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Systems of second order Stieltjes differential equations with three-point boundary conditions

Bianca Satco ✉

Stefan cel Mare University of Suceava, Faculty of Electrical Engineering and Computer Science, Universitatii 13, 720225, Suceava, Romania

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Abstract. In this paper, existence results are obtained for a system of second order boundary value problems

$$\begin{cases} u''_{\overline{g}}(t) = f(t, u(t), u'_{\overline{g}}(t)), \ \mu_{\overline{g}}\text{-a.e. } t \in [0,1) \\ u(0) = \overline{\alpha} \\ u(1) + ku(\theta) = \overline{\beta}, \end{cases}$$

with Stieltjes derivatives with respect to different derivators on different coordinates.

Using an appropriate Green function and Schauder's fixed point theorem, a very general existence result is obtained, thus overcoming several outcomes in literature.

Keywords: Stieltjes derivative, three-point boundary value problem, second order equation.

2020 Mathematics Subject Classification: 34A06, 34B10, 26A24, 26A42.

1 Introduction

The interest for second order differential equations with multi-point boundary conditions has robust roots in engineering (e.g. [17]) and it was, therefore, intensively studied. We refer the reader to [17, 31, 33] and the references therein and to [7] or [1, 24] for the framework of impulsive problems, respectively of dynamic equations on time scales.

At the same time, problems involving the Stieltjes derivative with respect to nondecreasing functions have recently found significant applications in studying real processes where stationary intervals and abrupt changes are equally present (see [11–13, 18] or [19]); this kind of behaviour was investigated through the theory of measure differential equations (e.g. [6]) and inclusions (for instance, [4] or [5]). Starting with the paper [16] (following an idea in [32]), a more convenient, equivalent writing using the Stieltjes derivative became increasingly popular (see [8, 9, 11, 12, 20, 23, 27, 28] for the single-valued case or [26, 29] for the set-valued setting).

✉ Email: bisatco@usm.ro

Second order equations with Stieltjes derivative were considered, to the best of the author's knowledge, only in the linear case with initial value boundary conditions in [8, 9] and in the nonlinear case with periodic conditions in [22].

When speaking about systems of differential problems, as described in [12, 14, 19] or [21], it is important to allow the variation speed of different unknown variables to depend on different nondecreasing functions and thus, to have different impulsive moments or stationary intervals of time; this means to consider a vector-valued derivator.

With all these in mind, we hereby develop a study of systems of nonlinear second order Stieltjes differential equations with three-point boundary conditions and several derivators:

$$\begin{cases} u''_{\overline{g}}(t) = f(t, u(t), u'_{\overline{g}}(t)), \ \mu_{\overline{g}}\text{-a.e. } t \in [0,1) \\ u(0) = \overline{\alpha} \\ u(1) + ku(\theta) = \overline{\beta}. \end{cases}$$

Inspired by [33], a Green's function is adapted to the present framework, allowing one to get existence and uniqueness for single-valued linear second order equations on the real line. A fixed point theorem is then applied to get an existence result in the nonlinear case for systems of second order Stieltjes differential problems with three-point boundary conditions and several derivators.

It is well-known that the theory of Stieltjes differential equations is strongly related to other types of problems: generalized differential equations ([30]), impulsive differential equations ([16]) or dynamic problems on time scales ([6]); consequently, the outcomes presented here can be used to get existence and uniqueness results for systems of second-order equations with three-point boundary conditions in the mentioned settings.

2 Notations and auxiliary results.

Let $g : [0,1] \to \mathbb{R}$ be a nondecreasing left-continuous function. Without any loss of generality, one may suppose that $g(0) = 0$. The g-measurability means the measurability with respect to (shortly, w.r.t.) the σ-algebra defined by g, μ_g is the Stieltjes measure generated by g (see [10]) and the Lebesgue–Stieltjes (shortly, LS-) integrability w.r.t. g is the abstract Lebesgue integrability w.r.t. the measure μ_g. Let $L^1_g([0,1])$ be the space of real LS-integrable functions w.r.t. g with its natural topological structure given by the norm

$$\|f\|_1 = \int_{[0,1)} |f(t)| dg(t).$$

Consider the following sets:

$$D_g = \{t \in [0,1] : g(t+) - g(t) > 0\}$$

and

$$C_g = \{t \in [0,1] : g \text{ is constant on } (t-\varepsilon, t+\varepsilon) \text{ for some } \varepsilon > 0\}$$

along with

$$N_g = \{u_n, v_n : n \in \mathbb{N}\} \setminus D_g,$$

where $C_g = \bigcup_{n \in \mathbb{N}} (u_n, v_n)$ with $(u_n, v_n)_n$ pairwise disjoint. Let $N_g^- = \{u_n : n \in \mathbb{N}\} \setminus D_g$ and $N_g^+ = \{v_n : n \in \mathbb{N}\} \setminus D_g$. As $\mu_g(C_g) = \mu_g(N_g) = 0$ (proved in [16]), these two sets are irrelevant in the study of differential equations. Note also that $D_g \cap C_g = \emptyset$.

Let us now recall the notion of differentiability related to Stieltjes integrals introduced in [8] (which extends that in [16] in such a way that the points in C_g are also covered).

Definition 2.1. Let $g : [0,1] \to \mathbb{R}$ be a nondecreasing left-continuous function such that $0 \notin N_g^-$ and $1 \notin C_g \cup N_g^+$. The derivative with respect to g (or the g-derivative) of $f : [0,1] \to \mathbb{R}$ at $\bar{t} \in [0,1]$ is defined by

$$f'_g(\bar{t}) = \lim_{t \to \bar{t}} \frac{f(t) - f(\bar{t})}{g(t) - g(\bar{t})} \quad \text{if } t \notin D_g \cup C_g,$$

$$f'_g(\bar{t}) = \lim_{t \to \bar{t}+} \frac{f(t) - f(\bar{t})}{g(t) - g(\bar{t})} \quad \text{if } t \in D_g,$$

$$f'_g(\bar{t}) = \lim_{t \to v_n+} \frac{f(t) - f(v_n)}{g(t) - g(v_n)} \quad \text{if } t \in (u_n, v_n) \subseteq C_g,$$

if the limits exist. In this case f is said to be g-differentiable at t.

The points of N_g must be approached in the following manner:

$$f'_g(\bar{t}) = \lim_{t \to \bar{t}+} \frac{f(t) - f(\bar{t})}{g(t) - g(\bar{t})} \quad \text{if } t \in N_g^+,$$

$$f'_g(\bar{t}) = \lim_{t \to \bar{t}-} \frac{f(t) - f(\bar{t})}{g(t) - g(\bar{t})} \quad \text{if } t \in N_g^-.$$

Note that if $t \in D_g$, the g-derivative $f'_g(t)$ exists if and only if the right limit $f(t+)$ exists, and in this case

$$f'_g(t) = \frac{f(t+) - f(t)}{g(t+) - g(t)},$$

while if $t \in (u_n, v_n) \subseteq C_g$ the g-derivative $f'_g(t)$ exists if and only if there exists the right g-derivative at v_n.

The g-derivative is very useful when trying to solve many interesting problems where abrupt modifications (corresponding to discontinuity points of g) and stationary times (corresponding to intervals where g is constant) are both part of the state behaviour, such as in [11, 12] or [13].

Connecting Stieltjes integrals and the Stieltjes derivative is one of the main technical issues of the theory; such connections are provided by Fundamental Theorems of Calculus ([16, Theorems 5.4, 6.2, 6.5]).

In order to state them, let us remind the reader (e.g. [16]) that f is called g-absolutely continuous ($f \in \mathcal{AC}_g([0,1])$) if for every $\varepsilon > 0$ there exists $\delta_\varepsilon > 0$ such that

$$\sum_{j=1}^{m} |f(b_j) - f(a_j)| < \varepsilon$$

for any set $\{(a_j, b_j); j = 1, \ldots, m\}$ of disjoint subintervals of $[0,1]$ satisfying

$$\sum_{j=1}^{m} (g(b_j) - g(a_j)) < \delta_\varepsilon.$$

Theorem 2.2.

a) ([16, Theorem 2.4]) Let $f : [0,1] \to \mathbb{R}$ be LS-integrable with respect to g. Then

$$F(t) = \int_{[0,t)} f(s)\, dg(s), \quad t \in [0,1]$$

defines a map g-absolutely continuous and μ_g-a.e. g-differentiable on $[0,1)$ with the property that $F'_g(t) = f(t)$, μ_g-a.e.

b) ([16, Theorem 5.4], see also [11, Theorem 5.1]) If $F : [0,1] \to \mathbb{R}$ is g-absolutely continuous, then F'_g exists μ_g-a.e. and

$$F(t) = F(0) + \int_{[0,t)} F'_g(s)\, dg(s) \quad \textit{for every } t \in [0,1].$$

Remark 2.3. It easily follows that if $f \in L^1_g([0,1])$,

$$\tilde{F}(t) = \int_{[t,1)} f(s)\, dg(s), \quad t \in [0,1)$$

defines a map μ_g-a.e. g-differentiable with the property that $\tilde{F}'_g(t) = -f(t)$, μ_g-a.e.

Let us also recall ([11]) that a map $f : [0,1] \to \mathbb{R}$ is g-continuous at a point $t \in [0,1]$ if for every $\varepsilon > 0$ one can find $\delta_{t,\varepsilon} > 0$ such that

$$s \in [0,1],\ |g(t) - g(s)| < \delta_{t,\varepsilon} \ \Rightarrow\ |f(t) - f(s)| < \varepsilon$$

and that g-continuity on $[0,1]$ means g-continuity at every $t \in [0,1]$.

Any g-absolutely continuous function is g-continuous and it was proved in [16, Proposition 5.3] that g-absolutely continuous functions are left-continuous and constant on the intervals where g is constant.

Note that g-continuous functions are not necessarily bounded, this is the reason to consider the space $\mathcal{BC}_g([0,1])$ of functions which are bounded and g-continuous ([11]). It is a Banach space when endowed with the norm

$$\|u\|_C = \sup_{t \in [0,1]} |u(t)|$$

and $\mathcal{AC}_g([0,1]) \subset \mathcal{BC}_g([0,1])$.

We add to Theorem 2.2 the result below.

Lemma 2.4. *Let $F : [0,1] \to \mathbb{R}$ be g-absolutely continuous such that there exists $f \in \mathcal{BC}_g([0,1])$ satisfying*

$$F'_g(t) = f(t), \quad \mu_g\text{-a.e.}$$

Then F is g-differentiable everywhere and $F'_g t) = f(t)$ for every $t \in [0,1]$.

Proof. By Theorem 2.2.b), for every $t \in [0,1]$,

$$F(t) = F(0) + \int_{[0,t)} F'_g(t) dg(t)$$

whence, by hypothesis,

$$F(t) = F(0) + \int_{[0,t)} f(t) dg(t), \quad \text{for every } t \in [0,1].$$

Applying [8, Lemma 3.14] we obtain that F is g-differentiable everywhere and $F'_g t) = f(t)$ for every $t \in [0,1]$. □

The following consequence of [8, Proposition 3.9] (see [9, Remark 2.11]) provides a useful rule for the g-derivative of a product.

Lemma 2.5. *Let $f_1, f_2 : [0,1] \to \mathbb{R}$ be g-continuous and g-differentiable at the point $t \in [0,1]$. Then*

$$(f_1 f_2)'_g(t) = (f_1)'_g(t) f_2(t) + f_1(t)(f_2)'_g(t) + (f_1)'_g(t)(f_2)'_g(t)\Delta g(t^*),$$

where $\Delta g(t^*) = g(t^*+) - g(t^*)$ is the jump of g at the point

$$t^* = \begin{cases} t, & \text{if } t \notin C_g \\ v_n, & \text{if } t \in (u_n, v_n) \subset C_g. \end{cases}$$

The recent works [8, 9] opened the way to study higher order differential equations with Stieltjes derivative. The space $\mathcal{BC}^1_g([0,1])$ was defined as the space of functions $f : [0,1] \to \mathbb{R}$ such that f is g-differentiable everywhere on $[0,1]$ and its g-derivative f'_g is bounded and g-continuous. Endowed with the norm

$$\|f\|_{\mathcal{BC}^1_g([0,1])} = \|f\|_C + \|f'_g\|_C$$

it is, by [8, Theorem 3.15], a Banach space.

The matter of compactness in $\mathcal{BC}_g([0,1])$ was addressed in [11].

Theorem 2.6 ([11, Proposition 5.6])**.** *Let $\mathcal{S} \subset \mathcal{AC}_g([0,1])$ be such that $\{u(0) : u \in \mathcal{S}\}$ is bounded and there exists $\phi \in L^1_g([0,1])$satisfying*

$$|u'_g(t)| \le \phi(t), \quad \mu_g\text{-a.e. in } (0,1] \text{ and for all } u \in \mathcal{S}.$$

Then $\mathcal{S}$ is relatively compact in $\mathcal{BC}_g([0,1])$.

Besides,

Lemma 2.7 ([15, Lemma 3.8])**.** *Let $(u_k)_k \subset \mathcal{AC}_g([0,1])$ be pointwise convergent to $u : [0,1] \to \mathbb{R}$. If there exists $\phi \in L^1_g([0,1])$ such that for all $k \in \mathbb{N}$,*

$$|(u_k)'_g(t)| \le \phi(t), \quad \mu_g\text{-a.e. in } (0,1],$$

then u is also g-absolutely continuous.

3 Main results

3.1 Second order linear Stieltjes differential equations with a single derivator

The aim of this subsection is to study the linear single-valued setting, namely the problem

$$\begin{cases} u''_g(t) = f(t), \ \mu_g\text{-a.e. } t \in [0,1) \\ u(0) = \alpha \\ u(1) + ku(\theta) = \beta \end{cases} \tag{3.1}$$

where $g : [0,1] \to \mathbb{R}$ is a nondecreasing left-continuous function, $k \in \mathbb{R}$ and $\theta \in (0,1)$ satisfy $g(1) + kg(\theta) \neq 0$ and $\alpha, \beta \in \mathbb{R}$.

Definition 3.1. A function $u \in \mathcal{BC}_g^1([0,1))$ is a solution of (3.1) if it verifies the boundary conditions $u(0) = \alpha$ and $u(1) + ku(\theta) = \beta$, its first g-derivative u'_g is g-absolutely continuous on $[0,1)$ and its second g-derivative (which is well defined μ_g-a.e. by Theorem 2.2) satisfies

$$u''_g(t) = f(t), \quad \mu_g\text{-a.e. in } [0,1).$$

The existence result given below involves a Green-type function appropriate for the present framework, inspired by [33].

Theorem 3.2. *Let $f : [0,1] \to \mathbb{R}$ be LS-integrable with respect to the nondecreasing left-continuous function $g : [0,1] \to \mathbb{R}$. Then $u_f : [0,1] \to \mathbb{R}$,*

$$\begin{aligned} u_f(t) = &\int_{[0,1)} G(t,s)f(s)dg(s) - \chi_{(\theta,1]}(t) \cdot A(g(t) - g(\theta)) \\ &+ \frac{g(1) - g(t) + k(g(\theta) - g(t))}{g(1) + kg(\theta)}\alpha + \frac{g(t)}{g(1) + kg(\theta)}\tilde{\beta} \end{aligned}$$

is a solution of (3.1).

Here $G : [0,1] \times [0,1] \to \mathbb{R}$ is defined as follows: if $t \le \theta$,

$$G(t,s) = \begin{cases} \frac{-g(s)[g(1)-g(t)+k(g(\theta)-g(t))]}{g(1)+kg(\theta)} - \Delta g(s^*), & \text{if } 0 \le s < t \\ \frac{-g(t)[g(1)-g(s)+k(g(\theta)-g(s))]}{g(1)+kg(\theta)}, & \text{if } t \le s < \theta \\ \frac{-g(t)(g(1)-g(s))}{g(1)+kg(\theta)}, & \text{if } \theta \le s \le 1, \end{cases}$$

while if $t > \theta$,

$$G(t,s) = \begin{cases} \frac{-g(s)[g(1)-g(t)+k(g(\theta)-g(t))]}{g(1)+kg(\theta)} - \Delta g(s^*), & \text{if } 0 \le s < \theta \\ \frac{-g(s)(g(1)-g(t))+kg(\theta)(g(t)-g(s))}{g(1)+kg(\theta)} - \Delta g(s^*) & \text{if } \theta \le s < t \\ \frac{-g(t)(g(1)-g(s))}{g(1)+kg(\theta)}, & \text{if } t \le s \le 1. \end{cases}$$

Besides,

$$\chi_{(\theta,1]}(t) = \begin{cases} 0, & \text{if } 0 \le t \le \theta \\ 1, & \text{if } \theta < t \le 1, \end{cases}$$

$$A = \frac{-k}{g(1) + kg(\theta)} \int_{[0,\theta)} f(s)(g(s) + 1)dg(s)$$

and

$$\tilde{\beta} = \beta + \left(\int_{[0,1)} \Delta g(s^*)f(s)dg(s) + k \int_{[0,\theta)} \Delta g(s^*)f(s)dg(s) + A(g(1) - g(\theta)) \right).$$

Proof. Let us first note that if $t \le \theta$,

$$\begin{aligned} u_f(t) = &\int_{[0,t)} -g(s)f(s)dg(s) + \frac{g(t)}{g(1) + kg(\theta)} \int_{[0,t)} g(s)(1+k)f(s)dg(s) - \int_{[0,t)} \Delta g(s^*)f(s)dg(s) \\ &- \frac{g(t)}{g(1) + kg(\theta)} \int_{[t,\theta)} [g(1) - g(s) + k(g(\theta) - g(s))]f(s)dg(s) \\ &- \frac{g(t)}{g(1) + kg(\theta)} \int_{[\theta,1)} (g(1) - g(s))f(s)dg(s) \\ &+ \frac{g(1) - g(t) + k(g(\theta) - g(t))}{g(1) + kg(\theta)}\alpha + \frac{g(t)}{g(1) + kg(\theta)}\tilde{\beta}. \end{aligned}$$

Then, using Theorem 2.2.a) and Lemma 2.5, u_f is g-absolutely continuous on $[0,\theta]$ and μ_g-a.e. on $[0,\theta)$,

$$\begin{aligned}(u_f)'_g(t) = & -g(t)f(t) + \frac{1+k}{g(1)+kg(\theta)}\int_{[0,t)} g(s)f(s)dg(s) + \frac{g^2(t)}{g(1)+kg(\theta)}(1+k)f(t) \\ & + \frac{1}{g(1)+kg(\theta)}g(t)(1+k)f(t)\Delta g(t^*) - \Delta g(t^*)f(t) \\ & - \frac{1}{g(1)+kg(\theta)}\int_{[t,\theta)}[g(1)-g(s)+k(g(\theta)-g(s))]f(s)dg(s) \\ & + \frac{g(t)}{g(1)+kg(\theta)}[g(1)-g(t)+k(g(\theta)-g(t))]f(t) \\ & + \frac{g(1)-g(t)+k(g(\theta)-g(t))}{g(1)+kg(\theta)}f(t)\Delta g(t^*) \\ & - \frac{1}{g(1)+kg(\theta)}\int_{[\theta,1)}(g(1)-g(s))f(s)dg(s) - \frac{(1+k)\alpha-\tilde{\beta}}{g(1)+kg(\theta)}.\end{aligned}$$

Using Lemma 2.4 we get that for every $t\in[0,\theta)$,

$$\begin{aligned}(u_f)'_g(t) = & \frac{1+k}{g(1)+kg(\theta)}\int_{[0,t)} g(s)f(s)dg(s) \\ & - \frac{1}{g(1)+kg(\theta)}\int_{[t,\theta)}[g(1)-g(s)+k(g(\theta)-g(s)]f(s)dg(s) \\ & - \frac{1}{g(1)+kg(\theta)}\int_{[\theta,1)}(g(1)-g(s))f(s)dg(s) \\ & - \frac{(1+k)\alpha-\tilde{\beta}}{g(1)+kg(\theta)}.\end{aligned} \tag{3.2}$$

It is g-absolutely continuous on $[0,\theta]$ and for μ_g-almost every $t\in[0,\theta)$,

$$(u_f)''_g(t) = \frac{1+k}{g(1)+kg(\theta)}g(t)f(t) + \frac{g(1)-g(t)+kg(\theta)-kg(t)}{g(1)+kg(\theta)}f(t) = f(t).$$

Also, for $t>\theta$,

$$\begin{aligned}u_f(t) = & \int_{[0,\theta)} -g(s)f(s)dg(s) + \frac{g(t)}{g(1)+kg(\theta)}\int_{[0,\theta)}(g(s)-k)f(s)dg(s) \\ & - \int_{[0,\theta)}\Delta g(s^*)f(s)dg(s) + \int_{[\theta,t)} -g(s)f(s)dg(s) \\ & + \frac{g(t)}{g(1)+kg(\theta)}\int_{[\theta,t)}(g(s)+kg(\theta))f(s)dg(s) - \int_{[\theta,t)}\Delta g(s^*)f(s)dg(s) \\ & - \frac{g(t)}{g(1)+kg(\theta)}\int_{[t,1)}(g(1)-g(s))f(s)dg(s) \\ & - A(g(t)-g(\theta)) + \frac{g(1)-g(t)+k(g(\theta)-g(t))}{g(1)+kg(\theta)}\alpha + \frac{g(t)}{g(1)+kg(\theta)}\tilde{\beta}.\end{aligned}$$

Then u_f is g-absolutely continuous on $(\theta,1)$ and μ_g-a.e. on $(\theta,1)$,

$$\begin{aligned}(u_f)'_g(t) = & \frac{1}{g(1)+kg(\theta)}\int_{[0,\theta)}(g(s)-k)f(s)dg(s) - g(t)f(t)\\ & + \frac{1}{g(1)+kg(\theta)}\int_{[\theta,t)}(g(s)+kg(\theta))f(s)dg(s)\\ & + \frac{g(t)}{g(1)+kg(\theta)}(g(t)+kg(\theta))f(t) + \frac{(g(t)+kg(\theta))f(t)}{g(1)+kg(\theta)}\Delta g(t^*) - \Delta g(t^*)f(t)\\ & - \frac{1}{g(1)+kg(\theta)}\int_{[t,1)}(g(1)-g(s))f(s)dg(s) + \frac{g(t)}{g(1)+kg(\theta)}(g(1)-g(t))f(t)\\ & + \frac{(g(1)-g(t))f(t)}{g(1)+kg(\theta)}\Delta g(t^*) - A - \frac{(1+k)\alpha-\tilde{\beta}}{g(1)+kg(\theta)}.\end{aligned}$$

Again by Lemma 2.4 we infer that u_f is g-differentiable on $(\theta,1)$ and that

$$\begin{aligned}(u_f)'_g(t) = & \frac{1}{g(1)+kg(\theta)}\int_{[0,\theta)}(g(s)-k)f(s)dg(s)\\ & + \frac{1}{g(1)+kg(\theta)}\int_{[\theta,t)}(g(s)+kg(\theta))f(s)dg(s)\\ & - \frac{1}{g(1)+kg(\theta)}\int_{[t,1)}(g(1)-g(s))f(s)dg(s) - A - \frac{(1+k)\alpha-\tilde{\beta}}{g(1)+kg(\theta)}.\end{aligned} \tag{3.3}$$

Consequently, $(u_f)'_g$ is g-absolutely continuous on $(\theta,1)$ and for μ_g-a.e. $t\in(\theta,1)$,

$$(u_f)''_g(t) = \frac{1}{g(1)+kg(\theta)}(g(t)+kg(\theta))f(t) + \frac{1}{g(1)+kg(\theta)}(g(1)-g(t))f(t) = f(t).$$

On the other hand, if $t=\theta\in D_g$, then using (3.2), (3.3) we can see that

$$\begin{aligned}(u_f)''_g(\theta) &= \frac{(u_f)'_g(\theta+)-(u_f)'_g(\theta-)}{\Delta g(\theta)}\\ &= \frac{1}{\Delta g(\theta)}\left(f(\theta)\Delta g(\theta) - \frac{k}{g(1)+kg(\theta)}\cdot\int_{[0,\theta)}f(s)(1+g(s))dg(s) - A\right)\\ &= f(\theta).\end{aligned}$$

Since in the case where $\theta\notin D_g$, $\mu_g(\{\theta\})=0$, we can conclude that $(u_f)''_g(t)=f(t)$ for μ_g-a.e. $t\in[0,1)$.

Besides, as u_f and $(u_f)'_g$ are both continuous at θ if $\theta\notin D_g$, it follows that $u\in\mathcal{BC}^1_g([0,1])$ and $(u_f)'_g$ is g-absolutely continuous on $[0,1]$.

As for the boundary conditions, it can be easily checked that

$$u_f(0) = \int_{[0,1)} G(0,s)f(s)dg(s) + \alpha = \alpha$$

and

$$
\begin{aligned}
u_f(1)+ku_f(\theta) &= \int_{[0,1)} G(1,s)f(s)dg(s) - A(g(1)-g(\theta)) + \frac{k(g(\theta)-g(1))}{g(1)+kg(\theta)}\alpha + \frac{g(1)\tilde{\beta}}{g(1)+kg(\theta)} \\
&\quad + k\int_{[0,1)} G(\theta,s)f(s)dg(s) + k\frac{g(1)-g(\theta)}{g(1)+kg(\theta)}\alpha + k\frac{g(\theta)\tilde{\beta}}{g(1)+kg(\theta)} \\
&= \tilde{\beta} + \int_{[0,\theta)} \frac{-kg(s)(g(\theta)-g(1)))}{g(1)+kg(\theta)} f(s)dg(s) - \int_{[0,\theta)} \Delta g(s^*)f(s)dg(s) \\
&\quad + \int_{[\theta,1)} \frac{kg(\theta)(g(1)-g(s))}{g(1)+kg(\theta)} f(s)dg(s) \\
&\quad - \int_{[\theta,1)} \Delta g(s^*)f(s)dg(s) - A(g(1)-g(\theta)) \\
&\quad + k\int_{[0,\theta)} \frac{-g(s)(g(1)-g(\theta))}{g(1)+kg(\theta)} f(s)dg(s) \\
&\quad - k\int_{[0,\theta)} \Delta g(s^*)f(s)dg(s) + k\int_{[\theta,1)} \frac{-g(\theta)(g(1)-g(s))}{g(1)+kg(\theta)} f(s)dg(s) = \beta.
\end{aligned}
$$

□

Corollary 3.3. *A map $u : [0,1] \to \mathbb{R}$ is a solution of (3.1) if and only if*

$$
\begin{aligned}
u(t) = &\int_{[0,1)} G(t,s)f(s)dg(s) - \chi_{(\theta,1]}(t)\cdot A(g(t)-g(\theta)) \\
&+ \frac{g(1)-g(t)+k(g(\theta)-g(t))}{g(1)+kg(\theta)}\alpha + \frac{g(t)}{g(1)+kg(\theta)}\tilde{\beta}
\end{aligned}
$$

Proof. One implication is proved in Theorem 3.2; as for the other one, suppose u_1, u_2 are two solutions of the considered problem. Then

$$
\begin{cases}
(u_1-u_2)''_g(t) = 0, \ \mu_g\text{-a.e. } t \in [0,1) \\
(u_1-u_2)(0) = 0 \\
(u_1-u_2)(1) + k(u_1-u_2)(\theta) = 0.
\end{cases}
\tag{3.4}
$$

Using the g-absolute continuity of $(u_1-u_2)'_g$ one infers that for every $t \in [0,1]$,

$$
(u_1-u_2)'_g(t) = (u_1-u_2)'_g(0) + \int_{[0,t)} (u_1-u_2)''_g(s)dg(s) = (u_1-u_2)'_g(0)
$$

and so, by the g-absolute continuity of u_1-u_2,

$$
(u_1-u_2)(t) = (u_1-u_2)(0) + \int_{[0,t)} (u_1-u_2)'_g(s)dg(s) = (u_1-u_2)(0) + (u_1-u_2)'_g(0)\cdot g(t).
$$

Now the boundary conditions lead us to

$$
(u_1-u_2)(0) = 0
$$

respectively

$$
(u_1-u_2)'_g(0)\cdot(g(1)+kg(\theta)) = 0.
$$

It follows that $(u_1-u_2)'_g(0) = 0$ and, consequently, $u_1(t) = u_2(t)$ for every $t \in [0,1]$. □

Remark 3.4. Taking into account that, by Lemma 2.5, $s = s^*$ μ_g-almost everywhere, in Theorem 3.2 the following expression could have been used instead for $G(t,s)$:
if $t \leq \theta$,

$$G(t,s) = \begin{cases} \frac{-g(s)[g(1)-g(t)+k(g(\theta)-g(t))]}{g(1)+kg(\theta)} - \Delta g(s), & \text{if } 0 \leq s < t \\ \frac{-g(t)[g(1)-g(s)+k(g(\theta)-g(s))]}{g(1)+kg(\theta)}, & \text{if } t \leq s < \theta \\ \frac{-g(t)(g(1)-g(s))}{g(1)+kg(\theta)}, & \text{if } \theta \leq s \leq 1, \end{cases}$$

while if $t > \theta$,

$$G(t,s) = \begin{cases} \frac{-g(s)[g(1)-g(t)+k(g(\theta)-g(t))]}{g(1)+kg(\theta)} - \Delta g(s), & \text{if } 0 \leq s < \theta \\ \frac{-g(s)(g(1)-g(t))+kg(\theta)(g(t)-g(s))}{g(1)+kg(\theta)} - \Delta g(s), & \text{if } \theta \leq s < t \\ \frac{-g(t)(g(1)-g(s))}{g(1)+kg(\theta)}, & \text{if } t \leq s \leq 1. \end{cases}$$

3.2 Second order linear Stieltjes differential systems with several derivators

As described in [12, 19] or [21], differential systems with several derivators naturally occur when investigating physical processes; more precisely, it leads to studying the existence (and uniqueness) of $u : [0,1] \to \mathbb{R}^d$ such that

$$\begin{cases} u''_{\overline{g}}(t) = \overline{f}(t), \ \mu_{\overline{g}}\text{-a.e. } t \in [0,1) \\ u(0) = \overline{\alpha} \\ u(1) + ku(\theta) = \overline{\beta} \end{cases} \tag{3.5}$$

where $\overline{g} = (g_1, \dots, g_d) : [0,1] \to \mathbb{R}^d$ with g_i left-continuous and nondecreasing for each $i \in \{1, \dots, d\}$, $\overline{f} : [0,1] \to \mathbb{R}^d$, $\overline{\alpha} = (\alpha_1, \dots, \alpha_d), \overline{\beta} = (\beta_1, \dots, \beta_d) \in \mathbb{R}^d$ and $\theta \in (0,1)$ satisfy the assumption: $g_i(1) + kg_i(\theta) \neq 0$ for each $i \in \{1, \dots, d\}$.

The $\overline{g}$-derivative is to be understood as the vector $((u_1)'_{g_1}, \dots, (u_d)'_{g_d})$, thus the system (3.5) must be understood as:

$$\begin{cases} (u_i)''_{g_i}(t) = f_i(t), \ \mu_{g_i}\text{-a.e. } t \in [0,1) \\ u_i(0) = \alpha_i \\ u_i(1) + ku_i(\theta) = \beta_i \end{cases} \tag{3.6}$$

for every $i = 1, \dots, d$.

Denote (as in [19, page 6]) by

$$\mathcal{BC}_{\overline{g}}([0,1], \mathbb{R}^d) = \prod_{i=1}^{d} \mathcal{BC}_{g_i}([0,1]),$$

by

$$\mathcal{BC}^1_{\overline{g}}([0,1], \mathbb{R}^d) = \prod_{i=1}^{d} \mathcal{BC}^1_{g_i}([0,1]),$$

respectively by

$$\mathcal{AC}_{\overline{g}}([0,1], \mathbb{R}^d) = \prod_{i=1}^{d} \mathcal{AC}_{g_i}([0,1]).$$

Definition 3.5. A function $u \in \mathcal{BC}^1_{\overline{g}}([0,1],\mathbb{R}^d)$ is a solution of (3.5) if for each i it verifies the boundary conditions $u_i(0) = \alpha_i$ and $u_i(1) + ku_i(\theta) = \beta_i$, the first g_i-derivative $(u_i)'_{g_i}$ is g_i-absolutely continuous and its g_i-derivative (which is well defined μ_{g_i}-a.e.) satisfies

$$(u_i)''_{g_i}(t) = f_i(t), \quad \mu_{g_i}\text{-a.e. in } [0,1).$$

Reasoning componentwise, Theorem 3.2 and Corollary 3.3 involve the following result.

Theorem 3.6. *Let $\overline{g} = (g_1,\dots,g_d) : [0,1] \to \mathbb{R}^d$ where g_i is left-continuous and nondecreasing for each $i \in \{1,\dots,d\}$ and let $\overline{f} \in L^1_{\overline{g}}([0,1],\mathbb{R}^d) = \prod_{i=1}^d L^1_{g_i}([0,1])$. Then $u : [0,1] \to \mathbb{R}^d$ is a solution of (3.5) if and only if for every $i \in \{1,\dots,d\}$:*

$$\begin{aligned} u_i(t) = &\int_{[0,1)} G_i(t,s) f_i(s)dg(s) - \chi_{(\theta,1]}(t)\cdot A_i(g_i(t) - g_i(\theta)) \\ &+ \frac{g_i(1) - g_i(t) + k(g_i(\theta) - g_i(t))}{g_i(1) + kg_i(\theta)}\alpha_i + \frac{g_i(t)}{g_i(1) + kg_i(\theta)}\tilde{\beta}_i, \quad \forall\, t \in [0,1], \end{aligned}$$

where $G_i : [0,1] \times [0,1] \to \mathbb{R}$ is defined as follows: if $t \le \theta$,

$$G_i(t,s) = \begin{cases} \frac{-g_i(s)[g_i(1)-g_i(t)+k(g_i(\theta)-g_i(t))]}{g_i(1)+kg_i(\theta)} - \Delta g_i(s_i^*), & \text{if } 0 \le s < t \\ \frac{-g_i(t)[g_i(1)-g_i(s)+k(g_i(\theta)-g_i(s))]}{g_i(1)+kg_i(\theta)}, & \text{if } t \le s < \theta \\ \frac{-g_i(t)(g_i(1)-g_i(s))}{g_i(1)+kg_i(\theta)}, & \text{if } \theta \le s \le 1, \end{cases}$$

while if $t > \theta$,

$$G_i(t,s) = \begin{cases} \frac{-g_i(s)[g_i(1)-g_i(t)+k(g_i(\theta)-g_i(t))]}{g_i(1)+kg_i(\theta)} - \Delta g_i(s_i^*), & \text{if } 0 \le s < \theta \\ \frac{-g_i(s)(g_i(1)-g_i(t))+kg_i(\theta)(g_i(t)-g_i(s))}{g_i(1)+kg_i(\theta)} - \Delta g_i(s_i^*), & \text{if } \theta \le s < t \\ \frac{-g_i(t)(g_i(1)-g_i(s))}{g_i(1)+kg_i(\theta)}, & \text{if } t \le s \le 1, \end{cases}$$

$$A_i = \frac{-k}{g_i(1) + kg_i(\theta)} \int_{[0,\theta)} f_i(s)(g_i(s) + 1)dg_i(s)$$

and

$$\tilde{\beta}_i = \beta_i + \left(\int_{[0,1)} \Delta g_i(s_i^*) f_i(s) dg_i(s) + k \int_{[0,\theta)} \Delta g_i(s_i^*) f_i(s) dg_i(s) + A_i(g_i(1) - g_i(\theta)) \right).$$

3.3 Existence result for systems of nonlinear second order boundary value equations

In this subsection, we focus on nonlinear differential problems of second order with boundary value conditions

$$\begin{cases} u''_{\overline{g}}(t) = f(t, u(t), u'_{\overline{g}}(t)), \ \mu_{\overline{g}}\text{-a.e. } t \in [0,1) \\ u(0) = \overline{\alpha} \\ u(1) + ku(\theta) = \overline{\beta}, \end{cases} \tag{3.7}$$

with $\overline{g} = (g_1,\dots,g_d) : [0,1] \to \mathbb{R}^d$ such that g_i is left-continuous and nondecreasing for each $i \in \{1,\dots,d\}$, $\overline{\alpha} = (\alpha_1,\dots,\alpha_d), \overline{\beta} = (\beta_1,\dots,\beta_d) \in \mathbb{R}^d$ and $k \in \mathbb{R}$, $\theta \in (0,1)$ satisfy the assumption: $g_i(1) + kg_i(\theta) \neq 0$ for each $i \in \{1,\dots,d\}$.

Definition 3.7. A function $u \in \mathcal{BC}^1_{\overline{g}}([0,1],\mathbb{R}^d)$ is a solution of (3.7) if $u(0)=\overline{\alpha}$, $u(1)+ku(\theta)=\overline{\beta}$, its first $\overline{g}$-derivative $u'_{\overline{g}} \in \mathcal{AC}_{\overline{g}}([0,1],\mathbb{R}^d)$ and its second $\overline{g}$-derivative satisfies

$$u''_{\overline{g}}(t)=f(t,u(t),u'_{\overline{g}}(t)), \quad \mu_{\overline{g}}\text{-a.e. in } [0,1),$$

i.e. for every $i \in \{1,\dots,d\}$:

$$(u_i)''_{g_i}(t)=f_i(t,u(t),u'_{\overline{g}}(t)), \quad \mu_{g_i}\text{-a.e. in } [0,1).$$

Theorem 3.8. *Let $f:[0,1]\times\mathbb{R}^d\times\mathbb{R}^d\to\mathbb{R}^d$ satisfy, for each $i\in\{1,\dots,d\}$, the following conditions:*

(1) *there exists $M_i \in L^1_{g_i}([0,1])$ such that*

$$|f_i(t,x,y)| \le M_i(t), \quad \mu_{g_i}\text{-a.e. } t\in[0,1],\ \forall\, x,y\in\mathbb{R}^d;$$

(2) *$f_i(\cdot,x,y)$ is g_i-measurable, for every $x,y\in\mathbb{R}^d$;*

(3) *$f_i(t,\cdot,\cdot)$ is continuous on $\mathbb{R}^d\times\mathbb{R}^d$, for μ_{g_i}-almost every $t\in[0,1]$.*

Then the problem (3.7) has solutions.

Proof. Let us prove that the set

$$\mathcal{K}=\left\{u\in\mathcal{BC}^1_{\overline{g}}([0,1],\mathbb{R}^d);|(u_i)''_{g_i}(t)|\le M_i(t),\mu_{g_i}\text{-a.e., } \forall\, i \text{ and } u(0)=\overline{\alpha},\ u(1)+ku(\theta)=\overline{\beta}\right\}$$

is nonempty, convex and compact in $\mathcal{BC}^1_{\overline{g}}([0,1],\mathbb{R}^d)$.

First, note that any map $f\in L^1_{\overline{g}}([0,1],\mathbb{R}^d)$ such that $|f_i(t)|\le M_i(t)$ for all $i\in\{1,\dots,d\}$ provides, thanks to Theorem 3.2, an element of $\mathcal{K}$ (which is, therefore, nonempty) and that the convexity can be easily checked.

Then, let us see that $\mathcal{K}$ is compact in $\mathcal{BC}^1_{\overline{g}}([0,1],\mathbb{R}^d)$.

To this aim, consider a sequence $(u_n)_n\subset\mathcal{K}$. It can be seen from the proof of Theorem 3.2 that for each $i\in\{1,\dots,n\}$, if $t<\theta$,

$$\begin{aligned}((u_n)_i)'_{g_i}(t)=&\ \frac{1+k}{g_i(1)+kg_i(\theta)}\int_{[0,t)}g_i(s)((u_n)_i)''_{g_i}(s)dg_i(s)\\ &-\frac{1}{g_i(1)+kg_i(\theta)}\int_{[t,\theta)}[g_i(1)-g_i(s)+k(g_i(\theta)-g_i(s)]((u_n)_i)''_{g_i}(s)dg_i(s)\\ &-\frac{1}{g_i(1)+kg_i(\theta)}\int_{[\theta,1)}(g_i(1)-g_i(s))((u_n)_i)''_{g_i}(s)dg_i(s)-\frac{(1+k)\alpha_i-\tilde{\beta}_i}{g_i(1)+kg_i(\theta)},\end{aligned}$$

while if $t>\theta$,

$$\begin{aligned}((u_n)_i)'_{g_i}(t)=&\ \frac{1}{g_i(1)+kg_i(\theta)}\int_{[0,\theta)}(g_i(s)-k)((u_n)_i)''_{g_i}(s)dg(s)-g_i(t)((u_n)_i)''_{g_i}(t)\\ &+\frac{1}{g_i(1)+kg_i(\theta)}\int_{[\theta,t)}(g_i(s)+kg_i(\theta))((u_n)_i)''_{g_i}(s)dg_i(s)\\ &+\frac{g_i(t)}{g_i(1)+kg_i(\theta)}(g_i(t)+kg_i(\theta))((u_n)_i)''_{g_i}(t)\\ &-\frac{1}{g_i(1)+kg_i(\theta)}\int_{[t,1)}(g_i(1)-g_i(s))((u_n)_i)''_{g_i}(s)dg_i(s)\\ &+\frac{g_i(t)}{g_i(1)+kg_i(\theta)}(g_i(1)-g_i(t))((u_n)_i)''_{g_i}(t)-A_i-\frac{(1+k)\alpha_i-\tilde{\beta}_i}{g_i(1)+kg_i(\theta)}.\end{aligned}$$

Using Theorem 2.6 on each coordinate, $((u_n)'_{\overline{g}})_n$ is relatively compact in the space $\mathcal{BC}_{\overline{g}}([0,1],\mathbb{R}^d)$. One can choose a subsequence, denoted again by $(u_n)_n$ such that $\left((u_n)'_{\overline{g}}\right)_n$ is uniformly convergent to a function $v \in \mathcal{AC}_{\overline{g}}([0,1],\mathbb{R}^d)$.

In both cases, as $u_n(0) = \overline{\alpha}, \forall\, n$, $(u_n)_n$ satisfies as well the hypotheses of Theorem 2.6, therefore one can choose a further subsequence (not re-labelled) such that $(u_n)_n$ uniformly converges to a function $u \in \mathcal{AC}_{\overline{g}}([0,1],\mathbb{R}^d)$.

Since for every $i \in \{1,\ldots,d\}$, the sequence $\left(((u_n)_i)'_{g_i}\right)_n$ uniformly converges to v, we infer that for each $t \in [0,1]$:

$$\int_{[0,t)} ((u_n)_i)'_{g_i}(s)dg_i(s) \to \int_{[0,t)} v(s)dg_i(s)$$

and remark that, by Theorem 2.2,

$$\int_{[0,t)} ((u_n)_i)'_{g_i}(s)dg_i(s) = (u_n)_i(t) - \alpha_i \to u_i(t) - \alpha_i;$$

it follows that

$$v(t) = u'_{\overline{g}}(t).$$

Moreover, note that $((u_n)''_{\overline{g}})_n$ is, by hypothesis, relatively weakly compact in $L^1_{\overline{g}}([0,1],\mathbb{R}^d)$, so on convex combinations we may suppose that $((u_n)''_{\overline{g}})_n$ converges pointwise to a function $w \in L^1_{\overline{g}}([0,1],\mathbb{R}^d)$. Applying a dominated convergence theorem one gets for each $i \in \{1,\ldots,d\}$, if $t < \theta$,

$$\begin{aligned}((u_n)_i)'_{g_i}(t) \to \frac{1+k}{g_i(1)+kg_i(\theta)} &\int_{[0,t)} g_i(s)w_i(s)dg_i(s)\\ &- \frac{1}{g_i(1)+kg_i(\theta)}\int_{[t,\theta)}[g_i(1)-g_i(s)+k(g_i(\theta)-g_i(s)]w_i(s)dg_i(s)\\ &- \frac{1}{g_i(1)+kg_i(\theta)}\int_{[\theta,1)}(g_i(1)-g_i(s))w_i(s)dg_i(s) - \frac{(1+k)\alpha_i-\tilde{\beta}_i}{g_i(1)+kg_i(\theta)}.\end{aligned}$$

while if $t > \theta$,

$$\begin{aligned}((u_n)_i)'_{g_i}(t) \to &\frac{1}{g_i(1)+kg_i(\theta)}\int_{[0,\theta)}(g_i(s)-k)w_i(s)dg(s) - g_i(t)w_i(t)\\ &+ \frac{1}{g_i(1)+kg_i(\theta)}\int_{[\theta,t)}(g_i(s)+kg_i(\theta))w_i(s)dg_i(s)\\ &+ \frac{g_i(t)}{g_i(1)+kg_i(\theta)}(g_i(t)+kg_i(\theta))w_i(t)\\ &- \frac{1}{g_i(1)+kg_i(\theta)}\int_{[t,1)}(g_i(1)-g_i(s))w_i(s)dg_i(s)\\ &+ \frac{g_i(t)}{g_i(1)+kg_i(\theta)}(g_i(1)-g_i(t))w_i(t) - A_i - \frac{(1+k)\alpha_i-\tilde{\beta}_i}{g_i(1)+kg_i(\theta)}.\end{aligned}$$

On the other hand, it was seen that for each $i \in \{1,\ldots,d\}$, $\left(((u_n)_i)'_{g_i}\right)_n$ uniformly converges to $(u_i)'_{g_i}$ which is equal to

$$\begin{aligned}\frac{1+k}{g_i(1)+kg_i(\theta)}&\int_{[0,t)} g_i(s)(u_i)''_{g_i}(s)dg_i(s)\\ &- \frac{1}{g_i(1)+kg_i(\theta)}\int_{[t,\theta)}[g_i(1)-g_i(s)+k(g_i(\theta)-g_i(s)](u_i)''_{g_i}(s)dg_i(s)\\ &- \frac{1}{g_i(1)+kg_i(\theta)}\int_{[\theta,1)}(g_i(1)-g_i(s))(u_i)''_{g_i}(s)dg_i(s) - \frac{(1+k)\alpha_i-\tilde{\beta}_i}{g_i(1)+kg_i(\theta)}.\end{aligned}$$

if $t < \theta$, respectively to

$$\frac{1}{g_i(1)+kg_i(\theta)}\int_{[0,\theta)}(g_i(s)-k)(u_i)''_{g_i}(s)dg(s)-g_i(t)(u_i)''_{g_i}(t)$$
$$+\frac{1}{g_i(1)+kg_i(\theta)}\int_{[\theta,t)}(g_i(s)+kg_i(\theta))(u_i)''_{g_i}(s)dg_i(s)$$
$$+\frac{g_i(t)}{g_i(1)+kg_i(\theta)}(g_i(t)+kg_i(\theta))(u_i)''_{g_i}(t)$$
$$-\frac{1}{g_i(1)+kg_i(\theta)}\int_{[t,1)}(g_i(1)-g_i(s))(u_i)''_{g_i}(s)dg_i(s)$$
$$+\frac{g_i(t)}{g_i(1)+kg_i(\theta)}(g_i(1)-g_i(t))(u_i)''_{g_i}(t)-A_i-\frac{(1+k)\alpha_i-\tilde{\beta}_i}{g_i(1)+kg_i(\theta)}$$

if $t > \theta$.

Consequently,

$$w(t)=u''_{\overline{g}}(t),\quad \mu_{\overline{g}}\text{-almost everywhere on } [0,1).$$

Therefore, $|(u_i)''_{g_i}(t)| \le M_i(t), \mu_{g_i}$-a.e., $\forall\, i$ and so, $u \in \mathcal{K}$. The compactness of $\mathcal{K}$ is thus proved.

Let us now check that the operator $\Xi : \mathcal{K} \to \mathcal{K}$,

$$\Xi(u)=v\in\mathcal{K}\quad\text{such that } v''_{\overline{g}}(t)=f(t,u(t),u'_{\overline{g}}(t)),\ \mu_{\overline{g}}\text{-a.e.}$$

satisfies the conditions of Schauder's fixed point theorem.

At the beginning, note that Ξ is well-defined by Theorem 3.2 and Corollary 3.3, since as in [11, Lemma 7.2] it can be proved that for any $u \in \mathcal{K}$, $f(\cdot, u(\cdot), u'_{\overline{g}}(\cdot)) \in L^1_{\overline{g}}([0,1], \mathbb{R}^d)$.

All we have to check now is that Ξ is continuous.

Let $(u_n)_n \subset \mathcal{K}$ converge to $u \in \mathcal{K}$ and we aim to prove that $(\Xi(u_n))_n = (v_n)_n$ converges to $\Xi(u) = v$.

As $(u_n)_n$ uniformly converges to u and $\left((u_n)'_{\overline{g}}\right)_n$ uniformly converges to $u'_{\overline{g}}$, hypothesis (3) involves that for each $i \in \{1, \dots, d\}$,

$$\left(f_i(\cdot,u_n(\cdot),(u_n)'_{\overline{g}}(\cdot))\right)_n\quad\text{converges to } f_i(\cdot,u(\cdot),u'_{\overline{g}}(\cdot)),\ \mu_{g_i}\text{-a.e.}$$

By Theorem 3.2, for every $n \in \mathbb{N}$ and $i \in \{1, \dots, d\}$:

$$(v_n)_i(t)=\int_{[0,1)}G_i(t,s)f_i(s,u_n(s),(u_n)'_{\overline{g}}(s))dg_i(s)-\chi_{(\theta,1]}(t)\cdot A_i(g_i(t)-g_i(\theta))$$
$$+\frac{g_i(1)-g_i(t)+k(g_i(\theta)-g_i(t))}{g_i(1)+kg_i(\theta)}\alpha_i+\frac{g_i(t)}{g_i(1)+kg_i(\theta)}\tilde{\beta}_i$$

and

$$v_i(t)=\int_{[0,1)}G_i(t,s)f_i(s,u(s),u'_{\overline{g}}(s))dg_i(s)-\chi_{(\theta,1]}(t)\cdot A_i(g_i(t)-g_i(\theta))$$
$$+\frac{g_i(1)-g_i(t)+k(g_i(\theta)-g_i(t))}{g_i(1)+kg_i(\theta)}\alpha_i+\frac{g_i(t)}{g_i(1)+kg_i(\theta)}\tilde{\beta}_i$$

for all $t \in [0,1]$.

The hypothesis (1) ensures that the dominated convergence theorem can be applied on each coordinate and so,

$$(v_n)_n \to v\quad\text{uniformly on } [0,1].$$

Likewise, from the proof of Theorem 3.2 it can be seen that for each $i \in \{1, \dots, n\}$, if $t < \theta$,

$$\begin{aligned}((v_n)_i)'_{g_i}(t) &= \frac{1+k}{g_i(1)+kg_i(\theta)} \int_{[0,t)} g_i(s) f_i(s, u_n(s), (u_n)'_g(s)) dg_i(s) \\ &\quad - \frac{1}{g_i(1)+kg_i(\theta)} \int_{[t,\theta)} [g_i(1) - g_i(s) + k(g_i(\theta) - g_i(s)] f_i(s, u_n(s), (u_n)'_g(s)) dg_i(s) \\ &\quad - \frac{1}{g_i(1)+kg_i(\theta)} \int_{[\theta,1)} (g_i(1) - g_i(s)) f_i(s, u_n(s), (u_n)'_g(s)) dg_i(s) - \frac{(1+k)\alpha_i - \tilde{\beta}_i}{g_i(1)+kg_i(\theta)}\end{aligned}$$

while if $t > \theta$,

$$\begin{aligned}&((v_n)_i)'_{g_i}(t) \\ &\quad = \frac{1}{g_i(1)+kg_i(\theta)} \int_{[0,\theta)} (g_i(s) - k) f_i(s, u_n(s), (u_n)'_g(s)) dg(s) - g_i(t) f_i(t, u_n(t), (u_n)'_g(t)) \\ &\quad + \frac{1}{g_i(1)+kg_i(\theta)} \int_{[\theta,t)} (g_i(s) + kg_i(\theta)) f_i(s, u_n(s), (u_n)'_g(s)) dg_i(s) \\ &\quad + \frac{g_i(t)}{g_i(1)+kg_i(\theta)} (g_i(t) + kg_i(\theta)) f_i(t, u_n(t), (u_n)'_g(t)) \\ &\quad - \frac{1}{g_i(1)+kg_i(\theta)} \int_{[t,1)} (g_i(1) - g_i(s)) f_i(s, u_n(s), (u_n)'_g(s)) dg_i(s) \\ &\quad + \frac{g_i(t)}{g_i(1)+kg_i(\theta)} (g_i(1) - g_i(t)) f_i(t, u_n(t), (u_n)'_g(t)) - A_i - \frac{(1+k)\alpha_i - \tilde{\beta}_i}{g_i(1)+kg_i(\theta)}.\end{aligned}$$

Also, if $t < \theta$,

$$\begin{aligned}(v_i)'_{g_i}(t) &= \frac{1+k}{g_i(1)+kg_i(\theta)} \int_{[0,t)} g_i(s) f_i(s, u(s), u'_g(s)) dg_i(s) \\ &\quad - \frac{1}{g_i(1)+kg_i(\theta)} \int_{[t,\theta)} [g_i(1) - g_i(s) + k(g_i(\theta) - g_i(s)] f_i(s, u(s), u'_g(s)) dg_i(s) \\ &\quad - \frac{1}{g_i(1)+kg_i(\theta)} \int_{[\theta,1)} (g_i(1) - g_i(s)) f_i(s, u(s), u'_g(s)) dg_i(s) - \frac{(1+k)\alpha_i - \tilde{\beta}_i}{g_i(1)+kg_i(\theta)}\end{aligned}$$

while if $t > \theta$,

$$\begin{aligned}(v_i)'_{g_i}(t) &= \frac{1}{g_i(1)+kg_i(\theta)} \int_{[0,\theta)} (g_i(s) - k) f_i(s, u(s), u'_g(s)) dg(s) - g_i(t) f_i(t, u(t), u'_g(t)) \\ &\quad + \frac{1}{g_i(1)+kg_i(\theta)} \int_{[\theta,t)} (g_i(s) + kg_i(\theta)) f_i(s, u(s), u'_g(s)) dg_i(s) \\ &\quad + \frac{g_i(t)}{g_i(1)+kg_i(\theta)} (g_i(t) + kg_i(\theta)) f_i(t, u(t), u'_g(t)) \\ &\quad - \frac{1}{g_i(1)+kg_i(\theta)} \int_{[t,1)} (g_i(1) - g_i(s)) f_i(s, u(s), u'_g(s)) dg_i(s) \\ &\quad + \frac{g_i(t)}{g_i(1)+kg_i(\theta)} (g_i(1) - g_i(t)) f_i(t, u(t), u'_g(t)) - A_i - \frac{(1+k)\alpha_i - \tilde{\beta}_i}{g_i(1)+kg_i(\theta)}.\end{aligned}$$

In both cases, again by dominated convergence theorem it follows that $((v_n)'_g)_n$ uniformly converges to v'_g.

So, $\Xi(u_n) = v_n \to \Xi(u) = v$ in $\mathcal{BC}^1_g([0,1], \mathbb{R}^d)$ and the continuity of Ξ is proved.

In conclusion, Ξ has fixed points which are solutions of the considered problem. □

Remark 3.9. To the best of the author's knowledge, this is the first study of systems of second order equations involving Stieltjes derivatives (besides, with several derivators) and three-point conditions on the boundary.

When $g(t) = t$ on $[0,1]$ we cover or complement already known results with various boundary value conditions (e.g. [17,31,33] and their references, see also [2,3,25]). Note finally that the strong connection between the theory of Stieltjes differential equations and other types of differential problems implies that the outcomes presented in this work lead to new results for systems of second-order generalized differential equations, impulsive differential equations (e.g. [7]) and also dynamic problems on time scales (see [1,24]) with three-point boundary conditions.

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