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A systematic method for complete stability problem of a class of delayed neural networks in parameter space

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Abstract. This paper presents a systematic method to address the complete stability problem of delayed neural networks with heterogeneous free parameters. First, we adopt an algebraic method to investigate the complete stability problem with respect to the free delay parameter τ. Then, the stability analysis is extended to the scenario with additional free system parameters, denoted by a vector X. We can investigate the complete root classification for the auxiliary characteristic equation in the entire (X, τ)-space. As a result, we can analytically calculate the number of stability τ-intervals and characterize all classifications of stability property over the whole (X, τ)-space. Finally, we will give a systematic method for determining the stability set in the whole (X, τ)-space. Some representative examples show the effectiveness of the approach.

Keywords: neural networks, time delays, stability, complete root classification.

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1 Introduction

During the last four decades, the interest in investigating the stability of neural networks (NNs) has steadily increased since the pioneering work of Hopfield (see [19]). Based on the Hopfield NN model, Marcus and Westervelt incorporated time delays in an NN model (see [32]). Since then, various types of delayed NN models have been proposed, and various issues of these models (including stability, periodic solution, and bifurcation, etc.) have been studied extensively by many researchers (see e.g., [1–3, 7, 12, 15, 16, 18, 20, 23, 28, 40, 42, 44, 45, 47]).

The appearance of delays causes a dynamical system to be infinitely dimensional. Hence, the stability analysis for a time delay system will usually be much more complicated than that for a delay-free system (see e.g., [14, 34, 37]).

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From a practical point of view, besides a time delay, a system usually involves a number of parameters. One can design the system parameters to render the system asymptotically stable (see e.g., [36, 41] for the controller design, [22] for the economic system, [48] for the laser system, and [4] for the biological system).

The stability problem becomes much more complex if the system contains the delay free parameter as well as additional system free parameters (i.e., heterogeneous free parameters).

For delayed NNs, the stability set of delay τ subject to the form $\tau \in [0, \overline{\tau})$ is extensively addressed. Such a $\overline{\tau}$ is termed the delay margin (see [14]), a notion widely used in the field of time-delay systems. See, for instance, [5] and [21] for delayed Hopfield NNs, [9] and [38] for delayed bidirectional associative memory (BAM) NNs, and [8] and [46] for delayed NNs with an annular topology.

Recently, the complete stability problem w.r.t. the delay parameter τ* has been studied for NNs in [24]. It is found therein that a delayed NN may have more than one stability τ-interval including or excluding $\tau = 0$.

Nevertheless, the problem considered in [24] contains only a free delay parameter τ. In this paper, for the delayed NN, we will address the complete stability problem with heterogeneous free parameters, including a delay parameter τ and additional system parameters (throughout this paper, we denote them by a vector X).

It will be interesting to see in this paper that the stability parameter space of a delayed NN may have multiple disjoint parts.

As far as we know, the complete stability problem of delayed NNs with heterogeneous free parameters has not been well investigated. In our opinion, the difficulties in studying this problem mainly come from two aspects:

(i) The free parameter space is multiple-dimensional.

(ii) The free parameters are of heterogeneous types.

To the best of the authors' knowledge, the above two technical points can not be appropriately covered by the existing results for delayed NNs. In order to solve such a complete stability problem, we will develop a systematic approach.

First, for the case where τ is the only free parameter, an algebraic approach will be employed for the complete stability analysis w.r.t. τ of the NN.

Notably, the above approach covers the general case (to be explained in Remark 3.7).

Next, in the scenario where the free parameters contain delay parameter τ plus system parameter vector X, we will employ the discrimination system, a mathematical tool for polynomial algebra. Then, we can obtain the complete root classification (CRC) for the auxiliary characteristic equation.

Consequently, we are able to characterize the stability property over the entire (X, τ)-space. To be more precise, we can calculate the possible number of stability τ-intervals and analytically calculate them.

Combining the abovementioned results, we will develop a systematic method to identify the stability set in the whole (X, τ)-space.

This systematic method can contribute to a more refined design process. It offers a possible "stabilization" way such that a pair of (X, τ) can be found under which a practical NN system can be stable.

*We call the problem of analyzing the stability property along the entire positive τ-axis as the complete stability problem w.r.t. τ (see e.g., [26]).

We will present some examples with different NN architectures and show that all classifications of stability property can be characterized in the entire (X, τ)-space by using our approach. It will be seen in the examples that a very small change of the vector X may make the $NU(\tau)$ distribution ($NU(\tau)$ stands for the number of characteristic roots in $\mathbb{C}_+$) have a structural variation.

This paper is organized as follows. In Section 2, three types of delayed NNs and the corresponding characteristic functions are reviewed. The main results are proposed in Section 3. Illustrative examples are presented in Section 4. Finally, the paper concludes in Section 5.

Notations: Throughout the paper, the following standard notations are used. $\mathbb{R}$ ($\mathbb{R}_+$) is the set of (positive) real numbers. $\mathbb{N}$ stands for the set of non-negative integers. $\mathbb{C}$ represents the set of complex numbers. We use $\mathbb{C}_-$ and $\mathbb{C}_+$ respectively to denote the left half-plane and the right half-plane in $\mathbb{C}$. $\deg(\cdot)$ is the degree of a polynomial. Finally, $\lceil \gamma \rceil$ stands for the smallest integer greater than or equal to γ, where $\gamma \in \mathbb{R}$.

2 Neural network models and characteristic functions

We will review in this section three types of delayed NN models, including the Hopfield NN model, the bidirectional associative memory (BAM) NN model, and the NN model with an annular topology. In recent years, these NN models have attracted much attention.

We will show that the local stability is determined by the corresponding characteristic function $f(\lambda, \tau)$.

2.1 Delayed Hopfield neural network model

The first model to be recalled in this paper is the delayed Hopfield NN model (see e.g., [5] and [21]):

$$\dot{y}_i(t) = -\mu_i y_i(t) + \sum_{j=1}^{n} c_{ij} f(y_j(t - \tau_{ij})), \qquad i = 1, \ldots, n, \tag{2.1}$$

where $y_i(t)$ stands for the voltage on the input of the i-th neuron at time t; $\mu_i \in \mathbb{R}_+$ is a positive constant; $c_{ij} \in \mathbb{R}$ represents the connection weight of the unit j on the unit i; f is the activation function; $\tau_{ij} \in \mathbb{R}_+ \cup 0$ denotes the signal transmission delay.

Here, we consider a simplified Hopfield NN model, which has three neurons in series (see Fig. 2.1 (a) borrowed from [29]):

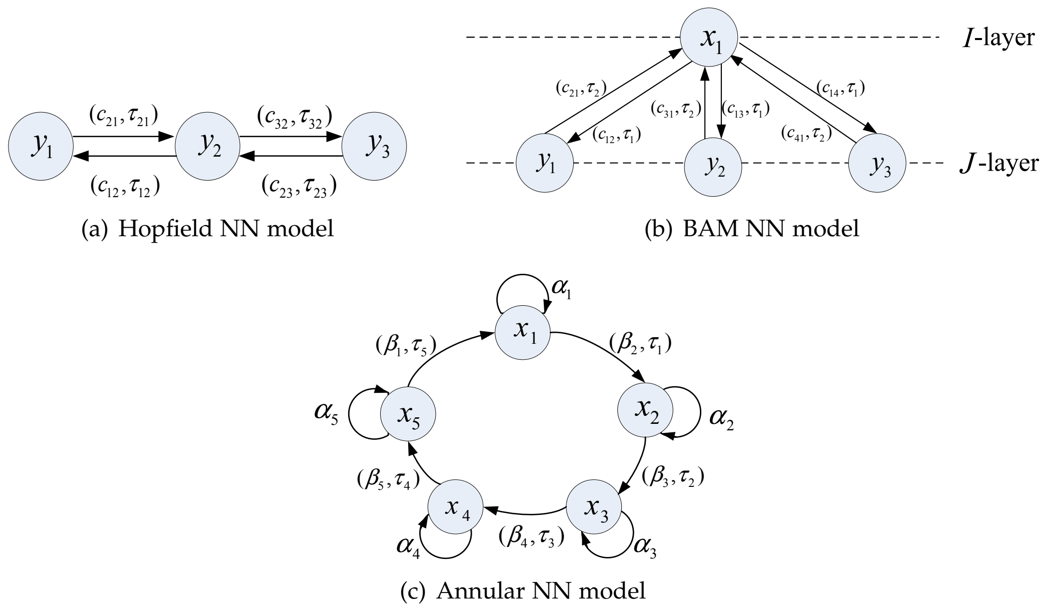
$$\begin{cases} \dot{y}_1(t) = -\mu_1 y_1(t) + c_{12} f(y_2(t - \tau_{12})), \\ \dot{y}_2(t) = -\mu_2 y_2(t) + c_{21} f(y_1(t - \tau_{21})) + c_{23} f(y_3(t - \tau_{23})), \\ \dot{y}_3(t) = -\mu_3 y_3(t) + c_{32} f(y_2(t - \tau_{32})), \end{cases} \tag{2.2}$$

where $c_{ij} = 0$ if $i = j$, $f \in C^1$, and $f(0) = 0$.

For the sake of simplicity, we assume that both the loops between neurons have the same sum of delays, i.e., $\tau_{12} + \tau_{21} = \tau_{23} + \tau_{32}$. We denote this value by τ.

It is not hard to see that for NN (2.2), the origin is the equilibrium and the linearization is

$$\begin{cases} \dot{y}_1(t) = -\mu_1 y_1(t) + h_{12} y_2(t - \tau_{12}), \\ \dot{y}_2(t) = -\mu_2 y_2(t) + h_{21} y_1(t - \tau_{21}) + h_{23} y_3(t - \tau_{23}), \\ \dot{y}_3(t) = -\mu_3 y_3(t) + h_{32} y_2(t - \tau_{32}), \end{cases}$$



(a) Hopfield NN model

(b) BAM NN model

(c) Annular NN model

Figure 2.1: Architectures of three delayed NN models.

where $h_{ij} = c_{ij} f'(0)$.

The corresponding characteristic function is

$$a_0(\lambda) + a_1(\lambda)e^{-\tau\lambda}, \tag{2.3}$$

where $a_0(\lambda) = \prod_{i=1}^{3}(\lambda + \mu_i)$, $a_1(\lambda) = -(h_{21}h_{12} + h_{23}h_{32})\lambda - h_{21}h_{12}\mu_3 - h_{23}h_{32}\mu_1$. It follows that $\deg(a_0(\lambda)) = 3$ and $\deg(a_1(\lambda)) = 1$.

For such a Hopfield NN model, the coefficients μ_i and h_{ij} are allowed to have free system parameters (as mentioned previously, we denote them by a vector X). In Example 4.1 of this paper, we will choose $X = (h_{21})$ and study the whole stability set in (h_{21}, τ)-plane.

2.2 Delayed bidirectional associative memory (BAM) neural network model

Next, we recall the BAM NNs (see e.g., [9] and [38]).

A delayed BAM NN is described by the model

$$\begin{cases} \dot{x}_i(t) = -\mu_i x_i(t) + \sum_{j=1}^{m} c_{ji} f_i(y_j(t-\tau_2)), \\ \dot{y}_j(t) = -\nu_j y_j(t) + \sum_{i=1}^{n} d_{ij} g_j(x_i(t-\tau_1)), \end{cases} \tag{2.4}$$

where $x_i(t)$ and $y_j(t)$ are the state of the neurons in the I-layer and the J-layer at time t, respectively ($i = 1, \ldots, n$, $j = 1, \ldots, m$, n and m denote the numbers of neurons); f_i and g_j are the activation functions; $c_{ji} \in \mathbb{R}$ and $d_{ij} \in \mathbb{R}$ are the connection weights; τ_1 and τ_2 are the signal transmission delays; $\mu_i \in \mathbb{R}_+$ and $\nu_j \in \mathbb{R}_+$ describe the stability of internal neuron process.

Here, we consider the case where $n = 1$ and $m = 3$ (the architecture of this model is described in Fig. 2.1 (b)).

For simplicity, we redefine $\mu_2 = \nu_1, \mu_3 = \nu_2, \mu_4 = \nu_3$, $c_{21} = c_{11}$, $c_{31} = c_{21}$, $c_{41} = c_{31}$,

$c_{12} = d_{11}$, $c_{13} = d_{12}$, $c_{14} = d_{13}$, $f_2 = g_1, f_3 = g_2, f_4 = g_3$. Then, the BAM NN model (2.4) reads:

$$\begin{cases} \dot{x}_1(t) = -\mu_1 x_1(t) + c_{21} f_1(y_1(t-\tau_2)) + c_{31} f_1(y_2(t-\tau_2)) + c_{41} f_1(y_3(t-\tau_2)), \\ \dot{y}_1(t) = -\mu_2 y_1(t) + c_{12} f_2(x_1(t-\tau_1)), \\ \dot{y}_2(t) = -\mu_3 y_2(t) + c_{13} f_3(x_1(t-\tau_1)), \\ \dot{y}_3(t) = -\mu_4 y_3(t) + c_{14} f_4(x_1(t-\tau_1)), \end{cases} \tag{2.5}$$

where $f_i \in C^1$ and $f_i(0) = 0, i = 1, 2, 3, 4$.

By letting $u_1(t) = x_1(t-\tau_1), u_2(t) = y_1(t), u_3(t) = y_2(t)$, $u_4(t) = y_3(t)$, and $\tau = \tau_1 + \tau_2$, the BAM NN model (2.5) may be rewritten as the following model

$$\begin{cases} \dot{u}_1(t) = -\mu_1 u_1(t) + c_{21} f_1(u_2(t-\tau)) + c_{31} f_1(u_3(t-\tau)) + c_{41} f_1(u_4(t-\tau)), \\ \dot{u}_2(t) = -\mu_2 u_2(t) + c_{12} f_2(u_1(t)), \\ \dot{u}_3(t) = -\mu_3 u_3(t) + c_{13} f_3(u_1(t)), \\ \dot{u}_4(t) = -\mu_4 u_4(t) + c_{14} f_4(u_1(t)). \end{cases} \tag{2.6}$$

Then, the linearization of the model (2.6) at the equilibrium $(0,0,0,0)$ is

$$\begin{cases} \dot{u}_1(t) = -\mu_1 u_1(t) + h_{21} u_2(t-\tau) + h_{31} u_3(t-\tau) + h_{41} u_4(t-\tau), \\ \dot{u}_2(t) = -\mu_2 u_2(t) + h_{12} u_1(t), \\ \dot{u}_3(t) = -\mu_3 u_3(t) + h_{13} u_1(t), \\ \dot{u}_4(t) = -\mu_4 u_4(t) + h_{14} u_1(t), \end{cases} \tag{2.7}$$

where $h_{ij} = c_{ij} f'_j(0)$.

The corresponding characteristic function is

$$a_0(\lambda) + a_1(\lambda) e^{-\tau\lambda}, \tag{2.8}$$

where $a_0(\lambda) = \lambda^4 + (\mu_1 + \mu_2 + \mu_3 + \mu_4)\lambda^3 + (\mu_1\mu_2 + \mu_3\mu_4 + \mu_1\mu_3 + \mu_1\mu_4 + \mu_2\mu_3 + \mu_2\mu_4)\lambda^2 + (\mu_1\mu_2\mu_3 + \mu_1\mu_2\mu_4 + \mu_1\mu_3\mu_4 + \mu_2\mu_3\mu_4)\lambda + \mu_1\mu_2\mu_3\mu_4$ and $a_1(\lambda) = -(h_{12}h_{21} + h_{13}h_{31} + h_{14}h_{41})\lambda^2 - (h_{12}h_{21}\mu_3 + h_{12}h_{21}\mu_4 + h_{13}h_{31}\mu_2 + h_{13}h_{31}\mu_4 + h_{14}h_{41}\mu_2 + h_{14}h_{41}\mu_3)\lambda - (h_{12}h_{21}\mu_3\mu_4 + h_{13}h_{31}\mu_2\mu_4 + h_{14}h_{41}\mu_2\mu_3)$. It follows that $\deg(a_0(\lambda)) = 4$ and $\deg(a_1(\lambda)) = 2$.

For this BAM NN model, the coefficients μ_i and h_{ij} are allowed to have free system parameters. In Examples 3.12 and 4.2 of this paper, we will choose $X = (h_{12}, h_{21})$ and study the whole stability set in (h_{12}, h_{21}, τ)-space. In Example 4.2, we will also obtain the stability set in the corresponding 4D parameter space.

2.3 Delayed annular neural network model

Finally, we recall the delayed annular NN model (see e.g., [8] and [46]). Now consider the following NN model with five neurons:

$$\begin{cases} \dot{x}_1(t) = -\mu_1 x_1(t) + \alpha_1 f(x_1(t)) + \beta_1 g(x_5(t-\tau_5)), \\ \dot{x}_2(t) = -\mu_2 x_2(t) + \alpha_2 f(x_2(t)) + \beta_2 g(x_1(t-\tau_1)), \\ \dot{x}_3(t) = -\mu_3 x_3(t) + \alpha_3 f(x_3(t)) + \beta_3 g(x_2(t-\tau_2)), \\ \dot{x}_4(t) = -\mu_4 x_4(t) + \alpha_4 f(x_4(t)) + \beta_4 g(x_3(t-\tau_3)), \\ \dot{x}_5(t) = -\mu_5 x_5(t) + \alpha_5 f(x_5(t)) + \beta_5 g(x_4(t-\tau_4)), \end{cases} \tag{2.9}$$

where $x_i(t)$ stands for the voltage on the input of the ith neuron at time t; $\mu_i \in \mathbb{R}_+$ denotes the ratio of the capacitance to the resistance; τ_i represents the signal transmission delay; α_i and β_i are the nonzero connection weights; the activation functions f and g are assumed to satisfy $f, g \in C^1$ and $f(0) = g(0) = 0$.

The above model (2.9) represents the dynamics of a ring of neurons, and the architecture of this model is described in Fig. 2.1 (c).

We now calculate the associated characteristic function.

For NN model (2.9), the origin is the equilibrium and the linearization is

$$\begin{cases} \dot{x}_1(t) = (s_1 - \mu_1)x_1(t) + h_1 x_5(t - \tau_5), \\ \dot{x}_2(t) = (s_2 - \mu_2)x_2(t) + h_2 x_1(t - \tau_1), \\ \dot{x}_3(t) = (s_3 - \mu_3)x_3(t) + h_3 x_2(t - \tau_2), \\ \dot{x}_4(t) = (s_4 - \mu_4)x_4(t) + h_4 x_3(t - \tau_3), \\ \dot{x}_5(t) = (s_5 - \mu_5)x_5(t) + h_5 x_4(t - \tau_4), \end{cases} \tag{2.10}$$

where $s_i = \alpha_i f'(0), h_i = \beta_i g'(0), i = 1, \ldots, 5$.

The corresponding characteristic function is

$$a_0(\lambda) + a_1(\lambda)e^{-\tau\lambda}, \tag{2.11}$$

where $a_0(\lambda) = \prod_{i=1}^{5}(\lambda + \mu_i - s_i)$, $a_1(\lambda) = -\prod_{i=1}^{5}(h_i)$, and $\tau = \sum_{i=1}^{5}\tau_i$. It follows that $\deg(a_0(\lambda)) = 5$ and $\deg(a_1(\lambda)) = 0$.

For such an annular NN model, the coefficients μ_i, s_i, and h_i are allowed to have free system parameters. In Example 4.3 of this paper, we will choose $X = (s_1)$ and study the whole stability set in (s_1, τ)-plane.

2.4 Characteristic function

It is clear that the characteristic functions for abovementioned three types of delayed NNs are subject to the same form

$$f(\lambda, \tau) = a_0(\lambda) + a_1(\lambda)e^{-\lambda\tau}, \tag{2.12}$$

where $a_0(\lambda)$ and $a_1(\lambda)$ ($\deg(a_0(\lambda)) > \deg(a_1(\lambda))$) are polynomials with real coefficients. The coefficients may contain free parameters.

The characteristic function $f(\lambda, \tau)$ is a quasipolynomial, and, unsurprisingly, it also covers many other types of delayed NNs.

The local asymptotic stability of delayed NNs may be analytically determined by the above characteristic function (2.12). More specifically, the delayed NN is locally asymptotically stable if all the characteristic roots are located in the left half-plane $\mathbb{C}_-$.

3 Main results

The complete stability problem w.r.t. the delay parameter τ has been studied for delayed NNs by using a frequency-sweeping approach in a very recent paper (see [24]). It is pointed out that a delayed NN may have more than one stability τ-interval including or excluding $\tau = 0$.

However, the frequency-sweeping approach, as a graphical one, is difficult to apply in the case with additional free system parameters. When a delayed NN has multiple free parameters, we need to study the parameter space and the difficulties mentioned in Introduction arise.

In order to study the complete stability problem of delayed NN with heterogeneous free parameters, we will fulfill a systematic analysis in the whole (X, τ)-space.

More specifically, when τ is the only free parameter, we will adopt an algebraic approach for the complete stability analysis (to be given in Subsection 3.1), and in the case where the free parameters include delay parameter τ plus system parameter vector X, we will adopt a tool for polynomial algebra and develop a method to achieve the CRC of effective W roots (to be given in Subsection 3.2). Finally, we will present a systematic method to detect the stability set in the whole (X, τ)-space (to be given in Subsection 3.3).

3.1 Complete stability problem w.r.t. τ

When τ is the only free parameter, we use $NU(\tau) \in \mathbb{N}$ to denote the number of characteristic roots in $\mathbb{C}_+$. $NU(\tau)$ may increase or decrease only when τ is a critical delay (CD) where the system has a critical imaginary root (CIR) $\lambda = j\omega$, $\omega \in \mathbb{R}$. We call the corresponding pair (λ, τ) a critical pair.

We denote the CIRs as $\lambda_\alpha = j\omega_\alpha$, $\alpha = 0, \ldots, \mathfrak{u} - 1$. Concerning a CIR $\lambda_\alpha = j\omega_\alpha$, the CDs are $\tau_{\alpha,k} = \tau_{\alpha,0} + \frac{2k\pi}{\omega_\alpha}, k \in \mathbb{N}$, where $\tau_{\alpha,0}$ is the minimum CD satisfying

$$e^{-\tau\omega_\alpha j} = -\frac{a_0(j\omega_\alpha)}{a_1(j\omega_\alpha)}. \tag{3.1}$$

For a critical pair $(\lambda_\alpha, \tau_{\alpha,k} > 0)$, the influence of the asymptotic behavior on $NU(\tau)$ can be reflected by the notation $\Delta NU_{\lambda_\alpha}(\tau_{\alpha,k}) \in \mathbb{N}$, representing the variation of $NU(\tau)$ due to λ_α as τ is increased from $\tau_{\alpha,k} - \varepsilon$ to $\tau_{\alpha,k} + \varepsilon$.

First, along the similar idea in [13], we may obtain the following lemma straightforwardly.

Lemma 3.1. *For the characteristic function $f(\lambda, \tau)$ of a delayed NN considered in this paper, there exists a critical imaginary root $\lambda = j\omega \neq 0$ if and only if (iff) the auxiliary characteristic equation $\mathcal{F}(W) = 0$ has a positive real root $W = \omega^2$, where*

$$\begin{aligned} \mathcal{F}(W) &= |a_0(j\omega)|^2 - |a_1(j\omega)|^2 \\ &= (\mathrm{Re}(a_0(j\omega)))^2 + (\mathrm{Im}(a_0(j\omega)))^2 - (\mathrm{Re}(a_1(j\omega)))^2 - (\mathrm{Im}(a_1(j\omega)))^2. \end{aligned} \tag{3.2}$$

Remark 3.2. The application of auxiliary characteristic function can be traced back to [13], with an almost 40-year history. It has been widely used for the stability analysis of time-delay systems. However, the existing methods have the restriction of not being able to address the multiple auxiliary characteristic roots. In the sequel, we will give some results covering the restriction.

For the auxiliary characteristic function (3.2), we are only concerned with the positive real W roots since $W = \omega^2$. Such roots are termed the *effective W roots*, denoted by W_α, $\alpha = 0, \ldots, \mathfrak{u} - 1$, with $W_\alpha = \omega_\alpha^2$. Without any loss of generality, assume that among W_α, there are $q_o \in \mathbb{N}$ ones with odd multiplicities (we denote them by $W_0^o, \ldots, W_{q_o-1}^o$), and $q_e \in \mathbb{N}$ ones with even multiplicities (we denote them by $W_0^e, \ldots, W_{q_e-1}^e$). We label them as:

$$W_0^o > \cdots > W_{q_o-1}^o > 0,\ W_0^e > \cdots > W_{q_e-1}^e > 0. \tag{3.3}$$

For each W_i^o and W_i^e, we use $(\lambda_i^o = j\omega_i^o, \tau_{i,k}^o)$, where $\omega_i^o = \sqrt{W_i^o} > 0$, and $(\lambda_i^e = j\omega_i^e, \tau_{i,k}^e)$, where $\omega_i^e = \sqrt{W_i^e} > 0, k \in \mathbb{N}$, respectively to denote critical pairs. The CDs are $\tau_{i,k}^o = \tau_{i,0}^o + \frac{2k\pi}{\omega_i^o}$ and $\tau_{i,k}^e = \tau_{i,0}^e + \frac{2k\pi}{\omega_i^e}$, respectively.

Lemma 3.3. *For any $\varepsilon > 0$, when W varies slightly near the effective W root of delayed NNs considered in this paper, one of the following three cases must happen:*

Case (1): For a $W = W_i^e$, when W increases from $W_i^e - \varepsilon$ to $W_i^e + \varepsilon$, the $\mathcal{F}(W)$ does not change sign.

Case (2): For a $W = W_i^o$ (i is even), when W increases from $W_i^o - \varepsilon$ to $W_i^o + \varepsilon$, the sign of $\mathcal{F}(W)$ changes from negative to positive.

Case (3): For a $W = W_i^o$ (i is odd), when W increases from $W_i^o - \varepsilon$ to $W_i^o + \varepsilon$, the sign of $\mathcal{F}(W)$ changes from positive to negative.

Proof of Lemma 3.3. In light of Lemma 3.1, it is clear that $\mathcal{F}(W) = 0$ is a polynomial equation. If the polynomial equation has a W_i^e root, the Case (1) is true. For delayed NNs considered in this paper, the characteristic function (2.12) is satisfied with $\deg(a_0(\lambda)) > \deg(a_1(\lambda))$. It can be seen that $\mathcal{F}(W) > 0$ when $W \to \infty$. Hence, Cases (2) and (3) are true. □

Lemma 3.4. *Suppose $\widetilde{W} = (\widetilde{\omega})^2$ is the effective W root of delayed NNs considered in this paper. $(\widetilde{\lambda} = j\widetilde{\omega}, \widetilde{\tau})$ is the critical pair with $f(\widetilde{\lambda}, \widetilde{\tau}) = 0$. As W increases from $\widetilde{W} - \varepsilon$ to $\widetilde{W} + \varepsilon$, we have the following three results.*

(i) $\Delta NU_{j\widetilde{\omega}}(\widetilde{\tau}_k) = 0$ *iff the $\mathcal{F}(W)$ does not change sign.*

(ii) $\Delta NU_{j\widetilde{\omega}}(\widetilde{\tau}_k) = +1$ *iff the sign of $\mathcal{F}(W)$ changes from negative to positive.*

(iii) $\Delta NU_{j\widetilde{\omega}}(\widetilde{\tau}_k) = -1$ *iff the sign of $\mathcal{F}(W)$ changes from positive to negative.*

Proof of Lemma 3.4. First, letting $z = e^{-\lambda\tau}$, we rewrite the characteristic function (2.12) as $p(\lambda, z) = a_0(\lambda) + a_1(\lambda)z$. For each λ, we have a solution of z such that $p(\lambda, z) = 0$. Hence, we can obtain that $z = -a_0(\lambda)/a_1(\lambda)$, denoted by $z(\lambda)$. Here, we suppose $\widetilde{\lambda} = j\widetilde{\omega}$ is a critical imaginary root (calculated by $\mathcal{F}(W) = 0$, $\widetilde{W} = \widetilde{\omega}^2$ is the effective W root). The corresponding critical delays are $\widetilde{\tau}_k = \widetilde{\tau}_0 + \frac{2k\pi}{\widetilde{\omega}}$, where $\widetilde{\tau}_0$ is the minimum critical delay satisfying $z(j\widetilde{\omega}) = e^{-\tau\widetilde{\omega}j} = -a_0(j\widetilde{\omega})/a_1(j\widetilde{\omega})$. It follows that $|z(j\widetilde{\omega})| = 1$.

Next, for a $\widetilde{\lambda}$, it is clear that under a small perturbation $+\varepsilon j$, we hold that $|z(j(\widetilde{\omega} + \varepsilon))| = |a_0(j(\widetilde{\omega} + \varepsilon)/a_1(j(\widetilde{\omega} + \varepsilon)|$. Hence, $|z(j(\widetilde{\omega} + \varepsilon))| - |z(j\widetilde{\omega})| > 0 (< 0)$ iff $|a_0(j(\widetilde{\omega} + \varepsilon))| - |a_1(j(\widetilde{\omega} + \varepsilon))| > 0 (< 0)$, which is the same sign as $\mathcal{F}((\widetilde{\omega} + \varepsilon)^2)$. The above analysis also applies when the small perturbation is $-\varepsilon j$.

Finally, as W increases from $\widetilde{W} - \varepsilon$ to $\widetilde{W} + \varepsilon$ (ω increases from $\widetilde{\omega} - \varepsilon$ to $\widetilde{\omega} + \varepsilon$), in accordance with Section III(B) of [26] and Theorem 1 of [30], we can obtain the value of $\Delta NU_{j\widetilde{\omega}}(\widetilde{\tau}_k)$ by the sign of $|z(j(\widetilde{\omega} + \varepsilon))| - |z(j\widetilde{\omega})|$ and $|z(j(\widetilde{\omega} - \varepsilon))| - |z(j\widetilde{\omega})|$. Then the Lemma 3.4 can be proved. □

Following from Lemma 3.3 and Lemma 3.4, we can obtain the algebraic criterion for the complete stability problem of the delayed NN as follows:

Lemma 3.5. *When $\widetilde{W} = (\widetilde{\omega}_i^+)^2$ ($+$ is "o" or "e") is the effective W root and τ is the only free parameter for the delayed NN considered in this paper, we have the following three results.*

(1) For a $\lambda_i^e = j\widetilde{\omega}_i^e$ ($\widetilde{W} = W_i^e = (\widetilde{\omega}_i^e)^2$), $\Delta NU_{j\widetilde{\omega}_i^e}(\tau_{i,k}^e) = 0$ for all $\tau_{i,k}^e > 0$.

(2) For a $\lambda_i^o = j\widetilde{\omega}_i^o$ ($\widetilde{W} = W_i^o = (\widetilde{\omega}_i^o)^2$) with i is even, $\Delta NU_{j\widetilde{\omega}_i^o}(\tau_{i,k}^o) = +1$ for all $\tau_{i,k}^o > 0$.

(3) For a $\lambda_i^o = j\widetilde{\omega}_i^o$ ($\widetilde{W} = W_i^o = (\widetilde{\omega}_i^o)^2$) with i is odd, $\Delta NU_{j\widetilde{\omega}_i^o}(\tau_{i,k}^o) = -1$ for all $\tau_{i,k}^o > 0$.

(The λ_i^o is also listed in order from largest to smallest, with λ_0^o ($i = 0$) being the largest.)

Remark 3.6. The study on the stability of time-delay systems can be roughly classified into two types, namely a time domain approach and a frequency-domain approach. For the current study by the time-domain approach, the Lyapunov–Krasovskii method [14] is widely used, but it usually yield conservative results and cannot deal with the case when the delay is a free parameter. Besides the Lyapunov-based methods, the CTCR method [35], the direct method [39], and the matrix pencil method [10] can effectively analyze the stability of systems. However, it is difficult to apply these methods to deal with the asymptotic behavior of multiple CIRs.

In this paper, we address the complete stability problem of the delayed NN. Based on the perspective of characteristic root, the stability of systems is studied analytically. The conclusion of our approach is almost non-conservative, and it is very easy to obtain the sufficient and necessary conditions.

Remark 3.7. For the stability analysis of time-delay systems, the asymptotic behavior of multiple CIRs is essential (see e.g., [6, 11, 25, 26, 33]). It was recently pointed out in [25] that the appearance of a multiple and degenerate CIR may cause the system to become 'asymptotically stable' from 'unstable'. As far as we know, in the earlier references of delayed NNs, the case with multiple and/or degenerate CIRs has not been studied. Our method covers the case with multiple and/or degenerate CIRs, as such information is not a technical constraint for Lemma 3.5 (unlike for previous results).

In view of Lemma 3.5, $\Delta NU_{\lambda_\alpha}(\tau_{\alpha,k})$ is a constant belonging to the set $\{-1, 0, +1\}$. Next, we are able to derive the expression of $NU(\tau)$ in view of the root continuity argument.

Theorem 3.8. *Consider a delayed NN under consideration in this paper. For any $\tau > 0$ which is not a critical delay, $NU(\tau)$ for the characteristic equation $f(\lambda, \tau) = 0$ can be explicitly expressed as:*

$$NU(\tau) = NU(+\varepsilon) + \sum\nolimits_{i=0}^{q_o-1} NU_i^o(\tau), \tag{3.4}$$

$$NU_i^o(\tau) = \begin{cases} 0, \ \tau < \tau_{i,0}^o, \\ 2(-1)^i \left\lceil \frac{\tau - \tau_{i,0}^o}{2\pi/\omega_i^o} \right\rceil, \ \tau > \tau_{i,0}^o, \end{cases} \quad \text{if } \tau_{i,0}^o \neq 0,$$

$$NU_i^o(\tau) = \begin{cases} 0, \ \tau < \tau_{i,1}^o, \\ 2(-1)^i \left\lceil \frac{\tau - \tau_{i,1}^o}{2\pi/\omega_i^o} \right\rceil, \ \tau > \tau_{i,1}^o, \end{cases} \quad \text{if } \tau_{i,0}^o = 0.$$

(In light of Theorem 5.1 in [26], one can calculate the value of $NU(+\varepsilon)$.)

The equilibrium corresponding to the characteristic equation $f(\lambda, \tau) = 0$ is locally asymptotically stable if τ is not a CD and belongs to the set with $NU(\tau) = 0$.

In view of the "$NU(\tau)$ vs. τ" plot, the equilibrium may undergo a Hopf bifurcation (see [17] for the Hopf Bifurcation Theorem) as τ is increased near a CD. The stability interval(s) and bifurcation values of τ can be exhaustively obtained in the whole positive τ-axis.

As mentioned, a delayed NN may contain multiple stability τ-intervals including or excluding $\tau = 0$. Hence, we introduce a notion from [27] that can be applied in the general case.

For the case where τ is the only free parameter, without loss of generality, we assume that the non-empty stability τ-set is subject to the form

$$\tau \in (\underline{\tau}_1, \overline{\tau}_1) \cup \cdots \cup (\underline{\tau}_s, \overline{\tau}_s), \tag{3.5}$$

plus possible $\tau = 0$, with $0 \leq \underline{\tau}_1 < \overline{\tau}_1 < \cdots < \underline{\tau}_s < \overline{\tau}_s$. $\overline{\tau}_s$ is termed the *generalized delay margin*.

Example 3.9. Consider the delayed BAM NN (2.5) with the activation functions $f_i(\cdot) = \tanh(\cdot)$. It is true that $h_{ij} = c_{ij}$. Here, we choose the coefficients as: $\mu_1 = 2.46$, $\mu_2 = 4.5769$, $\mu_3 = 0.8561$, $\mu_4 = 0.9669$, $h_{12} = 4.1$, $h_{13} = -0.3896$, $h_{14} = 2.3488$, $h_{21} = -4.57$, $h_{31} = -2.8466$, and $h_{41} = 0.7057$.

First, we calculate that $NU(0) = 0$ by solving the characteristic equation $f(\lambda, 0) = 0$.

Then, according to Lemma 3.1, we have $\mathcal{F}(W) = W^4 + 28.6674W^3 - 82.5660W^2 - 0.0347W + 69.9926$.

By calculating the roots of equation $\mathcal{F}(W) = 0$, we have two sets of CIRs: $\lambda_0^o = 1.4869j$ (associated with the CDs $\tau_{0,k}^o = 1.7802 + 4.2258k$, $k \in \mathbb{N}$) and $\lambda_1^o = 1.1136j$ (associated with the CDs $\tau_{1,k}^o = 2.6181 + 5.6420k$, $k \in \mathbb{N}$).

Furthermore, we can obtain that $\Delta NU_{\lambda_0^o}(\tau_{0,k}^o) = +1$ and $\Delta NU_{\lambda_1^o}(\tau_{1,k}^o) = -1$ in view of Lemma 3.5. Next, according to Theorem 3.8, we have the "$NU(\tau)$ vs. τ" plot, as shown in Fig. 3.1.

Consequently, we know that the equilibrium is locally asymptotically stable if $\tau \in [0, 1.7802) \cup (2.6181, 6.0059) \cup (8.2602, 10.2317) \cup (13.9022, 14.4575)$ and that a Hopf bifurcation may occur when $\tau = 1.7802$, 2.6181, 6.0059, 8.2602, 10.2317, 13.9022 and 14.4575. To verify the above analysis, we give the simulations for $\tau = 1.2$ (in the first stability τ-interval) in Fig. 3.2 (a), Fig. 3.2 (d) and Fig. 3.3 (a), $\tau = 2.3$ (between the first and the second stability τ-intervals) in Fig. 3.2 (b), Fig. 3.2 (e) and Fig. 3.3 (b), and $\tau = 5$ (in the second stability τ-interval) in Fig. 3.2 (c), Fig. 3.2 (f) and Fig. 3.3 (c).

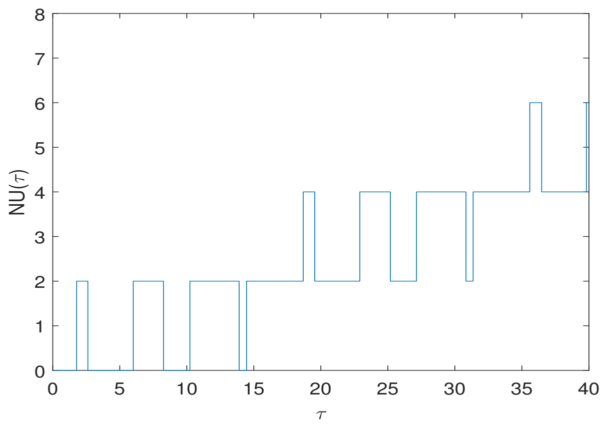
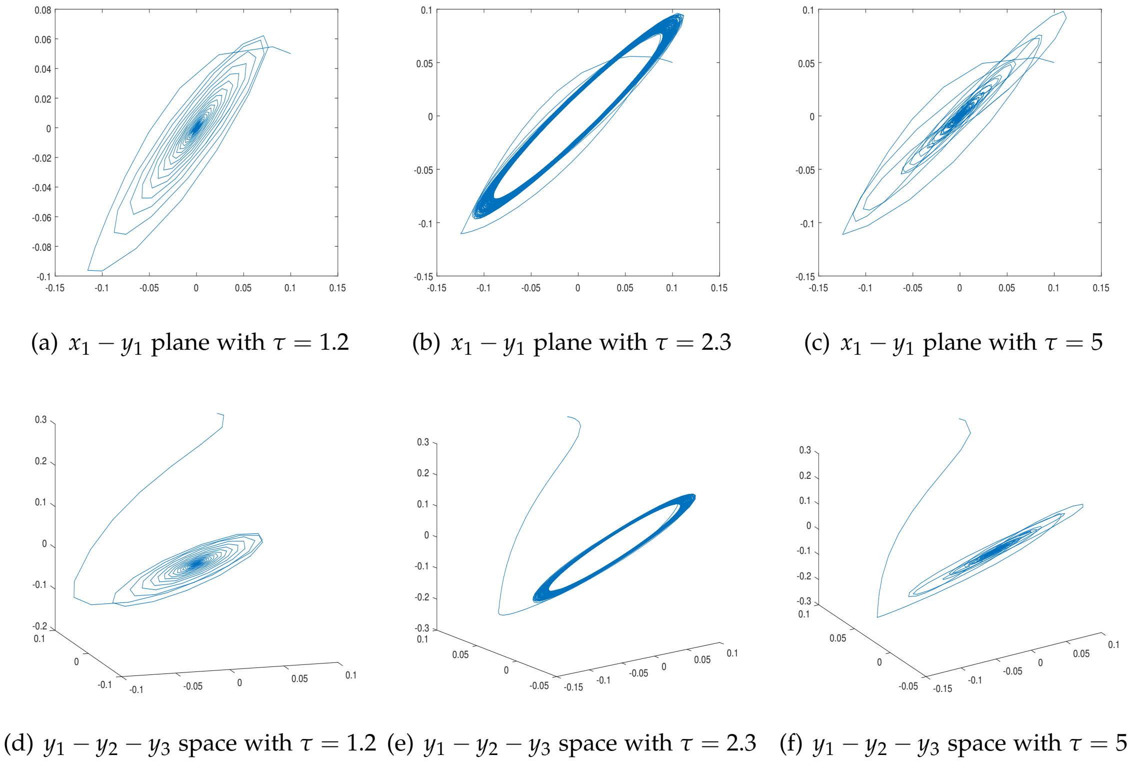


Figure 3.1: $NU(\tau)$ vs. τ for Example 3.9.

3.2 Complete root classification (CRC) of effective W roots

We proceed to address the scenario with additional free system parameters. In order to appropriately address such a case, we now investigate the real W root classification and the effective W root classification. The real (effective) W root classification refers to the information about the numbers and multiplicities of different real (effective) W roots. The complete root classification (CRC) of real (effective) W roots is the collection of all possible real (effective) W root classifications.

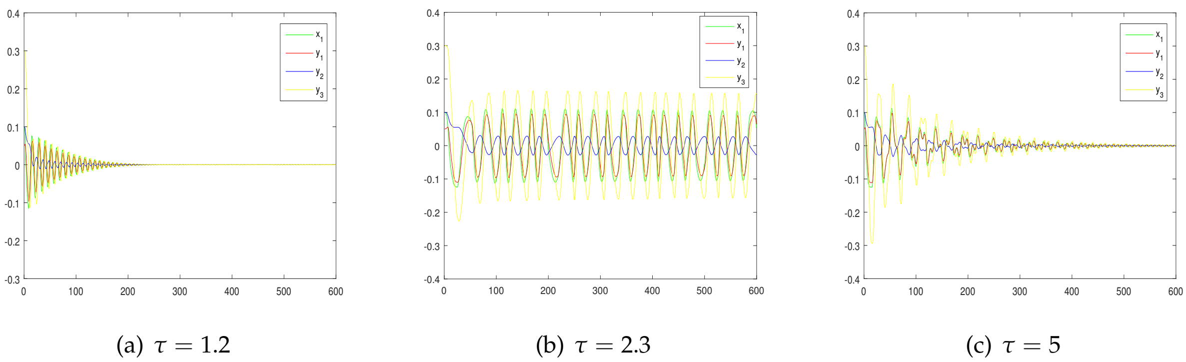
In the sequel, we recall a mathematical tool, termed the *discrimination system* (one may refer to [31] and [43] for details).



(a) $x_1 - y_1$ plane with $\tau = 1.2$ (b) $x_1 - y_1$ plane with $\tau = 2.3$ (c) $x_1 - y_1$ plane with $\tau = 5$

(d) $y_1 - y_2 - y_3$ space with $\tau = 1.2$ (e) $y_1 - y_2 - y_3$ space with $\tau = 2.3$ (f) $y_1 - y_2 - y_3$ space with $\tau = 5$

Figure 3.2: Phase portraits for Example 3.9.



(a) $\tau = 1.2$ (b) $\tau = 2.3$ (c) $\tau = 5$

Figure 3.3: Simulations for Example 3.9.

Let $\mathcal{Q}(z)$ be a polynomial with real coefficients,

$$\mathcal{Q}(z) = \zeta_q z^q + \zeta_{q-1} z^{q-1} + \cdots + \zeta_0, \zeta_q \neq 0. \tag{3.6}$$

The $2q \times 2q$ matrix $M =$

$$\begin{pmatrix}
\zeta_q & \zeta_{q-1} & \zeta_{q-2} & \cdots & \zeta_0 & & & \\
0 & q\zeta_q & (q-1)\zeta_{q-1} & \cdots & \zeta_1 & & & \\
 & \zeta_q & \zeta_{q-1} & \cdots & \zeta_1 & \zeta_0 & & \\
 & 0 & q\zeta_q & \cdots & 2\zeta_2 & \zeta_1 & & \\
 & & & \vdots & \vdots & & & \\
 & & & \zeta_q & \zeta_{q-1} & \zeta_{q-2} & \cdots & \zeta_0 \\
 & & & 0 & q\zeta_q & (q-1)\zeta_{q-1} & \cdots & \zeta_1
\end{pmatrix}$$

is the *discrimination matrix* of $\mathcal{Q}(z)$.

The determinant of the submatrix of M consisting of the first 2α rows and the first 2α columns $(\alpha = 1, \dots, q)$ is represented by D_α. We call the *discriminant sequence* of $\mathcal{Q}(z)$ as $D = [D_1, \dots, D_q]$.

In light of the signs of $D_1, \dots, D_q$, we can obtain the sign list $[\text{sign}(D_1), \dots, \text{sign}(D_q)]$ (sign(x)= 1(0)[-1] if $x > 0 (= 0)[< 0]$) and then the revised sign list for $\mathcal{Q}(z)$ (see [31] for more details).

Without loss of generality, let $\mathcal{F}(W)$ (3.2) be a qth-order polynomial

$$\mathcal{F}(W) = \psi_q W^q + \psi_{q-1} W^{q-1} + \cdots + \psi_0, \psi_q \neq 0, \tag{3.7}$$

where $\psi_q, \psi_{q-1}, \dots, \psi_0$ are coefficients. These coefficients are allowed to be functions of the free system parameters denoted by the vector X.

As for the delayed Hopfield NN model where $\deg(a_0(\lambda)) = 3$, the sign list of $\mathcal{F}(W)$ is [sign(D_1),sign(D_2),sign(D_3)]. If sign(D_2) = 0, the revised sign list is [sign(D_1), -1, sign(D_3)]. Otherwise, the revised sign list is exactly the sign list.

As for the delayed BAM NN model where $\deg(a_0(\lambda)) = 4$, the sign list of $\mathcal{F}(W)$ is [sign(D_1),sign(D_2),sign(D_3),sign(D_4)]. If sign(D_i) = 0, $i = 2,3$, the revised sign list is listed in Table 3.1. Otherwise, the revised sign list is exactly the sign list.

sign(D_i) = 0	The revised sign list
sign(D_2) = 0	[sign(D_1), -1, sign(D_3), sign(D_4)]
sign(D_3) = 0	[sign(D_1), sign(D_2), -1, sign(D_4)]
sign(D_2) = 0 and sign(D_3) = 0	[sign(D_1), -1, -1, sign(D_4)]

Table 3.1: The revised sign list with sign(D_i) = 0, $i = 2,3$ for the delayed BAM NN model.

As for the delayed annular NN model where $\deg(a_0(\lambda)) = 5$, the sign list of $\mathcal{F}(W)$ is [sign(D_1),sign(D_2),sign(D_3),sign(D_4),sign(D_5)]. If sign(D_i) = 0, $i = 2,3,4$, the revised sign list is listed in Table 3.2. Otherwise, the revised sign list is exactly the sign list.

sign(D_i) = 0	The revised sign list
sign(D_2) = 0	[sign(D_1), -1, sign(D_3), sign(D_4), sign(D_5)]
sign(D_3) = 0	[sign(D_1), sign(D_2), -1, sign(D_4), sign(D_5)]
sign(D_4) = 0	[sign(D_1), sign(D_2), sign(D_3), -1, sign(D_5)]
sign(D_2) = 0 and sign(D_3) = 0	[sign(D_1), -1, -1, sign(D_4), sign(D_5)]
sign(D_2) = 0 and sign(D_4) = 0	[sign(D_1), -1, sign(D_3), -1, sign(D_5)]
sign(D_3) = 0 and sign(D_4) = 0	[sign(D_1), sign(D_2), -1, -1, sign(D_5)]
sign(D_2) = 0, sign(D_3) = 0, and sign(D_4) = 0	[sign(D_1), -1, -1, 1, sign(D_5)]

Table 3.2: The revised sign list with sign(D_i) = 0, $i = 2,3$ for the delayed BAM NN model.

Proposition 3.10 ([43]). *Suppose that the revised sign list of $\mathcal{F}(W)$ has l non-vanishing members and the number of sign changes is v. It follows that the number of distinct real roots of $\mathcal{F}(W)$ is $l - 2v$.*

For example, if the revised sign list for $\mathcal{F}(W)$ is $[1, 1, -1]$, then $v = 1$, $l = 3$, and $\mathcal{F}(W)$ has $l - 2v = 1$ real W root. If the revised sign list for $\mathcal{F}(W)$ is $[1, 1, 1, 0]$, then $v = 0$, $l = 3$, and F(W) has $l - 2v = 3$ distinct real W roots. If the revised sign list for $\mathcal{F}(W)$ is $[1, -1, -1, 1, 0]$, then $v = 2$, $l = 4$, and $\mathcal{F}(W)$ has $l - 2v = 0$ real W root.

Applying the discrimination system to $\mathcal{F}(W)$ (3.7), one can obtain the discriminant sequence $D = [D_1, \ldots, D_q]$ and then the CRC of real W roots.

As earlier mentioned, we are only concerned with the positive real W roots (since $W = \omega^2$), i.e., the effective W roots. We will present a method to examine the CRC of effective W roots.

Theorem 3.11. *For the auxiliary characteristic function $\mathcal{F}(W)$ (3.7), consider a region partitioned in view of the CRC of real W roots. If the set $\psi_0 = 0$ separates this region into some subregions in X space, then the positive real W root classification keeps unchanged in each subregion. Otherwise, in the whole region, the positive real root classification remains unchanged.*

Proof of Theorem 3.11. For $\mathcal{F}(W)$, there exists a root $W = 0$ iff $\psi_0 = 0$. If a point in X space continuously moves in a region partitioned in view of the CRC of real W roots, all the real W roots' signs remain unchanged if it does not cross the set $\psi_0 = 0$. □

For each region partitioned in accordance with the CRC of real W roots, we can obtain the effective W root classifications. According to Theorem 3.11, for each region not separated (or subregion separated) by the set $\psi_0 = 0$, we can select any point to solve $\mathcal{F}(W) = 0$ and this point represents the effective W root classification for all the points in the region (subregion).

We follow the notations adopted in [43]. For example: "$\{1, 1, 1, 1\}$" stands for four different simple real W roots; "$\{2, 1, 1\}$" stands for one double real W root plus two simple real W roots; "$\{\ \}$" stands for no real W root. Furthermore, the effective W root classification is represented by $\{\cdot, \ldots, \cdot\}^+$. For example, if a real root classification $\{2, 1, 1\}$ involves one double *negative* real W root and two different simple *positive* real W roots, it is represented by $\{1, 1\}^+$ (two different simple effective W roots).

Example 3.12. Consider the delayed BAM NN of Example 3.9. Here, we choose h_{12} and h_{21} as free system parameters, i.e., $X = (h_{12}, h_{21})$. For simplicity, we study the case $(h_{12}, h_{21}) \in [1, 6] \times [-6, -1]$.

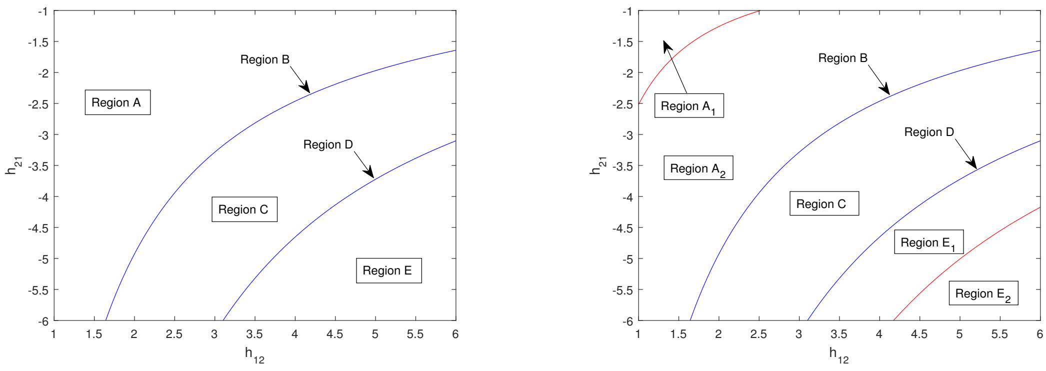
First, according to Lemma 3.1, we obtain $\mathcal{F}(W) = W^4 + 28.6674W^3 + (172.4842 - (h_{12}h_{21} + 2.7666)^2)W^2 + ((1.6555h_{12}h_{21} + 22.8053)(h_{12}h_{21} + 2.7666) - (1.8230h_{12}h_{21} + 15.1537)^2 + 229.9256)W + 86.8611 - (0.8278h_{12}h_{21} + 11.4027)^2$.

Then, applying the discrimination system, the discriminant sequence of $\mathcal{F}(W)$ is $[D_1, D_2, D_3, D_4]$, where $D_1 = 4$, $D_2 = 8h_{12}^2h_{21}^2 + 44.2653h_{12}h_{21} + 1146.8204$, $D_3 = 8h_{12}^6h_{21}^6 + 132.7960h_{12}^5h_{21}^5 - 316.9872h_{12}^4h_{21}^4 + 2920.3114h_{12}^3h_{21}^3 + 114921.2317h_{12}^2h_{21}^2 + 801885.8568h_{12}h_{21} + 8309682.1306$, and $D_4 = 0.1632h_{12}^{10}h_{21}^{10} + 11.7962h_{12}^9h_{21}^9 + 1602.5531h_{12}^8h_{21}^8 + 46407.3671h_{12}^7h_{21}^7 + 418378.3143h_{12}^6h_{21}^6 + 2094423.4796h_{12}^5h_{21}^5 + 46116871.4106h_{12}^4h_{21}^4 + 738951226.2778h_{12}^3h_{21}^3 + 6714650368.4108h_{12}^2h_{21}^2 + 40072389757.5584h_{12}h_{21} + 111789468423.0607$.

For this example, $D_1 > 0$, $D_2 > 0$ (one may easily prove it), D_3 and D_4 may represent different signs w.r.t. $X = (h_{12}, h_{21})$. We can obtain the CRC of real W roots and thereby partition the selected domain into five regions, as shown in Fig. 3.4 (a).

Region A with $\{1, 1, 1, 1\}$: $D_3 > 0 \cap D_4 > 0$; Region B with $\{2, 1, 1\}$: $D_3 > 0 \cap D_4 = 0$; Region C with $\{1, 1\}$: $(D_3 > 0 \cap D_4 < 0) \cup (D_3 = 0 \cap D_4 < 0) \cup (D_3 < 0 \cap D_4 < 0)$; Region D with $\{2, 1, 1\}$: $D_3 > 0 \cap D_4 = 0$; Region E with $\{1, 1, 1, 1\}$: $D_3 > 0 \cap D_4 > 0$.

Second, the set $\psi_0 = 0$, i.e., the set $86.8611 - (0.8278h_{12}h_{21} + 11.4027)^2 = 0$, corresponds to the red curves in Fig. 3.4 (b).



(a) Real W root classifications (b) Effective W root classifications

Figure 3.4: CRCs of real and effective W roots.

Region A is separated by the set $\psi_0 = 0$ into subregions A_1 and A_2, and Region E is separated by the set $\psi_0 = 0$ into subregions E_1 and E_2.

In accordance with Theorem 3.11, we have the following results: Region A_1: $\{1,1,1,1\} \to \{1\}^+$; Region A_2: $\{1,1,1,1\} \to \{\ \}^+$; Region B: $\{2,1,1\} \to \{\ \}^+$; Region C: $\{1,1\} \to \{\ \}^+$; Region D: $\{2,1,1\} \to \{2\}^+$; Region E_1: $\{1,1,1,1\} \to \{1,1\}^+$; Region E_2: $\{1,1,1,1\} \to \{1\}^+$.

Furthermore, we can combine Regions A_2, B, and C into one region, labeled by Region B'. Consequently, we may partition the domain into five regions: Regions A_1, B', D, E_1, and E_2. Region D is the boundary between Regions B' and E_1, whose analytic condition is $D_3 > 0 \cap D_4 = 0$.

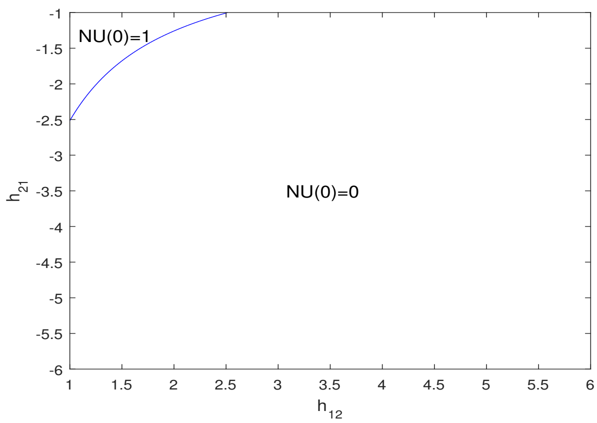


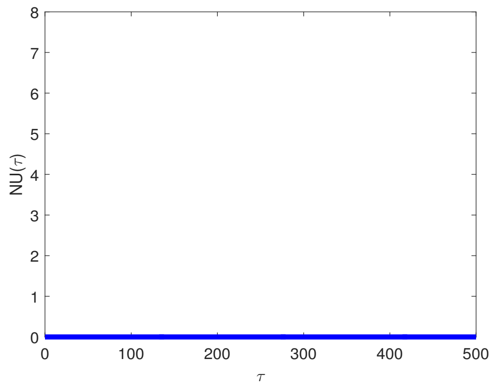
Figure 3.5: $NU(0)$ distribution w.r.t. (h_{12}, h_{21}) for Example 3.12.

Next, based on the equation $f(\lambda, 0) = 0$, $NU(0)$ distribution w.r.t. (h_{12}, h_{21}) is obtained (see Fig. 3.5).

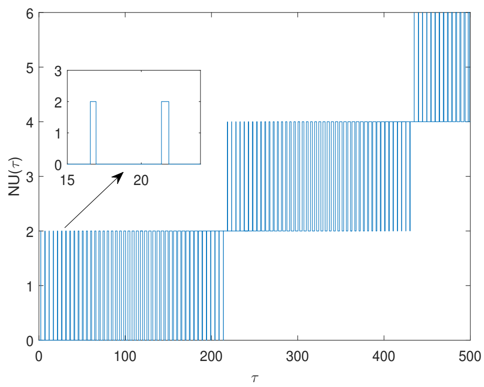
According to the above analysis, we can obtain the following results:

(1) In Region A_1 ($NU(0) = 1$): As there does not exist an effective W root, the system is not asymptotically stable for any $\tau \in [0, +\infty)$.

(2) In Region B' ($NU(0) = 0$): As there does not exist an effective W root, the system is delay-independently stable.



(a) $h_{21} = -4.5396$



(b) $h_{21} = -4.5397$

Figure 3.6: $NU(\tau)$ vs. τ for Example 3.12.

(3) On Region D ($NU(0) = 0$): As there exists a double effective W root, the system is asymptotically stable for all $\tau \geq 0$ except at the CDs.

(4) In Region E_1 ($NU(0) = 0$): As there exist two simple effective W roots, the system may contain more than one stability τ-interval including $\tau = 0$.

(5) In Region E_2 ($NU(0) = 0$): As there exists one simple effective W root, the system has only one stability τ-interval of the form $[0, \overline{\tau}_1)$.

For illustrations, we choose a point ($h_{12} = 4.1, h_{21} = -4.5396$) in Region B' and a point ($h_{12} = 4.1, h_{21} = -4.5397$) in Region E_1. The corresponding "$NU(\tau)$ vs. τ" plots are shown in Fig. 3.6 (a) and Fig. 3.6 (b), respectively. We can see that the $NU(\tau)$ distribution undergoes a structural variation due to a very small change of h_{21}.

Remark 3.13. The analytic condition for the boundary of different effective W root classifications is available. As the $NU(\tau)$ distribution does not have a structural variation inside a region with identical effective W root classification, we may study the $NU(\tau)$ distribution by employing the parameter-sweeping technique. It is worth noting that such a technique is not for the qualitative test, and hence it is not necessary to set a very fine grid in practice.

3.3 Systematic approach for determining stability set in (X, τ)-space

Based on Theorems 3.8 and 3.11, we now propose a systematic approach to investigate the stability of delayed NNs in the (X, τ)-space, consisting of the following steps.

Step 0: Linearize the delayed NN under consideration at the equilibrium and calculate the characteristic function $f(\lambda, \tau)$. Next, obtain the analytic expression of auxiliary characteristic function $\mathcal{F}(W)$.

Step 1: Obtain the CRC of real W roots in the entire X space with the aid of discrimination system.

Step 2: Determine all possible effective W root classifications in accordance with Theorem 3.11 in the whole X space.

Step 3: Obtain the $NU(0)$ distribution in the X space.

Step 4: Scan the X space. For each point, investigate the complete stability problem w.r.t. τ using Theorem 3.8.

With the steps above, we are able to determine the stability set in the entire (X,τ)-space. Moreover, we can characterize all classifications of stability property in the entire (X,τ)-space.

4 Illustrative examples

In this section, we will give some examples with different NN architectures to illustrate the proposed approach. We adopt the widely-used hyperbolic tangent functions as the activation functions, i.e., $f_i(\cdot) = \tanh(\cdot)$, $g_i(\cdot) = \tanh(\cdot)$, $i = 1,2,\dots,n$. Then, $f_i(0) = g_i(0) = 0$ and $f_i'(0) = g_i'(0) = 1$, $i = 1,2,\dots,n$.

We will show that an NN with different coefficients can exhibit different classifications of stability property. It is worth mentioning that a very small change of the vector X may make the $NU(\tau)$ distribution have a structural variation, which may alter the classification of stability property. This phenomenon demonstrates the necessity of dividing the parameter space.

Example 4.1. Consider the delayed Hopfield NN (2.2). With the activation functions adopted in this section, it is true that $h_{ij} = c_{ij}$. Here, we choose the coefficients as: $\mu_1 = 1.4147, \mu_2 = 1.0102, \mu_3 = 0.4610$, $h_{12} = 1.5176$, $h_{23} = -1.4432$, $h_{32} = 0.9080$. We let $X = (h_{21})$.

Step 0: The characteristic function is given by (2.3). In light of Lemma 3.1, we have $\mathcal{F}(W) = W^3 + 3.2344W^2 + (-2.3031h_{21}^2 + 3.9774h_{21} + 0.9674)W - 0.4895h_{21}^2 + 2.5940h_{21} - 3.0027$.

Step 1: The discriminant sequence is $[D_1, D_2, D_3]$, where $D_1 = 3$, $D_2 = 13.8187h_{21}^2 - 23.8644h_{21} + 15.1182$, and $D_3 = 48.8657h_{21}^6 - 253.1688h_{21}^5 + 490.2890h_{21}^4 - 623.2539h_{21}^3 + 742.0196h_{21}^2 - 443.8572h_{21} + 0.0105$.

For this example, $D_1 > 0$, $D_2 > 0$ (one may easily prove it), and D_3 may represent different signs w.r.t. h_{21}. We can obtain the CRC of real W roots: (1) {1,1,1} iff $h_{21} \in (-\infty, 0.00002373) \cup (1.265628, 1.766693) \cup (2.341787, +\infty)$ (where $D_3 > 0$); (2) {1} iff $h_{21} \in (0.00002373, 1.265628) \cup (1.766693, 2.341787)$ (where $D_3 < 0$); (3) {2,1} iff $h_{21} \in \{0.00002373, 1.265628, 1.766693, 2.341787\}$ (where $D_3 = 0$).

Step 2: When $\psi_0 = 0$, i.e., $-0.4895h_{21}^2 + 2.5940h_{21} - 3.0027 = 0$, we can calculate that $h_{21} = 1.708129$ or $h_{21} = 3.591537$. Hence, the intervals $(1.265628, 1.766693)$ and $(2.341787, +\infty)$ are respectively separated into $(1.265628, 1.708129)$, $(1.708129, 1.766693)$, $(2.341787, 3.591537)$, and $(3.591537, +\infty)$.

Furthermore, in light of Theorem 3.11, we can obtain the CRC of effective W roots:

(1) $h_{21} \in (-\infty, 1.708129) \cup (3.591537, +\infty)$, $\{1\}^+$;

(2) $h_{21} \in (1.708129, 2.341787)$, $\{\ \}^+$;

(3) $h_{21} = \{2.341787\}$, $\{2\}^+$;

(4) $h_{21} \in (2.341787, 3.591537)$, $\{1,1\}^+$.

Step 3: We analyze the $NU(0)$ distribution w.r.t. h_{21} by solving the characteristic equation $f(\lambda, 0) = 0$. We have that when $h_{21} \in (-\infty, 2.342233)$, $NU(0) = 0$.

In view of the above analysis, we have the following results:

(1) The system contains only 1 stability τ-interval of the form $[0, \overline{\tau}_1)$ iff $h_{21} \in (-\infty, 1.708129)$;

(2) The system is delay-independently stable iff $h_{21} \in (1.708129, 2.341787)$;

(3) The system is asymptotically stable for all $\tau \geq 0$ except at the CDs iff $h_{21} = 2.341787$;

(4) The system may contain more than one stability τ-interval including $\tau = 0$ iff $h_{21} \in (2.341787, 2.342233)$;

(5) The system may contain stability τ-interval(s) excluding $\tau = 0$ iff $h_{21} \in (2.342233, 3.591537)$;

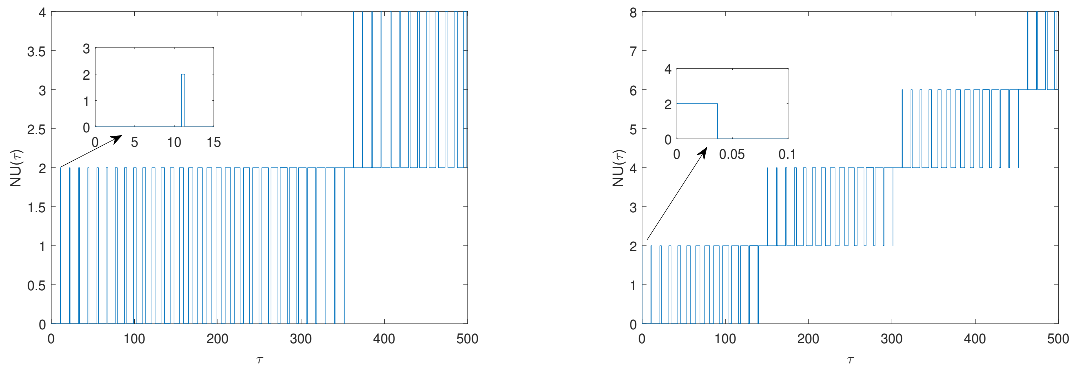
(6) The system has no stability τ-interval iff $h_{21} \in (3.591537, +\infty)$.

It can be seen that all classifications of stability property can be characterized in the whole (h_{21},τ)-plane. To illustrate the above results and to show the qualitative change of stability property, we next choose some values of h_{21} and give the detailed results.

At point $h_{21} = 1.70$, the system contains only 1 stability τ-interval $[0, 32.1647)$. At point $h_{21} = 1.71$, the system is delay-independently stable.

At point $h_{21} = 2.342$, the system has 32 stability τ-intervals including $\tau = 0$ and the generalized delay margin is 351.8083. At point $h_{21} = 2.343$, the system has 13 stability τ-intervals excluding $\tau = 0$ and the generalized delay margin is 139.7930. To intuitively show the difference, we give "$NU(\tau)$ vs. τ" plots in Fig. 4.1 (a) and Fig. 4.1 (b), respectively.

At point $h_{21} = 3.59$, the system contains only 1 stability τ-interval $(2.4170, 2.9154)$. At point $h_{21} = 3.60$, no stability τ-interval exists.



(a) $h_{21} = 2.342$ (b) $h_{21} = 2.343$

Figure 4.1: $NU(\tau)$ vs. τ for Example 4.1.

Step 4: We can address the whole (h_{21},τ)-plane and exhaustively determine the stability set. For a clear demonstration, we provide the stability set with $(h_{21}, \tau) \in [-2, 5] \times [0, 50]$ in Fig. 4.2. □

Example 4.2. For the delayed BAM NN (2.5) in Example 3.12, we here choose a domain $(h_{12}, h_{21}) \in [4.1, 4.7] \times [-4.9, -4.57]$ for a clear illustration. Scan the (h_{12}, h_{21})-plane, and for each point we calculate the stability set by using Theorem 3.8. Then we can exhaustively determine the stability set, as shown in Fig. 4.3.

It is interesting to see in Fig. 4.3 that the stability parameter space of the delayed BAM NN has multiple disjoint parts.

Using the approach proposed in Subsection 3.3, we can also obtain the stability set in the corresponding 4-D parameter space, as reflected in Fig. 4.4. We here choose a domain

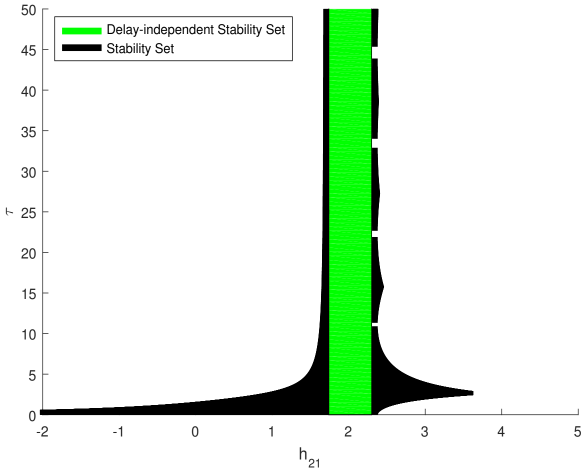


Figure 4.2: Stability set in the (h_{21}, τ)-plane for Example 4.1.

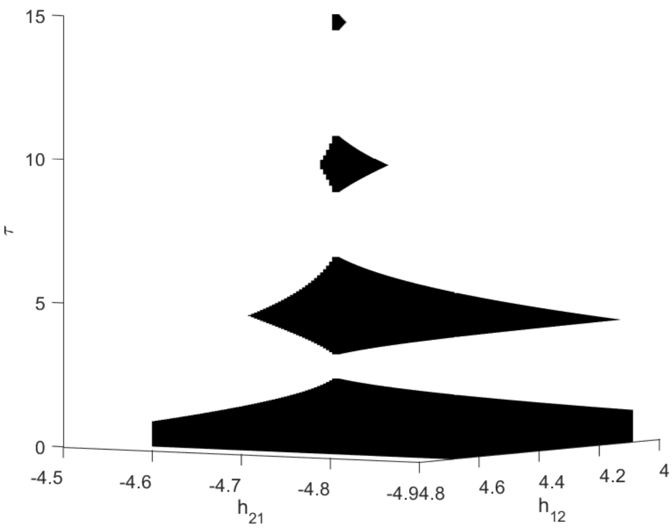


Figure 4.3: Stability set in the (h_{12}, h_{21}, τ)-space for Example 4.2.

$(h_{12}, h_{13}, h_{14}) \in [4.1, 4.2] \times [-0.38, -0.36] \times [2.2, 2.4]$ and the color information represents the generalized delay margin. Here, the maximum generalized delay margin is 14.5923. □

Example 4.3. Consider the delayed annular NN (2.9) with coefficients as in [8] and [46]: $\mu_i = 2, \alpha_i = 1, i = 1, \ldots, 5$, $\beta_1 = 2, \beta_2 = 1, \beta_3 = 1, \beta_4 = 1.3$, and $\beta_5 = -0.5$.

With the activation functions adopted in this section, it is true that $s_i = \alpha_i$ and $h_i = \beta_i$. Here, we let $X = (s_1)$.

Step 0: The characteristic function is given by (2.11). In light of Lemma 3.1, we have that $\mathcal{F}(W) = W^5 + (s_1^2 - 4s_1 + 8)W^4 + (4s_1^2 - 16s_1 + 22)W^3 + (6s_1^2 - 24s_1 + 28)W^2 + (4s_1^2 - 16s_1 + 17)W + s_1^2 - 4s_1 + 2.31$.

Step 1: The discriminant sequence is $[D_1, D_2, D_3, D_4, D_5]$, where $D_1 = 5$, $D_2 = 4s_1^4 - 32s_1^3 + 88s_1^2 - 96s_1 + 36$, $D_3 = 0$, $D_4 = -285.61s_1^4 + 2284.88s_1^3 - 6283.42s_1^2 + 6854.64s_1 - 2570.49$, and $D_5 = -1235.6631s_1^{10} + 24713.2621s_1^9 - 216241.0432s_1^8 + 1087383.5315s_1^7 - 3472213.3222s_1^6 +$

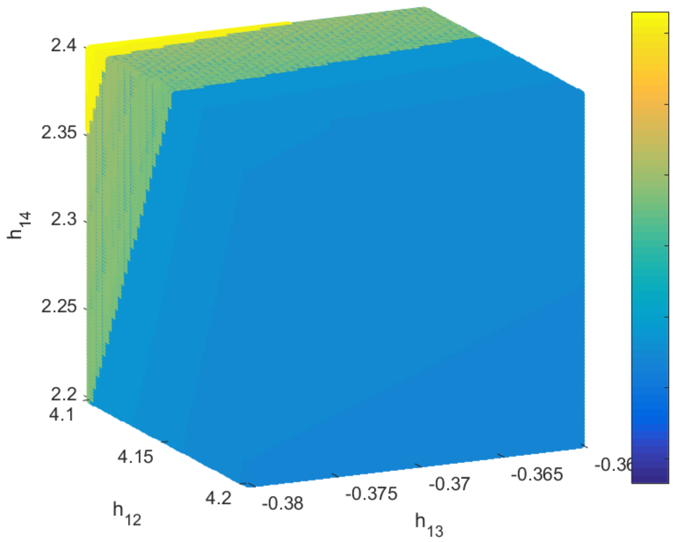


Figure 4.4: Generalized delay margin w.r.t. (h_{12}, h_{13}, h_{14}).

$7344781.4902s_1^5 - 10416639.9667s_1^4 + 9786451.7837s_1^3 - 5838508.1664s_1^2 + 2001774.2285s_1 - 274774.5492$.

For this example, $D_1 > 0$, $D_2 \geq 0$, $D_3 = 0$, $D_4 \leq 0$ (one may easily prove the properties concerning the signs of D_2 and D_4), and D_5 may represent different signs w.r.t. s_1. We can obtain the CRC of real W roots: (1) {1,1,1} iff $s_1 \in (-\infty, 0.3172) \cup (3.6828, +\infty)$ (in the case $D_1 > 0, D_2 > 0, D_3 = 0, D_4 < 0, D_5 < 0$); (2) {1} iff $s_1 \in (0.3172, 3.6828)$ (in the case $D_1 > 0, D_2 > 0, D_3 = 0, D_4 < 0, D_5 > 0$ or $D_1 > 0, D_2 = 0, D_3 = 0, D_4 = 0, D_5 > 0$); (3) {2,1} iff $s_1 \in \{0.3172, 3.6828\}$ (in the case $D_1 > 0, D_2 > 0, D_3 = 0, D_4 < 0, D_5 = 0$).

Step 2: When $\psi_0 = 0$, i.e., $s_1^2 - 4s_1 + 2.31 = 0$, we can calculate that $s_1 = 0.7$ or $s_1 = 3.3$. Hence, the interval $(0.3172, 3.6828)$ is separated into $(0.3172, 0.7)$, $(0.7, 3.3)$, and $(3.3, 3.6828)$.

Furthermore, in light of Theorem 3.11, we can obtain the CRC of effective W roots:

(1) $s_1 \in (0.7, 3.3)$, $\{1\}^+$;

(2) $s_1 \in (-\infty, 0.7) \cup (3.3, +\infty)$, $\{\}^+$.

Step 3: We analyze the $NU(\tau)$ distribution w.r.t. s_1 through solving the characteristic equation $f(\lambda, 0) = 0$. We have that when $s_1 \in (-\infty, 1.6138356)$, $NU(0) = 0$.

In view of the above analysis, we have that there are three classifications of stability property as described below:

(1) The system is delay-independently stable iff $s_1 \in (-\infty, 0.7)$;

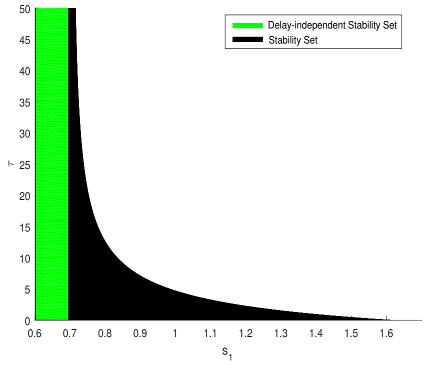
(2) The system contains only one stability τ-interval of the form $[0, \overline{\tau}_1)$ iff $s_1 \in (0.7, 1.6138356)$;

(3) The system has no stability τ-interval iff $s_1 \in (1.6138356, +\infty)$.

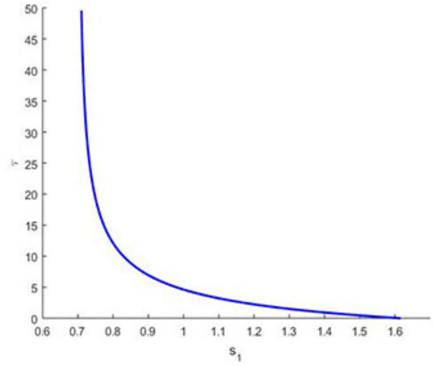
Step 4: We can study the whole (s_1,τ)-plane and exhaustively determine the stability set. For a clear demonstration, we provide the stability set with $(s_1, \tau) \in [0.6, 1.7] \times [0, 50]$ in Fig. 4.5 (a).

It can be seen that the system has one stability τ-interval iff $s_1 \in (0.7, 1.6138356)$. The boundary of the corresponding stability region is shown in Fig. 4.5 (b).

To illustrate our analysis, we choose some values of s_1 and analyze the stability of the system. We list the details in Table 4.1. □



(a) Stability set



(b) Boundary of stability set

Figure 4.5: Stability set and the boundary curve for Example 4.3.

s_1	stability τ-set	s_1	stability τ-set	s_1	stability τ-set
0.69	$[0, +\infty)$	0.99	$[0, 4.7956)$	1.61	$[0, 0.0139)$
0.71	$[0, 49.4106)$	1	$[0, 4.6174)$	1.62	ϕ
0.72	$[0, 33.4746)$	1.01	$[0, 4.4477)$	1.63	ϕ

Table 4.1: Stability τ-set for Example 4.3.

5 Conclusion and future work

In this paper, we considered the stability of delayed neural networks (NNs) with a free delay parameter τ and a free system parameter vector X. As far as we know, the complete stability problem of delayed NNs involving heterogeneous free parameters has not been well investigated.

We proposed a systematic method to investigate the stability in the (X, τ)-space. As a consequence, we can exhaustively determine the stability set in the whole (X, τ)-space. The effectiveness of the approach is illustrated by some numerical examples. It is interesting to see that the stability parameter space of a delayed NN may have multiple disjoint parts.

Based on the research of this paper, we may further investigate more general delayed NNs. In this paper, we focus on the case where the characteristic functions are in the form $f(\lambda, \tau) = a_0(\lambda) + a_1(\lambda)e^{-\tau\lambda}$. A more general form is $f(\lambda, \tau) = a_0(\lambda) + a_1(\lambda)e^{-\tau\lambda} + \cdots + a_q(\lambda)e^{-q\tau\lambda}$, where $a_0(\lambda), \ldots, a_q(\lambda)$, $q \in \mathbb{N}_+$, are polynomials. The stability analysis when $q > 1$ will be much more complicated. In the future, we would extend the approach to such a case.

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