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Nonlinear integral inequalities involving Ψ-Hilfer fractional integrals and iterated fractional integrals, with applications to Ψ-Caputo fractional differential equations

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Abstract. In this paper, modifications of the well-known desingularization method proposed by the first author are used to derive nonlinear versions of the Henry–Gronwall inequality for integral inequalities with the Ψ-Hilfer fractional integral, as well as nonlinear integral inequalities with iterated Ψ-Hilfer fractional integrals. The results are applied to obtain bounds for solutions of initial value problems for nonlinear Ψ-Caputo fractional differential equations, and sufficient conditions for the non-existence of blowing-up solutions.

Keywords: Ψ-Hilfer fractional integral inequality, generalized Henry–Gronwall inequality, iterated Ψ-Hilfer fractional integrals, Ψ-Caputo fractional derivative.

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1 Introduction and preliminaries

The monograph [5] by Dan Henry is devoted to the study of semilinear parabolic differential equations, where these equations are studied in the framework of so-called mild solutions of abstract evolution equations. The mild solutions are given as solutions of some Volterra integral equations with weakly singular kernels. One of the basic tools used there is a linear integral inequality with a weakly singular kernel, for which a generalization of the Gronwall inequality is proved (see [5, Lemma 7.1.1]). This result is being frequently quoted as Henry's lemma. Currently, this method is also applied in the theory of fractional differential equations. The first nonlinear version of the Henry's integral inequality is proved in the paper [12] by a

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new method, often referred to as the desingularization method. In this paper, we apply it to the following nonlinear integral inequalities:

$$u(t) \leq a(t) + b(t) \int_{t_0}^{t} [\Psi(t) - \Psi(s)]^{\alpha-1} \Psi'(s) F(s) \omega(u(s)) ds, \quad t \in [t_0, T]; \tag{1.1}$$

$$u(t) \leq a(t) + \sum_{i=1}^{n} b_i(t) \int_{t_0}^{t} [\Psi(t) - \Psi(s)]^{\alpha_i-1} \Psi'(s) F_i(s) \omega_i(u(s)) ds, \quad t \in [t_0, T]; \tag{1.2}$$

$$\begin{aligned} u(t) \leq a(t) &+ b_1(t) \int_{t_0}^{t} [\Psi(t) - \Psi(s)]^{\alpha-1} \Psi'(s) F_1(s) \omega_1(u(s)) ds \\ &+ b_2(t) \int_{t_0}^{t} [\Psi(t) - \Psi(s)]^{\alpha-1} \Psi'(s) c(s) \\ &\times \int_{t_0}^{s} [\Psi(s) - \Psi(\sigma)]^{\beta-1} \Psi'(\sigma) F_2(\sigma) \omega_2(u(\sigma)) d\sigma \, ds, \quad t \in [t_0, T]; \end{aligned} \tag{1.3}$$

$$\begin{aligned} u(t) \leq a(t) &+ b_1(t) \int_{t_0}^{t} [\Psi(t) - \Psi(s)]^{\alpha-1} \Psi'(s) F_1(s) \omega_1(u(s)) ds \\ &+ b_2(t) \int_{t_0}^{t} [\Psi(t) - \Psi(s)]^{\alpha-1} \Psi'(s) F_2(s) \\ &\times \omega_2 \left(b_3(s) \int_{t_0}^{s} [\Psi(s) - \Psi(\sigma)]^{\beta-1} \Psi'(\sigma) F_3(\sigma) \omega_3(u(\sigma)) d\sigma \right) ds, \quad t \in [t_0, T]. \end{aligned} \tag{1.4}$$

For the readers' convenience, we recall the following definitions from the tempered Ψ-fractional calculus.

Definition 1.1 ([4,16])**.** Let $\alpha > 0$, $\lambda \geq 0$, x be a continuous function on $[t_0, T]$, and $\Psi \in C^1[t_0, T]$ satisfy $\Psi'(t) > 0$ for all $t \in [t_0, T]$. Tempered Ψ-Hilfer fractional integral of order $\alpha > 0$ is defined by

$$I_{t_0}^{\alpha,\lambda,\Psi} x(t) = \frac{1}{\Gamma(\alpha)} \int_{t_0}^{t} (\Psi(t) - \Psi(s))^{\alpha-1} \, \mathrm{e}^{-\lambda(\Psi(t)-\Psi(s))} \, \Psi'(s) \, x(s) \, ds$$

for $t \in [t_0, T]$, where $\Gamma(\cdot)$ is the Euler gamma function.

Definition 1.2 ([19])**.** Let $\Psi \in C^n[t_0, T]$, $n \in \mathbb{N}$ be such that $\Psi'(t) > 0$ for all $t \in [t_0, T]$. For $n - 1 < \alpha < n$, $\lambda \geq 0$, tempered Ψ-Caputo fractional derivative of order α of $x \in C^{n-1}[t_0, T]$ is defined by

$${}^{C}D_{t_0}^{\alpha,\lambda,\Psi} x(t) = {}^{RL}D_{t_0}^{\alpha,\lambda,\Psi} \left[x(t) - \mathrm{e}^{-\lambda\Psi(t)} \sum_{k=0}^{n-1} \frac{x_{\lambda,\Psi}^{[k]}(t_0)}{k!} (\Psi(t) - \Psi(t_0))^k \right]$$

for $t \in [t_0, T]$ whenever the right side makes sense, where

$$x_{\lambda,\Psi}^{[n]}(t) = \left(\frac{1}{\Psi'(t)} \frac{d}{dt} \right)^n \left(\mathrm{e}^{\lambda\Psi(t)} \, x(t) \right)$$

and

$$\begin{aligned} {}^{RL}D_{t_0}^{\alpha,\lambda,\Psi} x(t) &= \mathrm{e}^{-\lambda\Psi(t)} \left(I_{t_0}^{n-\alpha,\lambda,\Psi} x(t) \right)_{\lambda,\Psi}^{[n]} \\ &= \frac{\mathrm{e}^{-\lambda\Psi(t)}}{\Gamma(n-\alpha)} \left(\frac{1}{\Psi'(t)} \frac{d}{dt} \right)^n \int_{t_0}^{t} (\Psi(t) - \Psi(s))^{n-\alpha-1} \Psi'(s) \, \mathrm{e}^{\lambda\Psi(s)} \, x(s) \, ds \end{aligned}$$

is the tempered Ψ-Riemann–Liouville fractional derivative of order α.

It is worth to note that if $x \in C^n[t_0,T]$, then ${}^C D_{t_0}^{\alpha,\lambda,\Psi} x(t)$ may be expressed as

$$ {}^C D_{t_0}^{\alpha,\lambda,\Psi} x(t) = \frac{\mathrm{e}^{-\lambda\Psi(t)}}{\Gamma(n-\alpha)} \int_{t_0}^{t} (\Psi(t)-\Psi(s))^{n-\alpha-1}\Psi'(s) x_{\lambda,\Psi}^{[n]}(s)\, ds $$

(see [16,19]). In this case, since $x_{\lambda,\Psi}^{[n]} \in C[t_0,T]$, Lemma 1 of [20] yields that ${}^C D_{t_0}^{\alpha,\lambda,\Psi} x \in C[t_0,T]$.

We refer the reader to papers [4,16,18–20] for the properties of these operators. The above notions generalize Ψ-Hilfer fractional integral from [7], $I_{t_0}^{\alpha,\Psi} = I_{t_0}^{\alpha,0,\Psi}$, and Ψ-Caputo fractional derivative from [1], ${}^C D_{t_0}^{\alpha,\Psi} = {}^C D_{t_0}^{\alpha,0,\Psi}$, which we use in this paper.

In the paper [18], the following integral inequality with the tempered Ψ-Hilfer fractional integral is studied:

$$ u(t) \le a(t) + b(t)\int_{t_0}^{t} [\Psi(t)-\Psi(s)]^{\alpha-1}\, \mathrm{e}^{-\lambda[\Psi(t)-\Psi(s)]}\, \Psi'(s)F(s)\omega(u(s))ds,\ t \in [t_0,T). $$

The obtained Henry–Gronwall inequality is proved under the assumption $\lambda > 0$. The method of desingularization applied in the proof is not suitable for the inequality with $\lambda = 0$, i.e., for inequality (1.1). In this case, we need to apply a modification of the desingularization method proposed in the paper [12].

In [21, Theorem 3], inequality (1.1) is studied for $\omega(u) = u$, $F(t) \equiv 1$ and an estimation of u is obtained in a form of a series. Here, we investigate nonlinear analogues (see Corollaries 2.5, 2.13). Similarly, Corollaries 2.7, 2.15 generalize e.g. [11, Theorem 1.4], where inequality (1.2) is investigated with $\Psi(t) = t$, $F_i(t) \equiv 1$ and $\omega_i(u) = u$ for each $i = 1,2,\dots,n$.

Inequality (1.1) with $\alpha \in (0,1)$, $\Psi(t) = t$, $a(t) = at^{-\mu}$, and $b(t) = bt^{-\nu}$ with various values of parameters μ, ν is studied in [6,23–25]. In [9], inequality (1.1) is investigated with $\Psi(t) = t$, $\alpha \in (0,1)$, but instead of non-decreasing ω, $\widehat{\omega}$ is considered such that $\widehat{\omega}^p(u) \le \omega(u^p)$ for $p > 1/\alpha$ and some non-decreasing ω. In [22], (1.1) is studied with $\Psi(t) = t$, $\alpha \in (0,1)$, $F(t) = t^{-\gamma}$, $\gamma \ge 0$ and $1-\alpha+\gamma < 1$; also the case $a(t) = at^{-\mu}$ is considered.

Other integral inequalities useful for fractional calculus can be found e.g. in [2,19].

Throughout the whole paper, for generality, we consider inequality (1.1) as well as others on an interval $[t_0,T]$ for some $t_0 < T \le \infty$, where the case $T = \infty$ is understood as $[t_0,\infty)$. We use the notation $\mathbb{N}$ for the set of all positive integers, and $\|\cdot\|$ for any vector norm in $\mathbb{R}^N$, $N \in \mathbb{N}$.

The paper is structured as follows. In the next section, we derive the main results of the paper – the nonlinear integral inequalities for weakly singular integrals. We apply two types of desingularization, namely with inserting an auxiliary exponential function and without it. The inequalities are applied in Section 3 to obtain boundedness results for solutions of nonlinear fractional differential equations, and sufficient conditions for the non-existence of solutions blowing-up in a finite time. Lastly, Section 4 concludes the results and sketches directions of future research.

2 Nonlinear integral inequalities and their corollaries

This section is devoted to integral inequalities (1.1)–(1.4). It is split in two subsections according to a method used to desingularize the fractional integral, i.e., to get rid of the singular kernel.

For the case of brevity, we introduce the notation for classes of functions,

$$\mathfrak{P}^k_M := \{\varphi \in C^k(M) \mid \varphi'(t) > 0 \text{ for all } t \in M\},$$
$$\mathfrak{O} := \{\varphi \in C([0,\infty),[0,\infty)) \mid \varphi(t) > 0\ \forall t > 0,\ \varphi \text{ is non-decreasing}\}.$$

In our proofs, we rely on a generalization of the well-known Bihari's inequality [3, 10] that was proved in [9]. Next, we provide its version sufficient for our purposes.

Theorem 2.1. *Let a, b, F be non-negative, continuous functions on $[t_0, T]$. Let $\omega \in \mathfrak{O}$ and u be a non-negative continuous function on $[t_0, T]$, satisfying*

$$u(t) \le a(t) + b(t) \int_{t_0}^{t} F(s)\omega(u(s))\, ds, \quad t \in [t_0, T].$$

Then

$$u(t) \le \Omega^{-1}\left(\Omega(A(t)) + B(t)\int_{t_0}^{t} F(s)\, ds\right), \quad t \in [t_0, t_1],$$

where $t_1 \in (t_0, T]$ is such that

$$\Omega(A(t_1)) + B(t_1)\int_{t_0}^{t_1} F(s)\, ds \in \operatorname{Dom}\Omega^{-1},$$

$\Omega(v) = \int_{v_0}^{v} \frac{1}{\omega(\sigma)} d\sigma$ for $v \ge v_0 > 0$, Ω^{-1} is the inverse of Ω, $A(t) = \sup_{s\in[t_0,t]} a(s)$, $B(t) = \sup_{s\in[t_0,t]} b(s)$.

It should be noted that in [9, Theorem 1] the latter result is proved for positive functions a and b, and with $F \in L^1[t_0, T]$ positive on a set of positive measure. Then, in [9, Remark 1], a is allowed to be non-negative if $\omega(t) > 0$ for all $t > 0$. Since we use continuous functions, if there does not exist a set of positive measure where F is positive, then $F \equiv 0$ and the statement holds trivially. Next, if $b(t_0) > 0$ then $B(t_0) > 0$ which is sufficient for the proof to work. If $b(t) = 0$ for all $t \in [t_0, t_0 + h]$ for some $h > 0$ then, again, the statement clearly holds for all those t. Moreover, if $\int_0 \frac{1}{\omega(\sigma)} d\sigma$ converges, one can set $v_0 = 0$. Therefore, we allow $v_0 \ge 0$ in this paper.

2.1 Parametric desingularization

Here we apply a parameter-dependent modification of the desingularization method from [12], where an exponential function was added and the Hölder inequality was applied to tame the weakly singular integral kernel. So, we are able to prove the following nonlinear Henry–Gronwall type inequality.

Theorem 2.2. *Let $\alpha \in (0,1)$, $\alpha q > 1$, $\mu > 0$, a, b and F be non-negative, continuous functions on $[t_0, T]$, and $\Psi \in \mathfrak{P}^1_{[t_0,T]}$. Let $\omega \in \mathfrak{O}$ and u be a non-negative continuous function on $[t_0, T]$, satisfying inequality (1.1). Then*

$$u(t) \le \left[\Omega^{-1}\left(\Omega(A(t)) + B(t)\int_{t_0}^{t} \Psi'(s)F(s)^q\, \mathrm{e}^{-q\mu\Psi(s)}\, ds\right)\right]^{1/q}, \quad t \in [t_0, t_1], \tag{2.1}$$

where $t_1 \in (t_0, T]$ is such that

$$\Omega(A(t_1)) + B(t_1)\int_{t_0}^{t_1} \Psi'(s)F(s)^q\, \mathrm{e}^{-q\mu\Psi(s)}\, ds \in \operatorname{Dom}\Omega^{-1}, \tag{2.2}$$

$\Omega(v) = \int_{v_0}^{v} \frac{1}{[\omega(\sigma^{1/q})]^q} d\sigma$ *for* $v \geq v_0 \geq 0$, Ω^{-1} *is the inverse of* Ω, *and*

$$A(t) = 2^{q-1} \sup_{s\in[t_0,t]} a(s)^q, \quad B(t) = 2^{q-1} M_{p,\mu,\alpha}^q \sup_{s\in[t_0,t]} \left\{ b(s)^q \, e^{q\mu\Psi(s)} \right\},$$
$$M_{p,\mu,\alpha} = \left(\frac{\Gamma(p(\alpha-1)+1)}{(p\mu)^{p(\alpha-1)+1}} \right)^{1/p}, \quad p = \frac{q}{q-1}. \tag{2.3}$$

In the proof of this theorem, we will need the following generalization of the inequality from [12, page 354].

Lemma 2.3. *Let* $\alpha \in (0,1)$, $p > 0$, $p(\alpha-1)+1 > 0$, $\mu > 0$, *and* $\Psi(t) \in \mathfrak{P}^1_{[t_0,T]}$. *Then*

$$\int_{t_0}^{t} [\Psi(t) - \Psi(s)]^{p(\alpha-1)} \, e^{p\mu\Psi(s)} \, \Psi'(s) ds \leq M_{p,\mu,\alpha}^p \, e^{p\mu\Psi(t)}, \quad t \in [t_0, T], \tag{2.4}$$

where $M_{p,\mu,\alpha}$ *is given by the formula* (2.3).

Proof. Using the substitution $\Psi(t) - \Psi(s) = \sigma$ one can rewrite the left-hand side of inequality (2.4) as

$$\int_{t_0}^{t} [\Psi(t) - \Psi(s)]^{p(\alpha-1)} \, e^{p\mu\Psi(s)} \, \Psi'(s) ds = \int_{0}^{\Psi(t)-\Psi(t_0)} \sigma^{p(\alpha-1)} \, e^{p\mu(\Psi(t)-\sigma)} \, d\sigma$$
$$= e^{p\mu\Psi(t)} \int_{t_0}^{\Psi(t)-\Psi(t_0)} \sigma^{p(\alpha-1)} \, e^{-p\mu\sigma} \, d\sigma \leq e^{p\mu\Psi(t)} \int_{0}^{\infty} \sigma^{p(\alpha-1)} \, e^{-p\mu\sigma} \, d\sigma = M_{p,\mu,\alpha}^p \, e^{p\mu\Psi(t)} . \square$$

Now, we present the proof of Theorem 2.2.

Proof of Theorem 2.2. First notice that $q > \frac{1}{\alpha} > 1$. Thus, $p(\alpha-1)+1 > 0$ and $M_{p,\mu,\alpha}$ is well defined. Since $\frac{1}{p} + \frac{1}{q} = 1$, we can write $\Psi'(t) = \Psi'(t)^{1/p}\Psi'(t)^{1/q}$. Using the Hölder inequality and Lemma 2.3, we derive

$$\begin{aligned}
&\int_{t_0}^{t} [\Psi(t) - \Psi(s)]^{\alpha-1} \Psi'(s) F(s) \omega(u(s)) ds \\
&\quad = \int_{t_0}^{t} [\Psi(t) - \Psi(s)]^{\alpha-1} \, e^{\mu\Psi(s)} \, \Psi'(s)^{1/p} F(s) \, e^{-\mu\Psi(s)} \, \Psi'(s)^{1/q} \omega(u(s)) ds \\
&\quad \leq \left(\int_{t_0}^{t} [\Psi(t) - \Psi(s)]^{p(\alpha-1)} \, e^{p\mu\Psi(s)} \, \Psi'(s) ds \right)^{1/p} \left(\int_{t_0}^{t} F(s)^q \, e^{-q\mu\Psi(s)} \, \Psi'(s) \omega(u(s))^q ds \right)^{1/q} \\
&\quad \leq M_{p,\mu,\alpha} \, e^{\mu\Psi(t)} \left(\int_{t_0}^{t} F(s)^q \, e^{-q\mu\Psi(s)} \, \Psi'(s) \omega(u(s))^q ds \right)^{1/q} .
\end{aligned} \tag{2.5}$$

Hence, this inequality along with (1.1) yield

$$u(t) \leq a(t) + b(t) M_{p,\mu,\alpha} \, e^{\mu\Psi(t)} \left(\int_{t_0}^{t} F(s)^q \, e^{-q\mu\Psi(s)} \, \Psi'(s) \omega(u(s))^q ds \right)^{1/q} . \tag{2.6}$$

Using the inequality $(b_1 + b_2)^q \leq 2^{q-1}(b_1^q + b_2^q)$ valid for any $b_1, b_2 \geq 0$, we obtain

$$\begin{aligned}
v(t) &\leq 2^{q-1} a(t)^q + 2^{q-1} b(t)^q M_{p,\mu,\alpha}^q \, e^{q\mu\Psi(t)} \int_{t_0}^{t} F(s)^q \, e^{-q\mu\Psi(s)} \, \Psi'(s) \omega(u(s))^q ds \\
&= 2^{q-1} a(t)^q + 2^{q-1} b(t)^q M_{p,\mu,\alpha}^q \, e^{q\mu\Psi(t)} \int_{t_0}^{t} F(s)^q \, e^{-q\mu\Psi(s)} \, \Psi'(s) \omega(v(s)^{1/q})^q ds,
\end{aligned}$$

where $v(t) = u(t)^q$. Consequently, Theorem 2.1 implies

$$v(t) \le \Omega^{-1}\left(\Omega(A(t)) + B(t)\int_{t_0}^{t} F(s)^q \,\mathrm{e}^{-q\mu\Psi(s)}\,\Psi'(s)ds\right),$$

where A, B, Ω and Ω^{-1} are as in the theorem. This proves inequality (2.1). □

Remark 2.4. Since functions A, B, Ω are non-decreasing and $\Psi'(t)F(t)^q\,\mathrm{e}^{-q\mu\Psi(t)} \ge 0$ for all $t \in [t_0, T]$, condition (2.2) is equivalent to

$$\Omega(A(t)) + B(t)\int_{t_0}^{t} \Psi'(s)F(s)^q\,\mathrm{e}^{-q\mu\Psi(s)}\,ds \in \operatorname{Dom}\Omega^{-1}$$

for all $t \in [t_0, t_1]$.

Some of corollaries of Theorem 2.2 are explicitly stated below.

Corollary 2.5. *Let the assumptions of Theorem 2.2 be fulfilled with $\omega(u) = u$, i.e., let u satisfy*

$$u(t) \le a(t) + b(t)\int_{t_0}^{t} [\Psi(t) - \Psi(s)]^{\alpha-1}\Psi'(s)F(s)u(s)ds, \quad t \in [t_0, T].$$

Then

$$u(t) \le A(t)^{1/q}\exp\left\{\frac{B(t)}{q}\int_{t_0}^{t} F(s)^q\,\mathrm{e}^{-q\mu\Psi(s)}\,\Psi'(s)ds\right\}, \quad t \in [t_0, T],$$

where functions A, B are given by (2.3).

Proof. Putting the particular form of ω in the statement of Theorem 2.2 immediately yields $\Omega(v) = \ln\frac{v}{v_0}$ for $v \ge v_0 > 0$, $\Omega^{-1}(v) = v_0\,\mathrm{e}^{v}$ for $v \ge 0$, $v_0 > 0$, and the corollary follows from (2.1). □

Corollary 2.6. *Let the assumptions of Theorem 2.2 be fulfilled with $\omega(u) = u^m$ for $m > 0$, $m \ne 1$, i.e., let u satisfy*

$$u(t) \le a(t) + b(t)\int_{t_0}^{t} [\Psi(t) - \Psi(s)]^{\alpha-1}\Psi'(s)F(s)u(s)^m ds, \quad t \in [t_0, T].$$

Then, the following assertions hold:

1. *if $0 < m < 1$, then*

$$u(t) \le \left(A(t)^{1-m} + (1-m)B(t)\int_{t_0}^{t} F(s)^q\,\mathrm{e}^{-q\mu\Psi(s)}\,\Psi'(s)ds\right)^{\frac{1}{q(1-m)}}, \quad t \in [t_0, T];$$

2. *if $m > 1$, then*

$$u(t) \le \frac{A(t)^{1/q}}{\left(1 - (m-1)A(t)^{m-1}B(t)\int_{t_0}^{t} F(s)^q\,\mathrm{e}^{-q\mu\Psi(s)}\,\Psi'(s)ds\right)^{\frac{1}{q(m-1)}}}, \quad t \in [t_0, t_1]$$

for any $t_1 \in (t_0, T]$ such that

$$\int_{t_0}^{t_1} F(s)^q\,\mathrm{e}^{-q\mu\Psi(s)}\,\Psi'(s)ds < \frac{1}{(m-1)A(t_1)^{m-1}B(t_1)};$$

where functions A, B are given by (2.3).

Proof. The statement can be obtained analogously to Corollary 2.5 with $\omega(u) = u^m$. Note that here we use the fact that functions A, B are non-decreasing. $\square$

Theorem 2.2 can also be easily extended to the case of multiple Ψ-Hilfer fractional integrals:

Corollary 2.7. *Let $n \in \mathbb{N}$, $\alpha_i \in (0,1)$ for $i = 1,2,\dots,n$, $q \min_{i=1,2,\dots,n} \alpha_i > 1$, $\mu > 0$, a, b_i and F_i be non-negative, continuous functions on $[t_0, T]$ for each $i = 1,2,\dots,n$, and $\Psi \in \mathfrak{P}^1_{[t_0,T]}$. Let $\omega_i \in \mathfrak{O}$ for each $i = 1,2,\dots,n$ and u be a non-negative continuous function on $[t_0, T]$, satisfying inequality (1.2). Then,*

$$u(t) \leq \left[\Omega^{-1}\left(\Omega(A(t)) + B(t)\int_{t_0}^{t} \Psi'(s)F(s)^q \,\mathrm{e}^{-q\mu\Psi(s)}\, ds\right)\right]^{1/q}, \quad t \in [t_0, t_1],$$

where $t_1 \in (t_0, T]$ is such that

$$\Omega(A(t_1)) + B(t_1)\int_{t_0}^{t_1} \Psi'(s)F(s)^q \,\mathrm{e}^{-q\mu\Psi(s)}\, ds \in \operatorname{Dom}\Omega^{-1},$$

$\Omega(v) = \int_{v_0}^{v} \frac{1}{[\omega(\sigma^{1/q})]^q} d\sigma$ for $v \geq v_0 \geq 0$, Ω^{-1} is the inverse of Ω, and

$$F = \max_{i=1,2,\dots,n} F_i, \quad \omega = \max_{i=1,2,\dots,n} \omega_i,$$

$$A(t) = 2^{q-1} \sup_{s\in[t_0,t]} a(s)^q, \quad B(t) = 2^{q-1} \sup_{s\in[t_0,t]} \left(\sum_{i=1}^{n} M_{p,\mu,\alpha_i} b_i(s)\,\mathrm{e}^{\mu\Psi(s)}\right)^q,$$

$$M_{p,\mu,\alpha} = \left(\frac{\Gamma(p(\alpha-1)+1)}{(p\mu)^{p(\alpha-1)+1}}\right)^{1/p}, \quad p = \frac{q}{q-1}.$$

Proof. The right-hand side of inequality (1.2) can be estimated from above by

$$\begin{aligned} a(t) &+ \sum_{i=1}^{n} b_i(t)\int_{t_0}^{t} [\Psi(t) - \Psi(s)]^{\alpha_i - 1}\Psi'(s)F(s)\omega(u(s))ds \\ &\leq a(t) + \sum_{i=1}^{n} b_i(t)M_{p,\mu,\alpha_i}\,\mathrm{e}^{\mu\Psi(t)}\left(\int_{t_0}^{t} F(s)^q\,\mathrm{e}^{-q\mu\Psi(s)}\,\Psi'(s)\omega(u(s))^q ds\right)^{1/q} \end{aligned}$$

as in (2.5). Then the proof is finished as the proof of Theorem 2.2. $\square$

Next, we give an estimation of function u satisfying inequality (1.3).

Theorem 2.8. *Let $\alpha, \beta \in (0,1)$, $\alpha + \beta < 1$, $\alpha q > 1$, $\mu > 0$, a, b_1, b_2, c, F_1 and F_2 be non-negative, continuous functions on $[t_0, T]$, and $\Psi \in \mathfrak{P}^1_{[t_0,T]}$. Let $\omega_1, \omega_2 \in \mathfrak{O}$ and u be a non-negative continuous function on $[t_0, T]$, satisfying inequality (1.3). Then,*

$$u(t) \leq \left[\Omega^{-1}\left(\Omega(A(t)) + B(t)\int_{t_0}^{t} \Psi'(s)G(s)^q \,\mathrm{e}^{-q\mu\Psi(s)}\, ds\right)\right]^{1/q}, \quad t \in [t_0, t_1],$$

where $t_1 \in (t_0, T]$ is such that

$$\Omega(A(t_1)) + B(t_1)\int_{t_0}^{t_1} \Psi'(s)G(s)^q \,\mathrm{e}^{-q\mu\Psi(s)}\, ds \in \operatorname{Dom}\Omega^{-1},$$

$\Omega(v) = \int_{v_0}^{v} \frac{1}{[\omega(\sigma^{1/q})]^q} d\sigma$ *for* $v \geq v_0 \geq 0$, Ω^{-1} *is the inverse of* Ω, *and*

$$G = \max\{F_1, F_2\}, \quad \omega = \max\{\omega_1, \omega_2\}, \quad A(t) = 2^{q-1} \sup_{s\in[t_0,t]} a(s)^q,$$

$$B(t) = 2^{q-1} \sup_{s\in[t_0,t]} \left\{ \mathrm{e}^{q\mu\Psi(s)} \left(b_1(s) M_{p,\mu,\alpha} + B(\alpha,\beta) b_2(s) M_{p,\mu,\alpha+\beta} \sup_{\sigma\in[t_0,s]} c(\sigma) \right)^q \right\}, \tag{2.7}$$

$$M_{p,\mu,\alpha} = \left(\frac{\Gamma(p(\alpha-1)+1)}{(p\mu)^{p(\alpha-1)+1}} \right)^{1/p}, \quad p = \frac{q}{q-1},$$

$B(\cdot,\cdot)$ *is the Euler beta function.*

Proof. By a trivial estimation using functions G and ω, and applying (2.5), we get

$$\begin{aligned}
u(t) &\leq a(t) + b_1(t) \int_{t_0}^{t} [\Psi(t) - \Psi(s)]^{\alpha-1} \Psi'(s) G(s) \omega(u(s)) ds \\
&\quad + b_2(t) \int_{t_0}^{t} [\Psi(t) - \Psi(s)]^{\alpha-1} \Psi'(s) c(s) \\
&\quad \times \int_{t_0}^{s} [\Psi(s) - \Psi(\sigma)]^{\beta-1} \Psi'(\sigma) G(\sigma) \omega(u(\sigma)) d\sigma\, ds \\
&\leq a(t) + b_1(t) M_{p,\mu,\alpha}\, \mathrm{e}^{\mu\Psi(t)} \left(\int_{t_0}^{t} G(s)^q\, \mathrm{e}^{-q\mu\Psi(s)}\, \Psi'(s) \omega(u(s))^q ds \right)^{1/q} \\
&\quad + b_2(t) \sup_{s\in[t_0,t]} c(s) \int_{t_0}^{t} [\Psi(t) - \Psi(s)]^{\alpha-1} \Psi'(s) \\
&\quad \times \int_{t_0}^{s} [\Psi(s) - \Psi(\sigma)]^{\beta-1} \Psi'(\sigma) G(\sigma) \omega(u(\sigma)) d\sigma\, ds.
\end{aligned}$$

Changing the order of integration and by the change of variable $\Psi(t) - \Psi(s) = \rho(\Psi(t) - \Psi(\sigma))$, we derive

$$\begin{aligned}
&\int_{t_0}^{t} [\Psi(t) - \Psi(s)]^{\alpha-1} \Psi'(s) \int_{t_0}^{s} [\Psi(s) - \Psi(\sigma)]^{\beta-1} \Psi'(\sigma) G(\sigma) \omega(u(\sigma)) d\sigma\, ds \\
&\quad = \int_{t_0}^{t} \Psi'(\sigma) G(\sigma) \omega(u(\sigma)) \left(\int_{\sigma}^{t} [\Psi(t) - \Psi(s)]^{\alpha-1} \Psi'(s) [\Psi(s) - \Psi(\sigma)]^{\beta-1} ds \right) d\sigma \\
&\quad = \int_{t_0}^{t} \Psi'(\sigma) G(\sigma) \omega(u(\sigma)) [\Psi(t) - \Psi(\sigma)]^{\alpha+\beta-1} \left(\int_{0}^{1} \rho^{\alpha-1} (1-\rho)^{\beta-1} d\rho \right) d\sigma \\
&\quad = B(\alpha,\beta) \int_{t_0}^{t} [\Psi(t) - \Psi(\sigma)]^{\alpha+\beta-1} \Psi'(\sigma) G(\sigma) \omega(u(\sigma)) d\sigma.
\end{aligned}$$

Now, we apply an estimation analogous to (2.5) to estimate the latter right-hand side by

$$B(\alpha,\beta) M_{p,\mu,\alpha+\beta}\, \mathrm{e}^{\mu\Psi(t)} \left(\int_{t_0}^{t} G(s)^q\, \mathrm{e}^{-q\mu\Psi(s)}\, \Psi'(s) \omega(u(s))^q ds \right)^{1/q}.$$

So, we arrive at

$$\begin{aligned}
u(t) &\leq a(t) + \left(b_1(t) M_{p,\mu,\alpha} + B(\alpha,\beta) b_2(t) M_{p,\mu,\alpha+\beta} \sup_{s\in[t_0,t]} c(s) \right) \mathrm{e}^{\mu\Psi(t)} \\
&\quad \times \left(\int_{t_0}^{t} G(s)^q\, \mathrm{e}^{-q\mu\Psi(s)}\, \Psi'(s) \omega(u(s))^q ds \right)^{1/q},
\end{aligned}$$

which is of the form of (2.6). Hence, the proof is finished by the same way as the proof of Theorem 2.2. □

Finally, we investigate inequality (1.4).

Theorem 2.9. *Let $\alpha, \beta \in (0,1)$, $q \min\{\alpha, \beta\} > 1$, $\mu > 0$, a, b_1, b_2, b_3, F_1, F_2, F_3 be non-negative, continuous functions on $[t_0, T]$, and $\Psi \in \mathfrak{P}^1_{[t_0,T]}$. Let $\omega_1, \omega_2, \omega_3 \in \mathfrak{O}$ and u be a non-negative continuous function on $[t_0, T]$, satisfying inequality (1.4). Then,*

$$u(t) \le \left[\Omega^{-1}\left(\Omega(A(t)) + B(t)\int_{t_0}^{t} G(s)^q\, e^{-q\mu\Psi(s)}\,\Psi'(s)ds \right.\right.$$
$$\left.\left. + C(t)\int_{t_0}^{t} F_2(s)^q\, e^{-q\mu\Psi(s)}\,\Psi'(s)ds\right)\right]^{1/q}, \quad t \in [t_0, t_1],$$

where $t_1 \in (t_0, T]$ is such that

$$\Omega(A(t_1)) + B(t_1)\int_{t_0}^{t_1} G(s)^q\, e^{-q\mu\Psi(s)}\,\Psi'(s)ds + C(t_1)\int_{t_0}^{t_1} F_2(s)^q\, e^{-q\mu\Psi(s)}\,\Psi'(s)ds \in \operatorname{Dom}\Omega^{-1},$$

$\Omega(v) = \int_{v_0}^{v} \frac{1}{[\omega(\sigma^{1/q})]^q} d\sigma$ *for $v \ge v_0 \ge 0$, Ω^{-1} is the inverse of Ω, and*

$$\begin{gathered} G = \max\{F_1, F_3\}, \quad \omega = \max\{\omega_1, \omega_2, \omega_3\}, \quad b = \max\{b_1, b_3\}, \\ A(t) = 3^{q-1}\sup_{s\in[t_0,t]} a(s)^q, \quad B(t) = 3^{q-1}M^q_{p,\mu}\sup_{s\in[t_0,t]}\left\{b(s)^q\, e^{q\mu\Psi(s)}\right\}, \\ C(t) = 3^{q-1}M^q_{p,\mu,\alpha}\sup_{s\in[t_0,t]}\left\{b_2(s)^q\, e^{q\mu\Psi(s)}\right\}, \quad M_{p,\mu} = \max\{M_{p,\mu,\alpha}, M_{p,\mu,\beta}\}, \\ M_{p,\mu,\alpha} = \left(\frac{\Gamma(p(\alpha-1)+1)}{(p\mu)^{p(\alpha-1)+1}}\right)^{1/p}, \quad p = \frac{q}{q-1}. \end{gathered} \tag{2.8}$$

To prove Theorem 2.9, we need the following auxiliary result.

Lemma 2.10. *Let a, b, c, f, g be non-negative, continuous functions on $[t_0, T]$. Let $\omega \in \mathfrak{O}$ and u be a non-negative continuous function on $[t_0, T]$, satisfying*

$$\begin{aligned} u(t) \le{}& a(t) + b(t)\int_{t_0}^{t} f(s)\omega(u(s))ds \\ &+ c(t)\int_{t_0}^{t} g(s)\omega\left(b(s)\int_{t_0}^{s} f(\sigma)\omega(u(\sigma))d\sigma\right)ds, \quad t \in [t_0, T]. \end{aligned} \tag{2.9}$$

Then,

$$u(t) \le \Omega^{-1}\left(\Omega(\bar{a}(t)) + \bar{b}(t)\int_{t_0}^{t} f(s)ds + \bar{c}(t)\int_{t_0}^{t} g(s)ds\right), \quad t \in [t_0, t_1], \tag{2.10}$$

where $t_1 \in (t_0, T]$ is such that

$$\Omega(\bar{a}(t_1)) + \bar{b}(t_1)\int_{t_0}^{t_1} f(s)ds + \bar{c}(t_1)\int_{t_0}^{t_1} g(s)ds \in \operatorname{Dom}\Omega^{-1},$$

$\Omega(v) = \int_{v_0}^{v} \frac{d\sigma}{\omega(\sigma)}$ *for $v \ge v_0 \ge 0$, Ω^{-1} is the inverse of Ω, and*

$$\bar{a}(t) = \sup_{s\in[t_0,t]} a(s), \quad \bar{b}(t) = \sup_{s\in[t_0,t]} b(s), \quad \bar{c}(t) = \sup_{s\in[t_0,t]} c(s).$$

Proof. Let $\widetilde{T} \in (t_0, T]$ be sufficiently small (this is specified later) and fixed. From (2.9), for any $t \in [t_0, \widetilde{T}]$,

$$\begin{aligned} u(t) &\le \bar{a}(\widetilde{T}) + \bar{b}(\widetilde{T}) \int_{t_0}^{t} f(s)\omega(u(s))ds \\ &\quad + \bar{c}(\widetilde{T}) \int_{t_0}^{t} g(s)\omega\left(\bar{b}(\widetilde{T}) \int_{t_0}^{s} f(\sigma)\omega(u(\sigma))d\sigma\right) ds =: z(t). \end{aligned}$$

It is easy to see that function z is non-negative, non-decreasing, C^1, and $z(t_0) = \bar{a}(\widetilde{T})$. Differentiating, we obtain

$$\begin{aligned} z'(t) &\le \bar{b}(\widetilde{T})f(t)\omega(u(t)) + \bar{c}(\widetilde{T})g(t)\omega\left(\bar{b}(\widetilde{T}) \int_{t_0}^{t} f(s)\omega(u(s))ds\right) \\ &\le \bar{b}(\widetilde{T})f(t)\omega(z(t)) + \bar{c}(\widetilde{T})g(t)\omega(z(t)) \end{aligned}$$

for all $t \in [t_0, \widetilde{T}]$. Integrating over $[t_0, t]$ yields

$$z(t) \le \bar{a}(\widetilde{T}) + \int_{t_0}^{t} \left(\bar{b}(\widetilde{T})f(s) + \bar{c}(\widetilde{T})g(s)\right) \omega(z(s))ds, \quad t \in [t_0, \widetilde{T}].$$

Now, applying Theorem 2.1 results in

$$u(t) \le z(t) \le \Omega^{-1}\left(\Omega(\bar{a}(\widetilde{T})) + \int_{t_0}^{t} \bar{b}(\widetilde{T})f(s) + \bar{c}(\widetilde{T})g(s)ds\right)$$

for all $t \in [t_0, \widetilde{T}]$. In particular, this estimation holds for $t = \widetilde{T}$ and we obtain (2.10). Now, one can see that $\widetilde{T}$ at the beginning of the proof had to be smaller than or equal to t_1. □

Proof of Theorem 2.9. Using notation (2.8) and estimation (2.5), inequality (1.4) gives

$$\begin{aligned} u(t) &\le a(t) + b(t) \int_{t_0}^{t} [\Psi(t) - \Psi(s)]^{\alpha-1}\Psi'(s)G(s)\omega(u(s))ds \\ &\quad + b_2(t) \int_{t_0}^{t} [\Psi(t) - \Psi(s)]^{\alpha-1}\Psi'(s)F_2(s) \\ &\quad \times \omega\left(b(s) \int_{t_0}^{s} [\Psi(s) - \Psi(\sigma)]^{\beta-1}\Psi'(\sigma)G(\sigma)\omega(u(\sigma))d\sigma\right) ds \\ &\le a(t) + b(t)M_{p,\mu}\, e^{\mu\Psi(t)} \left(\int_{t_0}^{t} G(s)^q\, e^{-q\mu\Psi(s)}\, \Psi'(s)\omega(u(s))^q ds\right)^{1/q} \\ &\quad + b_2(t)M_{p,\mu,\alpha}\, e^{\mu\Psi(t)} \left(\int_{t_0}^{t} F_2(s)^q\, e^{-q\mu\Psi(s)}\, \Psi'(s)\right. \\ &\quad \left. \times\, \omega\left(b(s)M_{p,\mu}\, e^{\mu\Psi(s)} \left(\int_{t_0}^{s} G(\sigma)^q\, e^{-q\mu\Psi(\sigma)}\, \Psi'(\sigma)\omega(u(\sigma))^q d\sigma\right)^{1/q}\right)^q ds\right)^{1/q}. \end{aligned}$$

Using the inequality $(a_1 + a_2 + a_3)^q \le 3^{q-1}(a_1^q + a_2^q + a_3^q)$ valid for any $a_1, a_2, a_3 \ge 0$, and the estimation

$$b(s)M_{p,\mu}\, e^{\mu\Psi(s)} \le \left(3^{q-1}b(s)^q M_{p,\mu}^q\, e^{q\mu\Psi(s)}\right)^{1/q},$$

we obtain

$$
\begin{aligned}
v(t) \le\ & 3^{q-1}a(t)^q + 3^{q-1}b(t)^q M_{p,\mu}^q \,\mathrm{e}^{q\mu\Psi(t)} \int_{t_0}^t G(s)^q \,\mathrm{e}^{-q\mu\Psi(s)}\, \Psi'(s)\omega(v(s)^{1/q})^q ds \\
& + 3^{q-1}b_2(t)^q M_{p,\mu,\alpha}^q \,\mathrm{e}^{q\mu\Psi(t)} \int_{t_0}^t F_2(s)^q \,\mathrm{e}^{-q\mu\Psi(s)}\, \Psi'(s) \\
& \times \omega\left(\left(3^{q-1}b(s)^q M_{p,\mu}^q \,\mathrm{e}^{q\mu\Psi(s)} \int_{t_0}^s G(\sigma)^q \,\mathrm{e}^{-q\mu\Psi(\sigma)}\, \Psi'(\sigma)\omega(v(\sigma)^{1/q})^q d\sigma\right)^{1/q}\right)^q ds
\end{aligned}
$$

for all $t \in [t_0, T]$, where $v(t) = u(t)^q$. This is of the form of (2.9), and the statement follows from Lemma 2.10. □

2.2 Non-exponential desingularization

In this part, we apply the desingularization method without inserting $\mathrm{e}^{\mu\Psi(s)}\,\mathrm{e}^{-\mu\Psi(s)}$ prior to estimating by the Hölder inequality. This can be considered as a limit case $\mu = 0$. Notice that here, Lemma 2.3 can not be used, since $\lim_{\mu\to 0^+} M_{p,\mu,\alpha} = \infty$. The results are analogous to those of the previous section, therefore the proofs are rather sketchy and only the main differences are mentioned. It is shown in the next section, that both approaches, $\mu > 0$ and $\mu = 0$, might give useful and different estimations.

First, we provide an alternative to Theorem 2.2.

Theorem 2.11. *Let all the assumptions of Theorem 2.2 be satisfied. Then,*

$$
u(t) \le \left[\Omega^{-1}\left(\Omega(A(t)) + \mathcal{B}(t)\int_{t_0}^t \Psi'(s)F(s)^q ds\right)\right]^{1/q}, \quad t \in [t_0, t_1], \tag{2.11}
$$

where $t_1 \in (t_0, T]$ is such that

$$
\Omega(A(t_1)) + \mathcal{B}(t_1)\int_{t_0}^{t_1} \Psi'(s)F(s)^q ds \in \operatorname{Dom}\Omega^{-1},
$$

$\Omega(v) = \int_{v_0}^v \frac{1}{[\omega(\sigma^{1/q})]^q} d\sigma$ *for $v \ge v_0 \ge 0$, Ω^{-1} is the inverse of Ω, and*

$$
\begin{aligned}
A(t) &= 2^{q-1}\sup_{s\in[t_0,t]} a(s)^q, \quad \mathcal{B}(t) = 2^{q-1}\sup_{s\in[t_0,t]} \{b(s)^q \mathcal{M}_{p,\alpha}(s)^q\}, \\
\mathcal{M}_{p,\alpha}(t) &= \left(\frac{[\Psi(t)-\Psi(t_0)]^{p(\alpha-1)+1}}{p(\alpha-1)+1}\right)^{1/p}, \quad p = \frac{q}{q-1}.
\end{aligned} \tag{2.12}
$$

Proof. The idea of the proof follows the proof of Theorem 2.2. This time, instead of Lemma 2.3, we employ the change of the dummy variable $\Psi(t) - \Psi(s) = \sigma$ to get the equality

$$
\int_{t_0}^t [\Psi(t)-\Psi(s)]^{p(\alpha-1)}\Psi'(s)ds = \int_0^{\Psi(t)-\Psi(t_0)} \sigma^{p(\alpha-1)} d\sigma = \mathcal{M}_{p,\alpha}(t)^p.
$$

So, instead of (2.5), we have

$$
\begin{aligned}
& \int_{t_0}^t [\Psi(t)-\Psi(s)]^{\alpha-1}\Psi'(s)F(s)\omega(u(s))ds \\
& \le \left(\int_{t_0}^t [\Psi(t)-\Psi(s)]^{p(\alpha-1)}\Psi'(s)ds\right)^{1/p}\left(\int_{t_0}^t F(s)^q\Psi'(s)\omega(u(s))^q ds\right)^{1/q} \\
& = \mathcal{M}_{p,\alpha}(t)\left(\int_{t_0}^t F(s)^q\Psi'(s)\omega(u(s))^q ds\right)^{1/q}.
\end{aligned} \tag{2.13}
$$

Therefore, instead of (2.6), we have

$$u(t) \le a(t) + b(t)\mathcal{M}_{p,\alpha}(t)\left(\int_{t_0}^{t} F(s)^q \Psi'(s)\omega(u(s))^q ds\right)^{1/q}.$$

The proof is then completed as the proof of Theorem 2.2. □

Remark 2.12. Let us assume that $\alpha \ge 1$. Then the integral in (1.1) does not contain a singular kernel, so we have

$$u(t) \le a(t) + b(t)(\Psi(t) - \Psi(t_0))^{\alpha-1} \int_{t_0}^{t} \Psi'(s)F(s)\omega(u(s))ds, \quad t \in [t_0, T]$$

and Theorem 2.1 yields

$$u(t) \le \Omega^{-1}\left(\Omega(A(t)) + B(t)\int_{t_0}^{t} \Psi'(s)F(s)ds\right), \quad t \in [t_0, t_1], \tag{2.14}$$

where $t_1 \in (t_0, T]$ is such that

$$\Omega(A(t_1)) + B(t_1)\int_{t_0}^{t_1} \Psi'(s)F(s)ds \in \operatorname{Dom}\Omega^{-1},$$

$\Omega(v) = \int_{v_0}^{v} \frac{d\sigma}{\omega(\sigma)}$ for $v \ge v_0 \ge 0$, Ω^{-1} is the inverse of Ω, and

$$A(t) = \sup_{s\in[t_0,t]} a(s), \quad B(t) = \sup_{s\in[t_0,t]} \{b(s)(\Psi(s) - \Psi(t_0))^{\alpha-1}\}.$$

On the other side, the proof of Theorem 2.11 now works for any $q > 1$, i.e., estimation (2.11) holds for any $q > 1$. Taking the limit $q \to 1^+$ (i.e., $p \to \infty$) in (2.11) gives exactly (2.14), since

$$\begin{aligned}\lim_{q\to 1^+} \mathcal{M}_{p,\alpha}(s)^q &= \lim_{q\to 1^+} \frac{(\Psi(s) - \Psi(t_0))^{q\alpha-1}}{(p(\alpha-1)+1)^{q-1}} \\ &= \lim_{q\to 1^+} \frac{(\Psi(s) - \Psi(t_0))^{q\alpha-1}}{\left(\frac{q\alpha-1}{q-1}\right)^{q-1}} = (\Psi(s) - \Psi(t_0))^{\alpha-1}\end{aligned}$$

if $\alpha > 1$, and

$$\lim_{q\to 1^+} \mathcal{M}_{p,\alpha}(s)^q = (\Psi(s) - \Psi(t_0))^{q-1} = 1$$

if $\alpha = 1$.

Analogues of Corollaries 2.5, 2.6, and 2.7 are stated below.

Corollary 2.13. *If all the assumptions of Corollary 2.5 are satisfied, then*

$$u(t) \le A(t)^{1/q} \exp\left\{\frac{\mathcal{B}(t)}{q}\int_{t_0}^{t} F(s)^q \Psi'(s)ds\right\}, \quad t \in [t_0, T],$$

where functions A, $\mathcal{B}$ are given by (2.12).

Proof. The statement follows from Theorem 2.11 by setting $\omega(u) = u$ (cf. the proof of Corollary 2.5). □

Corollary 2.14. *If all the assumptions of Corollary 2.6 are satisfied, then the following assertions hold:*

1. *if* $0 < m < 1$, *then*

$$u(t) \leq \left(A(t)^{1-m} + (1-m)\mathcal{B}(t)\int_{t_0}^{t} F(s)^q \Psi'(s)ds\right)^{\frac{1}{q(1-m)}}, \quad t \in [t_0, T];$$

2. *if* $m > 1$, *then*

$$u(t) \leq \frac{A(t)^{1/q}}{\left(1-(m-1)A(t)^{m-1}\mathcal{B}(t)\int_{t_0}^{t} F(s)^q \Psi'(s)ds\right)^{\frac{1}{q(m-1)}}}, \quad t \in [t_0, t_1]$$

for any $t_1 \in (t_0, T]$ *such that*

$$\int_{t_0}^{t_1} F(s)^q \Psi'(s)ds < \frac{1}{(m-1)A(t_1)^{m-1}\mathcal{B}(t_1)};$$

where functions A, $\mathcal{B}$ *are given by* (2.12).

Proof. The statement follows from Theorem 2.11 by setting $\omega(u) = u^m$ (cf. the proof of Corollary 2.6). □

Corollary 2.15. *If all the assumptions of Corollary 2.7 are satisfied, then*

$$u(t) \leq \left[\Omega^{-1}\left(\Omega(A(t)) + \mathcal{B}(t)\int_{t_0}^{t} \Psi'(s)F(s)^q ds\right)\right]^{1/q}, \quad t \in [t_0, t_1],$$

where $t_1 \in (t_0, T]$ *is such that*

$$\Omega(A(t_1)) + \mathcal{B}(t_1)\int_{t_0}^{t_1} \Psi'(s)F(s)^q ds \in \operatorname{Dom}\Omega^{-1},$$

$\Omega(v) = \int_{v_0}^{v} \frac{1}{[\omega(\sigma^{1/q})]^q} d\sigma$ *for* $v \geq v_0 \geq 0$, Ω^{-1} *is the inverse of* Ω, *and*

$$F = \max_{i=1,2,\dots,n} F_i, \quad \omega = \max_{i=1,2,\dots,n} \omega_i,$$

$$A(t) = 2^{q-1}\sup_{s\in[t_0,t]} a(s)^q, \quad \mathcal{B}(t) = 2^{q-1}\sup_{s\in[t_0,t]}\left(\sum_{i=1}^{n} b_i(s)\mathcal{M}_{p,\alpha_i}(s)\right)^q,$$

$$\mathcal{M}_{p,\alpha}(t) = \left(\frac{[\Psi(t)-\Psi(t_0)]^{p(\alpha-1)+1}}{p(\alpha-1)+1}\right)^{1/p}, \quad p = \frac{q}{q-1}.$$

Proof. Analogously to the proof of Corollary 2.7, but using estimation (2.13) instead of (2.5), the right-hand side of (1.2) can be estimated by

$$a(t) + \sum_{i=1}^{n} b_i(t)\mathcal{M}_{p,\alpha_i}(t)\left(\int_{t_0}^{t} F(s)^q \Psi'(s)\omega(u(s))^q ds\right)^{1/q}.$$

Then the proof is completed as the proof of Theorem 2.2, by taking the q-th power and using Theorem 2.1. □

Next, we give a result on inequality (1.3).

Theorem 2.16. *Let all the assumptions of Theorem 2.8 be satisfied. Then,*

$$u(t) \le \left[\Omega^{-1}\left(\Omega(A(t)) + \mathcal{B}(t)\int_{t_0}^{t}\Psi'(s)G(s)^q ds\right)\right]^{1/q}, \quad t \in [t_0, t_1],$$

where $t_1 \in (t_0, T]$ is such that

$$\Omega(A(t_1)) + \mathcal{B}(t_1)\int_{t_0}^{t_1}\Psi'(s)G(s)^q ds \in \operatorname{Dom}\Omega^{-1},$$

$\Omega(v) = \int_{v_0}^{v}\frac{1}{[\omega(\sigma^{1/q})]^q}d\sigma$ *for $v \ge v_0 \ge 0$, Ω^{-1} is the inverse of Ω, and*

$$\begin{aligned} G &= \max\{F_1, F_2\}, \quad \omega = \max\{\omega_1, \omega_2\}, \quad A(t) = 2^{q-1}\sup_{s\in[t_0,t]} a(s)^q, \\ \mathcal{B}(t) &= 2^{q-1}\sup_{s\in[t_0,t]}\left\{b_1(s)\mathcal{M}_{p,\alpha}(s) + B(\alpha,\beta)b_2(s)\mathcal{M}_{p,\alpha+\beta}(s)\sup_{\sigma\in[t_0,s]} c(\sigma)\right\}^q, \\ \mathcal{M}_{p,\alpha}(t) &= \left(\frac{[\Psi(t)-\Psi(t_0)]^{p(\alpha-1)+1}}{p(\alpha-1)+1}\right)^{1/p}, \quad p = \frac{q}{q-1}. \end{aligned} \tag{2.15}$$

Proof. We follow the proof of Theorem 2.8. Instead of (2.5), we apply estimation (2.13) to arrive at

$$\begin{aligned} u(t) &\le a(t) + b_1(t)\mathcal{M}_{p,\alpha}(t)\left(\int_{t_0}^{t} G(s)^q\Psi'(s)\omega(u(s))^q ds\right)^{1/q} \\ &\quad + B(\alpha,\beta)b_2(t)\sup_{s\in[t_0,t]} c(s)\int_{t_0}^{t}[\Psi(t)-\Psi(\sigma)]^{\alpha+\beta-1}\Psi'(\sigma)G(\sigma)\omega(u(\sigma))d\sigma \\ &\le a(t) + b_1(t)\mathcal{M}_{p,\alpha}(t)\left(\int_{t_0}^{t} G(s)^q\Psi'(s)\omega(u(s))^q ds\right)^{1/q} \\ &\quad + B(\alpha,\beta)b_2(t)\mathcal{M}_{p,\alpha+\beta}(t)\sup_{s\in[t_0,t]} c(s)\left(\int_{t_0}^{t} G(s)^q\Psi'(s)\omega(u(s))^q ds\right)^{1/q}. \end{aligned}$$

The proof is completed as the proof of Theorem 2.2. □

A result on inequality (1.4) follows.

Theorem 2.17. *Let all the assumptions of Theorem 2.9 be satisfied. Then,*

$$\begin{aligned} u(t) \le \Bigg[\Omega^{-1}\Bigg(&\Omega(A(t)) + \mathcal{B}(t)\int_{t_0}^{t} G(s)^q\Psi'(s)ds \\ &+ C(t)\int_{t_0}^{t} F_2(s)^q\Psi'(s)ds\Bigg)\Bigg]^{1/q}, \quad t \in [t_0, t_1], \end{aligned}$$

where $t_1 \in (t_0, T]$ is such that

$$\Omega(A(t_1)) + \mathcal{B}(t_1)\int_{t_0}^{t_1} G(s)^q\Psi'(s)ds + C(t_1)\int_{t_0}^{t_1} F_2(s)^q\Psi'(s)ds \in \operatorname{Dom}\Omega^{-1},$$

$\Omega(v) = \int_{v_0}^{v} \frac{1}{[\omega(\sigma^{1/q})]^q} d\sigma$ *for* $v \geq v_0 \geq 0$, Ω^{-1} *is the inverse of* Ω, *and*

$$\begin{gathered} G = \max\{F_1, F_3\}, \quad \omega = \max\{\omega_1, \omega_2, \omega_3\}, \quad b = \max\{b_1, b_3\}, \\ A(t) = 3^{q-1} \sup_{s\in[t_0,t]} a(s)^q, \quad \mathcal{B}(t) = 3^{q-1} \sup_{s\in[t_0,t]} \left\{ b(s)\mathcal{M}_p(s) \right\}^q, \\ C(t) = 3^{q-1} \sup_{s\in[t_0,t]} \left\{ b_2(s)\mathcal{M}_{p,\alpha}(s) \right\}^q, \quad p = \frac{q}{q-1}, \\ \mathcal{M}_p = \max\{\mathcal{M}_{p,\alpha}, \mathcal{M}_{p,\beta}\}, \quad \mathcal{M}_{p,\alpha}(t) = \left(\frac{[\Psi(t) - \Psi(t_0)]^{p(\alpha-1)+1}}{p(\alpha-1)+1} \right)^{1/p}. \end{gathered} \tag{2.16}$$

Proof. We follow the proof of Theorem 2.9, but now, we apply estimation (2.13) instead of (2.5) to obtain

$$\begin{aligned} u(t) &\leq a(t) + b(t)\mathcal{M}_p(t) \left(\int_{t_0}^{t} G(s)^q \Psi'(s) \omega(u(s))^q ds \right)^{1/q} \\ &\quad + b_2(t)\mathcal{M}_{p,\alpha}(t) \left(\int_{t_0}^{t} F_2(s)^q \Psi'(s) \right. \\ &\quad \left. \times \omega \left(b(s)\mathcal{M}_p(s) \left(\int_{t_0}^{s} G(\sigma)^q \Psi'(\sigma) \omega(u(\sigma))^q d\sigma \right)^{1/q} \right)^q ds \right)^{1/q}. \end{aligned}$$

Taking the q-th power of this inequality and an application of Lemma 2.10 completes the proof, similarly to the proof of Theorem 2.9. □

3 Applications to fractional differential equations with Ψ-Caputo derivative

In this section, we apply our integral inequalities to various classes of initial value problems (IVPs) for fractional differential equations with Ψ-Caputo derivative, to derive results on the boundedness of the solutions and on the non-existence of blowing-up solutions. We note that in our results, the existence of solutions is assumed. For results on the existence we refer the reader to [1,20].

Nevertheless, we start with the following theoretical example to show that, in different values of t, each one of Theorems 2.2 and 2.11 might give a better estimation than the other.

Example 3.1. Let a and u be non-negative continuous functions on $[0, T]$ for some $0 < T \leq \infty$, $\omega \in \mathfrak{O}$, and u satisfy

$$u(t) \leq a(t) + \mathrm{e}^{-t} \int_0^t (t-s)^{-\frac{1}{4}} \omega(u(s)) ds, \quad t \in [0, T]. \tag{3.1}$$

In this case, $\alpha = 3/4$, $b(t) = \mathrm{e}^{-t}$, $\Psi(t) = t$, $F(t) \equiv 1$, and $t_0 = 0$. Let us take $q = 2$, implying $p = 2$, and $\mu > 0$. Let Ω, Ω^{-1}, A be as in Theorem 2.2. By (2.3), $M_{p,\mu,\alpha} = \sqrt[4]{\frac{\pi}{2\mu}}$ and

$$B(t) = 2\sqrt{\frac{\pi}{2\mu}} \sup_{s\in[0,t]} \left\{ \mathrm{e}^{-2s} \, \mathrm{e}^{2\mu s} \right\} = \begin{cases} \sqrt{\frac{2\pi}{\mu}}, & 0 < \mu \leq 1, \\ \sqrt{\frac{2\pi}{\mu}} \, \mathrm{e}^{2(\mu-1)t}, & \mu > 1. \end{cases}$$

Hence, Theorem 2.2 gives

$$\Omega(u(t)^2) - \Omega(A(t)) \le B(t) \int_0^t e^{-2\mu s}\, ds = \begin{cases} \sqrt{\frac{\pi}{2\mu}} \frac{1-e^{-2\mu t}}{\mu}, & 0 < \mu \le 1, \\ \sqrt{\frac{\pi}{2\mu}} \frac{(e^{2\mu t}-1)\, e^{-2t}}{\mu}, & \mu > 1 \end{cases} \tag{3.2}$$

for all $t \in [0, t_1]$, where $t_1 \in (0, T]$ is such that

$$\Omega(A(t_1)) + \sqrt{\frac{\pi}{2\mu}} \frac{1 - e^{-2\mu t_1}}{\mu} \in \operatorname{Dom} \Omega^{-1}$$

if $0 < \mu \le 1$, and

$$\Omega(A(t_1)) + \sqrt{\frac{\pi}{2\mu}} \frac{(e^{2\mu t_1} - 1)\, e^{-2t_1}}{\mu} \in \operatorname{Dom} \Omega^{-1}$$

if $\mu > 1$.

On the other side, in the notation of (2.12), $\mathcal{M}_{p,\alpha}(t) = \sqrt[4]{4t}$ and

$$\mathcal{B}(t) = 2 \sup_{s\in[0,t]} \{2\sqrt{s}\, e^{-2s}\} = 4 \sup_{s\in[0,t]} \{\sqrt{s}\, e^{-2s}\} = \begin{cases} 4\sqrt{t}\, e^{-2t}, & 0 \le t \le \frac{1}{4}, \\ \frac{2}{\sqrt{e}}, & t \ge \frac{1}{4}. \end{cases}$$

Therefore, by Theorem 2.11,

$$\Omega(u(t)^2) - \Omega(A(t)) \le \mathcal{B}(t) \int_0^t 1 ds = \begin{cases} 4t^{\frac{3}{2}}\, e^{-2t}, & 0 \le t \le \frac{1}{4}, \\ \frac{2t}{\sqrt{e}}, & t > \frac{1}{4} \end{cases} \tag{3.3}$$

for all $t \in [0, t_1]$, where $t_1 \in (0, T]$ is such that

$$\Omega(A(t_1)) + 4t_1^{\frac{3}{2}}\, e^{-2t_1} \in \operatorname{Dom} \Omega^{-1}$$

if $t_1 \le \frac{1}{4}$, and

$$\Omega(A(t_1)) + \frac{2t_1}{\sqrt{e}} \in \operatorname{Dom} \Omega^{-1}$$

if $t_1 > \frac{1}{4}$.

Let us denote $E_1(t)$, $E_2(t)$ the right-hand sides of estimations (3.2), (3.3), respectively. For better illustration, graphs of these functions are given in Figure 3.1.

Having a concrete function $\omega \in \mathfrak{O}$ in (3.1), one obtains an explicit estimation of u. For instance, if $\omega(u) = u$, then function $\Omega(v) = \ln \frac{v}{v_0}$ (see the proof of Corollary 2.5) maps $[v_0, \infty)$ for $v_0 > 0$ onto $[0, \infty)$. Hence,

$$u(t) \le \left(v_0\, e^{\ln A(t) - \ln v_0 + E_i(t)}\right)^{1/2} = \sqrt{A(t)\, e^{E_i(t)}}, \quad t \in [0, T]$$

for each $i = 1, 2$. As another example, consider $\omega(u) = \sqrt{u}$. Then, $\Omega(v) = 2(\sqrt{v} - \sqrt{v_0})$ maps $[v_0, \infty)$ for $v_0 \ge 0$ onto $[0, \infty)$. Thus, $\Omega^{-1}(v) = (\sqrt{v_0} + v/2)^2$ and

$$u(t) \le \sqrt{v_0} + \frac{1}{2}\left(2(\sqrt{A(t)} - \sqrt{v_0}) + E_i(t)\right) = \sqrt{A(t)} + \frac{E_i(t)}{2}, \quad t \in [0, T]$$

for each $i = 1, 2$.

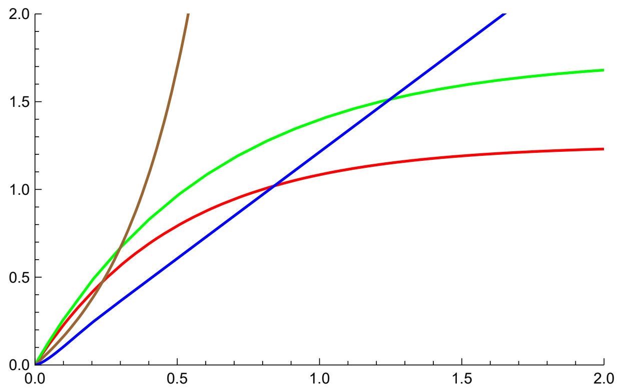

Figure 3.1: Graphs of the right-hand sides of (3.2) ($\mu = 0.8$ – green, $\mu = 1$ – red, $\mu = 3$ – brown) and (3.3) (blue).

Now, we investigate the IVPs with Ψ-Caputo fractional derivative. First, we consider the following one

$$ {}^{C}D_{t_0}^{\alpha,\Psi}x(t) = f(t, x(t)), \quad t > t_0, \tag{3.4} $$

$$ x(t_0) = x_0 \tag{3.5} $$

for some constant $x_0 \in \mathbb{R}^N$, $N \in \mathbb{N}$, where $\alpha \in (0,1)$, $\Psi \in \mathfrak{P}^1_{[t_0,\infty)}$, and $f \in C([t_0,\infty) \times \mathbb{R}^N, \mathbb{R}^N)$.

Definition 3.2. A mapping $x \in C([t_0, t_0 + h], \mathbb{R}^N)$, $0 < h \le \infty$ is a solution of IVP (3.4), (3.5), if ${}^{C}D_a^{\alpha,\Psi}x(t)$ exists and is continuous on $(t_0, t_0 + h]$, x fulfills equation (3.4) and initial condition (3.5). The solution x is called blowing-up, if it is defined on $[t_0, \tau)$ for some $t_0 < \tau < \infty$ and $\lim_{t\to\tau^-} \|x(t)\| = \infty$.

Theorem 3.3. *Let F be a non-negative continuous function defined on $[t_0, T]$, and $\omega \in \mathfrak{O}$ be such that*

$$ \|f(t,x)\| \le F(t)\omega(\|x\|), \quad t \in [t_0, T], \ x \in \mathbb{R}^N. $$

If x is a solution of (3.4), (3.5) defined on $[t_0, T]$, $\alpha q > 1$, $\mu > 0$, then it satisfies

1. *the inequality*

$$ \|x(t)\| \le \left[\Omega^{-1} \left(\Omega(2^{q-1}\|x_0\|^q) + \frac{2^{q-1}M_{p,\mu,\alpha}^q \,\mathrm{e}^{q\mu\Psi(t)}}{\Gamma(\alpha)^q} \int_{t_0}^{t} \Psi'(s)F(s)^q \,\mathrm{e}^{-q\mu\Psi(s)}\, ds \right) \right]^{1/q} $$

for all $t \in [t_0, t_1]$, where $M_{p,\mu,\alpha}$ is given by (2.3) and $t_1 \in (t_0, T]$ is such that

$$ \Omega(2^{q-1}\|x_0\|^q) + \frac{2^{q-1}M_{p,\mu,\alpha}^q \,\mathrm{e}^{q\mu\Psi(t_1)}}{\Gamma(\alpha)^q} \int_{t_0}^{t_1} \Psi'(s)F(s)^q \,\mathrm{e}^{-q\mu\Psi(s)}\, ds \in \operatorname{Dom}\Omega^{-1}; $$

2. *the inequality*

$$ \|x(t)\| \le \left[\Omega^{-1} \left(\Omega(2^{q-1}\|x_0\|^q) + \frac{2^{q-1}\mathcal{M}_{p,\alpha}(t)^q}{\Gamma(\alpha)^q} \int_{t_0}^{t} \Psi'(s)F(s)^q ds \right) \right]^{1/q} $$

for all $t \in [t_0, t_1]$, where $\mathcal{M}_{p,\alpha}$ is given by (2.12) and $t_1 \in (t_0, T]$ is such that

$$ \Omega(2^{q-1}\|x_0\|^q) + \frac{2^{q-1}\mathcal{M}_{p,\alpha}(t_1)^q}{\Gamma(\alpha)^q} \int_{t_0}^{t_1} \Psi'(s)F(s)^q ds \in \operatorname{Dom}\Omega^{-1}, $$

where $\Omega(v) = \int_{v_0}^{v} \frac{1}{[\omega(\sigma^{1/q})]^q} d\sigma$ *for* $v \geq v_0 \geq 0$, *and* Ω^{-1} *is the inverse of* Ω.

Proof. From [1, Theorem 2] (see also [17, Theorem 2]), we know that x satisfies the integral equation

$$x(t) = x_0 + \frac{1}{\Gamma(\alpha)} \int_{t_0}^{t} [\Psi(t) - \Psi(s)]^{\alpha-1} \Psi'(s) f(s, x(s)) ds$$

for all $t \in [t_0, T]$. Then, for the norm of the solution, we have

$$\begin{aligned} \|x(t)\| &\leq \|x_0\| + \frac{1}{\Gamma(\alpha)} \int_{t_0}^{t} [\Psi(t) - \Psi(s)]^{\alpha-1} \Psi'(s) \|f(s, x(s))\| ds \\ &\leq \|x_0\| + \frac{1}{\Gamma(\alpha)} \int_{t_0}^{t} [\Psi(t) - \Psi(s)]^{\alpha-1} \Psi'(s) F(s) \omega(\|x(s)\|) ds \end{aligned}$$

for any $t \in [t_0, T]$. Applying Theorems 2.2, 2.11 proves the first and the second statement, respectively. □

Theorem 3.4. *Let all the assumptions of Theorem 3.3 be fulfilled with* $T = \infty$ *and*

$$\int_{v_0}^{\infty} \frac{d\sigma}{[\omega(\sigma^{1/q})]^q} = \infty \tag{3.6}$$

for some $v_0 > 0$. *Then* (3.4), (3.5) *does not possess a blowing-up solution.*

Proof. Let x be a blowing-up solution of (3.4), (3.5) which is defined on $[t_0, b)$ for some $t_0 < b < \infty$, i.e., $\lim_{t \to b^-} \|x(t)\| = \infty$. Notice that condition (3.6) means $\Omega(\infty) = \infty$ for Ω as in the Theorem 3.3. In other words, $[0, \infty) \subset \Omega^{-1}$, and Statement 1 of Theorem 3.3 holds for any $t_1 \in [t_0, b)$, $\mu = 1$. But this is equivalent to

$$\Omega(\|x(t)^q\|) \leq \Omega(2^{q-1}\|x_0\|^q) + \frac{2^{q-1} M_{p,1,\alpha}^q \, e^{q\Psi(t)}}{\Gamma(\alpha)^q} \int_{t_0}^{t} \Psi'(s) F(s)^q \, e^{-q\Psi(s)} \, ds.$$

Taking the limit $t \to b^-$ gives a contradiction, since the right-hand side is bounded. Therefore, $b = \infty$ and the proof is complete. □

The following example illustrates the use of the latter result.

Example 3.5. Let x be a solution of initial value problem (3.4), (3.5), and the assumptions of Theorem 3.3 be satisfied with

$$\omega(u) = u^{\frac{q-1}{q}} \left(\ln(1+u)\right)^{\frac{1}{q}}.$$

Then x is not blowing-up.

Indeed, for $v_0 > 0$, we have

$$\begin{aligned} \int_{v_0}^{\infty} \frac{d\sigma}{[\omega(\sigma^{1/q})]^q} &= q \int_{v_0^{1/q}}^{\infty} \frac{\tau^{q-1} d\tau}{[\omega(\tau)]^q} = q \int_{v_0^{1/q}}^{\infty} \frac{\tau^{q-1} d\tau}{\tau^{q-1} \ln(1+\tau)} \\ &= q \int_{v_0^{1/q}}^{\infty} \frac{d\tau}{\ln(1+\tau)} \geq q \int_{v_0^{1/q}}^{\infty} \frac{d\tau}{\tau} = \infty. \end{aligned}$$

So, the assumptions of Theorem 3.4 are satisfied.

Next, we consider the IVP with a Ψ-Hilfer fractional integral on the right side,

$$ {}^{C}D_{t_0}^{\alpha,\Psi}x(t) = f(t,x(t)) + c(t)I_{t_0}^{\beta,\Psi}g(t,x(t)), \quad t > t_0, \tag{3.7}$$

$$ x(t_0) = x_0 \tag{3.8}$$

for $x_0 \in \mathbb{R}^N$, $N \in \mathbb{N}$, where $\alpha, \beta \in (0,1)$, $\alpha+\beta < 1$, $\Psi \in \mathfrak{P}^1_{[t_0,\infty)}$, $c \in C([t_0,\infty),\mathbb{R})$, and $f, g \in C([t_0,\infty)\times\mathbb{R}^N,\mathbb{R}^N)$.

Theorem 3.6. *Let F, G be non-negative continuous functions defined on $[t_0,T]$, and $\omega, \omega_2 \in \mathfrak{O}$ be such that*

$$\begin{aligned} \|f(t,x)\| &\le F(t)\omega_1(\|x\|), \\ \|g(t,x)\| &\le G(t)\omega_2(\|x\|), \end{aligned} \qquad t \in [t_0,T],\ x \in \mathbb{R}^N.$$

If x is a solution of (3.7), (3.8) defined on $[t_0,T]$, $\alpha q > 1$, $\mu > 0$, then it satisfies

1. *the inequality*

$$\|x(t)\| \le \left[\Omega^{-1}\left(\Omega(2^{q-1}\|x_0\|^q) + B(t)\int_{t_0}^{t}\Psi'(s)H(s)^q\,\mathrm{e}^{-q\mu\Psi(s)}\,ds\right)\right]^{1/q}$$

for all $t \in [t_0,t_1]$, where

$$B(t) = 2^{q-1}\sup_{s\in[t_0,t]}\left\{\mathrm{e}^{q\mu\Psi(s)}\left(\frac{M_{p,\mu,\alpha}}{\Gamma(\alpha)} + \frac{M_{p,\mu,\alpha+\beta}}{\Gamma(\alpha+\beta)}\sup_{\sigma\in[t_0,s]}|c(\sigma)|\right)^q\right\},$$

$M_{p,\mu,\alpha}$ is given by (2.7), and $t_1 \in (t_0,T]$ is such that

$$\Omega(2^{q-1}\|x_0\|^q) + B(t_1)\int_{t_0}^{t_1}\Psi'(s)H(s)^q\,\mathrm{e}^{-q\mu\Psi(s)}\,ds \in \operatorname{Dom}\Omega^{-1};$$

2. *the inequality*

$$\|x(t)\| \le \left[\Omega^{-1}\left(\Omega(2^{q-1}\|x_0\|^q) + \mathcal{B}(t)\int_{t_0}^{t}\Psi'(s)H(s)^q ds\right)\right]^{1/q}$$

for all $t \in [t_0,t_1]$, where

$$\mathcal{B}(t) = 2^{q-1}\sup_{s\in[t_0,t]}\left\{\frac{\mathcal{M}_{p,\alpha}(s)}{\Gamma(\alpha)} + \frac{\mathcal{M}_{p,\alpha+\beta}(s)}{\Gamma(\alpha+\beta)}\sup_{\sigma\in[t_0,s]}|c(\sigma)|\right\}^q,$$

$\mathcal{M}_{p,\alpha}$ is given by (2.15), and $t_1 \in (t_0,T]$ is such that

$$\Omega(2^{q-1}\|x_0\|^q) + \mathcal{B}(t_1)\int_{t_0}^{t_1}\Psi'(s)H(s)^q ds \in \operatorname{Dom}\Omega^{-1},$$

where $H = \max\{F,G\}$, $\omega = \max\{\omega_1,\omega_2\}$, $\Omega(v) = \int_{v_0}^{v}\frac{1}{[\omega(\sigma^{1/q})]^q}d\sigma$ for $v \ge v_0 \ge 0$, and Ω^{-1} is the inverse of Ω.

Proof. As in the proof of Theorem 3.3, we use the integral equation for the solution of (3.7), (3.8), and we estimate its norm using the assumptions on f, g to obtain

$$\begin{aligned}\|x(t)\| \le \|x_0\| &+ \frac{1}{\Gamma(\alpha)}\int_{t_0}^{t}[\Psi(t)-\Psi(s)]^{\alpha-1}\Psi'(s)F(s)\omega_1(\|x(s)\|)ds\\ &+ \frac{1}{\Gamma(\alpha)}\int_{t_0}^{t}[\Psi(t)-\Psi(s)]^{\alpha-1}|c(s)|\\ &\times \frac{1}{\Gamma(\beta)}\int_{t_0}^{s}[\Psi(s)-\Psi(\sigma)]^{\beta-1}G(\sigma)\omega_2(\|x(\sigma)\|)d\sigma\, ds\end{aligned}$$

for any $t \in [t_0, T]$. The application of Theorems 2.8 and 2.16 proves the statements. □

Theorem 3.7. *Let all the assumptions of Theorem 3.6 be fulfilled with $T = \infty$ and $\Omega(\infty) = \infty$ for Ω as in Theorem 3.6. Then (3.7), (3.8) does not possess a blowing-up solution.*

Proof. The statement can be proved exactly as Theorem 3.4 with the use of Theorem 3.6 instead of Theorem 3.3. □

Finally, we consider the more general equation with the right-hand side depending on the fractional integral of the solution,

$$^{C}D_{t_0}^{\alpha,\Psi}x(t) = f(t, x(t), I_{t_0}^{\beta,\Psi}g(t, x(t))), \quad t > t_0, \tag{3.9}$$

$$x(t_0) = x_0 \tag{3.10}$$

for $x_0 \in \mathbb{R}^N$, $N \in \mathbb{N}$, where $\alpha, \beta \in (0, 1)$, $\Psi \in \mathfrak{P}^1_{[t_0,\infty)}$, and $f \in C([t_0, \infty) \times \mathbb{R}^N \times \mathbb{R}^N, \mathbb{R}^N)$, $g \in C([t_0, \infty) \times \mathbb{R}^N, \mathbb{R}^N)$.

Theorem 3.8. *Let F, G be non-negative continuous functions defined on $[t_0, T]$, and $\omega_1, \omega_2, \omega_3 \in \mathfrak{O}$ be such that*

$$\begin{aligned}\|f(t, x, y)\| &\le F(t)(\omega_1(\|x\|) + \omega_2(\|y\|)), \quad t \in [t_0, T],\ x, y \in \mathbb{R}^N,\\ \|g(t, x)\| &\le G(t)\omega_3(\|x\|), \quad t \in [t_0, T],\ x \in \mathbb{R}^N.\end{aligned}$$

If x is a solution of (3.9), (3.10) defined on $[t_0, T]$, $q \min\{\alpha, \beta\} > 1$, $\mu > 0$, then it satisfies

1. *the inequality*

$$\begin{aligned}\|x(t)\| \le \Bigg[\Omega^{-1}\Bigg(&\Omega(3^{q-1}\|x_0\|^q)\\ &+ 3^{q-1}M_{p,\mu}^q\, e^{q\mu\Psi(t)} \max\left\{\frac{1}{\Gamma(\alpha)}, \frac{1}{\Gamma(\beta)}\right\}^q \int_{t_0}^{t} H(s)^q\, e^{-q\mu\Psi(s)}\, \Psi'(s)ds\\ &+ \frac{3^{q-1}M_{p,\mu,\alpha}^q\, e^{q\mu\Psi(t)}}{\Gamma(\alpha)^q}\int_{t_0}^{t} F(s)^q\, e^{-q\mu\Psi(s)}\, \Psi'(s)ds\Bigg)\Bigg]^{1/q}\end{aligned}$$

for all $t \in [t_0, t_1]$, where $M_{p,\mu} = \max\{M_{p,\mu,\alpha}, M_{p,\mu,\beta}\}$, $M_{p,\mu,\alpha}$ is given by (2.8), and $t_1 \in (t_0, T]$ is such that

$$\begin{aligned}\Omega(3^{q-1}\|x_0\|^q) + 3^{q-1}M_{p,\mu}^q\, e^{q\mu\Psi(t_1)} \max\left\{\frac{1}{\Gamma(\alpha)}, \frac{1}{\Gamma(\beta)}\right\}^q \int_{t_0}^{t_1} H(s)^q\, e^{-q\mu\Psi(s)}\, \Psi'(s)ds\\ + \frac{3^{q-1}M_{p,\mu,\alpha}^q\, e^{q\mu\Psi(t_1)}}{\Gamma(\alpha)^q}\int_{t_0}^{t_1} F(s)^q\, e^{-q\mu\Psi(s)}\, \Psi'(s)ds \in \operatorname{Dom}\Omega^{-1};\end{aligned}$$

2. *the inequality*

$$\|x(t)\| \le \Bigg[\Omega^{-1}\Bigg(\Omega(3^{q-1}\|x_0\|^q) + 3^{q-1}\mathcal{M}_p(t)^q \max\left\{\frac{1}{\Gamma(\alpha)}, \frac{1}{\Gamma(\beta)}\right\}^q \int_{t_0}^{t} H(s)^q \Psi'(s) ds + \frac{3^{q-1}\mathcal{M}_{p,\alpha}(t)^q}{\Gamma(\alpha)^q} \int_{t_0}^{t} F(s)^q \Psi'(s) ds\Bigg)\Bigg]^{1/q}$$

for all $t \in [t_0, t_1]$*, where* $\mathcal{M}_p = \max\{\mathcal{M}_{p,\alpha}, \mathcal{M}_{p,\beta}\}$*,* $\mathcal{M}_{p,\alpha}$ *is given by* (2.16)*, and* $t_1 \in (t_0, T]$ *is such that*

$$\Omega(3^{q-1}\|x_0\|^q) + 3^{q-1}\mathcal{M}_p(t_1)^q \max\left\{\frac{1}{\Gamma(\alpha)}, \frac{1}{\Gamma(\beta)}\right\}^q \int_{t_0}^{t_1} H(s)^q \Psi'(s) ds + \frac{3^{q-1}\mathcal{M}_{p,\alpha}(t_1)^q}{\Gamma(\alpha)^q} \int_{t_0}^{t_1} F(s)^q \Psi'(s) ds \in \operatorname{Dom}\Omega^{-1}$$

where $H = \max\{F, G\}$*,* $\omega = \max\{\omega_1, \omega_2, \omega_3\}$*,* $\Omega(v) = \int_{v_0}^{v} \frac{1}{[\omega(\sigma^{1/q})]^q} d\sigma$ *for* $v \ge v_0 \ge 0$*, and* Ω^{-1} *is the inverse of* Ω*.*

Proof. Using the estimation

$$\|f(t, x(t), I_{t_0}^{\beta,\Psi} g(t, x(t)))\| \le F(t)\left(\omega_1(\|x(t)\|) + \omega_2(I_{t_0}^{\beta,\Psi}(G(t)\omega_3(\|x(t)\|)))\right)$$

following from the assumptions on f and g, and an appropriate integral equation (see [17, Theorem 2] or the proof of Theorem 3.3), we get

$$\|x(t)\| \le \|x_0\| + \frac{1}{\Gamma(\alpha)} \int_{t_0}^{t} [\Psi(t) - \Psi(s)]^{\alpha-1} \Psi'(s) F(s) \times \left(\omega_1(\|x(s)\|) + \omega_2\left(\frac{1}{\Gamma(\beta)} \int_{t_0}^{s} [\Psi(s) - \Psi(\sigma)]^{\beta-1} G(\sigma)\omega_3(\|x(\sigma)\|) d\sigma\right)\right) ds$$

for any $t \in [t_0, T]$. The statements then follow from Theorems 2.9 and 2.17, respectively. □

Theorem 3.9. *Let all the assumptions of Theorem 3.8 be fulfilled with* $T = \infty$ *and* $\Omega(\infty) = \infty$ *for* Ω *as in Theorem 3.8. Then* (3.9)*,* (3.10) *does not possess a blowing-up solution.*

Proof. The statement can be proved exactly as Theorem 3.4 with the use of Theorem 3.8 instead of Theorem 3.3. □

4 Conclusions and discussion

In this paper, the parametrized desingularization method and the desingularization method without an exponential function have been applied to derive new estimations of functions satisfying integral inequalities involving integrals with a weakly singular kernel, namely Ψ-Hilfer fractional integrals of order from $(0, 1)$. More precisely, nonlinear Henry–Gronwall integral inequalities (Theorems 2.2, 2.11), integral inequalities with an iterated fractional integral (Theorems 2.8, 2.16), and integral inequalities with a fractional integral of a function of a fractional integral (Theorems 2.9, 2.17) have been proved.

It should be mentioned, that all of the results of Section 2 are valid also for $\alpha \geq 1$, $\beta \geq 1$, or $\alpha + \beta \geq 1$ in Theorems 2.8, 2.16, although, in these cases, better estimations might exist, since the integrals are no more singular.

The integral inequalities have been applied to IVPs for fractional differential equations with Ψ-Caputo derivatives to obtain bounds for the solutions and to prove sufficient conditions for the non-existence of blowing-up solutions.

Further applications of our integral inequalities may be found in the study of stability or existence of solutions of generalizations of evolution equations investigated in [8, 13–15].

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