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Second order neutral delay differential equations and their oscillatory criteria derived by linearization

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Abstract. In this paper, we investigate the oscillatory behavior of second-order neutral delay differential equations with a canonical operator of the form

$$\left(a(x)\left(v'(x)\right)^{m}\right)' + C(x, u(\varphi(x))) = 0.$$

We introduce new monotonicity properties of the non-oscillatory solutions of these equations, which are then used to linearize the equations and derive new oscillatory criteria. The presented results significantly improve upon existing criteria.

Keywords: second order neutral delay differential equations, asymptotic properties, linearization, oscillation theory.

2020 Mathematics Subject Classification: 34K40, 34K25, 34K06, 34K11.

1 Introduction

The aim of our paper is to investigate the asymptotic and oscillatory behavior of solutions for second-order neutral delay differential equations

$$\left(a(x)\left(v'(x)\right)^{m}\right)' + C(x, u(\varphi(x))) = 0, \tag{1.1}$$

where $v(x) = u(x) + b(x)u(\psi(x))$.

We make the following assumptions throughout the paper:

(H1) $a(x) \in C([x_0, \infty), (0, \infty))$ and m is a ratio of two positive odd integers,

(H2) $c(x), b(x) \in C([x_0, \infty), (0, \infty))$ and $\frac{C(x,u(x))}{u^m(x)} \geq c(x)$, $b(x) \leq 1$,

(H3) $\psi(x) \in C([x_0, \infty))$ and $\psi(x) \leq x$, $\lim_{x \to \infty} \psi(x) = \infty$,

(H4) $\varphi(x) \in C([x_0, \infty))$ and $\varphi(x) \leq x$, $\lim_{x \to \infty} \varphi(x) = \infty$.

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Additionally, we require the canonical condition

$$A(x) = \int_{x_0}^{x} \frac{dy}{a^{1/m}(y)} \to \infty \quad \text{as } x \to \infty \tag{1.2}$$

which ensures proper asymptotic behavior of solutions.

A solution of (1.1) is defined to be a function $u(x) \in C^1([T_0, \infty))$ with $T_0 \geq x_0$, satisfying $a(x)(v'(x))^m \in C^1([T_0, \infty))$ and equation (1.1) on $[T_0, \infty)$. We only consider solutions $u(x)$ of (1.1) that satisfy $\sup\{|u(x)| : x \geq T\} > 0$ for all $T \geq T_0$, and assume that such solutions exist. A solution of (1.1) is classified as oscillatory if it has arbitrarily large zeros on $[T_0, \infty)$, and non-oscillatory otherwise. The equation is said to be oscillatory if all of its solutions are oscillatory.

Second-order delay differential equations have been the focus of significant research for several decades. Numerous papers have been dedicated to the problem of establishing oscillatory criteria for various types of differential equations. Notable monographs in this area include those by Blanka Baculíková and Josef Džurina [4], Agarwal et al. [1], Došly and Řehák [5], Erbe et al. [8], Kiguradze and Chanturia [10], Ladde et al. [13], and Győri and Ladas [9], as well as numerous works [2,3,6,7,11,14,15].

Such a criterion for the second-order delay differential equations, as presented by Koplatadze et al. [11], is based on the monotonic properties $u(x) \uparrow$ and $\frac{u(x)}{x} \downarrow$ of positive solutions of $u''(x) + c(x)u(\varphi(x)) = 0$. These properties have been widely used in the literature to establish oscillatory behavior in various types of differential equations.

In this paper, we aim to establish new comparison theorems and oscillatory criteria for the investigation of second-order neutral delay differential equations by utilizing the monotonic properties of non-oscillatory solutions. Building upon the previous work of Blanka Baculíková and Josef Džurina [4], we present novel monotonic properties that can be utilized to linearize the equation, which in turn allows us to deduce the oscillation of the original equation from its linear forms. Our first objective is to establish new comparison theorems for equation (1.1). The second objective is to provide oscillatory criteria that consider the linear forms of (1.1). Finally, we seek to test the strength of the general criteria with an example. The presented results in this paper significantly enhance the existing oscillatory criteria.

Our work builds upon previous research in the field and provides a novel and useful contribution to the theory of second-order neutral delay differential equations.

2 Preliminary results

We start with some useful lemmas concerning monotonic properties of nonoscillatory solutions for studied equation.

Lemma 2.1. *Let $u(x)$ be a positive solution of* (1.1). *Then*

(i) $a(x)(v'(x))^m > 0$,

(ii) $\frac{v(x)}{A(x)}$ *is decreasing*

for $x \geq x_1 \geq x_0$. Moreover if

$$\int_{x_0}^{\infty} A^m(\varphi(y))(1 - b(\varphi(y)))^m c(y)dy = \infty \tag{2.1}$$

holds, then

(iii) $\lim_{x\to\infty} \frac{v(x)}{A(x)} = 0$.

Proof. Suppose that $u(x)$ is a positive solution of (1.1). Then

$$\left(a(x)\left(v'(x)\right)^m\right)' < 0,$$

and there exists $x_1 \geq x_0$ such that $a(x)(v'(x))^m$ has a constant sign for $x \geq x_1$. Now assume the opposite, i.e., $a(x)(v'(x))^m < 0$. This implies the existence of a constant $k > 0$ such that $a(x)(v'(x))^m \leq -k < 0$. Integrating the previous inequality from x_1 to x and using (1.2), we obtain

$$v(x) \leq v(x_1) - k^{1/m} A(x) \to -\infty \quad \text{as } x \to \infty.$$

Hence, we have shown that assuming $a(x)(v'(x))^m < 0$ leads to a contradiction, and therefore we can conclude that $a(x)(v'(x))^m > 0$.

Using the monotonicity of $a^{1/m}(x)v'(x)$, we get

$$v(x) \geq \int_{x_1}^{x} \frac{a^{1/m}(y)v'(y)}{a^{1/m}(y)} dy \geq a^{1/m}(x)v'(x)A(x), \tag{2.2}$$

which implies $(\frac{v(x)}{A(x)})' < 0$.

On the other hand, as $\frac{v(x)}{A(x)}$ is a positive and decreasing function, there exists a positive constant δ such that

$$\lim_{x\to\infty} \frac{v(x)}{A(x)} = \delta \geq 0.$$

Assume on the contrary that $\delta > 0$. Then $\frac{v(x)}{A(x)} \geq \delta$ for $x \geq x_1$. From the definition of $v(x)$ we have

$$\begin{aligned} u(x) &= v(x) - b(x)u(\psi(x)) \\ &\geq v(x) - b(x)v(\psi(x)) \\ &\geq v(x)(1 - b(x)). \end{aligned} \tag{2.3}$$

Integrating (1.1) from x_1 to x, we obtain

$$a(x_1)\left(v'(x_1)\right)^m \geq \delta^m \int_{x_1}^{x} c(y)A^m(\varphi(y))(1 - b(\varphi(y)))^m dy$$

which for $x \to \infty$ contradicts with (2.1). So that $\lim_{x\to\infty} \frac{v(x)}{A(x)} = 0$. The proof is completed. □

Since $A(x)$ is increasing, there exists $\lambda \geq 1$ such that

$$\frac{A(x)}{A(\varphi(x))} \geq \lambda. \tag{2.4}$$

Theorem 2.2. *Let* (2.1) *hold and there exist a positive constant n such that*

$$\frac{1}{m} A^m(\varphi(x))a^{1/m}(x)A(x)c(x)(1 - b(\varphi(x)))^m \geq n \quad \text{for } x \geq x_0. \tag{2.5}$$

If $u(x)$ is a positive solution of (1.1), *then*

(i) $\frac{v(x)}{A^{1-n}(x)}$ *is decreasing for* $x \geq x_1$,

(ii) $\frac{v(x)}{A^{n_0}(x)}$ *is increasing for* $x \geq x_1$, *where* $n_0 = n^{1/m}\lambda^n$.

Proof. Assume that $u(x)$ is a positive solution of (1.1). It is important to note that condition (iii) of Lemma 2.1 implies

$$\lim_{x\to\infty} a^{1/m}(x)v'(x) = 0. \tag{2.6}$$

Therefore, an integration of (1.1) yields

$$a^{1/m}(x)v'(x) = \left(\int_x^\infty C(y, u(\varphi(y)))dy\right)^{1/m}. \tag{2.7}$$

It is easy to see that

$$\left[\left(a^{1/m}(x)v'(x)\right)^m\right]' = m\left(a^{1/m}(x)v'(x)\right)^{m-1}\left(a^{1/m}(x)v'(x)\right)'. \tag{2.8}$$

Setting into (1.1), we have

$$\left(a^{1/m}(x)v'(x)\right)' + \frac{1}{m}\left(a^{1/m}(x)v'(x)\right)^{1-m} C(x, u(\varphi(x))) = 0. \tag{2.9}$$

Then $w(x) = a^{1/m}(x)v'(x)$ is positive decreasing and satisfies

$$w'(x) + \frac{1}{m}w^{1-m}(x)C(x, u(\varphi(x))) = 0,$$

and from (H2) we get

$$w'(x) + \frac{1}{m}w^{1-m}(x)c(x)u^m(\varphi(x)) \le 0. \tag{2.10}$$

On the other hand, (2.2) and (2.3) implies

$$v(x) \ge a^{1/m}(x)v'(x)A(x)$$
$$v(x) \ge w(x)A(x)$$
$$\frac{u(x)}{(1-b(x))} \ge w(x)A(x)$$

and so

$$u(\varphi(x)) \ge w(x)A(\varphi(x))(1-b(\varphi(x))). \tag{2.11}$$

Substituting the last inequality into (2.10), we get

$$w'(x) + \frac{1}{m}c(x)A^m(\varphi(x))(1-b(\varphi(x)))^m w(x) \le 0$$

and

$$w'(x) + \frac{n}{A(x)a^{1/m}(x)}w(x) \le 0$$

which implies

$$-w'(x)A(x) \ge \frac{n}{a^{1/m}(x)}w(x) = nv'(x).$$

We can skip the proof of the theorem as it is similar to [4, Theorem 2.3]. □

The previous results did not distinguish between the cases $m < 1$ and $m \ge 1$. However, in order to provide oscillatory criteria for (1.1), we need to consider these cases separately.

3 Oscillatory results for $m \geq 1$

To simplify our notation let us denote

$$\kappa = \frac{(1-n)^{1-m}\lambda^{n(m-1)}}{m}.$$

Now we will provide new comparison principles that significantly simplify the examination of neutral delay differential equations.

Theorem 3.1. *Let $m \geq 1$, and (2.1), (2.5) hold. Then (1.1) is oscillatory provided that*

$$\left(a^{1/m}(x)v'(x)\right)' + \kappa A^{m-1}(\varphi(x))c(x)(1-b(\varphi(x)))^m v(\varphi(x)) = 0 \tag{3.1}$$

is oscillatory.

Proof. Suppose the opposite of the desired result, namely that $u(x)$ is a positive solution of (1.1). Since $\frac{v(x)}{A^{1-n}(x)}$ is decreasing, we have the inequality

$$v(x) \geq \frac{a^{1/m}(x)v'(x)}{(1-n)}A(x), \tag{3.2}$$

which for $m \geq 1$ yields

$$v^{1-m}(x) \leq \frac{\left(a^{1/m}(x)v'(x)\right)^{1-m}}{(1-n)^{1-m}}A^{1-m}(x).$$

Hence

$$\left(a^{1/m}(x)v'(x)\right)^{1-m} \geq (1-n)^{1-m}\frac{v^{1-m}(x)}{A^{1-m}(x)}. \tag{3.3}$$

Using again the monotonic property of $\frac{v(x)}{A^{1-n}(x)}$, we get

$$v^{1-m}(x) \geq \frac{v^{1-m}(\varphi(x))}{A^{(1-m)(1-n)}(\varphi(x))}A^{(1-m)(1-n)}(x). \tag{3.4}$$

Substituting (3.4) from (3.3), we have in view of (2.4) that

$$\begin{aligned}\left(a^{1/m}(x)v'(x)\right)^{1-m} &\geq (1-n)^{1-m}\frac{A^{(1-m)(1-n)}(x)}{A^{(1-m)(1-n)}(\varphi(x))}\frac{v^{1-m}(\varphi(x))}{A^{1-m}(x)} \\ &\geq (1-n)^{1-m}\lambda^{n(m-1)}\frac{v^{1-m}(\varphi(x))}{A^{1-m}(\varphi(x))}.\end{aligned} \tag{3.5}$$

By combining (2.10) and (3.5), we can derive that $v(x)$ satisfies the linear differential inequality

$$\left(a^{1/m}(x)v'(x)\right)' + \kappa c(x)A^{m-1}(\varphi(x))(1-b(\varphi(x)))^m v(\varphi(x)) \leq 0. \tag{3.6}$$

Conversely, by [12, Corollary 1], it is ensured that the corresponding differential equation (3.1) possesses a positive solution, which contradicts the assumption made earlier, thus completing the proof. □

We will utilize the outcomes of the previous theorem to establish novel oscillatory criteria.

Theorem 3.2. *Let* $m \geq 1$, *and* (2.1), (2.5) *hold. If*

$$\limsup_{x\to\infty}\Big\{A^{n-1}(\varphi(x))\int_{x_1}^{\varphi(x)} c(y)A(y)A^{m-n}(\varphi(y))(1-b(\varphi(y)))^m dy$$
$$+A^n(\varphi(x))\int_{\varphi(x)}^{x} c(y)A^{m-n}(\varphi(y))(1-b(\varphi(y)))^m dy$$
$$+A^{1-n_0}(\varphi(x))\int_{x}^{\infty} c(y)A^{m+n_0-1}(\varphi(y))(1-b(\varphi(y)))^m dy\Big\} > \frac{1}{\kappa}$$

then (1.1) *is oscillatory.*

Proof. The proof of the theorem closely follows the approach taken in [4, Theorem 4.1], making it unnecessary to provide a full proof here. □

For practical applications, we often need criteria that are easier to verify. The next theorem provides such conditions:

Theorem 3.3. *Let* $m \geq 1$, *and* (2.1), (2.5) *hold. If*

$$\liminf_{x\to\infty}\int_{\varphi(x)}^{x} c(y)A^m(\varphi(y))(1-b(\varphi(y)))^m dy > \frac{1}{\kappa e}$$

then (1.1) *is oscillatory.*

Proof. Suppose, for the sake of contradiction, that (1.1) has a positive solution $u(x)$. By Theorem 3.2, equation (3.1) is nonoscillatory, and we can assume that it has an eventually positive solution $v(x)$. Then, we can define $w(x) = a^{1/m}(x)v'(x)$, which is positive and decreasing. Therefore, we have

$$v(x) = v(x_1) + \int_{x_1}^{x} v'(y)dy \geq \int_{x_1}^{x} \frac{a^{1/m}(y)v'(y)}{a^{1/m}(y)}dy$$
$$\geq a^{1/m}(x)v'(x)\int_{x_1}^{x}\frac{dy}{a^{1/m}(y)} = w(x)A(x).$$

By substituting the expression for $v(x)$ into (3.1), we obtain a first-order differential inequality of the form

$$w'(x) + \kappa A^m(\varphi(x))c(x)(1-b(\varphi(x)))^m w(\varphi(x)) \leq 0 \tag{3.7}$$

with positive solution $w(x)$. According to [13, Theorem 2.1.1], (3.7) has no positive solution if

$$\liminf_{x\to\infty}\int_{\varphi(x)}^{x} \kappa c(y)A^m(\varphi(y))(1-b(\varphi(y)))^m dy > \frac{1}{e}.$$

This contradicts our assumption that (1.1) has a positive solution. Hence, the proof is complete. □

4 Oscillatory results for $m \in (0,1)$

We now turn our attention to the case where $0 < m < 1$. This case requires different techniques due to the different nature of the nonlinearity. The following results parallel those of the previous section, but with important modifications to account for the changed parameter range.

To simplify the notation, we define

$$\omega = \frac{n^{\frac{1-m}{m}} \lambda^{1-m}}{m(1-n_0)^{\frac{1-m}{m}}}.$$

Theorem 4.1. *Let* $0 < m < 1$*, and* (2.1), (2.5) *hold. Then* (1.1) *is oscillatory provided that*

$$\left(a^{1/m}(x)v'(x)\right)' + \omega A^{m-1}(x)c(x)(1-b(\varphi(x)))^m v(\varphi(x)) = 0 \tag{4.1}$$

is oscillatory.

Proof. Assume the contrary that $u(x)$ is a positive solution of (1.1). Differentiating (2.7) with respect to x leads to the equation

$$\left(a^{1/m}(x)v'(x)\right)' + \frac{1}{m}\left(\int_x^\infty C(y,u(\varphi(y)))dy\right)^{\frac{1-m}{m}} C(x,u(\varphi(x))) = 0.$$

Employing the fact that $\frac{v(x)}{A^{n_0}(x)}$ is an increasing function, we can rewrite the previous inequality as follows:

$$\left(a^{1/m}(x)v'(x)\right)' + \frac{1}{m}\frac{v^{1-m}(\varphi(x))}{A^{(1-m)n_0}(\varphi(x))}\left(\int_x^\infty \frac{c(y)b_0^m(x)dy}{A^{-mn_0}(\varphi(y))}\right)^{\frac{1-m}{m}} \frac{c(x)v^m(\varphi(x))}{b_0^{-m}(x)} \le 0,$$

where $b_0(x) = 1 - b(\varphi(x))$. Consequently, we can conclude that $u(x)$ must satisfy the linear differential inequality

$$\left(a^{1/m}(x)v'(x)\right)' + \frac{1}{m}\left(\int_x^\infty \frac{c(y)b_0^m(y)dy}{A^{-mn_0}(\varphi(y))}\right)^{\frac{1-m}{m}} \frac{b_0^m(x)c(x)v(\varphi(x))}{A^{(1-m)n_0}(\varphi(x))} \le 0. \tag{4.2}$$

Additionally, by utilizing (2.4) and (2.5), we can derive

$$\begin{aligned}
\int_x^\infty \frac{c(y)(1-b(\varphi(y)))^m}{A^{-mn_0}(\varphi(y))}dy &\ge \int_x^\infty mn\frac{A^{mn_0}(\varphi(y))}{A^m(\varphi(y))a^{1/m}(y)A(y)}dy \\
&\ge mn\lambda^{m(n_0-1)}\int_x^\infty \frac{A^{m(n_0-1)-1}(y)}{a^{1/m}(y)}dy \\
&\ge \frac{n\lambda^{m(n_0-1)}}{1-n_0}A^{m(n_0-1)}(x).
\end{aligned}$$

Substituting (4.2), we get

$$\left(a^{1/m}(x)v'(x)\right)' + \frac{n^{\frac{1-m}{m}}\lambda^{1-m}}{m(1-n_0)^{\frac{1-m}{m}}}\frac{A^{(1-m)(n_0-1)}(x)}{A^{(1-m)n_0}(\varphi(x))}(1-b(\varphi(x)))^m c(x)v(\varphi(x)) \le 0.$$

By applying [12, Corollary 1], we can conclude that the corresponding differential equation (4.1) also has a positive solution. This contradicts the assumption that (4.1) has no positive solutions, and therefore the proof is complete. □

Theorem 4.2. *Let* $0 < m < 1$*, and* (2.1), (2.5) *hold. If*

$$\limsup_{x\to\infty}\Bigg\{A^{n-1}(\varphi(x))\int_{x_1}^{\varphi(x)} A^{1-n}(\varphi(y))A^{m}(y)c(y)(1-b(\varphi(y)))^{m}dy$$
$$+A^{n}(\varphi(y))\int_{\varphi(x)}^{x} A^{1-n}(\varphi(y))A^{m-1}(y)c(y)(1-b(\varphi(y)))^{m}dy$$
$$+A^{1-n_0}(\varphi(y))\int_{x}^{\infty} A^{n_0}(\varphi(y))A^{m-1}(y)c(y)(1-b(\varphi(y)))^{m}dy\Bigg\}$$
$$>\frac{1}{\omega},$$

then (1.1) *is oscillatory.*

Proof. The proof of the theorem closely follows the approach used in [4, Theorem 4.2]. □

Theorem 4.3. *Let* $0 < m < 1$*, and* (2.1), (2.5) *hold. If*

$$\liminf_{x\to\infty}\int_{\varphi(x)}^{x} A(\varphi(y))A^{m-1}(y)c(y)(1-b(\varphi(y)))^{m}dy>\frac{1}{\omega e}$$

then (1.1) *is oscillatory.*

Proof. Since the proof of the theorem closely resembles that of [4, Theorem 4.3], it may be omitted. □

5 Example

Example 5.1. Consider a second-order neutral delay differential equation in the form of a general Euler differential equation

$$\left(x^{k}\left(v'(x)\right)^{m}\right)'+\frac{c_0}{x^{(m-k+1)}}u^{m}(\varphi_0 x)=0, \tag{5.1}$$

where $v(x)=u(x)+b_0u(\psi(x))$ with $c_0>0$, $\varphi_0\in(0,1)$ and $0\le b_0<1$.

Then

$$A(x)=\frac{mx^{\left(1-\frac{k}{m}\right)}}{m-k},\quad A(\varphi_0 x)=\frac{m(\varphi_0 x)^{\left(1-\frac{k}{m}\right)}}{m-k},\quad \lambda=\frac{A(x)}{A(\varphi_0 x)}=\varphi_0^{\left(\frac{k}{m}-1\right)},$$

$$n=\frac{c_0\varphi_0^{(m-k)}}{m}\left(\frac{m}{m-k}\right)^{(m+1)}(1-b_0)^{m},\quad n_0=\frac{c_0^{\left(\frac{1}{m}\right)}m\varphi_0^{\left(1-\frac{k}{m}\right)(1-n)}}{(m-k)^{\left(1+\frac{1}{m}\right)}}(1-b_0)$$

$$\kappa=\frac{(1-n)^{(1-m)}\varphi_0^{n(1-m)\left(1-\frac{k}{m}\right)}}{m}$$

and

$$\omega=\frac{n^{\frac{1}{m}-1}\varphi_0^{(1-m)\left(\frac{k}{m}-1\right)}}{m(1-n_0)^{\frac{1}{m}-1}}.$$

Equation (5.1) with $m \geq 1$ is oscillatory provided that,

$$mn \left\{ \frac{\varphi_0^{n\left(\frac{k}{m}-1\right)}}{1-n} + \frac{\varphi_0^{n\left(\frac{k}{m}-1\right)} - 1}{n} + \frac{1}{1-n_0} \right\} > \frac{1}{\kappa}$$

by Theorem 3.2, or

$$n(m-k) \ln \frac{1}{\varphi_0} > \frac{1}{e\kappa}$$

by Theorem 3.3.

Equation (5.1) with $0 < m < 1$ is oscillatory provided that,

$$mn\varphi_0^{(1-m)\left(1-\frac{k}{m}\right)} \left\{ \frac{\varphi_0^{n\left(\frac{k}{m}-1\right)}}{1-n} + \frac{\varphi_0^{n\left(\frac{k}{m}-1\right)} - 1}{n} + \frac{1}{1-n_0} \right\} > \frac{1}{\omega}$$

by Theorem 4.2 or

$$n(m-k)\varphi_0^{(1-m)\left(1-\frac{k}{m}\right)} \ln \frac{1}{\varphi_0} > \frac{1}{e\omega}$$

by Theorem 4.3.

6 Conclusion

In this paper, we have established new oscillation criteria for second-order neutral delay differential equations by utilizing the monotonic properties of non-oscillatory solutions. Our approach generalizes previous results by considering a neutral term and a more general nonlinearity. The presented theorems provide effective tools for determining the oscillatory behavior of these equations.

It is worth noting that when we set $b(x) = 0$ (eliminating the neutral term) and $C(x, u(x)) = c(x)u^m(x)$, our results reduce to those obtained by Baculíková and Džurina [4]. This confirms the consistency of our approach while demonstrating its broader applicability to more complex equations.

Declaration of generative AI and AI-assisted technologies in the writing process

During the preparation of this work, the authors used ChatGPT in order to improve the readability and language of the article. After using this tool, the authors reviewed and edited the content as needed and take full responsibility for the content of the published article.

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