, known as Emden–Fowler type equation, many authors have studied the differential equation

$$\big(p(t)|x'(t)|^{\alpha-1}x'(t)\big)'\pm q(t)|x(t)|^{\beta-1}x(t)=0,$$

whose properties such as existence, uniqueness and continuity of the solution, oscillatory and nonoscillatory behavior of solutions have been examined in monographs [15,27] as well as in papers [9,10,22,35,49]. The discrete counterpart of this equation

$$\Delta(p(n)|\Delta x(n)|^{\alpha-1}\Delta x(n))\pm q(n)|x(n+1)|^{\beta-1}x(n+1)=0, \tag{1.3}$$

has also attracted many researchers, see e.g. [11–14] and monographs [1,3].

The qualitative analysis of second-order nonlinear difference equations was further extended to the study of two-dimensional first-order and second-order nonlinear difference systems [4,21,28,30,31], as well as to symmetric and close-to-symmetric systems of difference equations [6,41,42,44–46].

As a further development of the research on second-order difference equations of Emden–Fowler type, the consideration of cyclic second-order systems of difference equations was naturally imposed. The study of such systems was first suggested in [17], and further attention has been given to them and to the close-to-cyclic difference equations in [29,38–40,43,47], in addition to the already quoted papers. A system of the form (SE) could be observed in the analysis of numerical methods for heat and fluid transfer in layered materials (e.g., cylindrical thermal insulation or heat transfer through geological layers).

In the continuous case, this type of system has been studied by Jaroš and Kusano in [18–20]. However, in the existing literature, there are no results concerning asymptotic analysis of solutions of a cyclic system of second-order difference equations, except in the work of Kapešić [23]. In this regard, the first task will be to classify positive decreasing solutions based on their behavior at infinity. Finding the necessary and sufficient conditions for the existence of all possible types of positive decreasing solutions is the second task. The last and most challenging task is obtaining the precise asymptotic formulas of these solutions.

It should be mentioned that the obtained results can be applied to cyclic systems of N first-order difference equations, in the case N is even.

2 Classification of positive decreasing solutions

By a solution of (SE) we mean a vector sequence

$$\mathbf{x} = (x_1, x_2, \dots, x_N) \in {}^{\mathbb{N}}\mathbb{R} \times \dots \times {}^{\mathbb{N}}\mathbb{R}, \qquad x_i = \{x_i(n)\}_{n\in\mathbb{N}}$$

where ${}^{\mathbb{N}}\mathbb{R} = \{f \mid f : \mathbb{N} \to \mathbb{R}\}$, whose components $x_i = \{x_i(n)\}_{n\in\mathbb{N}}$, $i = \overline{1,N}$ satisfy (SE). In what following, we will observe sequences x_i for sufficiently large n, i.e. $n \geq n_0$, for some $n_0 \in \mathbb{N}$. Therefore, we introduced notation ${}^{\mathbb{N}_{n_0}}\mathbb{R} = \{f \mid f : \mathbb{N}_{n_0} \to \mathbb{R}\}$, where $\mathbb{N}_{n_0} = \{n \in \mathbb{N} \mid n \geq n_0\}$.

A solution $\mathbf{x}$ is called nonoscillatory if all its components are eventually of one sign. Because of sign condition on the coefficients, if one component is nonoscillatory, then all components are nonoscillatory and eventually monotone, and therefore they have limit. A nonoscillatory solution is called positive if all its components are eventually positive. Our aim is to study the existence and asymptotic behavior of positive decreasing solutions of (SE), that is, solutions whose components are both eventually positive and decreasing, i.e. satisfying

$$x_i(n) > 0, \quad \Delta x_i(n) < 0, \quad \text{for } n \geq n_0, \quad i = \overline{1,N}. \tag{2.1}$$

We denote by $\mathcal{DS}$ the set of all the solutions of (SE) whose components are all eventually positive decreasing. From (2.1), for x_i, $i = \overline{1,N}$, one of the following two cases holds:

$$(i) \quad \lim_{n\to\infty} x_i(n) = k_i,\ k_i > 0 \quad \text{or} \quad (ii) \quad \lim_{n\to\infty} x_i(n) = 0.$$

For every component of any solution $\mathbf{x}$ of (SE), let we denote by $x_i^{[1]} = \{x_i^{[1]}(n)\}$ its quasi-difference $x_i^{[1]}(n) = p_i(n)|\Delta x_i(n)|^{\alpha_i - 1}\Delta x_i(n)$, $i = \overline{1,N}$. Then, for $x_i^{[1]}$, $i = \overline{1,N}$, one of the following two cases holds:

$$(iii) \quad \lim_{n\to\infty} x_i^{[1]}(n) = -\omega_i,\ \omega_i > 0 \quad \text{or} \quad (iv) \quad \lim_{n\to\infty} x_i^{[1]}(n) = 0.$$

However, if (I) holds, then only (iv) may hold. Indeed, if (iii) holds, then $-p_i(n)(-\Delta x_i(n))^{\alpha_i} \leq -\omega_i$, $n \geq n_0$ i.e. $x_i(n) \leq x_i(n_0) - \omega_i^{1/\alpha_i} \sum_{k=n_0}^{n-1} p_i(k)^{-1/\alpha_i}$. As the right-hand side tends to $-\infty$ contradicts positivity of x_i, we have the desired conclusion.

These leads to the following classification of positive decreasing solution: if (I) holds, each component x_i of positive decreasing solution $\mathbf{x}$ satisfies:

$(SD1)$ $\lim_{n\to\infty} x_i(n) = \lim_{n\to\infty} x_i^{[1]}(n) = 0$,

(AC) $\lim_{n\to\infty} x_i(n) = k_i > 0 \quad \Leftrightarrow \quad x_i(n) \sim k_i,\ n \to \infty$,

while if (II) holds each component x_i of positive decreasing solution $\mathbf{x}$ satisfies

$(SD2)$ $\lim_{n\to\infty} x_i(n) = \lim_{n\to\infty} x_i^{[1]}(n) = 0$,

$(P1)$ $\lim_{n\to\infty} x_i(n) = 0,\ \lim_{n\to\infty} x_i^{[1]}(n) = -\omega_i < 0$,

$$\Leftrightarrow \quad x_i(n) \sim W_i\,\pi_i(n),\ n \to \infty, \quad W_i = \omega_i^{1/\alpha_i}$$

(AC) $\lim_{n\to\infty} x_i(n) = k_i > 0 \quad \Leftrightarrow \quad x_i(n) \sim k_i,\ n \to \infty$,

where the following asymptotic relation has been used

$$f(n) \sim g(n),\ n \to \infty \ \Leftrightarrow\ \lim_{n\to\infty} \frac{f(n)}{g(n)} = 1.$$

Using the Stolz–Cesàro Theorem (see Theorem 4.5), solutions satisfying $(SD1)$ and $(SD2)$ can be characterized as

$$(SD1) \quad x_i(n) \prec P_i(n),\ n \to \infty, \qquad (SD2) \quad x_i(n) \prec \pi_i(n),\ n \to \infty,$$

where the following asymptotic relation has been used

$$f(n) \prec g(n),\ n \to \infty \ \Leftrightarrow\ \lim_{n\to\infty} \frac{f(n)}{g(n)} = 0.$$

Solutions of type $(P1)$ and (AC) are called *primitive solutions*, while solutions $(SD1)$, $(SD2)$ are called *strongly decreasing*. Necessary and sufficient conditions for the existence of primitive solutions will be established in the Section 5. On the other hand, the existence and precise asymptotic formulas of strongly decreasing solutions are not easy to determine in the general case. Therefore, in Section 6, we will assume that the coefficients of the system are regularly varying sequences and thus, restricting consideration to the class of regularly varying solutions, we provide necessary and sufficient conditions for the existence of such solutions.

3 Regularly varying sequences

The theory of regularly varying sequences, often called Karamata sequences (see [26]), was developed during the seventies by Galambos, Seneta and Bojanić in [8, 16]. However, until the appearance of the paper of Matucci and Řehák [32], the connection between regularly varying sequences and difference equations was not considered. In this paper, as well as in the following ones [33, 34, 36, 37], the theory of regularly varying sequences is further developed and applied in the asymptotic analysis of linear and half-linear difference equations of the second-order, giving necessary and sufficient conditions for the existence of regularly varying solutions of these equations. After this, further development of the discrete theory of regular variation, as well as its application to nonlinear difference equations of type Emden–Fowler type, can be found in [5], as well as in [24].

In this section, we present basic definitions and properties of regularly varying sequences that will be used in the main results. For a thorough discussion of regular variation, the reader is referred to Bingham et al. [7].

There are two main approaches in the basic theory of regularly varying sequences: the approach due to Karamata [26], based on a definition that can be understood as a direct discrete counterpart of simple and elegant continuous definition (Definition 3.1), and the approach due to Galambos and Seneta [16], based on purely sequential definition (Definition 3.2). Bojanić and Seneta have shown in [8] the equivalence of these two definitions.

Definition 3.1 (Karamata [26])**.** A positive sequence $y = \{y(k)\}$, $k \in \mathbb{N}$ is said to be *regularly varying of index* $\rho \in \mathbb{R}$ if

$$\lim_{k\to\infty} \frac{y([\lambda k])}{y(k)} = \lambda^{\rho} \quad \text{for } \forall\lambda > 0,$$

where $[n]$ denotes the integer part of n.

Definition 3.2 (Galambos and Seneta [16])**.** A positive sequence $y = \{y(k)\}, n \in \mathbb{N}$ is said to be *regularly varying of index* $\rho \in \mathbb{R}$ if there exists a positive sequence $\{\alpha(k)\}$ satisfying

$$\lim_{k\to\infty} \frac{y(k)}{\alpha(k)} = C, \quad 0 < C < \infty, \qquad \lim_{k\to\infty} k\,\frac{\Delta\alpha(k-1)}{\alpha(k)} = \rho\,.$$

If $\rho = 0$, then y is said to be *slowly varying*. The sets of regularly varying sequences with index ρ and slowly varying sequences are denoted $\mathcal{RV}(\rho)$ and $\mathcal{SV}$, respectively.

The concept of normalized regularly varying sequences was introduced by Matucci and Rehak in [33], where they also offered a modification of Definition 3.2, i.e. they proved that the second limit in Definition 3.2 can be replaced with

$$\lim_{k\to\infty} k\,\frac{\Delta\alpha(k)}{\alpha(k)} = \rho\,.$$

Definition 3.3. A positive sequence $y = \{y(k)\}, k \in \mathbb{N}$ is said to be *normalized regularly varying of index* $\rho \in \mathbb{R}$ if it satisfies

$$\lim_{k\to\infty} \frac{k\Delta y(k)}{y(k)} = \rho.$$

If $\rho = 0$, then y is called a *normalized slowly varying sequence.*

The notations $\mathcal{NRV}(\rho)$ and $\mathcal{NSV}$ are most commonly used to denote the set of all normalized regularly varying sequences of index ρ and the set of all normalized slowly varying sequences, respectively.

Typical examples are:

$$\{\log k\} \in \mathcal{NSV}, \quad \{k^{\rho}\log k\} \in \mathcal{NRV}(\rho), \quad \{1 + (-1)^k/k\} \in \mathcal{SV} \setminus \mathcal{NSV}\,.$$

In order to present results for a system of difference equations, we need to define a regularly varying vector $\mathbf{x} \in {}^{\mathbb{N}}\mathbb{R} \times \ldots \times {}^{\mathbb{N}}\mathbb{R}$, where ${}^{\mathbb{N}}\mathbb{R} = \{f \mid f : \mathbb{N} \to \mathbb{R}\}$.

Definition 3.4. A vector $\mathbf{x} \in {}^{\mathbb{N}}\mathbb{R} \times \ldots \times {}^{\mathbb{N}}\mathbb{R}$, $\mathbf{x} = (\{x_1(n)\}, \ldots, \{x_N(n)\})$ is said to be regularly varying of index $(\rho_1, \rho_2, \ldots, \rho_N)$ if $x_i = \{x_i(n)\} \in \mathcal{RV}(\rho_i)$ for $i = \overline{1,N}$. If all ρ_i are positive (or negative), then $\mathbf{x}$ is called regularly varying vector sequence of positive (or negative) index $(\rho_1, \rho_2, \ldots, \rho_N)$. The set of all regularly varying vectors of index $(\rho_1, \rho_2, \ldots, \rho_N)$ is denoted by $\mathcal{RV}(\rho_1, \rho_2, \ldots, \rho_N)$.

Various necessary and sufficient conditions for a sequence of positive numbers to be regularly varying have been established (see [8,16,32,33]) and consequently, any of these can be used to define regularly varying sequence. The one that is the most important is the following Representation theorem (see [8, Theorem 3]), while some other representation formula for regularly varying sequences was established in [33, Lemma 1].

Theorem 3.5 (Representation theorem). *A positive sequence $\{y(k)\}, k \in \mathbb{N}$ is said to be regularly varying of index $\rho \in \mathbb{R}$ if and only if there exists sequences $\{c(k)\}$ and $\{\delta(k)\}$ such that*

$$\lim_{k\to\infty} c(k) = c_0 \in (0,\infty) \quad \text{and} \quad \lim_{k\to\infty} \delta(k) = 0,$$

and

$$y(k) = c(k)\, k^{\rho} \exp\left(\sum_{i=1}^{k} \frac{\delta(i)}{i}\right).$$

In [8], a very useful embedding theorem was proved, which gives the possibility of using the continuous theory in developing a theory of regularly varying sequences. However, as noted in [8], such development is not generally straightforward and sometimes far from a simple imitation of arguments for regularly varying functions.

Theorem 3.6 (Embedding Theorem). *If $y = \{y(n)\}$ is regularly varying sequence of index $\rho \in \mathbb{R}$, then function $Y(t)$ defined on $[0,\infty)$ by $Y(t) = y([t])$ is a regularly varying function of index ρ. Conversely, if $Y(t)$ is a regularly varying function on $[0,\infty)$ of index ρ, then a sequence $\{y(k)\}$, $y(k) = Y(k)$, $k \in \mathbb{N}$ is regularly varying of index ρ.*

Next, we state some important properties of $\mathcal{RV}$ sequences useful for the development of the asymptotic behavior of solutions of (SE) in the subsequent sections (for more properties and proofs see [8,32]).

Theorem 3.7. *The following properties hold:*

(i) $y \in \mathcal{RV}(\rho)$ *if and only if* $y(k) = k^{\rho}\, l(k)$, *where* $l = \{l(k)\} \in \mathcal{SV}$.

(ii) *Let* $x \in \mathcal{RV}(\rho_1)$ *and* $y \in \mathcal{RV}(\rho_2)$. *Then,* $xy \in \mathcal{RV}(\rho_1+\rho_2)$, $x+y \in \mathcal{RV}(\rho)$, $\rho = \max\{\rho_1,\rho_2\}$ *and* $1/x \in \mathcal{RV}(-\rho_1)$.

(iii) *If* $y \in \mathcal{RV}(\rho)$, *then* $\lim_{k\to\infty} \frac{y(k+1)}{y(k)} = 1$.

(iv) *If* $l \in \mathcal{SV}$ *and* $l(k) \sim L(k)$, $k \to \infty$, *then,* $L \in \mathcal{SV}$.

(v) *If* $l \in \mathcal{SV}$, *then for any* $\varepsilon > 0$,

$$\lim_{k\to\infty} k^{\varepsilon} l(k) = \infty, \qquad \lim_{k\to\infty} k^{-\varepsilon} l(k) = 0,$$

(vi) *If* $y \in \mathcal{NRV}(\rho)$, *then* $\{k^{-\sigma} y(k)\}$ *is eventually increasing for each* $\sigma < \rho$ *and* $\{k^{-\mu} y(k)\}$ *is eventually decreasing for each* $\mu > \rho$.

The following theorem can be seen as *the discrete analog of the Karamata's integration theorem* and plays a central role in proving this paper main results. Proof of this theorem can be found in [24]. Also, some parts of this theorem's proof can be found in [8] and [36].

Theorem 3.8. *Let $l = \{l(n)\} \in \mathcal{SV}$.*

(i) If $\alpha > -1$, then $\lim_{n\to\infty} \frac{1}{n^{\alpha+1}l(n)} \sum_{k=1}^{n} k^{\alpha} l(k) = \frac{1}{1+\alpha}$;

(ii) If $\alpha < -1$, then $\lim_{n\to\infty} \frac{1}{n^{\alpha+1}l(n)} \sum_{k=n}^{\infty} k^{\alpha} l(k) = -\frac{1}{1+\alpha}$;

(iii) If $\sum_{k=1}^{\infty} \frac{l(k)}{k} < \infty$, then $S_\star(n) = \sum_{k=n}^{\infty} \frac{l(k)}{k}$, $S_\star \in \mathcal{SV}$ and $\lim_{n\to\infty} \frac{S_\star(n)}{l(n)} = \infty$;

(iv) If $\sum_{k=1}^{\infty} \frac{l(k)}{k} = \infty$, then $S^\star(n) = \sum_{k=1}^{n} \frac{l(k)}{k}$, $S^\star \in \mathcal{SV}$ and $\lim_{n\to\infty} \frac{S^\star(n)}{l(n)} = \infty$.

Remark 3.9. It is easy to see, in view of Theorem 3.7-(iii) and Theorem 3.8-(i), that for $l \in \mathcal{SV}$, if $\alpha > -1$, we have

$$\sum_{k=1}^{n-1} k^{\alpha} l(k) \sim \frac{(n-1)^{\alpha+1} l(n-1)}{\alpha+1} \sim \frac{n^{\alpha+1} l(n)}{\alpha+1} \sim \sum_{k=1}^{n} k^{\alpha} l(k), \qquad n \to \infty,$$

and since $\lim_{n\to\infty} \sum_{k=1}^{n-1} k^{\alpha} l(k) = \infty$, we also get

$$\sum_{k=n_0}^{n} k^{\alpha} l(k) \sim \sum_{k=1}^{n} k^{\alpha} l(k), \qquad n \to \infty.$$

If $\lim_{n\to\infty} \sum_{k=1}^{n} k^{-1} l(k) = \infty$, we have

$$\sum_{k=n_0}^{n} k^{-1} l(k) \sim \sum_{k=1}^{n} k^{-1} l(k), \qquad n \to \infty.$$

4 Basic concepts

In this section, we will state the basic notation and assertions necessary for the proofs of the main results in the following sections.

The existence of solutions will be demonstrated using fixed point techniques. Actually, the following two fixed point theorems will be applied throughout the paper.

Theorem 4.1 (Knaster–Tarski fixed point theorem [2])**.** *Let X be a partially ordered Banach space with ordering $\leq$. Let M be a subset of X with the following properties: The infimum of M belongs to M and every nonempty subset of M has a supremum which belongs to M. Let $\mathcal{F} : M \to M$ be an increasing mapping, i.e. $x \geq y$ implies $\mathcal{F}x \geq \mathcal{F}y$. Then $\mathcal{F}$ has a fixed point in M.*

Theorem 4.2 (Schauder–Tychonoff fixed point theorem [2])**.** *Let S be closed, convex, nonempty subset of a locally convex topological vector space X. Let T be a continuous mapping from S to itself, such that TS is relatively compact. Then T has a fixed point.*

To prove that appropriately constructed operator T is continuous, we will apply the following theorem.

Theorem 4.3 (Discrete Lebesgue's dominated convergence theorem [1])**.** *Let $\{a^{(m)}(k)\}$ be a double real sequence, $a^{(m)}(k) \geq 0$ for $m, k \in \mathbb{N}$ such that $\lim_{m\to\infty} a^{(m)}(k) = A(k)$, for every $k \in \mathbb{N}$. Assume that the series $\sum_{k=1}^{\infty} a^{(m)}(k)$ is totally convergent, that is, there exists a sequence $\{\alpha(k)\}$ such that $a^{(m)}(k) \leq \alpha(k)$ for all $m, k \in \mathbb{N}$, with $\sum_{k=1}^{\infty} \alpha(k) < \infty$. Then, the series $\sum_{k=1}^{\infty} A(k)$ converges and*

$$\lim_{m\to\infty} \sum_{k=1}^{\infty} a^{(m)}(k) = \sum_{k=1}^{\infty} A(k).$$

To apply the Schauder–Tychonoff fixed point theorem, relatively compactness of the set TS must be verified and for that purpose the following statement will be used. This theorem represent a discrete version of the Arzelà–Ascoli theorem and it is known as Cheng–Patula theorem (see [14]).

Theorem 4.4. *A bounded, uniformly Cauchy subset Ω of l^∞ is relatively compact.*

For the proof of regularity of solutions, Stolz–Cesàro Theorem will be used. Therefore, we recall two variants of the Stolz–Cesàro theorem (see [48]).

Theorem 4.5. *If $f = \{f(n)\}$ is a strictly increasing sequence of positive real numbers, such that $\lim_{n\to\infty} f(n) = \infty$, then for any sequence $g = \{g(n)\}$ of positive real numbers one has the inequalities:*

$$\liminf_{n\to\infty} \frac{\Delta f(n)}{\Delta g(n)} \le \liminf_{n\to\infty} \frac{f(n)}{g(n)} \le \limsup_{n\to\infty} \frac{f(n)}{g(n)} \le \limsup_{n\to\infty} \frac{\Delta f(n)}{\Delta g(n)}.$$

In particular, if the sequence $\{\Delta f(n)/\Delta g(n)\}$ has a limit, then

$$\lim_{n\to\infty} \frac{f(n)}{g(n)} = \lim_{n\to\infty} \frac{\Delta f(n)}{\Delta g(n)}. \tag{4.1}$$

Theorem 4.6. *Let $f = \{f(n)\}$, $g = \{g(n)\}$ be sequences of positive real numbers, such that*

(i) $\lim_{n\to\infty} f(n) = \lim_{n\to\infty} g(n) = 0$;

(ii) the sequence g is strictly monotone;

(iii) the sequence $\{\Delta f(n)/\Delta g(n)\}$ has a limit.

Then, a sequence $\{f(n)/g(n)\}$ is convergent and (4.1) holds.

In Section 6 analyzing the regularly varying solutions of the system with regularly varying coefficients, we assume $p_i \in \mathcal{RV}(\lambda_i)$, $q_i \in \mathcal{RV}(\mu_i)$, $i = \overline{1,N}$ and express them as follows:

$$p_i(n) = n^{\lambda_i} l_i(n), \quad q_i(n) = n^{\mu_i} m_i(n), \quad l_i, m_i \in \mathcal{SV}, \quad i = \overline{1,N}, \tag{4.2}$$

while components of the regularly varying solutions $\mathbf{x} \in \mathcal{RV}(\rho_1, \rho_2, \dots, \rho_N)$ of the observed system are expressed in the form

$$x_i(n) = n^{\rho_i} \xi_i(n), \quad \xi_i \in \mathcal{SV}, \quad i = \overline{1,N}. \tag{4.3}$$

We also assume that all sequences p_i, $i = \overline{1,N}$ satisfy either (I) or (II). Condition (I) is satisfied if and only if index of regularity λ_i satisfies either

$$\lambda_i < \alpha_i, \tag{4.4}$$

or

$$\lambda_i = \alpha_i \quad \text{and} \quad S_i = \sum_{n=1}^{\infty} n^{-1} l_i(n)^{-\frac{1}{\alpha_i}} = \infty. \tag{4.5}$$

Using Theorem 3.8, if (4.4) holds, the following asymptotic relation is obtained for the sequence $P_i = \{P_i(n)\}$ defined by (1.1):

$$P_i(n) \sim \frac{\alpha_i}{\alpha_i - \lambda_i} n^{\frac{\alpha_i - \lambda_i}{\alpha_i}} l_i(n)^{-\frac{1}{\alpha_i}}, \quad n \to \infty, \tag{4.6}$$

implying that $P_i \in \mathcal{RV}\big(\frac{\alpha_i-\lambda_i}{\alpha_i}\big)$. Clearly, if (4.5) holds, $P_i \in \mathcal{SV}$. Condition (II) is satisfied if and only if either

$$\lambda_i > \alpha_i, \tag{4.7}$$

or

$$\lambda_i = \alpha_i \quad \text{and} \quad S_i = \sum_{n=1}^{\infty} n^{-1} l_i(n)^{-\frac{1}{\alpha_i}} < \infty. \tag{4.8}$$

If (4.7) holds, using Theorem 3.8, the following asymptotic relation is obtained for the sequence $\pi_i = \{\pi_i(n)\}$ defined by (1.2):

$$\pi_i(n) \sim \frac{\alpha_i}{\lambda_i - \alpha_i} n^{\frac{\alpha_i-\lambda_i}{\alpha_i}} l_i(n)^{-\frac{1}{\alpha_i}}, \quad n \to \infty, \tag{4.9}$$

implying that $\pi_i \in \mathcal{RV}\big(\frac{\alpha_i-\lambda_i}{\alpha_i}\big)$. If (4.8) holds, $\pi_i \in \mathcal{SV}$.

Also, to simplify notation we denote $A_N = \alpha_1\alpha_2 \cdot \ldots \cdot \alpha_N$, $B_N = \beta_1\beta_2 \cdot \ldots \cdot \beta_N$ and use matrix

$$M = \begin{pmatrix} 1 & \frac{\beta_1}{\alpha_1} & \frac{\beta_1\beta_2}{\alpha_1\alpha_2} & \cdots & \frac{\beta_1\beta_2\cdot\ldots\cdot\beta_{N-2}}{\alpha_1\alpha_2\cdot\ldots\cdot\alpha_{N-2}} & \frac{\beta_1\beta_2\cdot\ldots\cdot\beta_{N-1}}{\alpha_1\alpha_2\cdot\ldots\cdot\alpha_{N-1}} \\ \frac{\beta_2\beta_3\cdot\ldots\cdot\beta_N}{\alpha_2\alpha_3\cdot\ldots\cdot\alpha_N} & 1 & \frac{\beta_2}{\alpha_2} & \cdots & \frac{\beta_2\beta_3\cdot\ldots\cdot\beta_{N-2}}{\alpha_2\alpha_3\cdot\ldots\cdot\alpha_{N-2}} & \frac{\beta_2\beta_3\cdot\ldots\cdot\beta_{N-1}}{\alpha_2\alpha_3\cdot\ldots\cdot\alpha_{N-1}} \\ \frac{\beta_3\beta_4\cdot\ldots\cdot\beta_N}{\alpha_3\alpha_4\cdot\ldots\cdot\alpha_N} & \frac{\beta_3\cdot\ldots\cdot\beta_N\beta_1}{\alpha_3\cdot\ldots\cdot\alpha_N\alpha_1} & 1 & \cdots & \frac{\beta_3\beta_4\cdot\ldots\cdot\beta_{N-2}}{\alpha_3\alpha_4\cdot\ldots\cdot\alpha_{N-2}} & \frac{\beta_3\beta_4\cdot\ldots\cdot\beta_{N-1}}{\alpha_3\alpha_4\cdot\ldots\cdot\alpha_{N-1}} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \frac{\beta_{N-1}\beta_N}{\alpha_{N-1}\alpha_N} & \frac{\beta_{N-1}\beta_N\beta_1}{\alpha_{N-1}\alpha_N\alpha_1} & \frac{\beta_{N-1}\beta_N\beta_1\beta_2}{\alpha_{N-1}\alpha_N\alpha_1\alpha_2} & \cdots & 1 & \frac{\beta_{N-1}}{\alpha_{N-1}} \\ \frac{\beta_N}{\alpha_N} & \frac{\beta_N\beta_1}{\alpha_N\alpha_1} & \frac{\beta_N\beta_1\beta_2}{\alpha_N\alpha_1\alpha_2} & \cdots & \frac{\beta_N\beta_1\cdot\ldots\cdot\beta_{N-2}}{\alpha_N\alpha_1\cdot\ldots\cdot\alpha_{N-2}} & 1 \end{pmatrix}, \tag{4.10}$$

whose elements will be denoted by $M = (M_{ij})$. The i-th row of (M_{ij}) is obtained by shifting the vector

$$\left(1, \frac{\beta_i}{\alpha_i}, \frac{\beta_i\beta_{i+1}}{\alpha_i\alpha_{i+1}}, \ldots, \frac{\beta_i\beta_{i+1}\cdots\beta_{i+(N-2)}}{\alpha_i\alpha_{i+1}\cdots\alpha_{i+(N-2)}}\right), \quad \alpha_{N+j} = \alpha_j,\ \beta_{N+j} = \beta_j,\ j = \overline{1, N-2}$$

$(i-1)$-times to the right cyclically, so that elements $M_{ij}, M_{ji}, i > j$, satisfy the relation

$$M_{ij}M_{ji} = \frac{\beta_1\beta_2 \cdot \ldots \cdot \beta_N}{\alpha_1\alpha_2 \cdot \ldots \cdot \alpha_N}, \quad i > j, \quad i = \overline{2, N}.$$

It is easy to see that elements of matrix M satisfy for $i = \overline{1, N}$, $j = \overline{1, N}$

$$M_{i+1,i}\frac{\beta_i}{\alpha_i} = \frac{B_N}{A_N}, \quad M_{i+1,j}\frac{\beta_i}{\alpha_i} = M_{ij} \quad \text{for } j \neq i,\ M_{N+1,j} = M_{1,j}. \tag{4.11}$$

The next matrix also plays important note in the proof of main results:

$$A = \begin{pmatrix} 1 & -\frac{\beta_1}{\alpha_1} & 0 & \cdots & 0 & 0 \\ 0 & 1 & -\frac{\beta_2}{\alpha_2} & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & & \vdots & \vdots \\ \vdots & \vdots & & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & -\frac{\beta_{N-1}}{\alpha_{N-1}} \\ -\frac{\beta_N}{\alpha_N} & 0 & 0 & \cdots & 0 & 1 \end{pmatrix}. \tag{4.12}$$

Since,

$$\det(A) = 1 - \frac{\beta_1\beta_2 \cdot \ldots \cdot \beta_N}{\alpha_1\alpha_2 \cdot \ldots \cdot \alpha_N} > 0,$$

the matrix A is invertible and its inverse matrix is given by

$$A^{-1} = \frac{A_N}{A_N - B_N} M. \tag{4.13}$$

Throughout the text, $n \geq n_0$ means that n is sufficiently large so that n_0 need not to be the same at each occurrence.

5 Existence of primitive decreasing solutions

The asymptotic behavior of primitive solutions is evident from the classification itself. The following theorems provide necessary and sufficient conditions for the existence of these solutions, using Fixed point theory and without the assumption that coefficients are regularly varying sequences.

Theorem 5.1. *The system* (SE) *has a solution* $\mathbf{x} \in \mathcal{DS}$ *with each component satisfying* (AC) *if and only if*

$$\sum_{n=1}^{\infty} \left(\frac{1}{p_i(n)} \sum_{k=n}^{\infty} q_i(k) \right)^{\frac{1}{\alpha_i}} < \infty, \quad i = \overline{1,N}. \tag{5.1}$$

Proof. The "only if" part: Let $\mathbf{x} = (x_1, x_2, \dots, x_N) \in \mathcal{DS}$ be a solution of (SE) with each component satisfying $\lim_{n\to\infty} x_i(n) = c_i$. Then, there exist $n_0 \in \mathbb{N}$ such that $x_i(n) \geq c_i$, $n \geq n_0$, $i = \overline{1,N}$.

If (I) holds, as previously shown $\lim_{n\to\infty} x_i^{[1]}(n) = 0$, $i = \overline{1,N}$. By summing equations of (SE) twice, first from n to ∞, and then from n_0 to ∞, we derive that

$$x_i(n_0) = c_i + \sum_{n=n_0}^{\infty} \left(\frac{1}{p_i(n)} \sum_{k=n}^{\infty} q_i(k) x_{i+1}(k+1)^{\beta_i} \right)^{\frac{1}{\alpha_i}} \geq c_{i+1}^{\frac{\beta_i}{\alpha_i}} \sum_{n=n_0}^{\infty} \left(\frac{1}{p_i(n)} \sum_{k=n}^{\infty} q_i(k) \right)^{\frac{1}{\alpha_i}},$$

for $i = \overline{1,N}$. From the previous equation, we see that condition (5.1) is satisfied.

If (II) holds, then, since $x_i^{[1]}$, $i = \overline{1,N}$ are eventually negative and increasing, we may assume that $x_i^{[1]}(n) \leq -\omega_i \leq 0$ for $n \geq n_0$, $i = \overline{1,N}$. By summing equations of (SE) from n_0 to n, we obtain for $i = \overline{1,N}$

$$-\omega_i - x_i^{[1]}(n_0) \geq x_i^{[1]}(n) - x_i^{[1]}(n_0) = \sum_{k=n_0}^{n} q_i(k) x_{i+1}(k+1)^{\beta_i} \geq c_{i+1}^{\beta_i} \sum_{k=n_0}^{n} q_i(k), \quad n \geq n_0.$$

Letting $n \to \infty$, we find

$$\sum_{n=n_0}^{\infty} q_i(n) < \infty, \quad i = \overline{1,N},$$

which together with (II) yields that the condition (5.1) is satisfied.

The "if" part: Suppose that (5.1) holds. Then, there exists $n_0 \in \mathbb{N}$ such that

$$\sum_{n=n_0}^{\infty} \left(\frac{1}{p_i(n)} \sum_{k=n}^{\infty} q_i(k) \right)^{\frac{1}{\alpha_i}} < 2^{1-\frac{\beta_i}{\alpha_i}}, \quad i = \overline{1,N}. \tag{5.2}$$

Denote by $\mathcal{L}_{n_0}$ the space of all vectors $\mathbf{x} = (x_1, x_2, \dots, x_N)$, such that $x_i = \{x_i(n)\} \in {}^{\mathbb{N}_{n_0}}\mathbb{R}$, $i = \overline{1,N}$ are bounded. Then, $\mathcal{L}_{n_0}$ is a Banach space endowed with the norm

$$\|\mathbf{x}\| = \max_{1\le i\le N}\left\{\sup_{n\ge n_0}|x_i(n)|\right\}. \tag{5.3}$$

Set

$$\Lambda_1 = \left\{\mathbf{x}\in\mathcal{L}_{n_0}\;\middle|\; c_i \le x_i(n) \le 2c_i,\quad n\ge n_0,\quad i=\overline{1,N}\right\}, \tag{5.4}$$

where c_i, $i=\overline{1,N}$ are positive constants such that

$$c_i \ge 2c_{i+1}^{\frac{\beta_i}{\alpha_i}},\quad i=\overline{1,N},\quad c_{N+1}=c_1. \tag{5.5}$$

Define operators $\mathcal{F}_i : {}^{\mathbb{N}_{n_0}}\mathbb{R} \to {}^{\mathbb{N}_{n_0}}\mathbb{R}$ by

$$\mathcal{F}_i x(n) = c_i + \sum_{k=n}^{\infty}\left(\frac{1}{p_i(k)}\sum_{s=k}^{\infty} q_i(s)x(s+1)^{\beta_i}\right)^{\frac{1}{\alpha_i}},\quad n\ge n_0,\quad i=\overline{1,N}, \tag{5.6}$$

and the mapping $\Theta : \Lambda_1 \to \mathcal{L}_{n_0}$ by

$$\Theta(x_1,x_2,\dots,x_N) = \left(\mathcal{F}_1x_2,\mathcal{F}_2x_3,\dots,\mathcal{F}_Nx_{N+1}\right),\quad x_{N+1}=x_1. \tag{5.7}$$

We will show that Θ has a fixed point by using Schauder–Tychonoff fixed point theorem. Namely, the operator Θ has the following properties:

(i) Θ *maps* Λ_1 *into itself*: Let $\mathbf{x}\in\Lambda_1$. Then, using (5.2), (5.4), (5.5) and (5.6), we see that

$$c_i \le \mathcal{F}_i x_{i+1}(n) \le c_i + (2c_{i+1})^{\frac{\beta_i}{\alpha_i}}\sum_{k=n}^{\infty}\left(\frac{1}{p_i(k)}\sum_{s=k}^{\infty}q_i(s)\right)^{\frac{1}{\alpha_i}} \le c_i + 2^{\frac{\beta_i}{\alpha_i}}\cdot\frac{c_i}{2}\cdot 2^{1-\frac{\beta_i}{\alpha_i}} = 2c_i,$$

for $i=\overline{1,N}$ and $n\ge n_0$.

(ii) Θ *is continuous*: Let $\varepsilon_i > 0, i=\overline{1,N}$ and $\{\mathbf{x}^{(m)}\}_{m\in\mathbb{N}} = \left\{(x_1^{(m)},x_2^{(m)},\dots,\ x_N^{(m)})\right\}_{m\in\mathbb{N}}$, be a sequence in Λ_1 which converges to $\mathbf{x}=(x_1,x_2,\dots,x_N)$ as $m\to\infty$. Since, Λ_1 is closed, $\mathbf{x}\in\Lambda_1$. The rest of the proof does not depend on i, so let $i\in\{1,2,\dots,N\}$ be arbitrary fixed. For every $n\ge n_0$, we have

$$\begin{aligned}&\left|\mathcal{F}_i x_{i+1}^{(m)}(n) - \mathcal{F}_i x_{i+1}(n)\right|\\ &\qquad\le \sum_{k=n}^{\infty}\frac{1}{p_i(k)^{\frac{1}{\alpha_i}}}\left|\left(\sum_{s=k}^{\infty}q_i(s)x_{i+1}^{(m)}(s+1)^{\beta_i}\right)^{\frac{1}{\alpha_i}} - \left(\sum_{s=k}^{\infty}q_i(s)x_{i+1}(s+1)^{\beta_i}\right)^{\frac{1}{\alpha_i}}\right|.\end{aligned}$$

Since (5.1) implies that $\sum_{n=n_0}^{\infty} q_i(n)$ is convergent, we conclude that $\sum_{n=n_0}^{\infty} q_i(n)\,x_{i+1}^{(m)}(n+1)^{\beta_i}$ is totally convergent, because $q_i(n)\,x_{i+1}^{(m)}(n+1)^{\beta_i} \le (2c_{i+1})^{\beta_i}q_i(n)$, for every $n\ge n_0, m\in\mathbb{N}$. Then, by a discrete analogue of Lebesgue dominated convergence theorem (Theorem 4.3), it holds for every $k\ge n_0$

$$\lim_{m\to\infty}\left|\left(\sum_{s=k}^{\infty}q_i(s)x_{i+1}^{(m)}(s+1)^{\beta_i}\right)^{\frac{1}{\alpha_i}} - \left(\sum_{s=k}^{\infty}q_i(s)x_{i+1}(s+1)^{\beta_i}\right)^{\frac{1}{\alpha_i}}\right| = 0,$$

which shows that

$$\lim_{m\to\infty}\sup_{n\ge n_0}\left|\mathcal{F}_i x_{i+1}^{(m)}(n) - \mathcal{F}_i x_{i+1}(n)\right| = 0.$$

Therefore, $\|\Theta\mathbf{x}^{(m)} - \Theta\mathbf{x}\| \to 0$ as $m \to \infty$, i.e. Θ is continuous.

(iii) $\Theta(\Lambda_1)$ *is relatively compact*: To show this, by Theorem 4.4, it is sufficient to show that $\Theta(\Lambda_1)$ is uniformly Cauchy in the topology of $\mathcal{L}_{n_0}$. For $\mathbf{x} \in \Lambda_1$ and $m > n \geq n_0$ we have

$$|\mathcal{F}_i x_{i+1}(m) - \mathcal{F}_i x_{i+1}(n)| = \left| \sum_{k=n}^{m-1} \left(\frac{1}{p_i(k)} \sum_{s=k}^{\infty} q_i(s) x_{i+1}(s+1)^{\beta_i} \right)^{\frac{1}{\alpha_i}} \right|$$
$$\leq \sum_{k=n}^{m-1} \frac{1}{p_i(k)^{\frac{1}{\alpha_i}}} \left| \sum_{s=k}^{\infty} q_i(s) x_{i+1}(s+1)^{\beta_i} \right|^{\frac{1}{\alpha_i}}$$
$$\leq (2c_{i+1})^{\frac{\beta_i}{\alpha_i}} \sum_{k=n}^{m-1} \left(\frac{1}{p_i(k)} \sum_{s=k}^{\infty} q_i(s) \right)^{\frac{1}{\alpha_i}}.$$

According to (5.1) it follows that $\Theta(\Lambda_1)$ is uniformly Cauchy.

Therefore, all the hypotheses of Schauder–Tychonoff fixed point theorem are fulfilled implying the existence of a fixed point $\mathbf{x} \in \Lambda_1$ of the mapping Θ, which satisfies

$$x_i(n) = c_i + \sum_{k=n}^{\infty} \left(\frac{1}{p_i(k)} \sum_{s=k}^{\infty} q_i(s) x_{i+1}(s+1)^{\beta_i} \right)^{\frac{1}{\alpha_i}}, \quad n \geq n_0, \quad i = \overline{1,N}.$$

It is clear that $\mathbf{x}$ is a positive decreasing solution of (SE) whose all components tend to constants. □

Theorem 5.2. *Let (II) hold. The system (SE) has a solution $\mathbf{x} \in \mathcal{DS}$ with each component satisfying $(P1)$ if and only if*

$$\sum_{n=1}^{\infty} q_i(n) \pi_{i+1}(n+1)^{\beta_i} < \infty, \quad i = \overline{1,N}. \tag{5.8}$$

Proof. The "only if" part: Let $\mathbf{x} = (x_1, x_2, \dots, x_N) \in \mathcal{DS}$ be a solution with each component satisfying $(P1)$. Since $x_i^{[1]}$, $i = \overline{1,N}$ are eventually negative and increasing, we may assume that $x_i^{[1]}(n) \leq -\omega_i < 0$ for $n \geq n_0$, $i = \overline{1,N}$. Then, for $n \geq n_0$, we have

$$x_i(n) = \sum_{k=n}^{\infty} \frac{\left(-x_i^{[1]}(k)\right)^{1/\alpha_i}}{p_i(k)^{1/\alpha_i}} \geq \omega_i^{\frac{1}{\alpha_i}} \pi_i(n), \quad i = \overline{1,N}.$$

Summing equations of the system (SE) from n_0 to n we get for $n \geq n_0$,

$$-\omega_i - x_i^{[1]}(n_0) \geq x_i^{[1]}(n+1) - x_i^{[1]}(n_0) = \sum_{k=n_0}^{n} q_i(k) x_{i+1}(k+1)^{\beta_i} \geq \omega_{i+1}^{\frac{\beta_i}{\alpha_{i+1}}} \sum_{k=n_0}^{n} q_i(k) \pi_{i+1}(k+1)^{\beta_i} \tag{5.9}$$

Letting that $n \to \infty$ in (5.9), we conclude that (5.8) holds.

The "if" part: Suppose that (5.8) holds. Then, there exists $n_0 \in \mathbb{N}$ such that

$$\sum_{n=n_0}^{\infty} q_i(n) \pi_{i+1}(n+1)^{\beta_i} < 2^{\alpha_i - \beta_i} \left(2^{\alpha_i} - 1\right), \quad i = \overline{1,N}. \tag{5.10}$$

Denote by $\mathcal{X}_{n_0}$ the space of all vectors $\mathbf{x} = (x_1, x_2, \dots, x_N)$, $x_i \in {}^{\mathbb{N}_{n_0}}\mathbb{R}$, $i = \overline{1,N}$, such that $\{x_i(n)/\pi_i(n)\}$, $i = \overline{1,N}$ are bounded. Then, $\mathcal{X}_{n_0}$ is a Banach space endowed with the norm

$$\|\mathbf{x}\| = \max_{1\le i\le N} \left\{ \sup_{n\ge n_0} \left| \frac{x_i(n)}{\pi_i(n)} \right| \right\}. \tag{5.11}$$

Further, $\mathcal{X}_{n_0}$ is partially ordered, with usual pointwise ordering $\le$: for $\mathbf{x}, \mathbf{y} \in \mathcal{X}_{n_0}$, $\mathbf{x} \le \mathbf{y}$ means $x_i(n) \le y_i(n)$ for all $n \ge n_0$ and $i = \overline{1,N}$. Set

$$\Lambda_2 = \left\{ \mathbf{x} \in \mathcal{X}_{n_0} \;\middle|\; c_i\pi_i(n) \le x_i(n) \le 2c_i\pi_i(n), \quad n \ge n_0, \quad i = \overline{1,N} \right\}, \tag{5.12}$$

where c_i, $i = \overline{1,N}$ are positive constants which satisfy (5.5). For any subset B of Λ_2, it is obvious that $\sup B \in \Lambda_2$ and $\inf B \in \Lambda_2$.

Let $\Theta : \Lambda_2 \to \mathcal{X}_{n_0}$ be the mapping given by (5.7), where $\mathcal{F}_i : {}^{\mathbb{N}_{n_0}}\mathbb{R} \to {}^{\mathbb{N}_{n_0}}\mathbb{R}$ is operator defined by

$$\mathcal{F}_i x(n) = \sum_{k=n}^{\infty} \left(\frac{1}{p_i(k)} \left(c_i^{\alpha_i} + \sum_{s=k}^{\infty} q_i(s) x(s+1)^{\beta_i} \right) \right)^{\frac{1}{\alpha\, i}}, \quad n \ge n_0, \quad i = \overline{1,N}, \tag{5.13}$$

We will show that Θ has a fixed point by using Theorem 4.1. Namely, the operator Θ has the following properties:

(i) Θ *maps* Λ_2 *into itself*: Let $\mathbf{x} \in \Lambda_2$. Then, using (5.5), (5.10), (5.12) and (5.13), we see that for $n \ge n_0$

$$\begin{aligned} c_i\pi_i(n) \le \mathcal{F}_i x_{i+1}(n) &\le \sum_{k=n}^{\infty} \left(\frac{1}{p_i(k)} \left(c_i^{\alpha_i} + (2c_{i+1})^{\beta_i} \sum_{s=k}^{\infty} q_i(s) \pi_{i+1}(s+1)^{\beta_i} \right) \right)^{\frac{1}{\alpha_i}} \\ &\le \sum_{k=n}^{\infty} \left(\frac{1}{p_i(k)} \left(c_i^{\alpha_i} + 2^{\alpha_i} \left(2^{\alpha_i} - 1\right) c_{i+1}^{\beta_i} \right) \right)^{\frac{1}{\alpha_i}} \\ &\le \sum_{k=n}^{\infty} \left(\frac{1}{p_i(k)} \left(c_i^{\alpha_i} + \left(2^{\alpha_i} - 1\right) c_i^{\alpha_i} \right) \right)^{\frac{1}{\alpha_i}} = 2c_i\pi_i(n), \end{aligned}$$

(ii) Θ *is increasing*, i.e. for any $\mathbf{x}, \mathbf{y} \in \Lambda_2$, $\mathbf{x} \le \mathbf{y}$ implies $\Theta\mathbf{x} \le \Theta\mathbf{y}$.

Thus, all the hypotheses of Theorem 4.1 are fulfilled, implying the existence of a fixed point $\mathbf{x} \in \Lambda_2$ of the mapping Θ. It is easy to see that $\mathbf{x}$ is a positive decreasing solution of the system (SE) and that its components satisfy $(P1)$. □

6 Existence and asymptotic behavior of strongly decreasing $\mathcal{RV}$-solutions

This section is devoted to the problem of existence and asymptotic behavior of strongly decreasing solutions for system (SE). Every strongly decreasing solution of (SE) is a solution of the system

$$x_i(n) = \sum_{k=n}^{\infty} \left(\frac{1}{p_i(k)} \sum_{j=k}^{\infty} q_i(j) x_{i+1}(j+1)^{\beta_i} \right)^{\frac{1}{\alpha_i}}, \quad i = \overline{1,N}, \quad n \ge n_0, \tag{6.1}$$

for some $n_0 \in \mathbb{N}$. We consider existence and asymptotic behavior of strongly decreasing $\mathcal{RV}$-solutions with a negative index of regularity. However, due to computing difficulties, the existence problem and asymptotic behavior of strongly decreasing slowly varying solutions will be excluded for the time being.

Assuming that condition (I) is satisfied and that coefficients are regularly varying, the following theorem gives the necessary and sufficient conditions for a system (SE) to posses a regularly varying solution $\mathbf{x}$ of a negative index $(\rho_1, \rho_2, \ldots, \rho_N)$ of regularity and moreover determine their asymptotic behavior at infinity precisely.

Theorem 6.1. *Let $p_i \in \mathcal{RV}(\lambda_i)$, $q_i \in \mathcal{RV}(\mu_i)$, $i = \overline{1,N}$. Suppose that (I) holds. The system (SE) has a regularly varying solution $\mathbf{x} \in \mathcal{RV}(\rho_1, \rho_2, \ldots, \rho_N)$ with $\rho_i < 0$, $i = \overline{1,N}$ if and only if*

$$\sum_{j=1}^{N} M_{ij} \frac{\alpha_j - \lambda_j + \mu_j + 1}{\alpha_j} < 0, \quad i = \overline{1,N} \tag{6.2}$$

holds, in which case ρ_i are uniquely determined by

$$\rho_i = \frac{A_N}{A_N - B_N} \sum_{j=1}^{N} M_{ij} \frac{\alpha_j - \lambda_j + \mu_j + 1}{\alpha_j}, \quad i = \overline{1,N} \tag{6.3}$$

and the asymptotic behavior of each component of any such solution is governed by the unique formula

$$x_i(n) \sim \left[\prod_{j=1}^{N} \left(\frac{n^{\frac{\alpha_j+1}{\alpha_j}} p_j(n)^{-\frac{1}{\alpha_j}} q_j(n)^{\frac{1}{\alpha_j}}}{D_j} \right)^{M_{ij}} \right]^{\frac{A_N}{A_N - B_N}}, \quad n \to \infty, \quad i = \overline{1,N}, \tag{6.4}$$

where

$$D_j = \left(\alpha_j - \lambda_j - \alpha_j \rho_j\right)^{\frac{1}{\alpha_j}} (-\rho_j), \quad j = \overline{1,N}. \tag{6.5}$$

Proof. The "only if" part: Let $\mathbf{x} \in \mathcal{RV}(\rho_1, \rho_2, \ldots, \rho_N)$ with all $\rho_i < 0$ be a solution of (SE). Then, by Theorem 3.7-(vi) and Theorem 3.7-(v) each x_i satisfies (2.1) and $x_i(n) \to 0, n \to \infty$. Also, since (I) holds, as shown in Section 2 we have that $x_i^{[1]}(n) \to 0$, as $n \to \infty$.

Using (4.2) and (4.3), and applying Theorem 3.7-(iii), we obtain

$$\begin{aligned} -x_i^{[1]}(n) &= \sum_{k=n}^{\infty} q_i(k) x_{i+1}(k+1)^{\beta_i} \sim \sum_{k=n}^{\infty} q_i(k) x_{i+1}(k)^{\beta_i} \\ &= \sum_{k=n}^{\infty} k^{\mu_i + \beta_i \rho_{i+1}} m_i(k) \xi_{i+1}(k)^{\beta_i}, \quad n \to \infty, \ i = \overline{1,N}. \end{aligned} \tag{6.6}$$

and

$$-\Delta x_i(n) = \left(\frac{1}{p_i(n)} \sum_{k=n}^{\infty} q_i(k) x_{i+1}(k+1)^{\beta_i} \right)^{\frac{1}{\alpha_i}} \sim n^{-\frac{\lambda_i}{\alpha_i}} l_i(n)^{-\frac{1}{\alpha_i}} H_i(n)^{\frac{1}{\alpha_i}}, \quad n \to \infty, \tag{6.7}$$

where $H_i(n) = \sum_{k=n}^{\infty} k^{\mu_i + \beta_i \rho_{i+1}} m_i(k) \xi_{i+1}(k)^{\beta_i}$. As the left-hand side of (6.6) tends to zero as $n \to \infty$, it must be $\mu_i + \beta_i \rho_{i+1} \leq -1$, $i = \overline{1,N}$. If $\mu_i + \beta_i \rho_{i+1} = -1$, for some i, then $H_i \in \mathcal{SV}$, so that summing (6.7) from n to ∞, we obtain

$$x_i(n) \sim \sum_{k=n}^{\infty} k^{-\frac{\lambda_i}{\alpha_i}} l_i(k)^{-\frac{1}{\alpha_i}} H_i(k)^{\frac{1}{\alpha_i}}, \quad n \to \infty. \tag{6.8}$$

Since $x_i(n) \to 0, n \to \infty$, using that either (4.4) or (4.5) is satisfied, it is only possible that $\lambda_i = \alpha_i$, in which case $x_i \in \mathcal{SV}$, contradicting the assumption that $\rho_i < 0$. Therefore, it follows that $\mu_i + \beta_i\rho_{i+1} < -1$ for all $i = \overline{1,N}$. Application of Theorem 3.8 to (6.7) gives for $i = \overline{1,N}$

$$-\Delta x_i(n) \sim \frac{n^{\frac{-\lambda_i+\mu_i+\beta_i\rho_{i+1}+1}{\alpha_i}} l_i(n)^{-\frac{1}{\alpha_i}} m_i(n)^{\frac{1}{\alpha_i}} \xi_{i+1}(n)^{\frac{\beta_i}{\alpha_i}}}{(-(\mu_i+\beta_i\rho_{i+1}+1))^{\frac{1}{\alpha_i}}}, \tag{6.9}$$

as $n \to \infty$. Summing (6.9) from n to ∞, we obtain

$$x_i(n) \sim \sum_{k=n}^{\infty} \frac{k^{\frac{-\lambda_i+\mu_i+\beta_i\rho_{i+1}+1}{\alpha_i}} l_i(k)^{-\frac{1}{\alpha_i}} m_i(k)^{\frac{1}{\alpha_i}} \xi_{i+1}(k)^{\frac{\beta_i}{\alpha_i}}}{(-(\mu_i+\beta_i\rho_{i+1}+1))^{\frac{1}{\alpha_i}}}, \quad n \to \infty. \tag{6.10}$$

Using that $x_i(n) \to 0, n \to \infty$, we conclude that $(-\lambda_i + \mu_i + \beta_i\rho_{i+1} + 1)/\alpha_i \le -1$, $i = \overline{1,N}$. All inequalities should be strict, because if the equality holds for some i, then (6.10) leads to the contradiction that $\rho_i = 0$. Therefore, $(-\lambda_i + \mu_i + \beta_i\rho_{i+1} + 1)/\alpha_i < -1$, $i = \overline{1,N}$. Applying Theorem 3.8 at (6.10), we conclude that

$$x_i(n) \sim \frac{n^{\frac{-\lambda_i+\mu_i+\beta_i\rho_{i+1}+1}{\alpha_i}+1} l_i(n)^{-\frac{1}{\alpha_i}} m_i(n)^{\frac{1}{\alpha_i}} \xi_{i+1}(n)^{\frac{\beta_i}{\alpha_i}}}{-\left(\frac{-\lambda_i+\mu_i+\beta_i\rho_{i+1}+1}{\alpha_i}+1\right)(-(\mu_i+\beta_i\rho_{i+1}+1))^{\frac{1}{\alpha_i}}}, \quad n \to \infty, \quad i = \overline{1,N}. \tag{6.11}$$

From the previous relation, we see that

$$\rho_i = \frac{-\lambda_i + \mu_i + \beta_i\rho_{i+1} + 1}{\alpha_i} + 1, \quad i = \overline{1,N}, \quad \rho_{N+1} = \rho_1 \tag{6.12}$$

which is equivalent to a linear cyclic system of equations

$$\rho_i - \frac{\beta_i}{\alpha_i}\rho_{i+1} = \frac{\alpha_i - \lambda_i + \mu_i + 1}{\alpha_i}, \quad i = \overline{1,N}, \quad \rho_{N+1} = \rho_1. \tag{6.13}$$

The matrix of the system (6.13) is given by (4.12). As shown in Section 4 the matrix A is invertible, implying that the system (6.13) has the unique solution $(\rho_1, \dots, \rho_N)$. Using (4.13), we derive that these ρ_i are given explicitly by (6.3). It is obvious that $\rho_i < 0$, $i = \overline{1,N}$ if and only if (6.2) holds.

Using (4.2) and (4.3), we can transform (6.11) into the following system of asymptotic relations

$$x_i(n) \sim \frac{n^{\frac{\alpha_i+1}{\alpha_i}} p_i(n)^{-\frac{1}{\alpha_i}} q_i(n)^{\frac{1}{\alpha_i}} x_{i+1}(n)^{\frac{\beta_i}{\alpha_i}}}{D_i}, \quad n \to \infty, \quad i = \overline{1,N}, \tag{6.14}$$

where D_i are given by (6.5). Without difficulty, we can obtain that each component x_i of regularly varying solution $\mathbf{x}$ satisfies the explicit asymptotic formula

$$\begin{aligned} x_i(n) &\sim \left[\prod_{j=1}^{N}\left(\frac{n^{\frac{\alpha_j+1}{\alpha_j}} p_j(n)^{-\frac{1}{\alpha_j}} q_j(n)^{\frac{1}{\alpha_j}}}{D_j}\right)^{M_{ij}}\right]^{\frac{A_N}{A_N-B_N}} \\ &= n^{\rho_i}\left[\prod_{j=1}^{N}\left(\frac{l_j(n)^{-\frac{1}{\alpha_j}} m_j(n)^{\frac{1}{\alpha_j}}}{D_j}\right)^{M_{ij}}\right]^{\frac{A_N}{A_N-B_N}}, \quad n \to \infty, \quad i = \overline{1,N}. \end{aligned} \tag{6.15}$$

It is clear from (6.15) that the regularity index of each x_i is exactly ρ_i.

The "if" part: Suppose now that (6.2) holds. Define ρ_i with (6.3) and sequences X_i, $i = \overline{1,N}$, by

$$X_i(n) = \left[\prod_{j=1}^{N} \left(\frac{n^{\frac{\alpha_j+1}{\alpha_j}} p_j(n)^{-\frac{1}{\alpha_j}} q_j(n)^{\frac{1}{\alpha_j}}}{D_j} \right)^{M_{ij}} \right]^{\frac{A_N}{A_N - B_N}} = n^{\rho_i} \chi_i(n), \quad i = \overline{1,N}, \tag{6.16}$$

where D_j, $j = \overline{1,N}$ are given by (6.5) and

$$\chi_i(n) = \left[\prod_{j=1}^{N} \left(\frac{l_j(n)^{-\frac{1}{\alpha_j}} m_j(n)^{\frac{1}{\alpha_j}}}{D_j} \right)^{M_{ij}} \right]^{\frac{A_N}{A_N - B_N}}, \quad \chi_i \in \mathcal{SV}.$$

From, (6.2) we have that $\rho_i < 0$, $i = \overline{1,N}$. First, we show that sequences $X_i \in \mathcal{RV}(\rho_i)$ satisfy the system of asymptotic relations

$$\sum_{k=n}^{\infty} \left(\frac{1}{p_i(k)} \sum_{s=k}^{\infty} q_i(s) X_{i+1}(s+1)^{\beta_i} \right)^{\frac{1}{\alpha_i}} \sim X_i(n), \quad n \to \infty, \quad i = \overline{1,N}, \tag{6.17}$$

where $X_{N+1} = X_1$. To that end, let us note that from "the only if" part we conclude that $(\rho_1, \dots, \rho_N)$, with ρ_i given by (6.3) is in fact the unique solution of the linear cyclic system of equations (6.13). In order to apply Theorem 3.8, it must hold that $\mu_i + \rho_{i+1}\beta_i < -1$, $i = \overline{1,N}$. From the fact $\rho_i < 0$ and (6.12) we have

$$\mu_i + \rho_{i+1}\beta_i = (\rho_i - 1)\alpha_i + \lambda_i - 1 < -\alpha_i + \lambda_i - 1 \le -1.$$

Therefore, using (6.12) and applying Theorem 3.8, we obtain

$$\left(\frac{1}{p_i(n)} \sum_{k=n}^{\infty} q_i(k) X_{i+1}(k+1)^{\beta_i} \right)^{\frac{1}{\alpha_i}} \sim \frac{n^{\rho_i - 1} l_i(n)^{-\frac{1}{\alpha_i}} m_i(n)^{\frac{1}{\alpha_i}} \chi_{i+1}(n)^{\frac{\beta_i}{\alpha_i}}}{(\alpha_i - \lambda_i - \alpha_i\rho_i)^{\frac{1}{\alpha_i}}}, \quad n \to \infty,$$

and

$$\sum_{k=n}^{\infty} \left(\frac{1}{p_i(k)} \sum_{s=k}^{\infty} q_i(s) X_{i+1}(s+1)^{\beta_i} \right)^{\frac{1}{\alpha_i}} \sim \frac{n^{\rho_i} l_i(n)^{-\frac{1}{\alpha_i}} m_i(n)^{\frac{1}{\alpha_i}} \chi_{i+1}(n)^{\frac{\beta_i}{\alpha_i}}}{D_i}, \quad n \to \infty. \tag{6.18}$$

Using relation (4.11) for matrix elements M_{ij}, the right-hand side of the relation (6.18) can be transformed as follows

$$\frac{l_i(n)^{-\frac{1}{\alpha_i}} m_i(n)^{\frac{1}{\alpha_i}}}{D_i} \chi_{i+1}(n)^{\frac{\beta_i}{\alpha_i}} = \frac{l_i(n)^{-\frac{1}{\alpha_i}} m_i(n)^{\frac{1}{\alpha_i}}}{D_i} \left[\prod_{j=1}^{N} \left(\frac{l_j(n)^{-\frac{1}{\alpha_j}} m_j(n)^{\frac{1}{\alpha_j}}}{D_j} \right)^{M_{i+1,j}\frac{\beta_i}{\alpha_i}} \right]^{\frac{A_N}{A_N - B_N}}$$

$$= \left[\prod_{j=1}^{N} \left(\frac{l_j(n)^{-\frac{1}{\alpha_j}} m_j(n)^{\frac{1}{\alpha_j}}}{D_j} \right)^{M_{ij}} \right]^{\frac{A_N}{A_N - B_N}} = \chi_i(n), \quad i = \overline{1,N}, \quad \chi_{N+1} = \chi_N,$$

so from (6.18), we obtain that X_i, $i = \overline{1,N}$ satisfy (6.17).

It follows from (6.17) that there exists $n_0 > 1$ such that for $n > n_0$ holds

$$\frac{1}{2}X_i(n) \le \sum_{k=n}^{\infty}\left(\frac{1}{p_i(k)}\sum_{s=k}^{\infty}q_i(s)X_{i+1}(s+1)^{\beta_i}\right)^{\frac{1}{\alpha_i}} \le 2X_i(n), \quad i = \overline{1,N}. \tag{6.19}$$

Let we choose positive constants ω_i and W_i so that

$$\omega_i \le \frac{1}{2}\omega_{i+1}^{\frac{\beta_i}{\alpha_i}}, \quad W_i \ge 2W_{i+1}^{\frac{\beta_i}{\alpha_i}}, \quad i = \overline{1,N}, \quad \omega_{N+1} = \omega_1, \quad W_{N+1} = W_1. \tag{6.20}$$

Consider the space Y_{n_0} of all vectors $\mathbf{x} = (x_1, x_2, \ldots, x_N)$, $x_i \in {}^{\mathbb{N}_{n_0}}\mathbb{R}$, $i = \overline{1,N}$, such that $\{x_i(n)/X_i(n)\}$, $i = \overline{1,N}$ are bounded. Then, Y_{n_0} is a Banach space endowed with the norm

$$\|\mathbf{x}\| = \max_{1\le i\le N}\left\{\sup_{n\ge n_0}\left|\frac{x_i(n)}{X_i(n)}\right|\right\}$$

Further, Y_{n_0} is partially ordered, with the usual pointwise ordering $\le$: For $\mathbf{x},\mathbf{y} \in Y_{n_0}$, $\mathbf{x} \le \mathbf{y}$ means $x_i(n) \le y_i(n)$ for all $n \ge n_0$ and $i = \overline{1,N}$. Define the subset $\mathcal{X} \subset Y_{n_0}$ with

$$\mathcal{X} = \left\{\mathbf{x} \in Y_{n_0} \;\middle|\; \omega_i X_i(n) \le x_i(n) \le W_i X_i(n), \quad n \ge n_0, \quad i = \overline{1,N}\right\}. \tag{6.21}$$

It is easy to see that for any $\mathbf{x} \in \mathcal{X}$, the norm of $\mathbf{x}$ is finite. Also, for any subset $B \subset \mathcal{X}$, it is obvious that $\inf B \in \mathcal{X}$ and $\sup B \in \mathcal{X}$. We will define the operators $\mathcal{F}_i : {}^{\mathbb{N}_{n_0}}\mathbb{R} \to {}^{\mathbb{N}_{n_0}}\mathbb{R}$ by

$$\mathcal{F}_i x(n) = \sum_{k=n}^{\infty}\left(\frac{1}{p_i(k)}\sum_{s=k}^{\infty}q_i(s)x(s+1)^{\beta_i}\right)^{\frac{1}{\alpha\, i}}, \quad n \ge n_0, \quad i = \overline{1,N}, \tag{6.22}$$

and define the mapping $\Phi : \mathcal{X} \to Y_{n_0}$ by

$$\Phi(x_1, x_2, \ldots, x_N) = \left(\mathcal{F}_1 x_2, \mathcal{F}_2 x_3, \ldots, \mathcal{F}_N x_{N+1}\right), \quad x_{N+1} = x_1. \tag{6.23}$$

We will show that Φ has a fixed point by using Theorem 4.1. Namely, the operator Φ has the following properties:

(i) Φ *maps* $\mathcal{X}$ *into itself*: Let $\mathbf{x} \in \mathcal{X}$. Then, using (6.19)–(6.22), we see that for $n \ge n_0$,

$$\mathcal{F}_i x_{i+1}(n) \le W_{i+1}^{\frac{\beta_i}{\alpha_i}}\sum_{k=n}^{\infty}\left(\frac{1}{p_i(k)}\sum_{s=k}^{\infty}q_i(s)X_{i+1}(s+1)^{\beta_i}\right)^{\frac{1}{\alpha_i}} \le 2W_{i+1}^{\frac{\beta_i}{\alpha_i}}X_i(n) \le W_i X_i(n),$$

and

$$\mathcal{F}_i x_{i+1}(n) \ge \omega_{i+1}^{\frac{\beta_i}{\alpha_i}}\sum_{k=n}^{\infty}\left(\frac{1}{p_i(k)}\sum_{s=k}^{\infty}q_i(s)X_{i+1}(s+1)^{\beta_i}\right)^{\frac{1}{\alpha_i}} \ge \frac{1}{2}\omega_{i+1}^{\frac{\beta_i}{\alpha_i}}X_i(n) \ge \omega_i X_i(n).$$

This shows that $\Phi\mathbf{x} \in \mathcal{X}$, that is, Φ is a self-map on $\mathcal{X}$.

(ii) Φ *is increasing*, i.e. for any $\mathbf{x},\mathbf{y} \in \mathcal{X}$, $\mathbf{x} \le \mathbf{y}$ implies $\Phi\mathbf{x} \le \Phi\mathbf{y}$.

Thus all the hypotheses of Theorem 4.1 are fulfilled implying the existence of a fixed point $\mathbf{x} \in \mathcal{X}$ of Φ, which satisfies

$$x_i(n) = \mathcal{F}_i x_{i+1}(n) = \sum_{k=n}^{\infty}\left(\frac{1}{p_i(k)}\sum_{s=k}^{\infty}q_i(s)x_{i+1}(s+1)^{\beta_i}\right)^{\frac{1}{\alpha\, i}}, \quad n \ge n_0, \quad i = \overline{1,N}.$$

It is easy to see that this solution is positive and decreasing. Since $\mathbf{x} \in \mathcal{X}$, it is clear that $x_i(n) \to 0$, $n \to \infty$. Furthermore, using (6.12), (6.16) and applying Theorem 3.8 we have

$$p_i(n)(-\Delta x_i(n))^{\alpha_i} = \sum_{k=n}^{\infty} q_i(k) x_{i+1}(k+1)^{\beta_i} \le W_{i+1}^{\beta_i} \sum_{k=n}^{\infty} q_i(k) X_{i+1}(k+1)^{\beta_i}$$
$$\sim W_{i+1}^{\beta_i} \frac{n^{(\rho_i-1)\alpha_i+\lambda_i} m_i(n) \chi_{i+1}(n)^{\beta_i}}{(1-\rho_i)\alpha_i - \lambda_i}, \quad n \to \infty, \quad i = \overline{1,N}.$$

Since $(\rho_i - 1)\alpha_i + \lambda_i < \lambda_i - \alpha_i \le 0$, we conclude that $x_i^{[1]}(n) \to 0$, $n \to \infty$. This shows that $\mathbf{x} \in \mathcal{X}$ is a strongly decreasing solution of the system (SE).

It remains to verify that $\mathbf{x} \in \mathcal{RV}(\rho_1, \rho_2, \dots, \rho_N)$. We define

$$u_i(n) = \sum_{k=n}^{\infty} \left(\frac{1}{p_i(k)} \sum_{s=k}^{\infty} q_i(s) X_{i+1}(s+1)^{\beta_i} \right)^{\frac{1}{\alpha_i}}, \quad i = \overline{1,N},$$

and put

$$r_i = \liminf_{n\to\infty} \frac{x_i(n)}{u_i(n)}, \quad R_i = \limsup_{n\to\infty} \frac{x_i(n)}{u_i(n)}.$$

Since $\omega_i X_i(n) \le x_i(n) \le W_i X_i(n)$, $n \ge n_0$, $i = \overline{1,N}$ and

$$u_i(n) \sim X_i(n), \quad n \to \infty, \quad i = \overline{1,N}, \tag{6.24}$$

it follows that $0 < r_i \le R_i < \infty$, $i = \overline{1,N}$. Using Lemma 4.5 we obtain

$$r_i \ge \liminf_{n\to\infty} \frac{\Delta x_i(n)}{\Delta u_i(n)} = \liminf_{n\to\infty} \frac{-\left(\frac{1}{p_i(n)} \sum_{k=n}^{\infty} q_i(k) x_{i+1}(k+1)^{\beta_i}\right)^{\frac{1}{\alpha_i}}}{-\left(\frac{1}{p_i(n)} \sum_{k=n}^{\infty} q_i(k) X_{i+1}(k+1)^{\beta_i}\right)^{\frac{1}{\alpha_i}}}$$
$$= \liminf_{n\to\infty} \left(\frac{\sum_{k=n}^{\infty} q_i(k) x_{i+1}(k+1)^{\beta_i}}{\sum_{k=n}^{\infty} q_i(k) X_{i+1}(k+1)^{\beta_i}} \right)^{\frac{1}{\alpha_i}} = \left(\liminf_{n\to\infty} \frac{\sum_{k=n}^{\infty} q_i(k) x_{i+1}(k+1)^{\beta_i}}{\sum_{k=n}^{\infty} q_i(k) X_{i+1}(k+1)^{\beta_i}} \right)^{\frac{1}{\alpha_i}}$$
$$\ge \left(\liminf_{n\to\infty} \frac{-q_i(n) x_{i+1}(n+1)^{\beta_i}}{-q_i(n) X_{i+1}(n+1)^{\beta_i}} \right)^{\frac{1}{\alpha_i}} = \liminf_{n\to\infty} \left(\frac{x_{i+1}(n+1)}{X_{i+1}(n+1)} \right)^{\frac{\beta_i}{\alpha_i}} = r_{i+1}^{\frac{\beta_i}{\alpha_i}}$$

where (6.24) has been used in the last step. Thus, r_i satisfy the cyclic system of inequalities

$$r_i \ge r_{i+1}^{\frac{\beta_i}{\alpha_i}}, \quad i = \overline{1,N}, \quad r_{N+1} = r_1. \tag{6.25}$$

Similarly, by taking the upper limits instead of the lower limits, we get that R_i satisfy the cyclic system of inequalities

$$R_i \le R_{i+1}^{\frac{\beta_i}{\alpha_i}}, \quad i = \overline{1,N}, \quad R_{N+1} = R_1. \tag{6.26}$$

From (6.25) and (6.26) we easily see that

$$r_i \ge r_i^{\frac{\beta_1\beta_2\cdots\beta_N}{\alpha_1\alpha_2\cdots\alpha_N}}, \quad R_i \le R_i^{\frac{\beta_1\beta_2\cdots\beta_N}{\alpha_1\alpha_2\cdots\alpha_N}},$$

whence, because of the hypothesis $\beta_1\beta_2 \dots \beta_N / \alpha_1\alpha_2 \dots \alpha_N < 1$, we find that $r_i \ge 1$ and $R_i \le 1$, $i = \overline{1,N}$. It follows therefore that $r_i = R_i = 1$ i.e. $\lim_{n\to\infty} x_i(n)/u_i(n) = 1$ for $i = \overline{1,N}$. Combined this with (6.24) implies that $x_i(n) \sim u_i(n) \sim X_i(n)$ as $n \to \infty$, which shows that each x_i is a regularly varying sequence of index ρ_i. Thus, the proof of the "if" part of Theorem 6.1. is completed. □

Assuming that condition (II) is satisfied, taking into account that (II) is satisfied if and only if (4.7) or (4.8) holds, the next two theorems provide the necessary and sufficient conditions for a system (SE) to posses a regularly varying solution $\mathbf{x}$ with the index of regularity $(\rho_1, \rho_2, \ldots, \rho_N)$ such that $\rho_i < \frac{\alpha_i - \lambda_i}{\alpha_i}$, $i = \overline{1,N}$ if (4.7) holds and such that $\rho_i < 0$, $i = \overline{1,N}$ if (4.8) holds, and moreover determine their asymptotic behavior at infinity precisely.

Theorem 6.2. *Let $p_i \in \mathcal{RV}(\lambda_i)$, $q_i \in \mathcal{RV}(\mu_i)$. Suppose that (4.7) holds. The system (SE) has a regularly varying solution $\mathbf{x} \in \mathcal{RV}(\rho_1, \rho_2, \ldots, \rho_N)$ with $\rho_i < \frac{\alpha_i - \lambda_i}{\alpha_i}$, $i = \overline{1,N}$ if and only if*

$$\sum_{j=1}^{N} M_{ij} \left(\frac{\mu_j + 1}{\alpha_j} + \frac{\beta_j(\alpha_{j+1} - \lambda_{j+1})}{\alpha_j \alpha_{j+1}} \right) < 0, \quad i = \overline{1,N}, \ \ \alpha_{N+1} = \alpha_1, \lambda_{N+1} = \lambda_1, \tag{6.27}$$

holds, in which case ρ_i are uniquely determined by (6.3) and the asymptotic behavior of each component of any such solution is governed by the unique formula (6.4) with D_j, $j = \overline{1,N}$ given by (6.5).

Proof. The "only if" part: Let $\mathbf{x} \in \mathcal{RV}(\rho_1, \rho_2, \ldots, \rho_N)$ with all $\rho_i < (\alpha_i - \lambda_i)/\alpha_i$ be a solution of (SE). Since from (4.7), ρ_i, $i = \overline{1,N}$ are negative, from Theorem 3.7-(v) we have that $x_i(n) \to 0$, $n \to \infty$. Also, we have that $\pi_i \in \mathcal{RV}(\frac{\alpha_i - \lambda_i}{\alpha_i})$, from the asymptotic relation (4.9) for π_i holding if (4.7) is satisfied. Therefore, we conclude that index of regularity of each x_i/π_i, $i = \overline{1,N}$ is less then zero, implying by Theorem 3.7-(v) that $\lim_{n\to\infty} x_i(n)/\pi_i(n) = 0$. Therefore, $\lim_{n\to\infty} x_i^{[1]}(n) = 0$. Using (4.2) and (4.3), and applying Theorem 3.7-(iii), we obtain (6.6). Then, $\lim_{n\to\infty} x_i^{[1]}(n) = 0$ implies that $\mu_i + \beta_i \rho_{i+1} \le -1$, $i = \overline{1,N}$. If $\mu_i + \beta_i \rho_{i+1} = -1$ for some i, then as in the proof of Theorem 6.1 we obtain that (6.8) holds. Since (4.7) holds, application of Theorem 3.8 gives

$$x_i(n) \sim \sum_{k=n}^{\infty} k^{-\frac{\lambda_i}{\alpha_i}} l_i(k)^{-\frac{1}{\alpha_i}} H_i(k)^{\frac{1}{\alpha_i}} \sim \frac{\alpha_i}{\lambda_i - \alpha_i} n^{\frac{\alpha_i - \lambda_i}{\alpha_i}} l_i(n)^{-\frac{1}{\alpha_i}} H_i(n)^{\frac{1}{\alpha_i}}, \quad n \to \infty,$$

where $H_i(n) = \sum_{k=n}^{\infty} k^{-1} m_i(k) \xi_{i+1}(k)^{\beta_i}$, $H_i \in \mathcal{SV}$. Thus, $x_i \in \mathcal{RV}(\frac{\alpha_i - \lambda_i}{\alpha_i})$, contradicting the hypothesis $\rho_i < \frac{\alpha_i - \lambda_i}{\alpha_i}$. Therefore, $\mu_i + \beta_i \rho_{i+1} < -1$ for all $i = \overline{1,N}$. Proceeding exactly as in the proof of Theorem 6.1, we get that (6.9) holds and conclude that $(-\lambda_i + \mu_i + \beta_i \rho_{i+1} + 1)/\alpha_i \le -1$ for all i. All inequalities should be strict, because if the equality holds for some i, then $\lambda_i - \alpha_i = \mu_i + \beta_i \rho_{i+1} + 1 < 0$, contradicting the assumption (4.7). Thus, (6.11) holds, which yields that $(\rho_1, \ldots, \rho_N)$ is the unique solution of the linear cyclic system of equations (6.13), so that ρ_i, $i = \overline{1,N}$ are uniquely determined by (6.3). That the asymptotic behavior of each component of solution $\mathbf{x} \in \mathcal{RV}(\rho_1, \rho_2, \ldots, \rho_N)$ with all $\rho_i < (\alpha_i - \lambda_i)/\alpha_i$ is governed by the unique formula (6.4) with D_j, $j = \overline{1,N}$ given by (6.5) is obtain as in the proof of Theorem 6.1.

To verify the condition (6.27), let we denote $d_i = \rho_i + \lambda_i/\alpha_i - 1$, $i = \overline{1,N}$. Then, the linear system of equations (6.13) is transformed into the system

$$d_i - \frac{\beta_i}{\alpha_i} d_{i+1} = \frac{\mu_i + 1}{\alpha_i} + \frac{\beta_i(\alpha_{i+1} - \lambda_{i+1})}{\alpha_i \alpha_{i+1}}, \quad i = \overline{1,N}, \quad d_{N+1} = d_1. \tag{6.28}$$

Matrix of the system (6.28) is given by (4.12). Since A is nonsingular, the system (6.28) has the unique solution $d_i, i = \overline{1,N}$, where

$$d_i = \sum_{j=1}^{N} M_{ij} \left(\frac{\mu_j + 1}{\alpha_j} + \frac{\beta_j(\alpha_{j+1} - \lambda_{j+1})}{\alpha_j \alpha_{j+1}} \right), \quad i = \overline{1,N}. \tag{6.29}$$

Since $\rho_i < \frac{\alpha_i - \lambda_i}{\alpha_i}$ if and only if $d_i < 0$, we conclude that the condition (6.27) holds.

The "if" part: Suppose that (6.27) holds. Define ρ_i with (6.3) and sequences $X_i \in \mathcal{RV}(\rho_i)$, $i = \overline{1, N}$, by (6.16). The indices of regularity of X_i, are the unique solution of the linear cyclic system (6.13). According to the assumption (6.27), we conclude that $\rho_i < (\alpha_i - \lambda_i)/\alpha_i < 0$, $i = \overline{1, N}$. Then, in a manner similar to that of the previous theorem proof, it can be demonstrated that X_i, $i = \overline{1, N}$, satisfy the asymptotic relation (6.17). Therefore, the "if" part of the theorem as well as asymptotic formulas for each component x_i of $\mathcal{RV}$-solution $\mathbf{x}$ can be obtained as in the "if" part proof of the previous theorem. □

Theorem 6.3. *Let $p_i \in \mathcal{RV}(\lambda_i)$, $q_i \in \mathcal{RV}(\mu_i)$. Suppose that (4.8) holds. The system (SE) has a regularly varying solution $\mathbf{x} \in \mathcal{RV}(\rho_1, \rho_2, \dots, \rho_N)$ with $\rho_i < 0$, $i = \overline{1, N}$ if and only if*

$$\sum_{j=1}^{N} M_{ij} \frac{\mu_j + 1}{\alpha_j} < 0, \quad i = \overline{1, N} \tag{6.30}$$

in which case ρ_i are uniquely determined by

$$\rho_i = \frac{A_N}{A_N - B_N} \sum_{j=1}^{N} M_{ij} \frac{\mu_j + 1}{\alpha_j}, \quad i = \overline{1, N} \tag{6.31}$$

and the asymptotic behavior of any such solution is governed by the unique formulas (6.4) with $D_j = \left(\alpha_j(-\rho_j)^{\alpha_j+1}\right)^{1/\alpha_j}$, $j = \overline{1, N}$.

Proof. The "only if" part: Suppose that the system (SE) has a solution $\mathbf{x} \in \mathcal{RV}(\rho_1, \rho_2, \dots, \rho_N)$, $\rho_i < 0$, $i = \overline{1, N}$. The assumption (4.8) implies that

$$\pi_i(n) = \sum_{k=n}^{\infty} k^{-1} l_i(k)^{-\frac{1}{\alpha_i}},$$

so that $\pi_i \in \mathcal{SV}$, implying that index of regularity of each x_i/π_i is $\rho_i < 0$ for $i = \overline{1, N}$. Therefore, by Theorem 3.7-(v), we have that $\lim_{n\to\infty} x_i(n) = 0$ and $\lim_{n\to\infty} x_i(n)/\pi_i(n) = 0$, implying that $\lim_{n\to\infty} x_i^{[1]}(n) = 0$. From (6.6), since the left-hand side tends to zero, we conclude that it must be $\mu_i + \beta_i\rho_{i+1} \leq -1$. If $\mu_i + \beta_i\rho_{i+1} = -1$ holds for some i, then as in the proof of Theorem 6.1 we obtain from (6.8) with $H_i \in \mathcal{SV}$, which due to the assumption (4.8) has the form

$$x_i(n) \sim \sum_{k=n}^{\infty} k^{-1} l_i(k)^{-\frac{1}{\alpha_i}} H_i(k)^{\frac{1}{\alpha_i}}, \quad n \to \infty, \quad i = \overline{1, N},$$

implying the contradiction $x_i \in \mathcal{SV}$. Therefore, $\mu_i + \beta_i\rho_{i+1} < -1$ for all $i = \overline{1, N}$. Then as in the proof of Theorem 6.1 we obtain (6.10) and since $x_i(n) \to 0$, $n \to \infty$ it must be

$$\frac{-\lambda_i + \mu_i + \beta_i\rho_{i+1} + 1}{\alpha_i} = -1 + \frac{\mu_i + \beta_i\rho_{i+1} + 1}{\alpha_i} \leq -1, \quad i = \overline{1, N}.$$

If equality holds for some i, then (6.10) yields the contradiction $x_i \in \mathcal{SV}$. Thus, $\mu_i + \beta_i\rho_{i+1} < -1$, $i = \overline{1, N}$, and applying Theorem 3.8 at (6.10), we conclude that

$$x_i(n) \sim \frac{n^{\frac{\mu_i + \beta_i\rho_{i+1} + 1}{\alpha_i}} l_i(n)^{-\frac{1}{\alpha_i}} m_i(n)^{\frac{1}{\alpha_i}} \xi_{i+1}(n)^{\frac{\beta_i}{\alpha_i}}}{-\frac{\mu_i + \beta_i\rho_{i+1} + 1}{\alpha_i} \left(-(\mu_i + \beta_i\rho_{i+1} + 1)\right)^{\frac{1}{\alpha_i}}}, \quad n \to \infty, \quad i = \overline{1, N}. \tag{6.32}$$

Therefore, (6.32) implies that $(\rho_1, \dots, \rho_N)$ is the unique solution of the linear cyclic system of equations

$$\rho_i - \frac{\beta_i}{\alpha_i}\rho_{i+1} = \frac{\mu_i + 1}{\alpha_i}, \quad i = \overline{1, N}, \quad \rho_{N+1} = \rho_1, \tag{6.33}$$

since the matrix of the system (6.33), given by (4.12), is invertible. Using (4.13), we obtain that $\rho_i, i = \overline{1, N}$ is given by (6.31). Proceeding exactly as in the proof of Theorem 6.1, we obtain that the unique asymptotic formula for each x_i is given by (6.4), where constants D_j are reduced to $D_j = \left(\alpha_j/(-\rho_i)^{\alpha_j+1}\right)^{1/\alpha_j}, j = \overline{1, N}$. Since all ρ_i are negative if and only if (6.30) holds, the "only if" part of the theorem is proved.

The "if" part of the theorem is proved and the explicit asymptotic formula for each component x_i of $\mathcal{RV}$-solution $\mathbf{x}$ can be obtained as in the proof of Theorem 6.1. □

Application. The one-dimensional system with positive coefficients $\{p(n)\}, \{q(n)\}$ is in fact the second-order difference equation of Emden–Fowler type (see (1.3)):

$$\Delta(p(n)|\Delta x(n)|^{\alpha-1}\Delta x(n)) = q(n)|x(n+1)|^{\beta-1}x(n+1), \tag{6.34}$$

This equation has been studied in [24, 25]. In this case, Theorems 6.1 and 6.2 reduce to Theorem 3.8 from [25]. However, since the case where the regularity index of the coefficient p is equal to α, was not considered neither in [25] nor in the existing literature so far, the new result is obtained as the corollary of Theorem 6.1 and Theorem 6.3. This result provides necessary and sufficient condition for the existence of $\mathcal{RV}$ strongly decreasing solutions of type $(SD1)$ if $S = \infty$ as well as of $\mathcal{RV}$ strongly decreasing solutions of type $(SD2)$ if $S < \infty$, where $S = \sum_{n=1}^{\infty} \frac{1}{p(n)^{\alpha}}$.

Corollary 6.4. *Let $\{p(n)\} \in \mathcal{RV}(\alpha)$ and $\{q(n)\} \in \mathcal{RV}(\mu)$. The equation (6.34) possesses a regularly varying solution of index $\rho < 0$ if and only if $\mu < -1$, in which case ρ is given by*

$$\rho = \frac{\mu + 1}{\alpha - \beta}$$

and the asymptotic behavior of any such solution x is governed by the unique formula

$$x(n) \sim \left[\frac{n^{\alpha+1}p(n)^{-1}q(n)}{\alpha(-\rho)^{\alpha+1}}\right]^{\frac{1}{\alpha-\beta}}, \quad n \to \infty.$$

Following example illustrate results obtained in the last corollary.

Example 6.5. Consider the difference equation

$$\Delta\left(-n^2\sqrt{\log n}(\Delta x(n))^2\right) = \frac{\gamma(n)}{n^6 \log^{17/6} n}\sqrt[3]{x(n+1)}, \quad n \geq 2, \tag{6.35}$$

where $\gamma(n)$ is positive real-valued sequence such that $\lim_{n\to\infty} \gamma(n) = \delta$. In this equation, $\alpha = 3$, $\beta = \frac{1}{3}$, $\{p(n)\} \in \mathcal{RV}(2)$, and $\{q(n)\} \in \mathcal{RV}(-6)$.

Since $\mu = -6 < -1$, by Corollary 6.4 we conclude that equation (6.35) has a strongly decreasing $\mathcal{RV}-$solution of index $\rho = -3$. The asymptotic behavior of such a solution is

$$x(n) \sim \left(\frac{\delta}{54}\right)^{\frac{3}{5}} n^{-3}(\log n)^{-2}, \quad n \to \infty.$$

If

$$\gamma(n) = n^2(n+1)(\log n)^{\frac{9}{2}}(\log(n+1))^{\frac{2}{3}}\left(\psi(n) - \left(\frac{n}{n+1}\right)^4 \frac{\sqrt{\log(n+1)}}{\log n}\psi(n+1)\right),$$

where

$$\psi(n) = \left(\left(\frac{n}{n+1}\right)^3 \frac{1}{(\log(n+1))^2} - \frac{1}{(\log n)^2}\right)^2,$$

then $\delta = 54$ and equation (6.35) has the exact solution $x(n) = n^{-3}(\log n)^{-2}$.

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